The Shape of Unlabeled Rooted Trees

Bernhard Gittenberger
joint work with **Michael Drmota**

Institute of Discrete Mathematics and Geometry
TU Wien, Austria

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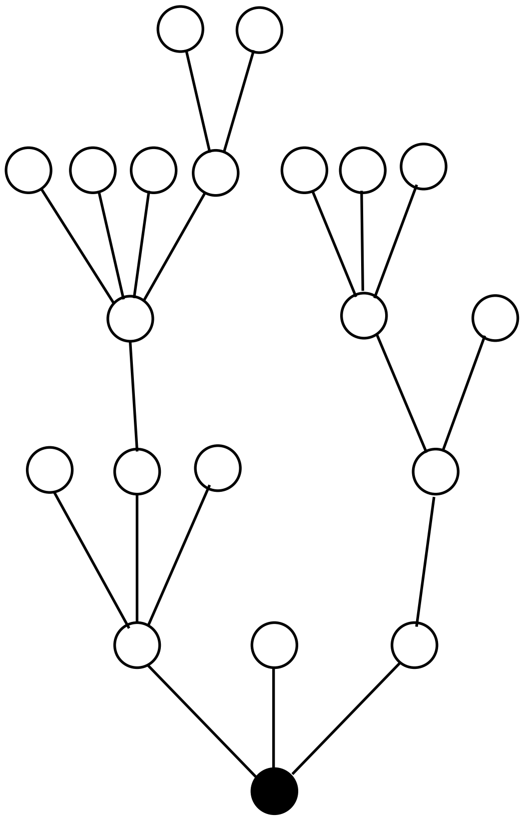
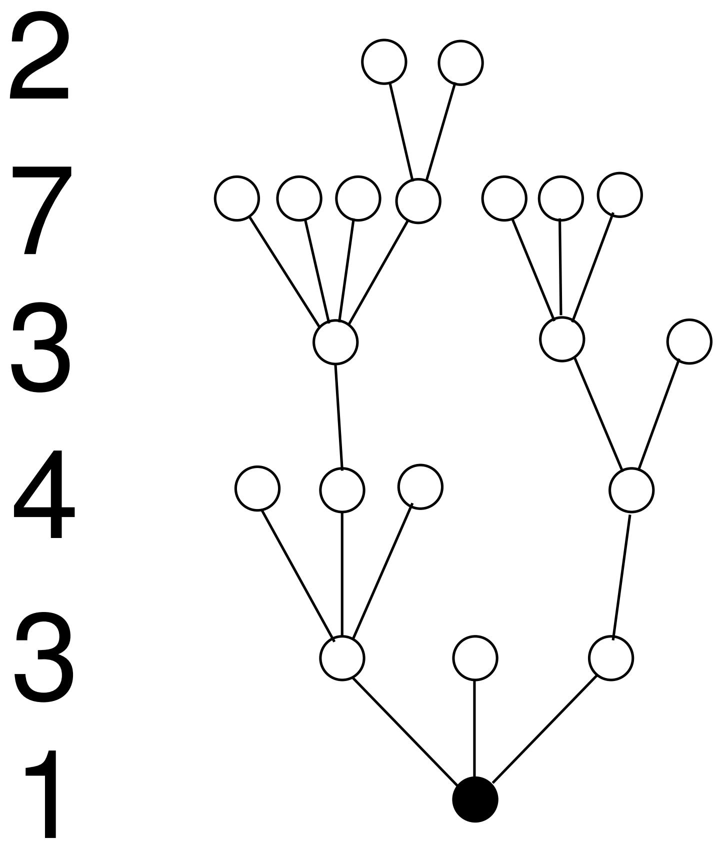
Introduction

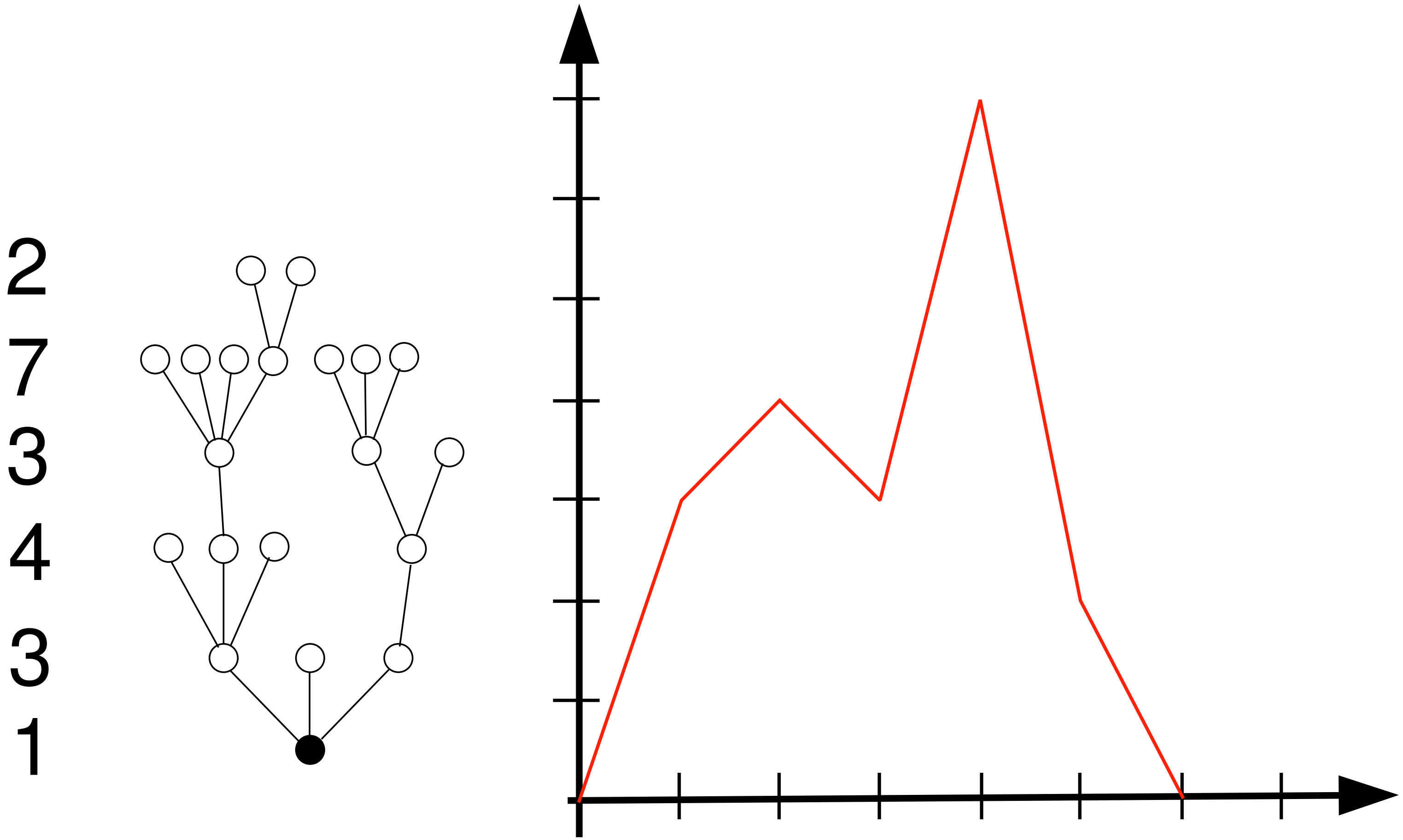


Introduction



Introduction

The profile:



Introduction

Two classes of trees

according to average height

1. $\log n$-trees: binary search trees, digital search trees, recursive trees, increasing trees

2. $\sqrt{n}$-trees: simply generated trees, Pólya trees

Historical Remarks

- Stepanov '69: Cayley trees, limiting distribution of level sizes, density and moments

- Kennedy '75, Kolchin '75, Takacs '91: simply generated trees, limiting density

- Aldous '91: simply generated trees, two conjectures for functional limit theorems, relation to Brownian excursion

- Drmota and G. '97, G. '99, Pitman '99: Proof of Aldous' conjectures

Historical Remarks

Representations of the Brownian excursion local time

- Getoor and Sharpe ’79: double Laplace transform approach directly via Brownian functionals

- Cohen and Hooghiemstra ’82: M/M/1 queues

- Knight ’82: convolution formula

- Louchard ’84: Laplace transform, Kac formula

Historical Remarks

Extensions:

- Pavlov 80ies, 90ies: random forests
- Drmota '96: nodes of fixed degree, functional limit theorems
- G. '02: functional limit theorem for simply generated forests

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Other tree classes:

- Chauvin, Drmota, Jabbour-Hattab 01, Drmota 04, Drmota and Hwang 2005: binary search trees
- Hwang et al. 2006, 2007: recursive trees, increasing trees

Brownian Excursion and Local Time

$W(t)$ ... Brownian motion (Wiener process)

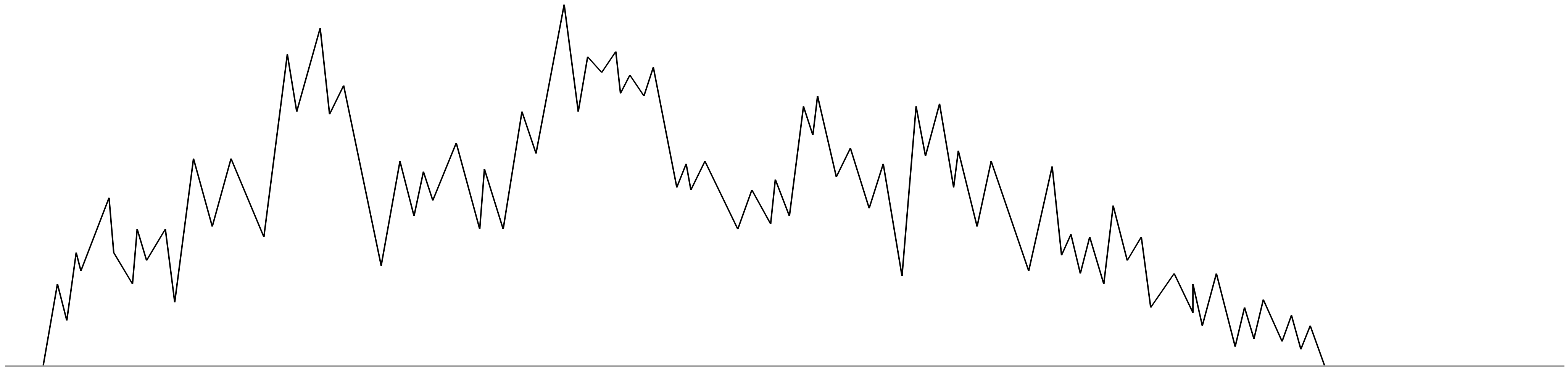
- $P[W(0) = 0] = 1$

- for $0 \le t_0 < t_1 < \cdots < t_k$ the increments $W(t_i) - W(t_{i-1})$ are independent

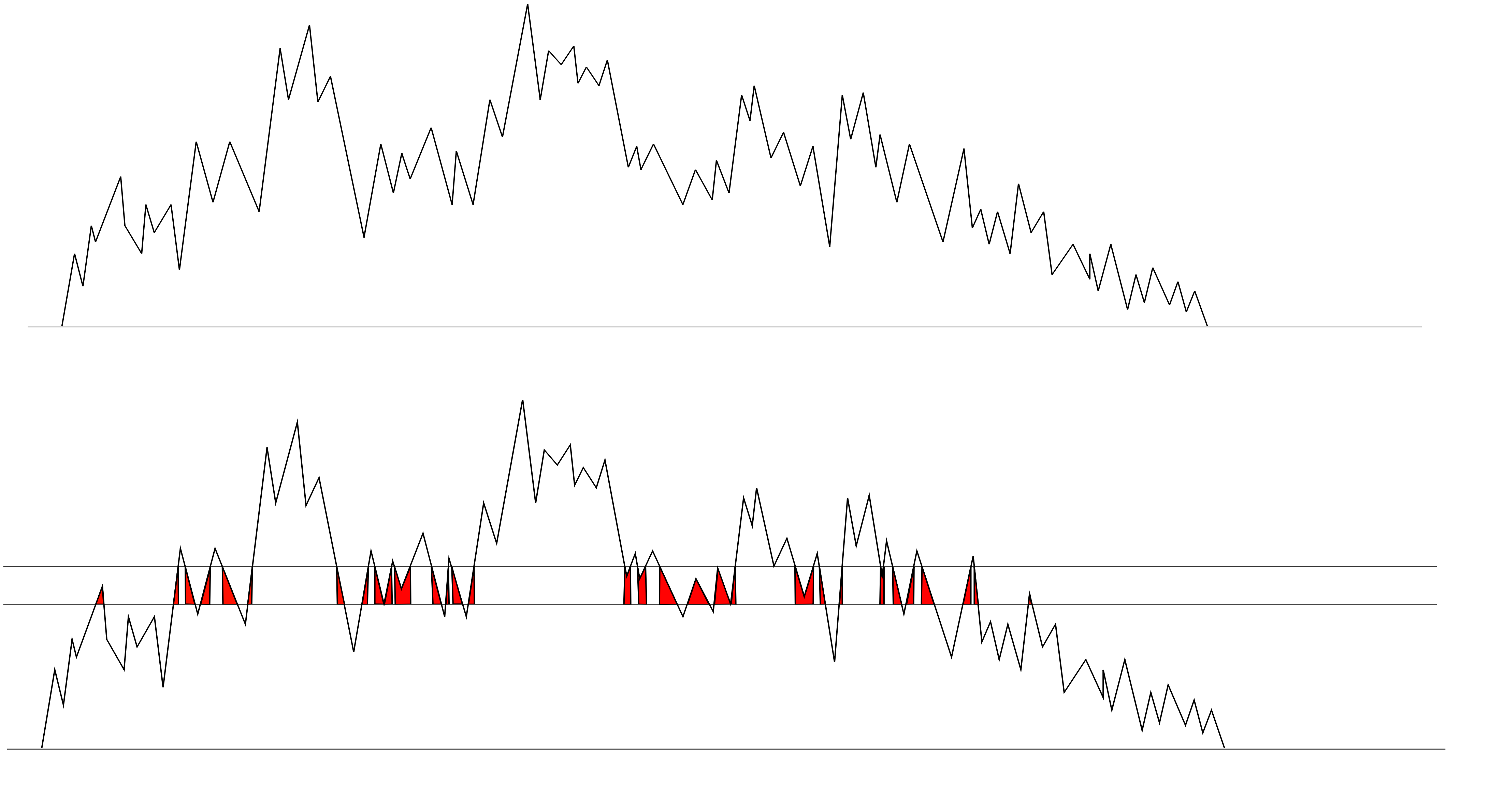
- for $0 \le s < t$ the random variable $W_t - W_s \sim \mathcal{N}(0, t-s)$

$e(t) \stackrel{d}{=} (W(t) | W(1) = 0)$ Brownian excursion

Brownian Excursion and Local Time



Brownian Excursion and Local Time



Brownian Excursion and Local Time

$\int_0^1 I_{[a,b]}(e(s))\,ds$ ... occupation time of $[a,b]$

local time:

$$\ell(\kappa) = \lim_{\varepsilon\to 0} \frac{1}{\varepsilon} \int_0^1 I_{[\kappa,\kappa+\varepsilon]}(e(s))\,ds$$

Brownian Excursion and Local Time

$\int_0^1 I_{[a,b]}(e(s))\,ds$ ... occupation time of $[a,b]$

local time:

$$\ell(\kappa) = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \int_0^1 I_{[\kappa,\kappa+\varepsilon]}(e(s))\,ds$$

$\phi_\kappa(t)$ ... characteristic function

Then

$$\phi_\kappa(t) = 1 + \frac{\sqrt{2}}{\sqrt{\pi}} \int_\gamma \frac{t\sqrt{-s}e^{-\kappa\sqrt{-2s}}}{\sqrt{-s}e^{\kappa\sqrt{-2s}} - it\sqrt{2}\sinh(\kappa\sqrt{-2s}\,)}\, e^{-s}\,ds$$

where γ is a straight line parallel to $\mathbb{R} \cdot i$

Generating Functions

$\mathcal{Y}_n$ class of trees of size n

y_n ... number of trees of size n

$y(x)$ ... generating function of the sequence y_n

$$y(x) = \sum_{n \geq 1} y_n x^n$$

$\mathcal{Y}_n$ simply generated trees

$$y(x) = x\varphi(y(x))$$

Pólya trees:

$$y(x) = xe^{y(x)} \exp\left(\sum_{i \geq 2} \frac{y(x^i)}{i}\right)$$

Generating Functions

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$$y(x) = xe^{y(x)} \exp\left(\sum_{i\geq 2} \frac{y(x^i)}{i}\right)$$

Pólya '37: radius of convergence ρ, $0 < \rho < 1$,
$x = \rho$ only singularity on circle of convergence

Otter '48: $y(\rho) = 1$

$$y(x) = 1 - b(\rho - x)^{1/2} + c(\rho - x) + d(\rho - x)^{3/2} + \cdots ,$$

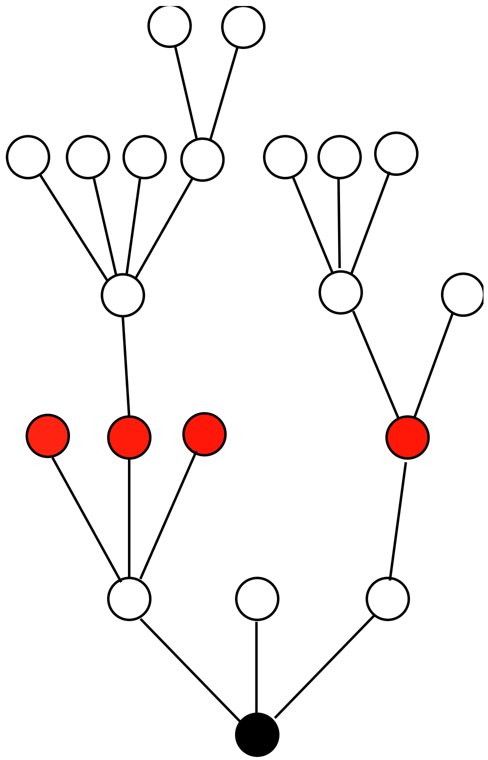
with $b \approx 2.681$, $c = b^2/3 \approx 2.396$, $\rho \approx 0.338$.

Hence

$$y_n \sim \frac{b\sqrt{\rho}}{2\sqrt{\pi}} n^{-3/2} \rho^{-n}$$

The Profile

$L_n(k)$ number of nodes at distance k from the root of an unlabeled rooted tree of size n



The Profile – Results

Theorem 1 *Let*

$$l_n(\kappa) = \frac{1}{\sqrt{n}} L_n\left(\kappa\sqrt{n}\right)$$

Then we have in $C[0,\infty)$

$$(l_n(t))_{t\geq 0} \xrightarrow{w} \frac{b\sqrt{\rho}}{2\sqrt{2}}\left(l\left(\frac{b\sqrt{\rho}}{2\sqrt{2}}\,t\right)\right)_{t\geq 0},$$

as $n\to\infty$*, where* b *and* ρ *are the constants in*

$$y(x) \sim 1 - b(\rho - x)^{1/2}, \qquad x \to \rho.$$

The Profile – Results

Theorem 2 *Let* $l_n(t)$ *as before.*

Then, as $n \to \infty$*, for any choice of fixed* $t_1, \ldots, t_d$

$$(l_n(\kappa_1), \ldots, l_n(\kappa_d)) \xrightarrow{w} \frac{b\sqrt{\rho}}{2\sqrt{2}} \left(l\left(\frac{b\sqrt{\rho}}{2\sqrt{2}}\kappa_1\right), \ldots, l\left(\frac{b\sqrt{\rho}}{2\sqrt{2}}\kappa_d\right)\right),$$

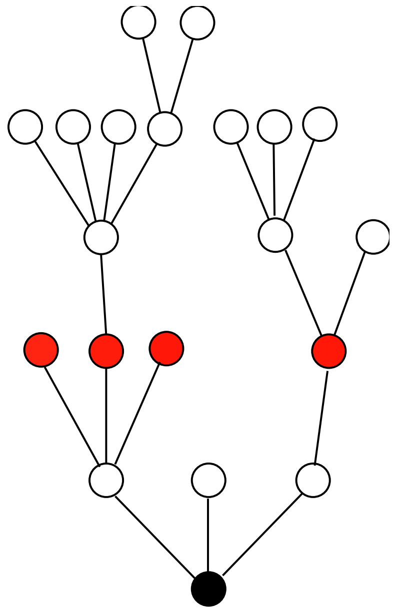
Theorem 3 *For all* $n, r, h \geq 0$ *and some* $C > 0$ *we have*

$$\mathbf{E}(L_n(r) - L_n(r+h))^4 \leq Ch^2 n$$

and therefore the processes $(l_n(t))_{t\geq 0}$ *are tight.*

The Profile – Combinatorial Setup

$L_n(k)$ number of nodes at distance k from the root of an unlabeled rooted tree of size n



y_{nm} ... number of trees in $\mathcal{Y}_n$ with m nodes in level k

$$y_k(x,u) = \sum_{n\geq 0}\sum_{m\geq 0} y_{nm} z^n u^m$$

The Profile – Combinatorial Setup

$\Longrightarrow$ the generating function $y_k(x,u)$ is given by

$$y_0(x,u) = uy(x)$$
$$y_{k+1}(x,u) = z \exp\left(y_k(x,u) + \sum_{i\ge 2} \frac{y_k(x^i,u^i)}{i}\right),\; k \ge 0$$

char. function of $\frac{1}{\sqrt{n}} L_n(k)$:

$$\phi_{kn}(t) = \frac{1}{y_n}[x^n] y_k\left(x, e^{it/\sqrt{n}}\right)$$

The Profile – Combinatorial Setup

the sequence $y_k(x,u)$ satisfies

$$\lim_{k\to\infty} y_k(x,u) = y(x)$$

hence define

$$w_k(x,u) = y_k(x,u) - y(x)$$

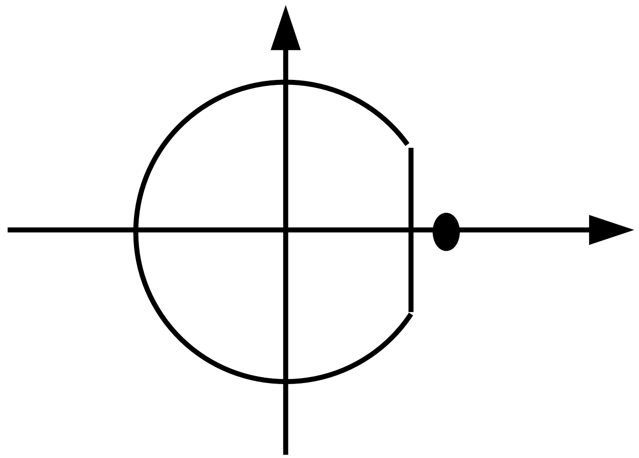
Then

$$\lim_{k\to\infty} w_k(x,u) = 0, \quad w_k(x,1) = 0$$

The Profile

Get more precise information and use

singularity analysis to determine



$$\phi_{kn}(t) = \frac{1}{y_n}[x^n] y_k\left(x, e^{it/\sqrt{n}}\right)$$

$$= \frac{1}{2\pi i y_n} \oint \frac{y_k\left(x, e^{it/\sqrt{n}}\right)}{x^{n+1}}\, dx$$

Notice:

$$y_k(x,u) = y(x) + w_k(x,u)$$

The Profile – Local Behaviour of w_k

Theorem 3 *Set* $x = \rho\left(1 + \frac{s}{n}\right)$, $u = e^{it/\sqrt{n}}$, $k = \kappa\sqrt{n}$. *Assume* $s = \mathcal{O}\left(\log^n\right)$, $t = \mathcal{O}\left(1\right)$, $n \to \infty$.

Then $w_k(x,u)$ *admits the local representation*

$$w_k(x,u) = \frac{y(x)^k w_0}{1 - w_0\left(\frac{1-y(x)^k}{2(1-y(x))} + f_k(x,u)\right)(1 + \mathcal{O}\left(w_0\right))}$$

where

$$f_k(x,u) = \sum_{\ell=0}^{k} \frac{\sum_{i \geq 2} \frac{w_\ell(x^i,u^i)}{i}}{w_\ell(x,u)^2}\, y(x)^\ell$$

$$w_0 = (u-1)y(x), \quad y(x) \sim 1 - b\sqrt{\rho}\sqrt{\rho - x}$$

The Profile – Idea of the Proof

<u>Idea of the Proof</u>

$$w_{k+1}(x,u) = y(x)\left(\exp\left(w_k(z,u) + \sum_{i\ge 2}\frac{w_k(z^i,u^i)}{i}\right) - 1\right)$$

Lemma 1 $|x| \le \rho^2 + \varepsilon$, $|u| = 1 \Longrightarrow$ *there exists* $0 < L < 1$ *with*

$$|w_k(x,u)| \le C|u-1| \cdot |x| \cdot L^k$$

Consequence: sums like

$$\sum_{i\ge 2}\frac{w_\ell(x^i,u^i)}{i}$$

behave relatively nicely.

The Profile – Idea of the Proof

Lemma 2 *Under the assumption of Theorem 2 we have*

$$|w_k(z,u)| \le C_1 |w_0| |y(z)|^k$$

and

$$|w_k(z,u)| \ge C_2 |w_0| |y(z)|^k (1 - C_3\varepsilon)^k$$

$\Longrightarrow w_0(x,u) f_k(x,u)$ is bounded

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$\Longrightarrow w_0(x,u) f_k(x,u)$ is bounded

and

w_k and f_k have square root type singularities w.r.t. x, in particular,

$$w_0 f_k(x,u) = (u-1) y(x) f_k(x,u) = C_1(x,u) + C_2(x,u) \sqrt{1 - \frac{x}{\rho}}$$

and

$$\lim_{k \to \infty} f_k(x,u) = c_0$$

The Profile – Idea of the Proof

Lemma 3 *For* $k \sim \kappa\sqrt{n}$ *we have* $L_n(k) \xrightarrow{w} A\ell(B\kappa)$.

$$A = \frac{1}{(1-c_0)b\sqrt{2\rho}}, \qquad B = \frac{b\sqrt{\rho}}{2\sqrt{2}}$$

The Profile – Normalisation

We have

$$\mathbf{E}L_n(k) = [x^n]\gamma_k(x)$$

with

$$\gamma_k(x) = \frac{\partial}{\partial u} y_k(x,u)\Big|_{u=1}$$

It can be shown that $\gamma_k(x) \sim C(x)y(x)^k$.

But

$$\mathbf{E}\sum_k L_n(k) = n \qquad \Longrightarrow \qquad C(x) \sim xy'(x)(1-y(x))$$

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$\Longrightarrow$ $\lim_{n\to\infty} \mathbf{E}L_n(k) = \mathbf{E}(A\ell(Bt))$ with $A = B = \frac{b\sqrt{\rho}}{2\sqrt{2}}$

Tightness

Tightness follows from

$$\mathbf{E}\,(L_n(r) - L_n(r+h))^\beta \le C\,(h\sqrt{n})^\alpha$$

for all non-negative integers n, r, h and some $\alpha > 1$, $\beta > 0$

We also know that showing

$$\mathbf{E}\,(L_n(r) - L_n(r+h))^4 \le C\,h^2 n$$

doable for simply generated trees.

Tightness

GF encoding the bivariate distribution of $L_n(k), L_n(k+m)$ is $y_{km}(x,u,v)$ with

$$\tilde{y}_{0,m}(x,u,v) = u y_m(x,v)$$
$$\tilde{y}_{k+1,m}(x,u,v) = x \exp\left(\sum_{i\ge 1} \frac{\tilde{y}_{k,m}(x^i,u^i,v^i)}{i}\right), \qquad k \ge 0.$$

Hence tightness follows from

$$[x^n]\left[\left(\frac{\partial}{\partial u} + 7\frac{\partial^2}{\partial u^2} + 6\frac{\partial^3}{\partial u^3} + \frac{\partial^4}{\partial u^4}\right)\tilde{y}_{r,h}\left(x,u,\frac{1}{u}\right)\right]_{u=1} \le C\,\frac{h^2}{\sqrt{n}}\rho^{-n}$$

and therefore from

$$\left[\left(\frac{\partial}{\partial u} + 7\frac{\partial^2}{\partial u^2} + 6\frac{\partial^3}{\partial u^3} + \frac{\partial^4}{\partial u^4}\right)\tilde{y}_{r,h}\left(x,u,\frac{1}{u}\right)\right]_{u=1} = \mathcal{O}\left(\frac{h^2}{1-|y(x)|}\right)$$

for $x \in \Delta$.

The Height

$y_n^{(k)}$ ... number of trees with n nodes and height at most k

$y_k(x) = \sum_{n\ge1} y_n^{(k)} x^n$

Then

$$y_0(x) = 0$$
$$y_{k+1}(x) = x \exp\left(\sum_{i\ge1} \frac{y_k(x^i)}{i} \right), \qquad k \ge 0.$$

$\Longrightarrow y_k(x) = y_k(x, 0)$

$$w_k(x) := w_k(x, 0) = y(x) - y_k(x) = \sum_{n\ge0} \mathbf{P}\{H_n > k\}\, y_n x^n$$

The Height

Theorem 4 *Let $k = \kappa\sqrt{n}$. Then, as $n \to \infty$, we have*

$$w_k(x) = \frac{-y(x)^k}{\frac{1}{2}\frac{1-y(x)^k}{1-y(x)} + \mathcal{O}\left(\sqrt{k}\right)}$$

uniformly for $|x - \rho| = \mathcal{O}\left(1/\sqrt{n}\right)$ such that $x \in \Delta$.

Thanks to Flajolet, Odlyzko '82
and Flajolet, Gao, Odlyzko, Richmond '95 this is enough.

The Height

Theorem 5 *Let* H_n *denote the height of a tree of size* n*. Then*

$$\mathbf{E}H_n \sim \frac{2\sqrt{\pi}}{b\sqrt{\rho}}\sqrt{n}.$$

Moreover, if we set $\beta = \frac{2\sqrt{n}}{hb\sqrt{\rho}}$*, then*

$$\mathbf{P}\{H_n = h\} = \frac{y_n^{(h)}}{y_n} \sim 4b\sqrt{\frac{\rho\pi^5}{n}}\beta^4 \sum_{m\geq 1} m^2(2(m^2\pi^2\beta^2 - 3)e^{-m^2\pi^2\beta^2}$$

The Height

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Large deviation results by Broutin & Flajolet 2008

THANK YOU!