Counting reducible and singular bivariate polynomials

Joachim von zur Gathen
Bonn

Four "accidents" can happen to a bivariate (or multivariate) polynomial over a field:

- a nontrivial factor,
- a square factor,
- a factor over an extension field,
- a singular root, where all partial derivatives also vanish.

We have a ground field F. The accidents may occur at two places:

- in F ("rational"),
- in an algebraic closure of F ("absolute").

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Squareful polynomials

Relatively irreducible polynomials

Singular Polynomials

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Taxonomy of views on polynomials over finite fields

1 variable   2 variables ≥ 2 variables

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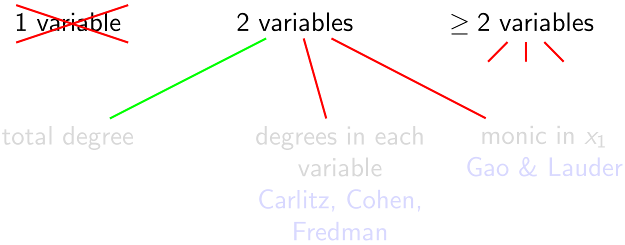

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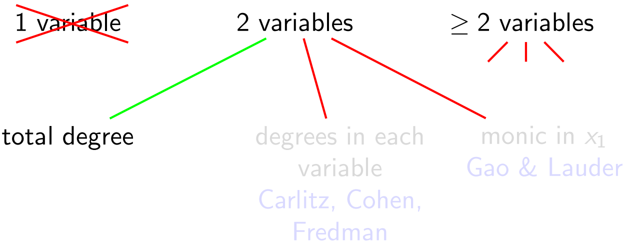

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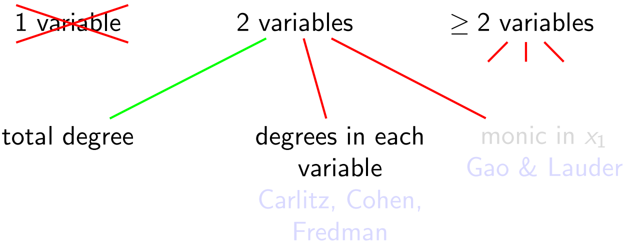

Taxonomy of views on polynomials over finite fields

- ~~1 variable~~
- 2 variables
  - total degree
  - degrees in each variable
    Carlitz, Cohen, Fredman
  - monic in x_1
    Gao & Lauder
- ≥ 2 variables

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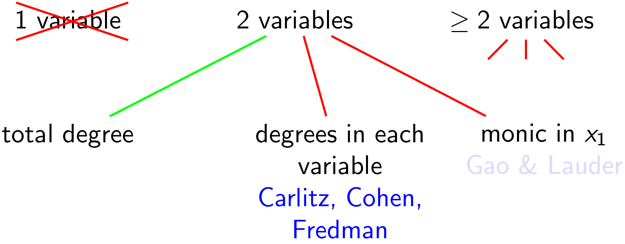

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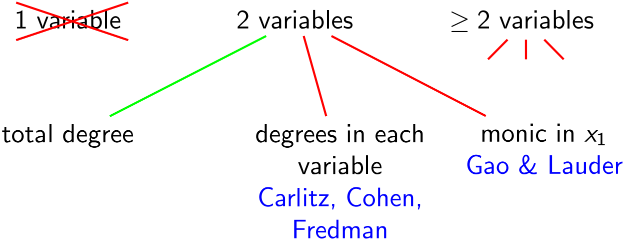

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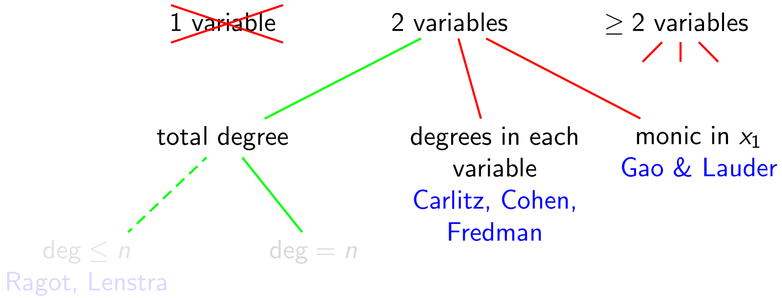

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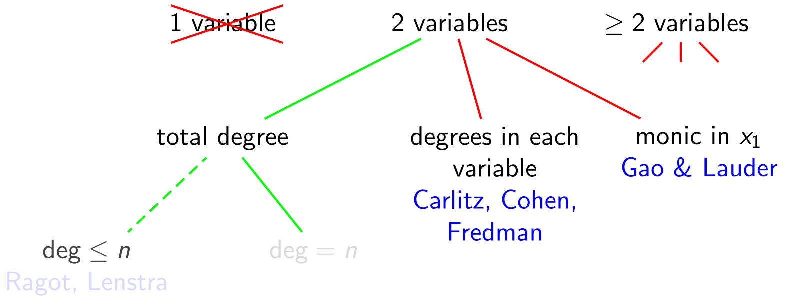

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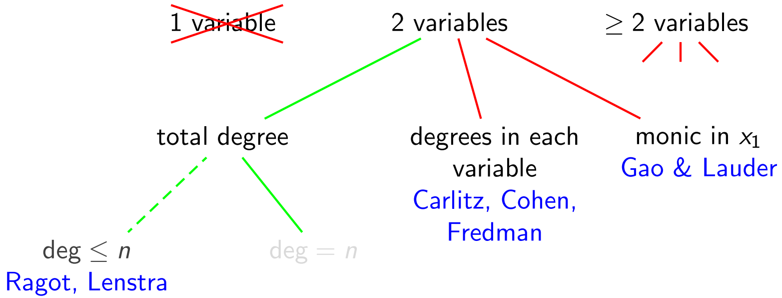

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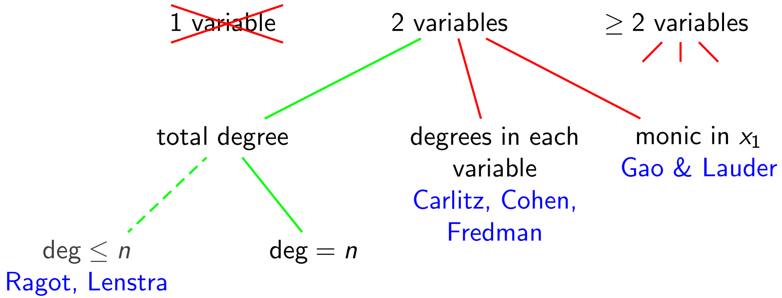

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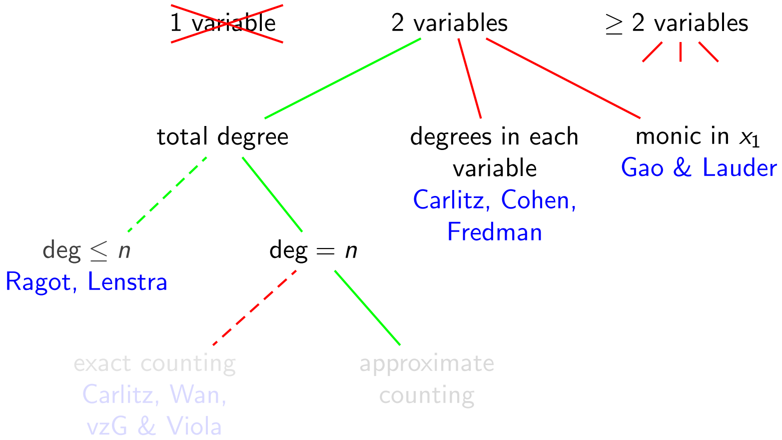

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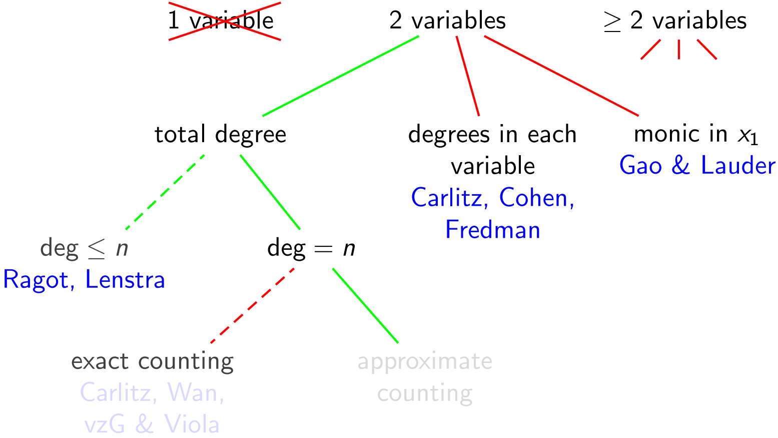

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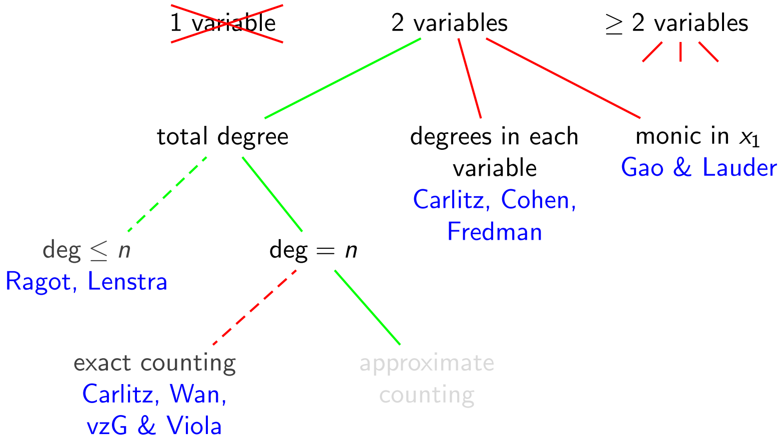

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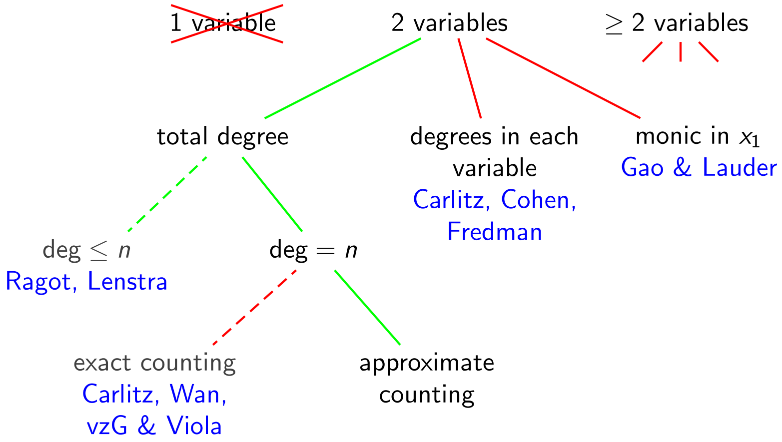

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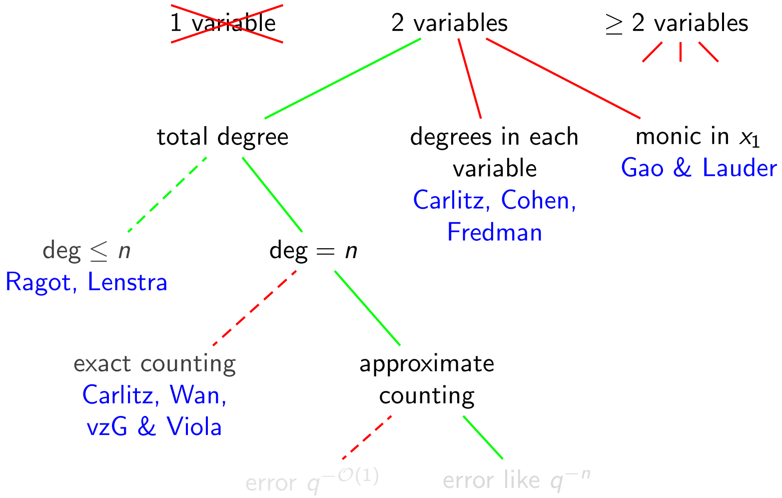

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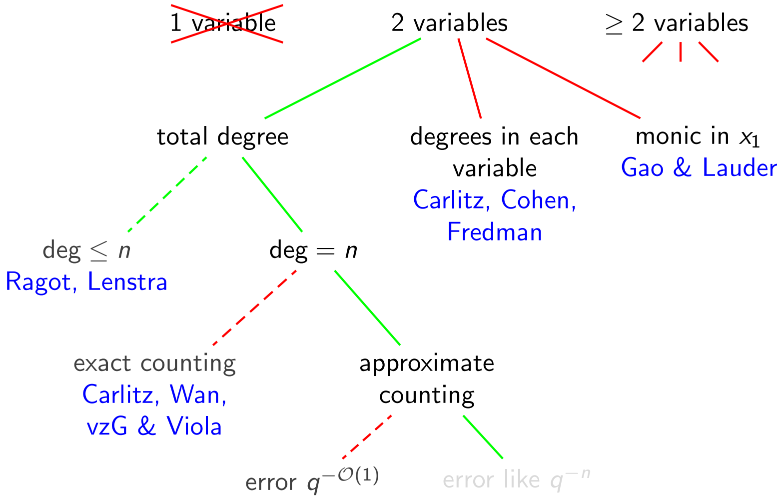

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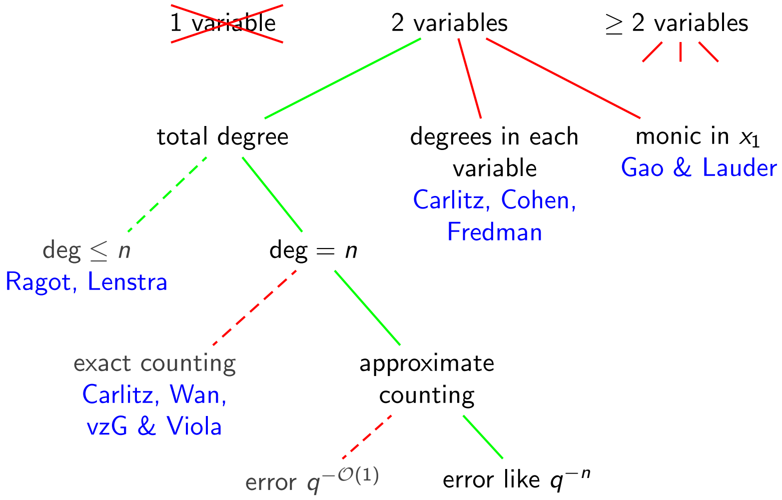

Notation:

- $B_n(F) \subseteq F[x, y]$: bivariate polynomials with total degree $\leq n$.
- Certain natural sets $A_n(F) \subseteq B_n(F)$.

Two different languages: geometric and combinatorial.

- Geometry: $B_n(F)$ affine space over F, $A_n(F)$ union of images of polynomial maps, thus (reducible) subvariety. Geometric goal: determine the codimension of $A_n(F)$ = codimension of irreducible components of maximal dimension.
- Combinatorial goal: $F = \mathbb{F}_q$ for a prime power q, find functions $\alpha_n(q)$ and $\beta_n(q)$ so that

$$\left| \frac{\#A_n(\mathbb{F}_q)}{\#B_n(\mathbb{F}_q)} - \alpha_n(q) \right| \leq \alpha_n(q) \cdot \beta_n(q),$$

with $\beta_n(q)$ tending to zero as q and n grow.

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Thus a random element of $B_n(\mathbb{F}_q)$ is in $A_n(\mathbb{F}_q)$ with probability about $\alpha_n(q)$.

- Best results: $\beta_n(q)$ goes to zero like q^{-n}.
- Simpler results: $\alpha_n(q) = q^{-m}$ with $\beta_n(q) = O(q^{-1})$.
- Weil bounds: $\beta_n(q) = n^{O(1)} q^{-1/2}$.

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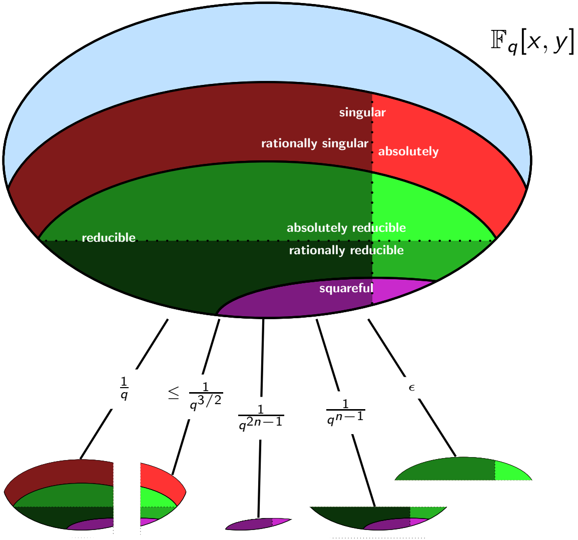
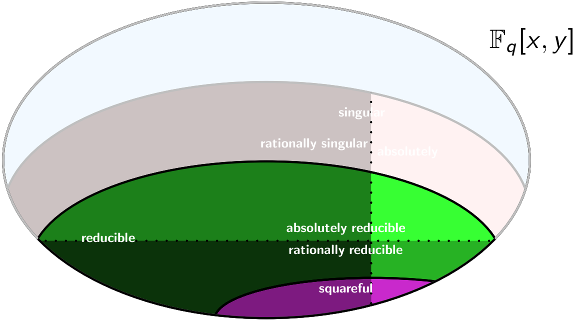


$\mathbb{F}_q[x, y]$
singular
rationally singular
absolutely
reducible
absolutely reducible
rationally reducible
squareful
$\frac{1}{q}$
$\leq \frac{1}{q^{3/2}}$
$\frac{1}{q^{2n-1}}$
$\frac{1}{q^{n-1}}$
ϵ

Reducible polynomials



n	all	reducibles
1	$q^3 - q$	0
2	$q^6 - q^3$	$(q^5 + q^4 - q^2 - q)/2$
3	$q^{10} - q^6$	$(3q^8 + 2q^7 - 2q^6 - 3q^5 - q^4 + 2q^3 - q)/3$
4	$q^{15} - q^{10}$	$(4q^{12} + 6q^{11} - 2q^{10} - 5q^9 - 7q^8 + 6q^6 - 2q^4 - q^3$ $+q^2)/4$
5	$q^{21} - q^{15}$	$(5q^{17} + 5q^{16} + 5q^{15} - 10q^{13} - 15q^{12} - 6q^{11}$ $+11q^{10} + 10q^9 - 5q^7 - q^6 + q^5 + q^3 - q)/5$
6	$q^{28} - q^{21}$	$(6q^{23} + 6q^{22} + 6q^{20} + 3q^{19} - 3q^{18} - 21q^{17}$ $-23q^{16} - 10q^{15} + 18q^{14} + 32q^{13} + 10q^{12} - 15q^{11}$ $-12q^{10} + 3q^8 - q^7 + 2q^5 - 3q^3 + q^2 + q)/6$

The numbers of reducible polynomials of degrees up to 6

Theorem

Consider polynomials of degree $n \geq 2$.

1. $\{reducibles\}$ *is a subvariety of codimension* $n-1$ *in* $\{all\}$.
2. *Let* $\rho_n(q) = (q+1)q^{-n}$. *Then for* $n \geq 3$

$$\left|\frac{\#\{reducibles\}}{\#\{all\}} - \rho_n(q)\right| \leq \rho_n(q) \cdot 2q^{-n+3},$$

$$\text{at degree 2:} \quad \frac{\#\{reducibles\}}{\#\{all\}} = \frac{\rho_2(q)}{2}.$$

3. *For* $n \geq 6$, *we have*

$$\left|\frac{\#\{reducibles\}}{\#\{all\}} - q^{-n+1}\right| \leq 2q^{-n}.$$

For $1 \leq k < n$: multiplication map

$$\mu_{k,n}\colon \begin{array}{rcl} \{\text{degree } k\} \times \{\text{degree } n-k\} & \longrightarrow & \{\text{degree } n\}, \\ (g,h) & \longmapsto & g \cdot h, \end{array}$$

$$\{\text{reducibles}\} = \bigcup_{1 \leq k \leq n/2} \operatorname{im} \mu_{k,n}.$$

Multiplication by units gives fiber dimension ≥ 1
$\Longrightarrow$ Zariski closure of im $\mu_{k,n}$ is a proper irreducible subvariety
$\Longrightarrow$ complement (= irreducible polynomials) is dense.

g, h irreducible
$\Longrightarrow$ fiber dimension $= 1$.
$\Longrightarrow$ generic fiber dimension is 1,
$b_n = \dim\{\text{polynomials of degree } n\}$.

$$\dim \operatorname{im} \mu_{k,n} = b_k + b_{n-k} - 1 = b_n - k(n-k).$$

Maximum at $k = 1$: $b_n - n + 1$. Hence codim $= n - 1$.

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Let $n \geq 3$. Each fiber of $\mu_{k,n}$ has at least $q-1$ elements.

$$\begin{aligned}
\#\,\mathrm{im}\,\mu_{k,n} &\leq \frac{1}{q-1} \cdot \#\{\text{degree } k\} \cdot \#\{\text{degree } n-k\} \\
&< \frac{q^{b_k}(1-q^{-k-1}) \cdot q^{b_{n-k}}}{q-1} \\
&= \frac{\rho_n(q) \cdot \{\text{all}\} \cdot q^{n-1-k(n-k)}(1-q^{-k-1})}{(1-q^{-2})(1-q^{-n-1})}.
\end{aligned}$$

- Some calculation gives the upper bound for $q \geq 3$.
- More calculation for $q = 2$ and $n \geq 8$.
- Even more for $q = 2$ and $n \neq 6$.
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Corollary

We have for $n \geq 2$

$$\#\{\textit{irreducibles}\} \geq q^{b_n} \cdot (1 - (q+2)q^{-n}).$$

Lower bound: g, h irreducible, $k < n/2$,
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$\Longrightarrow$ lower bound on reducibles.

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$\implies$ fiber size is $q - 1$,
$\implies$ lower bound on reducibles.

Corollary

We have for $n \geq 2$

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- Carlitz 1963:

$$\text{fraction of irreducibles} - 1 = O((q-1)q^{-n-1}).$$

"As the referee pointed out, [it] can be proved by a crude counting argument" that

$$1 - \frac{q^{-n+4}}{(q-1)^3} \leq \text{fraction of irreducibles} \leq 1.$$

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Upper bound on reducibles

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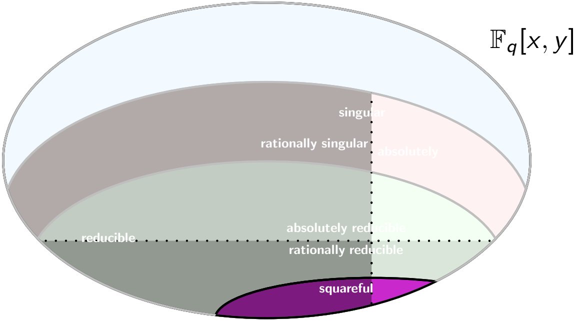
Squareful polynomials



n	squareful polynomials
1	0
2	$q^3 - q$
3	$q^5 + q^4 - q^3 - q^2$
4	$q^8 + q^7 + q^6 - 2q^5 - 2q^4 + q^2$
5	$q^{12} + q^{11} - q^7 - 2q^6 - q^5 + q^4 + q^3$
6	$q^{17} + q^{16} - q^{12} + q^{10} - q^9 - 4q^8 - q^7 + 2q^6 + 3q^5 - q^3$

The number of squareful polynomials of degrees up to 6.

Theorem

Let $n \geq 1$*.*

1. *For* $n \geq 2$, $\{squareful\}$ *is a subvariety of codimension* $2n-1$*.*
2. *Let*

$$\eta_n(q) = \frac{(q+1)q^{-2n}(1-q^{-n+1})}{1-q^{-n-1}}.$$

Then

$$|\text{fraction of squareful} - \eta_n(q)| \leq \eta_n(q) \cdot 3q^{-2n+6},$$

and for $n \leq 3$

$$\text{fraction of squareful} = \eta_n(q).$$

Cohen 1970: fraction of r-power-free polynomials is $1 - q^{-rm} + O(q^{-nm})$ among polynomials of degrees at most $m \leq n$ in x, y, respectively.

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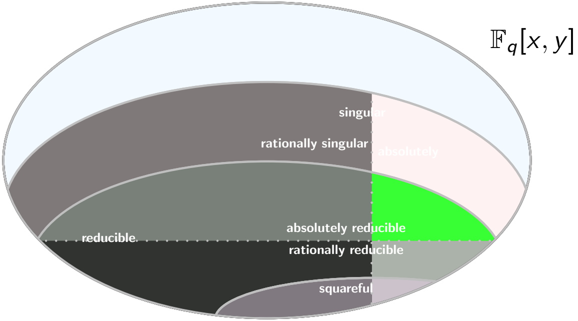
Relatively irreducible polynomials



An irreducible bivariate polynomial is *relatively irreducible* if it is not absolutely irreducible. Then it is the product of all conjugates of an irreducible polynomial over some extension field.

Application: algorithms for curves: point finding, estimating the size. Huang & Ierardi, 1993; von zur Gathen, Shparlinski & Karpinski, 1993, 1996; von zur Gathen & Shparlinski 1995, 1998; Matera & Cafure 2006.

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n	relatively irreducibles
2	$(q^5 - q^4 - q^2 + q)/2$
3	$(q^7 - q^6 + q^4 - 2q^3 + q)/3$
4	$(2q^{11} - 2q^{10} + q^9 - q^8 - 2q^6 + 2q^4 + q^3 - q^2)/4$
5	$(q^{11} - q^{10} + q^6 - q^5 - q^3 + q)/5$
6	$(3q^{19} - 3q^{18} + 3q^{17} - q^{16} - 2q^{15} - 2q^{13} + 2q^{12}$
	$-3q^{11} + 3q^8 + q^7 - 2q^5 + 3q^3 - q^2 - q)/6$

The numbers of relatively irreducible polynomials of degrees up to 6.

Theorem

Let $n \geq 2$, let $l \geq 2$ be the smallest prime divisor of n,

$$\varepsilon_n(q) = \frac{q^{-n^2(l-1)/2l}(1-q^{-1})}{l(1-q^{-l})(1-q^{-n-1})},$$

$$\delta_n(q) = \begin{cases} 2q^{-2n+2} & \text{if } n \text{ is prime,} \\ 2q^{-n+l+1} & \text{otherwise.} \end{cases}$$

Then

1. $\left| \text{fraction of rel irred} - \varepsilon_n(q) \right| \leq \varepsilon_n(q) \cdot \delta_n(q)$.
2. $\varepsilon_n(q) \leq q^{-n^2/4}/2$.
3. *If n is prime, then* $\varepsilon_n(q) \leq q^{-n(n-1)/2}/n$ *and*

$$\#\{\text{rel irred}\} = (q-1)(q^{2n} + q^n - q^2 - q)/n.$$

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Singular polynomials

$f \in F[x,y], P = (u,v) \in F^2:$

$$f(P) = 0 \iff P \text{ is on the curve } V(f) \subseteq F^2$$
$$\iff f \in m_p = (x-u, y-v) \subseteq F[x,y]$$

maximal ideal.

$$f(P) = \frac{\partial f}{\partial x}(P) = \frac{\partial f}{\partial y}(P) = 0 \iff P \text{ is singular on } V(f)$$
$$\iff f \text{ is singular at } P$$
$$\iff f \in s_p = m_p^2.$$

Quotient ring

$$F[x, y]/s_P = F + (x - u)F + (y - v)F$$

is a 3-dimensional vector space over F.

$$\operatorname{codim}_{F[x,y]} s_P = 3.$$

Affine Hilbert function of s_P:

$$\text{codim } s_P = 3$$

at degree n for n large enough.

Ragot 1997, 1999:

$$\text{fraction of singular} = 1 - (1 - q^{-3})^{q^2} \qquad (1)$$

for $n > 4q - 2$.
Similar result for multivariate polynomials.

Theorem
(Lenstra 2006): (1) $\iff n \geq 3q - 2$.

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$R = \mathbb{F}_q[x, y]$:

$P \in \mathbb{F}_q^2$, random polynomial:

$$\text{prob(singular at } P) = q^{-3}$$
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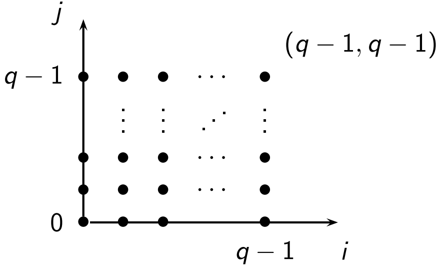
$\prod_{P \in \mathbb{F}_q^2} R/s_P$, random polynomial:  $\times \quad \cdots \quad \times$ 

$$\text{prob(nonsingular at all } P) = (1 - q^{-3})^{q^2}$$

Independence: Chinese Remainder Theorem

$$
\begin{aligned}
\prod_{P \in \mathbb{F}_q^2} R/m_P &= \prod_{u,v \in \mathbb{F}_q} R/(x-u, y-v) \\
&= R/\prod_{u,v \in \mathbb{F}_q} (x-u, y-v) \\
&= R/(x^q - x, y^q - y)
\end{aligned}
$$

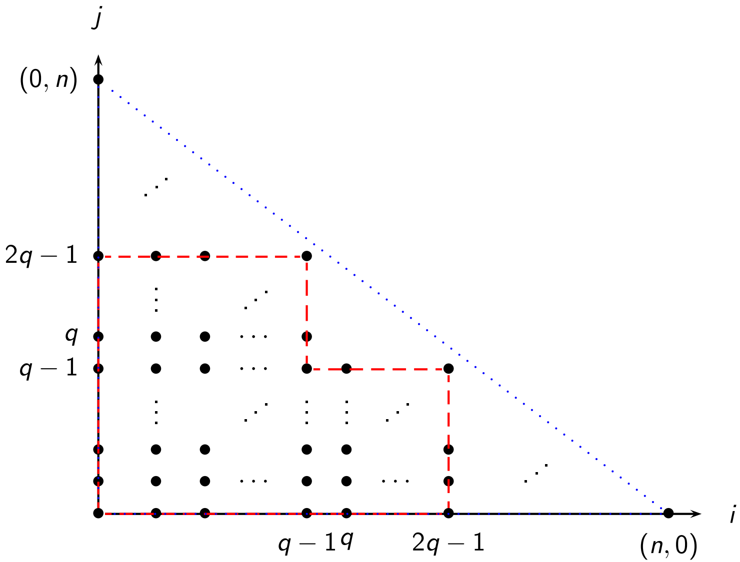
Monomial $x^i y^j \leftrightarrow (i,j)$:



Representatives for $R/(x^q - x, y^q - y)$

Representation of

$$\begin{aligned}\prod_{P\in\mathbb{F}_q^2} R/s_P &= \prod_{P\in\mathbb{F}_q^2} R/m_P^2 \\ &= R/(\prod_{P\in\mathbb{F}_q^2} m_P^2) = R/(x^q - x, y^q - y)^2 \\ &= R/((x^q - x)^2, (x^q - x)(y^q - y), (y^q - y)^2).\end{aligned}$$



j
(0, n)
2q − 1
q
q − 1
i
q − 1
q
2q − 1
(n, 0)

$$\begin{aligned}
\text{(1) holds} \quad &\Leftrightarrow \quad \text{degree } n \to R/(x^q - x, y^q - y)^2 \text{ surjective} \\
&\Leftrightarrow \quad n \geq 3q - 2. \quad \square
\end{aligned}$$

Small n?

$$\begin{aligned} 1-(1-q^{-3})^{q^2} &= \binom{q^2}{1}q^{-3} - \binom{q^2}{2}q^{-6} + - \cdots \\ &\approx q^{-1} - \frac{1}{2}q^{-2} + - \cdots \end{aligned}$$

Theorem

1. *$\{singular\}$ is an irreducible subvariety with codimension* 1*.*
2. *For $q, n \geq 3$, we have*

$$q^{-1} - \frac{1}{2}q^{-2} \leq \textit{fraction of singular} \leq q^{-1}.$$

Theorem

The fraction τ of absolutely singular and rationally nonsingular polynomials satisfies

$$\tau < 13n^{13}q^{-3/2}.$$

Conjecture

$|\tau - q^{-2}| = O(q^{-3})$.

Current work

- Exact counting, generating functions (alas, nowhere convergent), multivariate polynomials (with Alfredo Viola).
- Estimates for curves in higher dimensional spaces (with Guillermo Matera).
- Decomposable polynomials.

Thank you!