Asymptotic probability of Boolean functions over implication

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Outline

- Boolean expressions and trees
- A restricted propositional calculus
- Tautologies
- Probability and complexity of a Boolean function
- Main result: sketch of proof
- Extensions and open questions

Boolean expressions

$((x \vee \bar{x}) \wedge x) \wedge (\bar{x} \vee (x \vee \bar{x}))$

$(x \vee (y \wedge \bar{x})) \vee (((z \wedge \bar{y}) \vee (x \vee \bar{u})) \wedge (x \vee y))$

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Probability that a "random" expression on n boolean variables is a tautology (always true)?

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Probability that a "random" expression on n boolean variables is a tautology (always true)?

- $n = 1$: 4 boolean functions; $Proba(True) = 0.2886$
- $n = 2$: 16 boolean functions; $Proba(True) = 0.209$
- $n = 3$: 256 boolean functions; $Proba(True) = 0.165$

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- $n \to +\infty$: 2^{2^n} boolean functions

$$Proba(True) \sim ?$$

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$$Proba(True) \sim ?$$

$Proba(f)$ for any boolean function f?

Boolean expressions and trees

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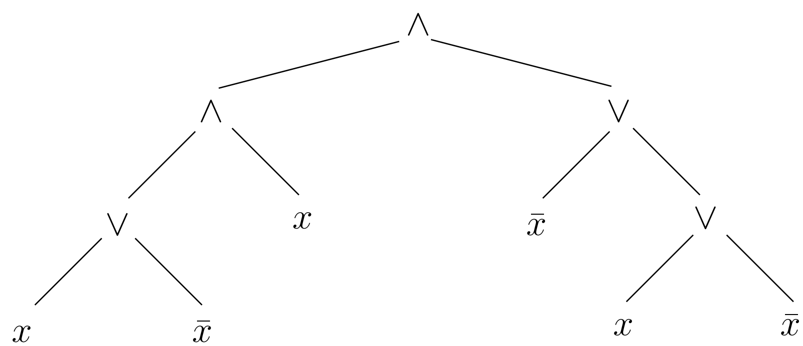


Consider a well-formed boolean expression

- Choose set of logical connectors, with arities
  $\leftrightarrow$ Choose labels and arities for internal nodes

- Choose set of boolean literals for the leaves
  $\leftrightarrow$ Choose labels for leaves

Boolean expressions and trees

- Expression $\sim$ labelled tree
- Random expression $\sim$ random labelled tree
- What notion of randomness on trees?
  - Choose size m of the tree; assume all trees of same size are equiprob.

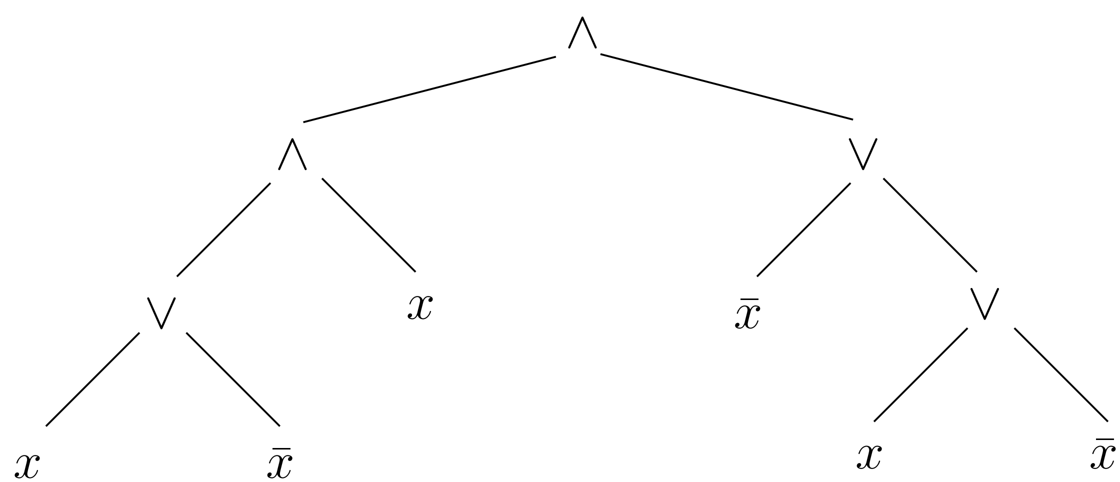
    Then let $m \to +\infty$
  - Choose tree at random (e.g., by a branching process): size is also random. Then label tree at random.

Boolean expressions and trees

- Expression $\sim$ labelled tree
- Random expression $\sim$ random labelled tree
- Two notions of randomness on trees/boolean expressions
- Each boolean expression computes a boolean function
- A boolean function is represented by an *infinite number of expressions*
- Can we use random boolean expressions to define a *probability distribution* on boolean functions?

Former work : And/Or trees

- One of the most studied models for random boolean expressions
- Binary trees; no simple node
- Internal nodes are labelled by $\vee$ or $\wedge$
- Leaves are labelled by the literals: $x_1, ..., x_n, \bar{x_1}, ..., \bar{x_n}$



And/Or trees

- Paris et al. 94: first definition of a tree distribution on boolean functions
- Lefman and Savicky 97:
  - Proof of existence of a tree distribution (by pruning)
  - Tree complexity of f: $L(f)$= size of smallest tree that computes f
  - $\frac{1}{4}\left(\frac{1}{8n}\right)^{L(f)} \leqslant P(f) \leqslant e^{-cL(f)/n^3}(1+O(1/n))$
- Chauvin et al. 04: alternative definition of probability by generating functions; improvment on upper bound: $P(f) \leqslant e^{-cL(f)/n^2}(1+O(1/n))$
- For tautologies:
  - Woods 05: Asymptotic probability $P(True) \sim 1/4n$ and probable shape of tautologies: $l \vee \ldots \vee \bar{l} \vee \ldots$
  - Kozik 08: Alternative derivation of asymptotic probability and shape

And/Or trees: probability and complexity

To sum up:

- definition of a tree-induced probability distribution on boolean functions
- probability of constant functions $True$ and $False$: known
- probability of a non-constant function:
  - lower bound $(1/4)\ (8n)^{-L(f)}$ (not that bad; order looks right)
  - upper bound $e^{-cL(f)/n^2}(1+O(1/n))$ (probably not tight)

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- Partial results. Can we go further?

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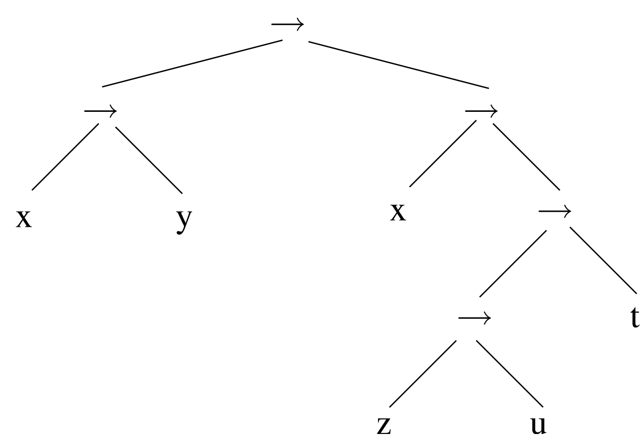
- Partial results. Can we go further?
- Consider a simpler system

A restricted propositional calculus

- Finite number of boolean variables : $x_1, x_2, \ldots, x_n$; **no** negative literals.
- A single connector $\rightarrow$ ($x_1 \rightarrow x_2$ is also $\overline{x_1} \vee x_2$).
- Expressions are *binary trees*: $(x \rightarrow y) \rightarrow (x \rightarrow (z \rightarrow u) \rightarrow t)$



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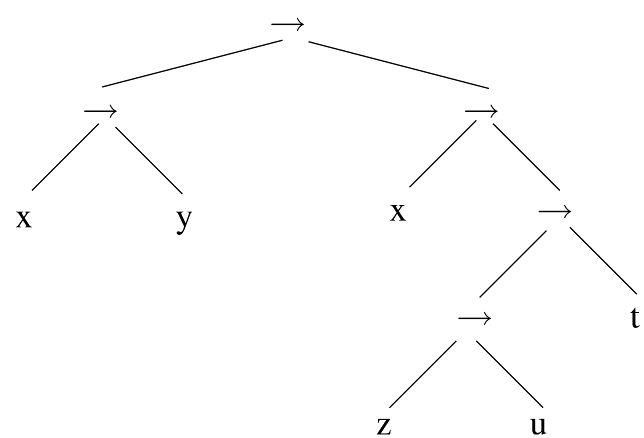

- An expression is a (possibly empty) sequence of expressions: premises, followed by a variable: goal.

A restricted propositional calculus

- Finite number of boolean variables : $x_1, x_2, \ldots, x_n$; no negative literals.
- A single connector $\rightarrow$
- "Simple" system: may hope for a detailed study of random expressions and boolean functions.
- Relevance to intuitionnistic logic:

  Tautology $\sim$ proof of a goal from premises

Boolean functions and expressions

An expression (a tree) computes a boolean function on k variables.

- What is the set of boolean functions that can be computed?

  $\Rightarrow$ Post set $S_0 = \{x \vee g(x_1, ..., x_k)\}$

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- Many different expressions compute the same boolean function.

  Probability that a "random" expression computes a specific function?

Probability of a boolean function

- Informally, it is the ratio of trees that compute f to the total number of trees (assuming this ratio can be defined).

- Define the size of a formula (tree) as the number of variable occurrences (leaves).

- Define $A_m = \{\text{trees of size } m\}$; $A_m(f) = \{\text{trees in } A_m \text{ that compute } f\}$.

  Assume a uniform distribution on A_m.

- Probability that a tree of size m computes f:

$$P_m(f) = \frac{|A_m(f)|}{|A_m|}$$

- For any boolean function f, $lim_{m \to +\infty} P_m(f)$ exists?

Probability of a boolean function

Existence of a limit $P(f) = lim_{m \to +\infty} P_m(f)$?

- Enumerate trees by size: g.f. $\Phi(z) = \sum_m |A_m| z^m = (1 - \sqrt{1 - 4nz})/2$
- Enumerate the set $A(f)$ of trees computing a specific function f:

  Generating function $\phi_f(z)$?

  Consider *all* boolean functions

  $A(f) = \cup_{g,h} (A(g), \to, A(h)) \Rightarrow \phi_f = \sum_{g,h} \phi_g \, \phi_h$

  $\Rightarrow$ write a system of algebraic equations for the enumerating functions

  $\Rightarrow$ Drmota-Lalley-Woods theorem gives asymptotics of $[z^m]\phi_f(z)$
- Putting all this together proves the existence of the prob. distribution P

  For any boolean function f, we compute

$$P(f) = \lim_{m \to +\infty} \frac{[z^m]\phi_f(z)}{[z^m]\Phi(z)}$$

Probability of a boolean function

- We have proved the existence of $P(f)$ for any f

  $(f \notin S_0$: $P(f) = 0)$

- *Can we compute explicitly the probability of a boolean function?*

Probability of a boolean function

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- *Can we compute explicitly the probability of a boolean function?*

- The complexity of a function f is the smallest size of a tree that computes f.

- *What is the relation beween the complexity and the probability of a boolean function?*

- *What is the typical shape of a tree that computes a specific function?*

- *What is the average complexity of a random boolean function?*

Tautologies

We begin with the simplest function: the constant $True$

- Simple tautology: a premise is equal to the goal.
- We know the probability of simple tautologies:

$$\frac{4n+1}{(2n+1)^2} \sim \frac{1}{n}$$

- Almost all tautologies are simple (Fournier et al. 07)
- Hence $P(True) \sim 1/n$
- Consequence: almost all tautologies in the system of implication and positive literals are intuitionnistic tautologies.

Probability of boolean functions

We know a.s. the shape of a random tautology.

We can compute the probability of $True$.

Can we extend this to a non-constant boolean function f?

Probability of boolean functions

- *True*: $1/n + O\left(1/n^2\right)$
- Literal x: $1/2n^2 + O\left(1/n^3\right)$
- Function $x \rightarrow y$: $9/16n^3 + O\left(1/n^4\right)$
- For all $f \in S_0 \setminus \{1\}$:

$$P(f) = \frac{\lambda(f)}{4^{L(f)} n^{L(f)+1}} \left(1 + O(1/n)\right)$$

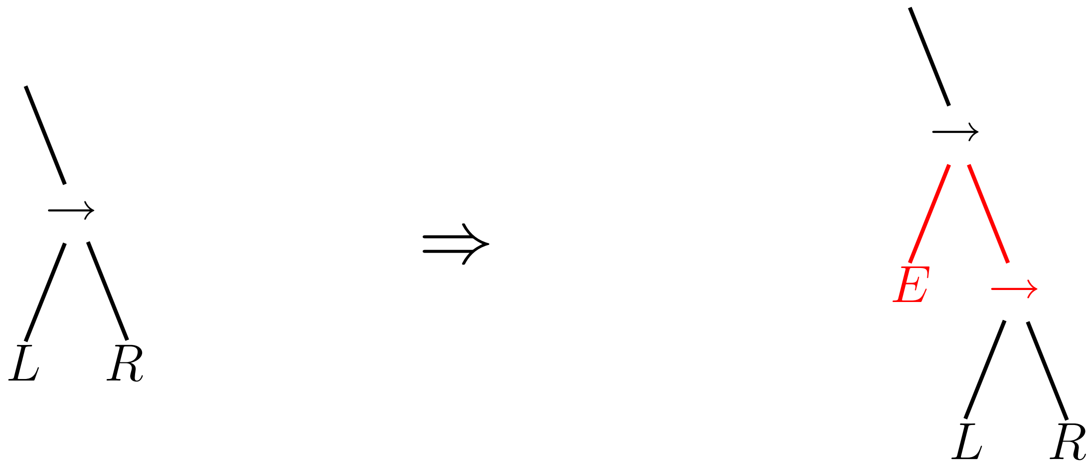
  - $\lambda(f)$ is related to the minimal trees for f
  - The trees of $A(f)$ are simple: a.s. obtained from a minimal tree by a single expansion

Sketch of proof

- Start from the set of minimal trees that compute f.
- Define extension rules: we obtain a larger (infinite) set of trees, still computing f; we can compute the probability of this set.
- Probability of this new set is related to the sizes of the initial trees, i.e. to the tree complexity of f.
- Do we obtain a.s. all the trees that compute f?
- If so, we know the probability of f, and we can express it in terms of its complexity.

Extensions of minimal trees

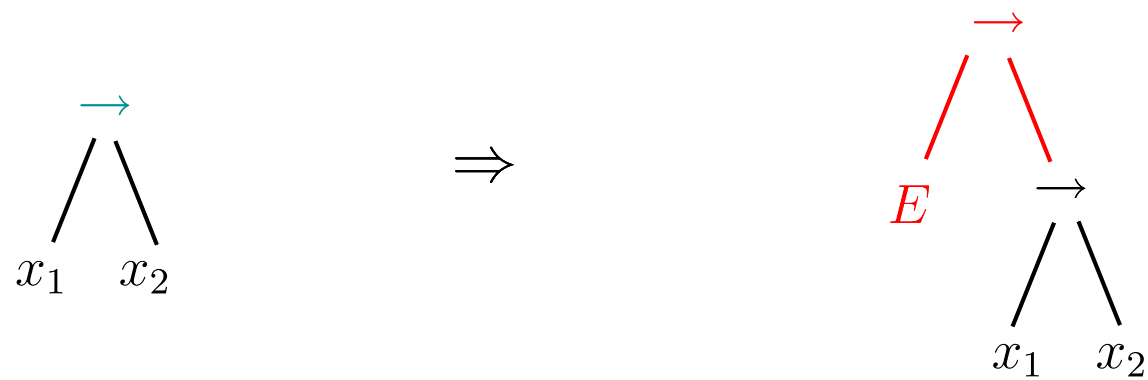
Consider a tree A that computes f, and a node of A



When can we expand a node of A, and still get a tree that computes f??

Extensions of minimal trees: example

$f = x_1 \rightarrow x_2$ has a unique minimal tree A_{min}:



- E is a tautology
- E has goal x_1
- E has a premise x_2

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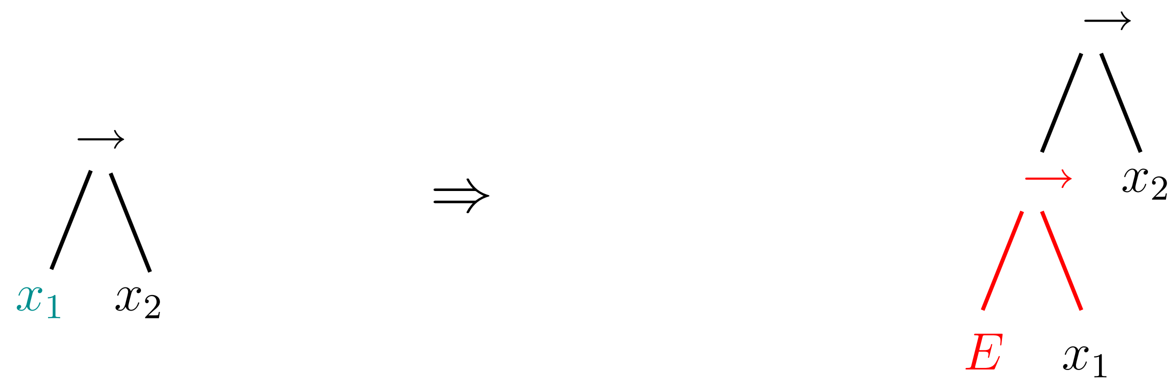

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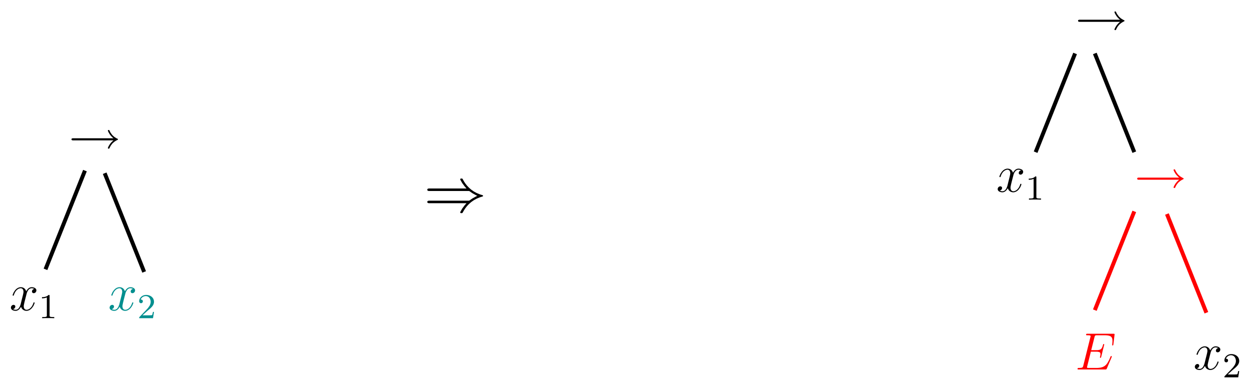

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Extensions of minimal trees: example

$f = x_1 \rightarrow x_2$ has a unique minimal tree A_{min}

- Nine possible types of expansion $\Rightarrow$ set $\mathcal{E}(A_{min})$ of trees computing f
- We can compute the probability of $\mathcal{E}(A_{min})$:

$$\frac{9}{16n^3} + O\left(\frac{1}{n^4}\right)$$

- This is the probability of f

Extensions of minimal trees

- Define extensions for minimal trees
- Compute probability of the set $\mathcal{E}(f)$ obtained by one extension
- Compute probability of the set $\mathcal{E}^+(f)$ obtained by a finite number of extensions
- Compute probability of $A(f) \setminus \mathcal{E}^+(f)$:
  - Define pruning rules: inverses of expansion rules
  - Any tree of $A(f)$ can be pruned into an irreducible tree
  - $\{$Minimal trees$\} \subset \{$Irreducible trees$\}$
  - Almost all trees of f can be pruned into irreducible trees.

Probability of a boolean function f

- Expression of the probability

$$P(f) = \frac{\lambda(f)}{4^{L(f)} n^{L(f)+1}} \, (1 + O(1/n))$$

- We obtain almost all the trees by a single expansion of a minimal tree

$$P(f) = Proba(\mathcal{E}(f) \, (1 + o(1))$$

- The number of possible expansions is related to properties of minimal trees:
  - m = number of minimal trees for f
  - e = number of essential variables of f

  Then

$$2(2m-1)L(f) \leq \lambda(f) \leq (1+2e)(2L(f)-1)m$$

Possible extensions

- Computation of the constant factor $\lambda(f)$?

  Done for read-once functions; for other functions?

- Result can be adapted when trees are obtained by a growing process

- What if we allow negative literals?

- What if we choose a different set of connectors?

Average complexity of a boolean function

- For a uniform distribution on boolean functions, maximal and average tree complexity is $2^k / \log k$ (Shannon, Lupanov...)
- What if the distribution is not uniform? for example, a tree distribution?
- We have computed the probability of a boolean function of *known* (hence, "fixed, small" and independent of k) complexity.
- What about the probability of a function of "large" (dependent on k) complexity?