The Smallest Component Size in Decomposable Structures

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We consider labelled structures built upon components by the multi-set construction. Let \( f_k \) (\( c_k \)) be the number of labeled structures (components) of size \( k \), and consider their exponential generating functions

\[
F(z) = \sum_{k \geq 0} f_k \frac{z^k}{k!}, \text{ and } C(z) = \sum_{k \geq 1} c_k \frac{z^k}{k!}.
\]

The Exponential Formula says

\[
F(z) = \exp(C(z)).
\]
We turn the set of all structures of size \( n \) into a probability space using the uniform distribution. Let \( X_n(r) \) be the size of the \( r \)th smallest component in a random structure of size \( n \). We will derive results about the limiting distribution of \( X_n(r) \) when the component generating function \( C(z) \) is of algebraic-logarithmic type.

To do that, we will study asymptotic properties of structures with a restricted pattern. When the component generating function \( C(z) \) is of logarithmic type, the smallest component size has been studied by Panario-Richmond (2001), Arratia-Barbour-Tavaré (2003), and Dong-Gao-Panario (2007).
Let $J$ be a set of positive integers, $N = \{0, 1, 2, \cdots \}$, and $S$ be a function from $J$ to $N$. We say that a decomposable structure has a **restricted pattern** $S$ when the number of components of size $j$ in the structure is specified as $S(j)$ for each $j \in J$. The following notation will be used throughout the talk.

- $|J|$ denotes the number of elements in $J$,
- $\hat{j} = \max\{j : j \in J\}$,
- $m = n - \sum_{j \in J} jS(j)$ denotes the degree of freedom of a structure of size $n$ and with a restricted pattern $S$.

We also use

$$C(z; J) = \sum_{j \in J} c_j \frac{z^j}{j!}.$$
The following is a simple extension of the Exponential Formula. 

**Lemma 1:** Let $S : J \mapsto N$ be a given restricted pattern. The exponential generating function of labeled structures with a restricted pattern $S$ is

$$F(z; S) = \exp \left( C(z) - C(z; J) \right) \prod_{j \in J} \frac{c_j^{S(j)} z^{jS(j)}}{(j!)^{S(j)} S(j)!}.$$
The $\triangle(\nu, \theta)$ region

For constants $\nu$ and $\theta$ with $\nu > 0$, $0 < \theta < \pi/2$, define the $\Delta$ region

$$\triangle(\nu, \theta) = \{z : |z| < 1 + \nu, z \neq 1, |\text{arg}(z - 1)| > \theta\}$$

Figure: The $\Delta$ region
In this talk we assume that our component generating function $C(z)$ is of *algebraic-logarithmic type* at singularity $\rho > 0$, that is, $C(\rho z)$ is analytic in $\Delta(\nu, \theta)$ and

$$C(\rho z) = c + d(1 - z)^\alpha \left( \ln \frac{1}{1 - z} \right)^\beta (1 + o(1)),$$

as $z \to 1$ in $\Delta(\nu, \theta)$.

Here $\alpha$ is called the algebraic exponent, and $\beta$ the logarithmic exponent.

The special case $\alpha = 0$, $\beta = 1$ (called the *logarithmic type*) has been studied extensively before.
We first consider the case that the algebraic exponent $\alpha$ satisfies $0 < \alpha < 1$. In this case,

$$C(\rho z) = c + d(1 - z)^\alpha \left( \ln \frac{1}{1 - z} \right)^\beta (1 + o(1)),$$

$$F(\rho z) = e^c \left( 1 + d(1 - z)^\alpha \left( \ln \frac{1}{1 - z} \right)^\beta \right).$$

Flajolet-Odlyzko’s transfer theorem gives

$$[z^n] C(\rho z) \sim \frac{d}{\Gamma(-\alpha)} (\log n)^\beta n^{-1-\alpha},$$

$$[z^n] F(\rho z) \sim \frac{d \exp(c)}{\Gamma(-\alpha)} (\log n)^\beta n^{-1-\alpha}.$$
More Notation

- \([z^n]G(z)\) denotes the coefficient of \(z^n\) in the generating function \(G(z)\).
- \(X_n(r)\) denotes the size of the \(r\)th smallest component of a random decomposable combinatorial structure of size \(n\).
- \(N_k = \{1, 2, \ldots, k\}\).
- \[K(S) = \prod_{j \in J} \frac{(c_j \rho^j / j!)^{S(j)}}{S(j)!} \exp \left(-c_j \rho^j / j!\right)\]
Theorem 1: Let $S$ be a restricted pattern such that $|J| = o\left(m(\log m)^{\frac{1-\alpha}{1-\alpha}}\right)$ and $\hat{j} = O(m/\log m)$. Then, as $m \to \infty$, the probability that a random structure of size $n$ has the pattern $S$ is given by

$$\frac{[z^n]F(z; S)}{[z^n]F(z)} \sim K(S) \left(\frac{\log m}{\log n}\right)^\beta \left(\frac{n}{m}\right)^{\alpha+1},$$

where the asymptotics is uniform over all $J$. 
The Probability of Having a Restricted Pattern

When the restricted pattern $S$ is small such that $m \sim n$, we have

$$\frac{f_n(S)}{f_n} \sim \prod_{j \in J} \left( \frac{c_j \rho^j / j!}{S(j)!} \right)^{S(j)} \exp \left( -\frac{c_j \rho^j}{j!} \right).$$

This result can be restated as follows.

**Corollary 1:** Let $Z_n(j)$ be the number of components of size $j$ in a random structure of size $n$. Suppose $|J| = o \left( n (\log n)^{\frac{1}{1-\alpha}} \right)$ and $\hat{j} = O(n / \log n)$, then $(Z_n(j) : j \in J)$ are asymptotically independent Poisson random variables with mean $c_j \rho^j / j!$ for each $j \in J$.

Similar result for the logarithmic type was obtained by Arratia-Stark-Tavaré (1995) and Dong-Gao-Panario (2007).
Sketch of the proof of Theorem 1

From Lemma 1, we have

\[ [z^n]F(z; S) = \rho^{-n}[z^n]F(\rho z; S) \]
\[ = \rho^{-n}K(S)[z^m] \exp (C(\rho z) - C(\rho z; J) + C(\rho; J)) . \]

Using Cauchy’s integral formula and the conditions on \( J \), one can prove that

\[ [z^m] \exp (C(\rho z) - C(\rho z; J) + C(\rho; J)) \sim [z^m] \exp (C(\rho z)) . \]

Thus

\[ [z^n]F(z; S) \sim K(S) \frac{d \exp(c)}{\Gamma(-\alpha)} (\log m)^\beta m^{-1-\alpha} \rho^{-n} . \]
Sketch of the proof of Theorem 1

Figure: The contour
The $r$th Smallest Component Size

To keep track of the $r$th smallest component size, we consider a pattern $S : N_k \mapsto N$, where $S(j)$ is specified below. We note that each structure with its $r$th smallest component size greater than $k$ corresponds to a structure with a pattern $S$ such that $\sum_{i \in N_k} S(i) \leq r - 1$. Hence we have

$$P(X_n(r) > k) = \sum \left\{ \frac{f_n(S)}{f_n} : \sum_{i \in N_k} S(i) \leq r - 1 \right\}.$$

When $r = O(\log n)$ and $k = o\left(n(\log n)^{\frac{-1}{1-\alpha}}\right)$, $S$ satisfies the conditions of Theorem 1, and $m \sim n$. 
**Corollary 2:** Suppose $r = O(\log n)$ and $k = o \left( n(\log n)^{-\frac{1}{1-\alpha}} \right)$. Then, as $n \to \infty$,

$$P(X_n(r) > k) \sim \exp(-C(\rho; N_k)) \sum_{j=0}^{r-1} \frac{C(\rho; N_k)^j}{j!}.$$ 

**Corollary 3:** The expected size of the smallest component is asymptotic to $ne^{-c}$. 
In the following we consider the case

\[ C(z) = d(1 - z/\rho)^{-p} + b \ln \frac{1}{1 - z/\rho} + c + o(1) \text{ as } z \to \rho. \]

For convenience we define \( h(z) = d(1 - z)^{-p} + b \ln \frac{1}{1 - z} \). The asymptotics of \([z^n] \exp(h(z))\) has been studied by Wright (1949) and Hayman (1956). In particular, it is known that, for \(0 < p < 2\),

\[ [z^n] \exp(h(z)) \sim \frac{1}{\sqrt{2\pi p(p + 1)d}} \exp \left( (1 + p)d \left( \frac{n}{pd} \right)^{\frac{p}{p+1}} \right) \]

\[ + \frac{pd}{2} \left( \frac{n}{pd} \right)^{\frac{p-1}{p+1}} + \frac{b - 1}{p + 1} \ln \frac{n}{pd} \]
The Case $\alpha = -p < 0$

Figure: The contour through the saddle point $R$, where

$$R(1 - R)^{-p-1} = \frac{m}{pd}.$$
The Case $\alpha = -p < 0$

We can extend the result of Wright and Hayman to generating functions including a restricted pattern $S$, using the saddle point method.

**Theorem 2:** For each $0 < p < 2$, there is a positive constant $0 < \eta < 1/(p + 1)$ such that if a pattern $S$ satisfies $\hat{\lambda} = O(m^\eta)$, then

$$[z^n]F(z; S) \sim \frac{K(S)}{\sqrt{2\pi p(p + 1)d}} \exp \left( (1 + p)d \left( \frac{m}{pd} \right)^{\frac{p}{p+1}} \right)$$

$$+ \frac{pd}{2} \left( \frac{m}{pd} \right)^{\frac{p-1}{p+1}} + \frac{b - 1}{p + 1} \ln \left( \frac{m}{pd} + c \right)$$
Corollary 4: Let $Z_n(j)$ be the number of components of size $j$ in a random structure of size $n$. Suppose $\hat{j} = O(n^\eta)$, then $(Z_n(j) : j \in J)$ are asymptotically independent Poisson random variables with mean $c_j \rho^j / j!$ for each $j \in J$.

Corollary 5: Suppose $r = O(\log n)$ and $k = O(n^\eta)$. Then, as $n \to \infty$, we have

$$P(X_n(r) > k) \sim \exp(-C(\rho; N_k)) \sum_{j=0}^{r-1} C(\rho; N_k)^j / j!.$$

Corollary 6: The expected size of the smallest component is asymptotic to the constant

$$\sum_{k \geq 0} \exp(-C(\rho; N_k)).$$
Examples

- A rooted labeled tree consists of a set of components (subtrees). The component generating function $C(z)$ is of alg-log type at the singularity $1/e$ with algebraic exponent $\alpha = 1/2$:

$$C(z) = 1 - \sqrt{2}(1 - ez)^{1/2} + O(1 - ez).$$

The expected size of the smallest subtree is asymptotic to $n/e$.

- A *fragmented permutation* is a set of permutations. The component generating function for fragmented permutations is $C(z) = \frac{z}{1 - z} = \frac{1}{1 - z} - 1$. We have $C(\rho, N_k) = k$, and the expected size of the smallest component is asymptotic to $\sum_{k \geq 0} e^{-k} = \frac{e}{e - 1}$.
Let $Y_n$ denote the number of components in a random structure of size $n$, and $X_n$ be the smallest component size.

- When $C(z)$ is of logarithmic type, $Y_n$ is asymptotically normal with expected value and variance both proportional to $\ln n$. $E(X_n)$ is also proportional to $\ln n$.

- When $C(z)$ is of alg-log type with algebraic exponent $0 < \alpha < 1$, we have $P(Y_n = k) \sim \frac{e^{-c}c^{k-1}}{(k-1)!}$. That is, $Y_n - 1$ is asymptotically Poisson with mean $c$. $E(X_n)$ is asymptotic to $ne^{-c}$.

- When $C(z)$ is of alg-log type with algebraic exponent $\alpha < 0$, $Y_n$ is also asymptotically normal with mean and variance both proportional to $n^{p/(p+1)}$. $E(X_n)$ is asymptotic to a constant.