Combinatorial Enumeration

Jason Z. Gao
Carleton University, Ottawa, Canada

Counting Combinatorial Structures

We are interested in counting combinatorial (discrete) structures of a given "size". For example, binary sequences of length n, trees with n vertices (or edges), permutations of n elements, number of ways to partition a set with n elements, number of ways to partition a positive integer n, polynomials of degree n over a finite field.

Labelled structures are structures involving labels, e.g., permutations, set partitions, and labelled trees.

Unlabelled structures are structures not involving labels, e.g., integer partitions, polynomials, and unlabelled trees.

Two Basic Counting Principles

Let $\mathcal{S}$ and $\mathcal{T}$ be two finite sets. Then $|\mathcal{S}|$ denotes the cardinality (the number of elements) of $\mathcal{S}$, and $\mathcal{S} \times \mathcal{T}$ denotes the Cartesian product, that is, the set of ordered pairs (s, t) with $s \in \mathcal{S},\ t \in \mathcal{T}$.

The Addition Principle: Suppose $\mathcal{S}$ and $\mathcal{T}$ are two disjoint sets, then

$$|\mathcal{S} \cup \mathcal{T}| = |\mathcal{S}| + |\mathcal{T}|.$$

The Multiplication Principle:

$$|\mathcal{S} \times \mathcal{T}| = |\mathcal{S}| \times |\mathcal{T}|.$$

Generating Functions

Let $\mathcal{A}$ be a set of structures, and $\mathcal{A}_n$ be the set of structures in $\mathcal{A}$ of size n. We assume that $\mathcal{A}_n$ is a finite set. Let a_n be the number of elements in $\mathcal{A}$ of size n.
The formal power series $A(x) = \sum_{n\geq 0} a_n x^n$ is called the ordinary generating function (OGF) of $\mathcal{A}$, and $\hat{A}(x) = \sum_{n\geq 0} a_n x^n/n!$ is called the exponential generating function (EGF) of $\mathcal{A}$.

Generating Functions

We will see that generating functions are very powerful tools in combinatorial enumeration, which link discrete structures with continuous functions. Ordinary generating functions are usually linked with unlabelled structures and exponential generating functions are usually linked with labelled structures.

Examples of Generating Functions

Let $\mathcal{A}$ be the set of binary sequences, including the empty sequence. Here the size of a sequence is the length of the sequence. It is easy to see that $|\mathcal{A}_n| = 2^n$. Hence

$$\begin{aligned} A(x) &= \sum_{n\geq 0} 2^n x^n = \frac{1}{1-2x}, \\ \hat{A}(x) &= \sum_{n\geq 0} 2^n x^n/n! = \exp(2x). \end{aligned}$$

Examples of Generating Functions

Let $\mathcal{A}$ be the set of permutations, including the empty permutation. The size here is the number of elements of a permutation, and it is clear that $|\mathcal{A}_n| = n!$. Thus we have

$$\begin{aligned} A(x) &= \sum_{n \geq 0} n! x^n, \\ \hat{A}(x) &= \sum_{n \geq 0} n! x^n / n! = \frac{1}{1-x}. \end{aligned}$$

The Product Formula for OGF of Unlabelled Structures

Let $A(x)$ be the OGF for $\mathcal{A}$ and $B(x)$ be the OGF for $\mathcal{B}$. Suppose $\mathcal{A} \cap \mathcal{B} = \emptyset$, then it is easy to see that $A(x) + B(x)$ is the OGF for $\mathcal{A} \cup \mathcal{B}$.
Suppose that the size of $(a, b) \in \mathcal{A} \times \mathcal{B}$ is the sum of the sizes of a and b. Then the number of structures in $\mathcal{A} \times \mathcal{B}$ of size n is equal to $\displaystyle\sum_{j=0}^{n} a_j b_{n-j}$. Thus the OGF for $\mathcal{A} \times \mathcal{B}$ is given by

$$\sum_{n \geq 0} \left(\sum_{j=0}^{n} a_j b_{n-j} \right) x^n = A(x)B(x).$$

Counting Unlabelled Structures with OGF

Example

Let $\mathcal{A}$ be the set of all binary sequences. We note that each nonempty binary sequence is uniquely decomposed into an ordered pair (a, b), where a is the first bit of the sequence and b consists of the remaining bits of the sequence. We can write

$$\mathcal{A} = \mathcal{A}_0 \cup (\mathcal{A}_1 \times \mathcal{A}).$$

Hence we have $A(x) = 1 + (2x)A(x)$, where 1 is the OGF for $\mathcal{A}_0$ and $2x$ is the OGF for $\mathcal{A}_1$.
Thus we have

$$A(x) = \frac{1}{1-2x} = \sum_{n \geq 0} 2^n x^n.$$

Counting Unlabelled Structures with OGF

Example

Let $\mathcal{A}$ be the set of binary sequences with no adjacent 0's. We have

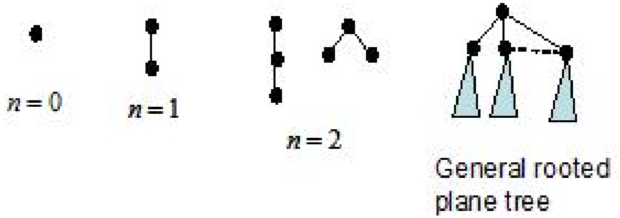
$$\mathcal{A} = \mathcal{A}_0 \cup (\{1\} \times \mathcal{A}) \cup \{0\} \cup (\{01\} \times \mathcal{A}).$$

Thus $A(x) = 1 + xA(x) + x + x^2 A(x)$, or

$$\begin{aligned} A(x) &= \frac{1+x}{1-x-x^2} \\ &= \sum_{n\geq 0} \frac{1}{\sqrt{5}} \left(\left(\frac{1+\sqrt{5}}{2} \right)^{n+2} - \left(\frac{1-\sqrt{5}}{2} \right)^{n+2} \right) x^n. \end{aligned}$$

Counting Unlabelled Structures with OGF

Let $\mathcal{A}_n$ be the set of rooted plane trees with n edges.



Counting Unlabelled Structures with OGF

First Approach: We note

$$\mathcal{A} = \cup_{k\geq 0}(\mathcal{A}_1 \times \mathcal{A})^k.$$

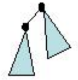
That is, an element in $\mathcal{A}$ is decomposed into a *sequence* of elements in $\mathcal{A}_1 \times \mathcal{A}$. Hence

$$A(x) = \sum_{k\geq 0}(xA(x))^k = \frac{1}{1 - xA(x)}.$$

So we have

$$A(x) = \frac{1 - \sqrt{1-4x}}{2x} = \sum_{n\geq 0} \frac{1}{n+1}\binom{2n}{n}x^n.$$

Second Approach: Decompose a rooted plane tree with at least one edge into three components: the left-most subtree, the left-most edge incident with the root vertex, and the rest.



Rooted plane tree with at least one edge

This gives

$$A(x) = 1 + xA(x)A(x).$$

The Product Formula for EGF of Labelled Structures

Now we consider labelled structures such that each structure of size n is also associated with n standard labels $1, 2, \ldots, n$.
Let $\hat{A}(x)$ be the EGF for $\mathcal{A}$ and $\hat{B}(x)$ be the EGF for $\mathcal{B}$.
Suppose $\mathcal{A} \cap \mathcal{B} = \emptyset$, then $\hat{A}(x) + \hat{B}(x)$ is the EGF for $\mathcal{A} \cup \mathcal{B}$.
Now we consider labelled structures formed by ordered pairs (a, b) of structures in $\mathcal{A}$ and $\mathcal{B}$. There is an extra factor to be considered here. That is the distribution of labels among a and b.
We again assume that the size of (a, b) is the sum of the sizes of a and b.

The Product Formula for EGF of Labelled Structures

Now a structure (a, b) of size n is obtained by taking a structure $a \in \mathcal{A}_k$, a structure $b \in \mathcal{B}_{n-k}$, **and** a distribution of labels $1, 2, \ldots, n$ among a and b. Since there are $\binom{n}{k}$ ways to distribute k labels to a and $n-k$ remaining labels to b, the number of structures (a, b) of size n is equal to

$$\sum_{k=0}^{n} \binom{n}{k} a_k b_{n-k}.$$

The Product Formula for EGF of Labelled Structures

Let $\mathcal{A} \otimes \mathcal{B}$ denote the set of (a, b) with a distribution of labels. Then the EGF for structures $\mathcal{A} \otimes \mathcal{B}$ is

$$
\begin{aligned}
& \sum_{n \geq 0} \left(\sum_{k=0}^{n} \binom{n}{k} a_k b_{n-k} \right) x^n / n! \\
= \ & \sum_{n \geq 0} \left(\sum_{k=0}^{n} \frac{a_k}{k!} \frac{b_{n-k}}{(n-k)!} \right) x^n \\
= \ & \hat{A}(x) \hat{B}(x).
\end{aligned}
$$

Counting Labelled Structures with EGF

Example Derangements are permutations with no fixed elements. Let $\mathcal{S}$ be the set of all permutations, $\mathcal{D}$ be the set of derangements, and $\mathcal{F}$ be the set of sets (of fixed points). We note that each permutation is decomposed uniquely into a set of fixed points and a derangement, that is, $\mathcal{S} = \mathcal{D} \otimes \mathcal{F}$. Also the EGF for $\mathcal{F}$ is

$$\hat{F}(x) = \sum_{n \geq 0} x^n/n! = e^x.$$

Thus

$$e^x \hat{D}(x) = \hat{S}(x) = \frac{1}{1-x},$$

$$\hat{D}(x) = \frac{1}{1-x} e^{-x}.$$

Counting Labelled Structures with EGF

Recall that each permutation is decomposed uniquely into an *unordered* collection of cycles. For each $k \geq 1$, let $\hat{A}_k(x)$ denote the EGF for permutations with exactly k cycles. Since there are $(n-1)!$ cyclic permutations of size n, we have

$$\hat{A}_1(x) = \sum_{n\geq 1}(n-1)!x^n/n! = \sum_{n\geq 1} x^n/n = \ln\frac{1}{1-x}.$$

Noting that the order of the cycles does not matter, we obtain

$$\hat{A}_2(x) = (1/2)\hat{A}_1(x)\hat{A}_1(x) = (1/2)\left(\ln\frac{1}{1-x}\right)^2.$$

Counting Labelled Structures with EGF

In general we have

$$\hat{A}_k(x) = (1/k!)(\hat{A}_1(x))^k = (1/k!).$$

Now suppose $\hat{D}_k(x)$ denotes the EGF for derangements with exactly exact k cycles. Since a derangement has no cycle of length 1, we have

$$\hat{D}_1(x) = \sum_{n\geq 2}(n-1)!x^n/n! = \ln\frac{1}{1-x} - x, \; \hat{D}_k(x) = (1/k!)(\hat{D}_1(x))^k,$$

and

$$\hat{D}(x) = \sum_{k\geq 1}(1/k!)(\hat{D}_1(x))^k = \exp(\hat{D}_1(x)) = \frac{1}{1-x}e^{-x}.$$

The Exponential Formula for Labelled Structures

We saw that permutations are decomposed into an unordered collection of cycles. Many combinatorial structures have similar decompositions. For example, a graph is decomposed into connected components, a rooted tree is decomposed into subtrees.

Let $\mathcal{C}$ be a set of labelled structures, called the *components*.

Let $\mathcal{F}$ be the set of labelled structures obtained by taking any unordered collection of structures in $\mathcal{C}$, and by distributing the labels. We say that $\mathcal{F}$ is constructed from $\mathcal{C}$ by the *multi-set construction*.

The Exponential Formula for Labelled Structures

Let $\hat{C}(x)$ be the EGF for the components (structures in $\mathcal{C}$, and $\hat{F}(x)$ be the EGF for the structures in $\mathcal{F}$. We have

$$\hat{F}(x) = \sum_{k \geq 0} (1/k!)(\hat{C}(x))^k = \exp(\hat{C}(x)).$$

This is called the *Exponential Formula* for labelled structures.

The Exponential Formula for Labelled Structures

Example

Let $\hat{G}(x)$ be the EGF of labelled graphs, and $\hat{C}(x)$ be the EGF of labelled connected graphs, where the size of a graph is the number of vertices. We have

$$\hat{G}(x) = \exp(\hat{C}(x)), \quad \hat{C}(x) = \ln \hat{G}(x).$$

Since there are $2^{\binom{n}{2}}$ labelled graphs with n vertices, we have

$$\hat{G}(x) = \sum_{n \geq 0} 2^{\binom{n}{2}} x^n / n!.$$

The Exponential Formula for Labelled Structures

Example

Let $\hat{A}(x)$ be the EGF of partitions of $\{1, 2, \ldots, n\}$, and $\hat{C}(x)$ be the EGF of sets. Then we have

$$\hat{C}(x) = \sum_{n \geq 1} x^n/n! = \exp(x) - 1,$$

$$\hat{A}(x) = \exp(\hat{C}(x)) = \exp(\exp(x) - 1).$$

The Exponential Formula for Labelled Structures

Example

Let $\hat{A}(x)$ be the EGF of rooted labelled trees, where the size of a tree is the number of vertices. We note that each rooted tree is decomposed into a vertex and an unordered collection of rooted subtrees. Thus

$$\hat{A}(x) = x \exp(\hat{A}(x)).$$

One may apply Lagrange inversion formula to obtain

$$\hat{A}(x) = \sum_{n \geq 0} n^{n-1} x^n / n!.$$

This implies that there are n^{n-1} rooted labelled trees with n vertices.

The Exponential Formula for Unlabelled Structures

Now we consider unlabelled structures consisting of a multi-set of components. Let $\mathcal{F}$ be the set of unlabelled structures which are built by taking a multi-set of components from $\mathcal{C}$. Let c_k be the number of structures in $\mathcal{C}$ of size k. For each $c \in \mathcal{C}$, let $SEQ(c)$ denote the set of sequences of c. Then

$$\mathcal{F} = \prod_{c \in \mathcal{C}} SEQ(c),$$

and hence

$$F(x) = \prod_{j \geq 1} \prod_{c \in \mathcal{C}_j} \frac{1}{1 - x^j} = \prod_{j \geq 1} \left(\frac{1}{1 - x^j} \right)^{c_j}.$$

The Exponential Formula for Unlabelled Structures

We note

$$\begin{aligned} F(x) &= \prod_{j\geq 1} \frac{1}{(1-x^j)^{c_j}} = \exp\left(\sum_{j\geq 1} c_j \ln(1-x^j)^{-1}\right) \\ &= \exp\left(\sum_{j\geq 1} c_j \sum_{k\geq 1} x^{kj}/k\right) = \exp\left(\sum_{k\geq 1} (1/k) \sum_{j\geq 1} c_j x^{kj}\right) \\ &= \exp\left(\sum_{k\geq 1} (1/k) C(x^k)\right). \end{aligned}$$

This is called the Exponential Formula for unlabelled structures.

The Exponential Formula for Unlabelled Structures

Example

A partition of a positive integer n is a multi-set of positive integers whose sum is n. Here the OGF of components is

$$C(x) = \sum_{n \geq 1} x^n = \frac{x}{1-x}.$$

Hence the OGF of integer partitions is

$$P(x) = \exp\left(\sum_{k \geq 1} \frac{1}{k} \frac{x^k}{1-x^k} \right).$$

The Exponential Formula for Unlabelled Structures

Example

Let $\mathbb{F}_q$ denote the finite field with q elements. Let $C(x)$ be the OGF for monic irreducible polynomials, and $F(x)$ be the OGF for all monic polynomials over $\mathbb{F}_q$. Then we have

$$\exp\left(\sum_{k\geq 1}(1/k)C(x^k)\right) = F(x) = \sum_{n\geq 0} q^n x^n = \frac{1}{1-qx},$$

and hence $\sum_{k\geq 1}(1/k)C(x^k) = \ln\frac{1}{1-qx}$. Using Möbius inversion, we obtain

$$C(x) = \sum_{r\geq 1}\frac{\mu(r)}{r}\ln\frac{1}{1-qx^r}.$$

Bivariate Generating Functions

We may use a bivariate generating function $F(x, y)$ for decomposable structures so that x marks the size of the structure and y marks the number of components of the structure. Let $\mathcal{F}_{n,k}$ denote the set of structures of size n with exactly k components. For labelled structures, we use

$$F(x, y) = \sum_{n \geq 0} \sum_{k \geq 0} |\mathcal{F}_{n,k}| x^n y^k / n!.$$

For unlabelled structures, we use

$$F(x, y) = \sum_{n \geq 0} \sum_{k \geq 0} |\mathcal{F}_{n,k}| x^n y^k.$$

The Exponential Formula for Bivariate Generating Functions

For labelled structures, we have

$$F(x,y) = \sum_{k\geq 0} y^k (\hat{C}(x))^k / k! = \exp(y\hat{C}(x)).$$

For unlabelled structures, we have

$$F(x,y) = \prod_{j\geq 1} \left(1 - yx^j\right)^{-c_j} = \exp\left(\sum_{k\geq 1} (1/k) y^k C(x^k)\right).$$

Use Bivariate Generating Functions to Compute Moments

We may turn $\mathcal{F}_n$ into a probability space by assigning a uniform distribution so that each structure in $\mathcal{F}_n$ has probability $1/|\mathcal{F}_n|$. Let X_n be the number of components of a random structure in $\mathcal{F}_n$. We may use the bivariate generating function to compute $E(X_n)$ and $E(X_n(X_n - 1))$.

$$E(X_n) = \frac{[x^n]F_y(x,1)}{[x^n]F(x,1)}, \quad E(X_n(X_n-1)) = \frac{[x^n]F_{yy}(x,1)}{[x^n]F(x,1)}.$$

Use Bivariate Generating Functions to Compute Moments

Example

Let X_n be the number of cycles of a random permutation of n elements. We had EGF for the components

$$\hat{C}(x) = \ln \frac{1}{1-x}, \ F(x, y) = \exp\left(y \ln \frac{1}{1-x} \right).$$

Hence

$$F_y(x, 1) = \ln \frac{1}{1-x} \exp\left(\ln \frac{1}{1-x} \right) = \frac{1}{1-x} \ln \frac{1}{1-x}.$$

$$[x^n] F_y(x, 1) = \sum_{k=1}^{n} \frac{1}{k} = \ln n + \gamma + o(1),$$

Use Bivariate Generating Functions to Compute Moments

Example

Let X_n be the number of irreducible factors of a random monic polynomial of degree n over $\mathbb{F}_q$. We have

$$F(x,y) = \exp(yC(x)),\ \ C(x) = \sum_{r \geq 1} \frac{\mu(r)}{r} \ln \frac{1}{1-qx^r},$$

$$F(x,1) = \frac{1}{1-qx},\ \ F_y(x,1) = C(x)\exp(C(x)) = \frac{1}{1-qx} C(x).$$

One can show that

$$[x^n]F_y(x,1) \sim [x^n]\left(\frac{1}{1-qx} \ln \frac{1}{1-qx}\right) = q^n \sum_{k=1}^{n} \frac{1}{k}.$$

Use Bivariate Generating Functions to Compute Moments

Example

Let $A(x, y)$ be the OGF for binary sequences with no adjacent 0's such that x marks the length of the sequence and y marks the number of 1's in the sequence. We have

$$A(x, y) = 1 + x + yxA(x, y) + yx^2 A(x, y).$$

Hence

$$A(x, y) = \frac{1 + x}{1 - xy - x^2 y}.$$

$$A(x, 1) = \frac{1 + x}{1 - x - x^2} = \frac{3 + \sqrt{5}}{2\sqrt{5}} \frac{1}{1 - r_1 x} - \frac{3 - \sqrt{5}}{2\sqrt{5}} \frac{1}{1 - r_2 x}.$$

The Binary Sequence Example Continued

Example

$$
\begin{aligned}
A_y(x,1) &= \frac{(1+x)(x+x^2)}{(1-x-x^2)^2} \\
&= \frac{2+\sqrt{5}}{5}\frac{1}{(1-r_1x)^2} - \frac{10+7\sqrt{5}}{25}\frac{1}{1-r_1x} \\
&\quad + \frac{2-\sqrt{5}}{5}\frac{1}{(1-r_2x)^2} - \frac{10-7\sqrt{5}}{25}\frac{1}{1-r_2x},
\end{aligned}
$$

where $r_1 = \dfrac{1+\sqrt{5}}{2}$, $r_2 = \dfrac{1-\sqrt{5}}{2}$. Noting $r_1 > 1$ and $|r_2| < 1$, we have

$$
\frac{[x^n]F_y(x,1)}{[x^n]F(x,1)} \sim \frac{5+\sqrt{5}}{10}n.
$$

Structures with Distinct Components: The Set Construction

Let $D(x, y)$ be the OGF for unlabelled structures with distinct components. Then we have

$$\begin{aligned} D(x,y) &= \prod_{j\geq 1}\left(1+yx^j\right)^{c_j} \\ &= \exp\left(\sum_{k\geq 1}(1/k)(-y)^k(-C(x^k))\right) \\ &= \exp\left(\sum_{k\geq 1}(1/k)(-1)^{k-1}y^kC(x^k)\right). \end{aligned}$$

Compare this with the exponential formula on slide 29.

Structures with Distinct Components: Integer Partitions

The component generating function for integer partitions is

$$C(x) = \frac{x}{1-x}.$$

Hence the OGF for integer partitions with distinct parts is

$$D(x,y) = \prod_{j\geq 1}\left(1+yx^j\right) = \exp\left(\sum_{k\geq 1}(1/k)(-1)^{k-1}y^k\frac{x^k}{1-x^k}\right)$$

Structures with Distinct Components: Polynomials

Recall that the OGF for irreducible polynomials over $\mathbb{F}_q$ is

$$C(x) = \sum_{r \geq 1} \frac{\mu(r)}{r} \ln \frac{1}{1 - qx^r}.$$

Hence the OGF for monic polynomials over $\mathbb{F}_q$ with distinct irreducible factors is

$$D(x, y) = \exp\left(\sum_{k \geq 1} (1/k)(-1)^{k-1} y^k C(x^k) \right)$$

It will be shown later that the asymptotic behavior of $D(x, y)$ is mainly determined by the first term $(k = 1)$.

Structures with Restricted Component Sizes: Integer Partitions

Let $P_o(x)$ be the OGF for integer partitions with odd parts only, and $P_d(x)$ be the OGF for integer partitions with distinct parts. Then we have

$$P_o(x) = \prod_{j \geq 1} \frac{1}{1 - x^{2j-1}} = \prod_{j \geq 1} \frac{1 - x^{2j}}{1 - x^j} = \prod_{j \geq 1} (1 + x^j) = P_d(x).$$