On the number of subtrees on the fringe of random trees

(partly joined with Huilan Chang)

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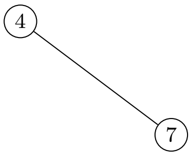

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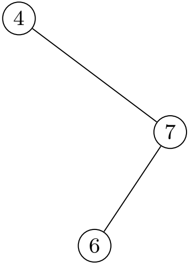

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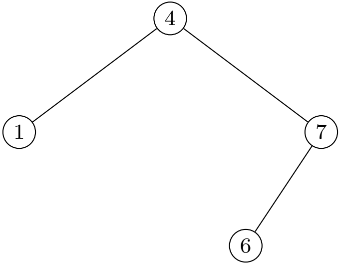

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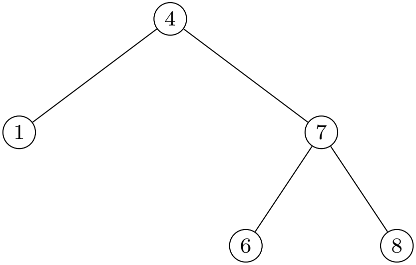

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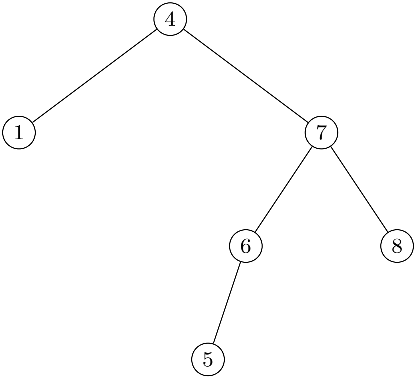

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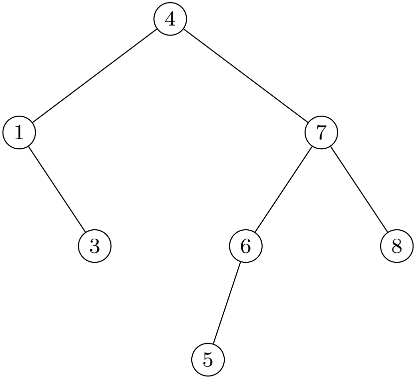

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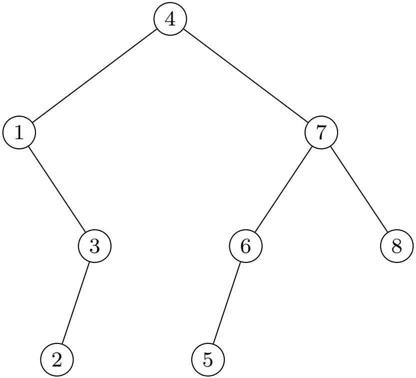

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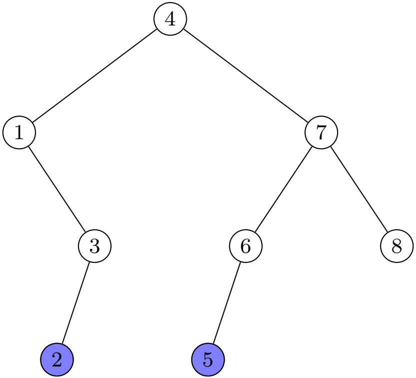

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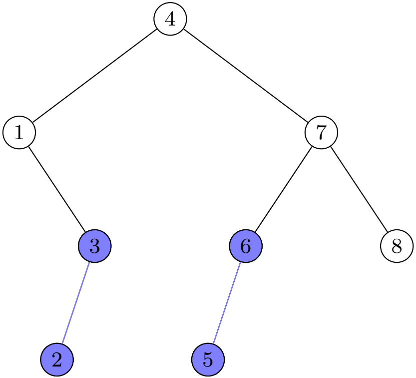

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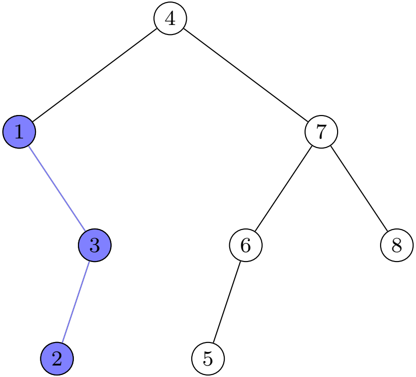

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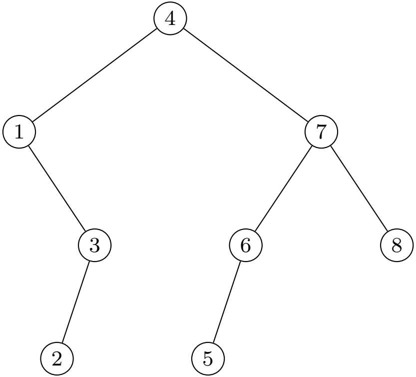

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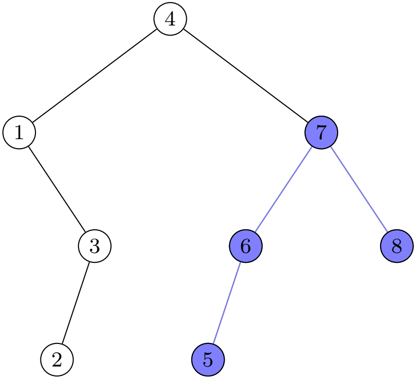

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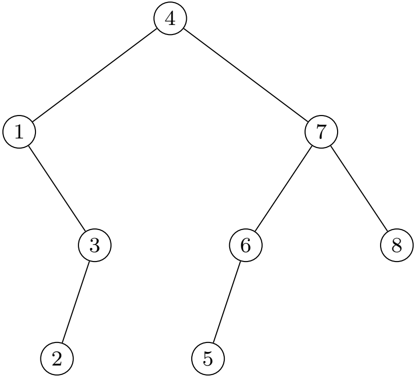

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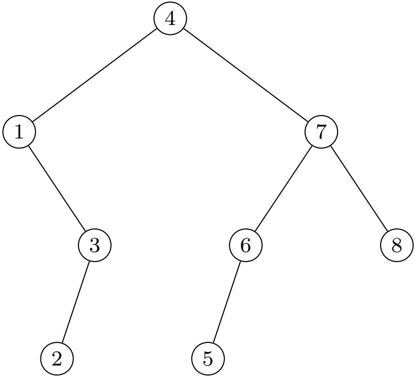

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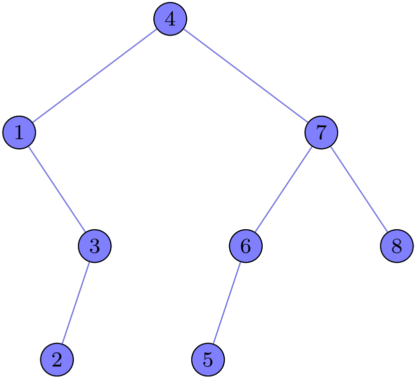

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Mean value and variance

$X_{n,k}$ satisfies

$$X_{n,k} \stackrel{d}{=} X_{I_n,k} + X^*_{n-1-I_n,k},$$

where $X_{k,k} = 1$, $X_{I_n,k}$ and $X^*_{n-1-I_n,k}$ are conditionally independent given I_n, and $I_n = \text{Unif}\{0, \ldots, n-1\}$.

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This yields

$$\mu_{n,k} := \mathbf{E}(X_{n,k}) = \frac{2(n+1)}{(k+1)(k+2)}, \qquad (n > k),$$

and

$$\sigma^2_{n,k} := \mathrm{Var}(X_{n,k}) = \frac{2k(4k^2+5k-3)(n+1)}{(k+1)(k+2)^2(2k+1)(2k+3)}$$

for $n > 2k+1$.

Some previous results

- Aldous (1991): Weak law of large numbers

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$\longrightarrow$ All the above results are for fixed k.

Results for $k = k_n$

Theorem (Feng, Mahmoud, Panholzer (2008))

(i) *(Normal range) Let* $k = o\left(\sqrt{n}\right)$ *and* $k \to \infty$ *as* $n \to \infty$. *Then,*

$$\frac{X_{n,k} - \mu_{n,k}}{\sqrt{2n/k^2}} \xrightarrow{d} \mathcal{N}(0,1).$$

(ii) *(Poisson range) Let* $k \sim c\sqrt{n}$ *as* $n \to \infty$. *Then,*

$$X_{n,k} \xrightarrow{d} \mathrm{Poisson}(2c^{-2}).$$

(iii) *(Degenerate range) Let* $k < n$ *and* $\sqrt{n} = o(k)$ *as* $n \to \infty$. *Then,*

$$X_{n,k} \xrightarrow{L_1} 0.$$

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- $X_{n,k}$ is related to parameters arising in genetics.

Yule generated random genealogical trees

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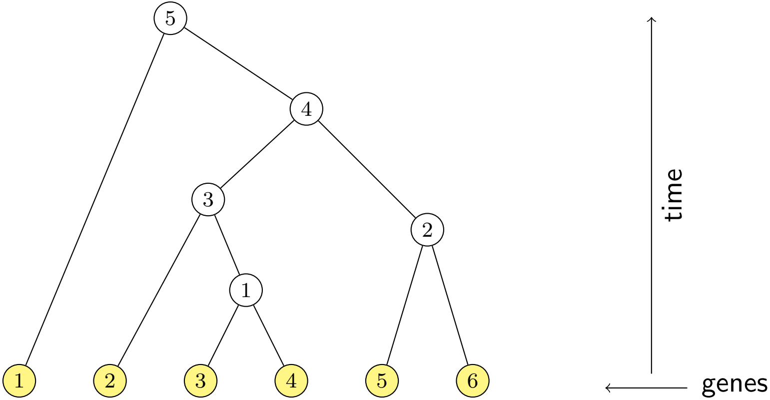

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Random model:

At every time point, two yellow nodes uniformly coalescent.

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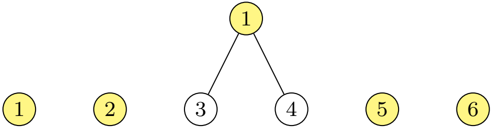

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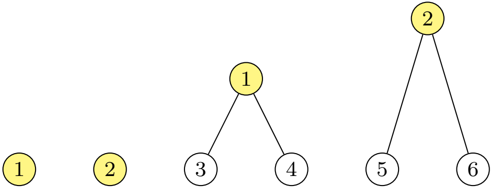

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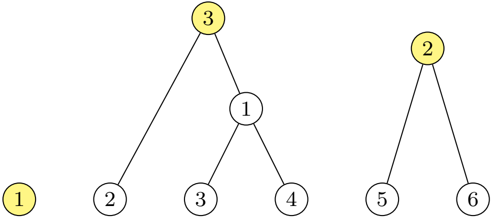

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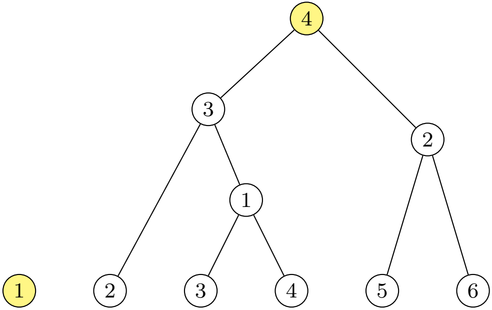

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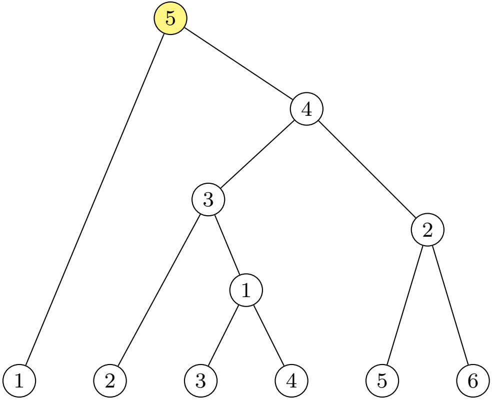

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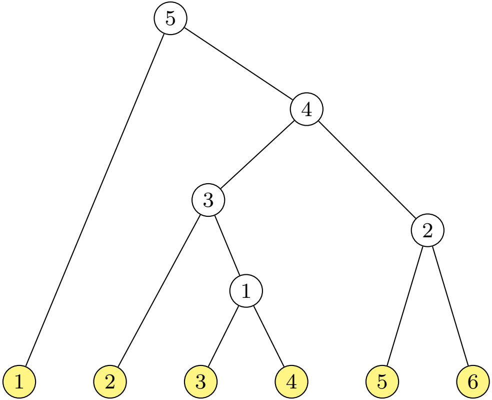

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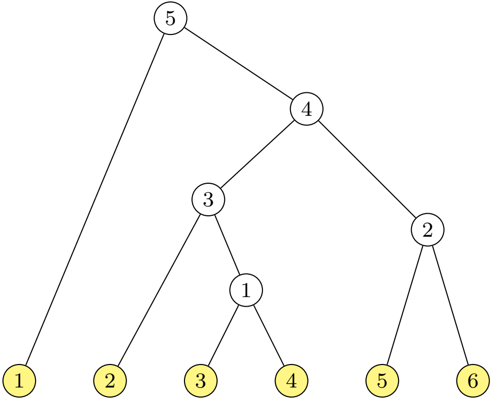

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Same model as random binary search tree model!

Shape parameters of genealogical trees

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- Nodes with minimal clade size k (Blum and François (2005)):

  If $k \geq 3$, then they are internal nodes with induced subtree of size $k-1$ and either an empty right subtree or empty left subtree.

Counting pattern in random binary search trees

Consider $X_{n,k}$ with

$$X_{n,k} \stackrel{d}{=} X_{I_n,k} + X^*_{n-1-I_n,k},$$

where $X_{k,k} = \text{Bernoulli}(p_k)$, $X_{I_n,k}$ and $X^*_{n-1-I_n,k}$ are conditionally independent given I_n, and $I_n = \text{Unif}\{0, \ldots, n-1\}$.

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Then,

p_k	shape parameter
1	# of $k+1$-pronged nodes
$2/k$	# of nodes with minimal clade size $k+1$
$2^{k-1}/k!$	# of $k+1$ caterpillars

Underlying recurrence and solution

All (centered or non-centered) moments satisfy

$$a_{n,k} = \frac{2}{n} \sum_{j=0}^{n-1} a_{j,k} + b_{n,k},$$

where $a_{k,k}$ is given and $a_{n,k} = 0$ for $n < k$.

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We have

$$a_{n,k} = \frac{2(n+1)}{(k+1)(k+2)} a_{k,k} + 2(n+1)\sum_{k<j<n} \frac{b_{j,k}}{(j+1)(j+2)} + b_{n,k},$$

where $n > k$.

Mean value and variance

We have

$$\mathbf{E}(X_{n,k}) = \frac{2(n+1)}{(k+1)(k+2)}p_k, \qquad (n > k),$$

and

$$\mathrm{Var}(X_{n,k}) = \frac{2p_k(4k^3+16k^2+19k+6-(11k^2+22k+6)p_k)(n+1)}{(k+1)(k+2)^2(2k+1)(2k+3)}$$

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Note that

$$\mathbf{E}(X_{n,k}) \sim \mathrm{Var}(X_{n,k}) \sim \frac{2p_k}{k^2} n$$

for $n > 2k+1$ and $k \to \infty$ as $n \to \infty$.

Higher moments

Denote by

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$$A_{n,k}^{(m)} = \frac{2}{n}\sum_{j=0}^{n-1} A_{j,k}^{(m)} + B_{n,k}^{(m)},$$

where

$$B_{n,k}^{(m)} := \sum_{\substack{i_1+i_2+i_3=m\\ 0\le i_1,i_2<m}} \binom{m}{i_1, i_2, i_3} \frac{1}{n}\sum_{j=0}^{n-1} A_{j,k}^{(i_1)} A_{n-1-j,k}^{(i_2)} \Delta_{n,j,k}^{i_3}$$

and

$$\Delta_{n,j,k} = \mathbf{E}(X_{j,k}) + \mathbf{E}(X_{n-1-j,k}) - \mathbf{E}(X_{n,k}).$$

Methods of moments

Theorem

Let Z_n, Z be random variables. Assume that, as $n \to \infty$,

$$\mathbf{E}(Z_n^m) \longrightarrow \mathbf{E}(Z^m)$$

for all $m \geq 1$ and Z is uniquely determined by its moment sequence. Then,

$$Z_n \xrightarrow{d} Z.$$

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Fuchs, Hwang, Neininger (2007): variation of the above scheme to study the profile of random binary search trees and random recursive trees.

Normal range

Proposition

Uniformly for $n, k, m \geq 1$ *and* $n > k$

$$A_{n,k}^{(m)} = \max\left\{\frac{2p_k n}{k^2}, \left(\frac{2p_k n}{k^2}\right)^{m/2}\right\}.$$

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Proposition

For $\mathbf{E}(X_{n,k}) \to \infty$ *as* $n \to \infty$,

$$A_{n,k}^{(2m-1)} = o\left(\left(\frac{2p_k n}{k^2}\right)^{m-1/2}\right), \qquad A_{n,k}^{(2m)} \sim g_m \left(\frac{2p_k n}{k^2}\right)^{m},$$

where

$$g_m = (2m)!/(2^m m!).$$

Poisson range

Consider

$$\bar{A}_{n,k}^{(m)} = \mathbf{E}(X_{n,k}(X_{n,k}-1)\cdots(X_{n,k}-m+1)).$$

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Then, similarly as before:

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$$\bar{A}_{n,k}^{(m)} = \max\left\{\frac{2p_k n}{k^2}, \left(\frac{2p_k n}{k^2}\right)^m\right\}.$$

(ii) *For* $\mathbf{E}(X_{n,k}) \to c$ *and* $k < n$ *as* $n \to \infty$,

$$\bar{A}_{n,k}^{(m)} \longrightarrow c^m.$$

The phase change

Theorem

(i) *(Normal range) Let* $\mathbf{E}(X_{n,k}) \to \infty$ *and* $k \to \infty$ *as* $n \to \infty$*. Then,*

$$\frac{X_{n,k} - \mathbf{E}(X_{n,k})}{\sqrt{2p_k n/k^2}} \xrightarrow{d} \mathcal{N}(0,1).$$

(ii) *(Poisson range) Let* $\mathbf{E}(X_{n,k}) \to c > 0$ *and* $k < n$ *as* $n \to \infty$*. Then,*

$$X_{n,k} \xrightarrow{d} \mathrm{Poisson}(c).$$

(iii) *(Degenerate range) Let* $\mathbf{E}(X_{n,k}) \to 0$ *as* $n \to \infty$*. Then,*

$$X_{n,k} \xrightarrow{L_1} 0.$$

A comparison of the phase change

For k-caterpillars, we have

$$\mathbf{E}(X_{n,k}) = \frac{2^{k-1}n}{(k+2)!}.$$

Note that either

$$\mathbf{E}(X_{n,k}) \to \infty \qquad \text{or} \qquad \mathbf{E}(X_{n,k}) \to 0.$$

So, there is no Poisson range.

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shape parameter	location	phase change
k-pronged nodes	$\sqrt{n}$	normal - poisson - degenerate
minimal clade size k	$\sqrt[3]{n}$	normal - poisson - degenerate
k-caterpillars	$\ln n/(\ln\ln n)$	normal - degenerate

Refined results (for # of subtrees)

Define

$$\phi_{n,k}(y) = e^{-\sigma_{n,k}^2 y^2/2}\mathbf{E}\left(e^{(X_{n,k}-\mu_{n,k})y}\right).$$

and

$$\phi_{n,k}^{(m)} = \left.\frac{d^m \phi_{n,k}(y)}{dy^m}\right|_{y=0}.$$

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Proposition

Uniformly for $n, k \geq 1$ *and* $m \geq 0$

$$|\phi_{n,k}^{(m)}| \leq m! A^m \max\left\{\frac{n}{k^2}, \left(\frac{n}{k^2}\right)^{m/3}\right\}$$

for a suitable constant A.

Characteristic function

Let $\varphi_{n,k}(y) = \mathbf{E}(\exp\{(X_{n,k} - \mu_{n,k})iy/\sigma_{n,k}\})$.

Proposition

Let $1 \le k = o(\sqrt{n})$.

(i) *For* n *large*

$$\varphi_{n,k}(y) = e^{-y^2/2}\left(1 + \mathcal{O}\left(|y|^3 \frac{k}{\sqrt{n}}\right)\right),$$

uniformly for y *with* $|y| \le \epsilon n^{1/6}/k^{1/3}$.

(ii) *For* n *large*

$$|\varphi_{n,k}(y)| \le e^{-\epsilon y^2/2},$$

where $|y| \le \pi\sigma_{n,k}$.

Berry-Esseen bound and LLT for the normal range

Theorem (Rate of convergency)

For $1 \leq k = o(\sqrt{n})$ *as* $n \to \infty$,

$$\sup_{x\in\mathbb{R}} \left| P\left(\frac{X_{n,k} - \mu_{n,k}}{\sigma_{n,k}} < x\right) - \Phi(x)\right| = \mathcal{O}\left(\frac{k}{\sqrt{n}}\right).$$

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Theorem (LLT)

For $1 \leq k = o(\sqrt{n})$ *as* $n \to \infty$,

$$P(X_{n,k} = \lfloor \mu_{n,k} + x\sigma_{n,k} \rfloor) = \frac{e^{-x^2/2}}{\sqrt{2\pi}\sigma_{n,k}} \left(1 + \mathcal{O}\left((1 + |x|^3) \frac{k}{\sqrt{n}} \right) \right),$$

uniformly in $x = o(n^{1/6}/k^{1/3})$.

LLT for the Poisson range

Define

$$\bar{\phi}_{n,k}(y) = e^{-\mu_{n,k}(y-1)}\mathbf{E}\left(y^{X_{n,k}}\right).$$

and

$$\phi_{n,k}^{(m)} = \left.\frac{d^m\bar{\phi}_{n,k}(y)}{dy^m}\right|_{y=1}.$$

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Proposition

Uniformly for $n > k$ and $m \geq 0$

$$|\bar{\phi}_{n,k}^{(m)}| \leq m!A^m\left(\frac{n}{k^3}\right)^{m/2}$$

for a suitable constant A.

Poisson approximation

Theorem (LLT)

For $k < n$ and $n \to \infty$,

$$P(X_{n,k} = l) = e^{-\mu_{n,k}} \frac{(\mu_{n,k})^l}{l!} + \mathcal{O}\left(\frac{n}{k^3}\right)$$

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uniformly in l.

Theorem (Poisson approximation)

Let $k < n$ and $k \to \infty$ as $n \to \infty$. Then,

$$d_{TV}(X_{n,k}, \mathrm{Poisson}(\mu_{n,k})) \longrightarrow 0.$$

Remark: A rate can be given as well.

Other types of random trees

- Random recursive trees

  Non-plane, labelled trees with every label sequence from the root to a leave increasing; random model is the uniform model.

  Methods works as well (with minor modifications) and similar results can be proved.

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  Non-plane, labelled trees with every label sequence from the root to a leave increasing; random model is the uniform model.

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- Plane-oriented recursive trees (PORTs)

  Plane, labelled trees with every label sequence from the root to a leave increasing; random model is the uniform model.

  Method works as well, but details more involved.

of subtrees for PORTs

$X_{n,k}$ satisfies

$$X_{n,k} \stackrel{d}{=} \sum_{i=1}^{N} X_{I_i,k}^{(i)},$$

where $X_{k,k} = 1$ and $X_{I_i,k}^{(i)}$ are conditionally independent given $(N, I_1, I_2, \ldots)$.

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This can be simplified to

$$X_{n,k} \stackrel{d}{=} X_{I_n,k} + X_{n-I_n,k}^{*} - \mathbf{1}_{\{n-I_n=k\}},$$

where $X_{k,k} = 1$, $X_{I_n,k}$ and $X_{n-I_n,k}^{*}$ are conditionally independent given I_n and

$$P(I_n = j) = \frac{2(n-j)C_j C_{n-j}}{nC_n}.$$

Underlying recurrence and solution

All (centered or non-centered) moments satisfy

$$a_{n,k} = 2\sum_{j=1}^{n-1} \frac{C_j C_{n-j}}{C_n} a_{j,k} + b_{n,k},$$

where $a_{k,k}$ is given and $a_{n,k} = 0$ for $n < k$.

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We have

$$a_{n,k} = \frac{C_k(n+1-k)C_{n+1-k}}{C_n} a_{k,k} + \sum_{k<j\leq n} \frac{C_j(n+1-j)C_{n+1-j}}{C_n} b_{n,k},$$

where $n > k$.

Mean value and variance of PORTs

We have,

$$\mu_{n,k} := \mathbf{E}(X_{n,k}) = \frac{2n-1}{4k^2-1}, \qquad (n > k).$$

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$$\mu_{n,k} := \mathbf{E}(X_{n,k}) = \frac{2n-1}{4k^2-1}, \qquad (n > k).$$

Moreover, for fixed k as $n \to \infty$,

$$\mathrm{Var}(X_{n,k}) \sim c_k n,$$

where

$$c_k = \frac{8k^2-4k-8}{(4k^2-1)^2} - \frac{((2k-3)!!)^2}{((k-1)!)^2 4^{k-1} k(2k+1)},$$

and, for $k < n$ and $k \to \infty$ as $n \to \infty$,

$$\mathbf{E}(X_{n,k}) \sim \mathrm{Var}(X_{n,k}) \sim \frac{n}{2k^2}.$$

The phase change

Theorem

(i) *(Normal range) Let* $k = o(\sqrt{n})$ *and* $k \to \infty$ *as* $n \to \infty$*. Then,*

$$\frac{X_{n,k} - \mu_{n,k}}{\sqrt{n/(2k^2)}} \xrightarrow{d} \mathcal{N}(0,1).$$

(ii) *(Poisson range) Let* $k \sim c\sqrt{n}$ *as* $n \to \infty$*. Then,*

$$X_{n,k} \xrightarrow{d} \mathrm{Poisson}((2c^2)^{-1}).$$

(iii) *(Degenerate range) Let* $k < n$ *and* $\sqrt{n} = o(k)$ *as* $n \to \infty$*. Then,*

$$X_{n,k} \xrightarrow{L_1} 0.$$

More results and future research

- Parameters of genealogical trees under different random models

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- Universality of the phase change for the number of subtrees

  Very simple classes of increasing trees and more general classes of increasing trees (polynomial varieties, mobile trees, etc.)

Polynomial varieties

Bergeron, Flajolet, Salvy (1992): classes of increasing trees with degree function $\phi(\omega)$ under the uniform model.

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$$\phi(\omega) = \phi_d \omega^d + \cdots + \phi_0,$$

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$$\phi(\omega) = \phi_d \omega^d + \cdots + \phi_0,$$

where $d \geq 2$ and $\phi_d, \phi_0 \neq 0$.

For mean value and variance of the number of subtrees,

$$\mathbf{E}(X_{n,k}) \sim \mathrm{Var}(X_{n,k}) \sim \frac{d}{d-1} \cdot \frac{n}{k^2},$$

where $k \to \infty$ as $n \to \infty$.

Polynomial varieties - phase change

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(i) *(Normal range) Let* $k = o\left(\sqrt{n}\right)$ *and* $k \to \infty$ *as* $n \to \infty$. *Then,*

$$\frac{X_{n,k} - \mu_{n,k}}{\sqrt{nd/((d-1)k^2)}} \xrightarrow{d} \mathcal{N}(0,1).$$

(ii) *(Poisson range) Let* $k \sim c\sqrt{n}$ *as* $n \to \infty$. *Then,*

$$X_{n,k} \xrightarrow{d} \mathrm{Poisson}(d/((d-1)c^2)).$$

(iii) *(Degenerate range) Let* $k < n$ *and* $\sqrt{n} = o(k)$ *as* $n \to \infty$. *Then,*

$$X_{n,k} \xrightarrow{L_1} 0.$$

Mobile trees - phase change

Theorem

(i) *(Normal range) Let* $k = o\left(\sqrt{n/\ln n}\right)$ *and* $k \to \infty$ *as* $n \to \infty$. *Then,*

$$\frac{X_{n,k} - \mu_{n,k}}{\sqrt{n/(k^2 \ln k)}} \xrightarrow{d} \mathcal{N}(0,1).$$

(ii) *(Poisson range) Let* $k \sim c\sqrt{n/\ln n}$ *as* $n \to \infty$. *Then,*

$$X_{n,k} \xrightarrow{d} \mathrm{Poisson}(2c^{-2}).$$

(iii) *(Degenerate range) Let* $k < n$ *and* $\sqrt{n/\ln n} = o(k)$ *as* $n \to \infty$. *Then,*

$$X_{n,k} \xrightarrow{L_1} 0.$$

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  Very simple classes of increasing trees and more general classes of increasing trees (polynomial varieties, mobile trees, etc.)
- Phase change results for the number of nodes with out-degree k

  Important in computer science.

  A phase change from normal to degenerate is expected (no Poisson range).