ON SOME FACTORIZATIONS OF RANDOM WORDS

PHILIPPE CHASSAING

INSTITUT ELIE CARTAN

&

ELAHE ZOHOORIAN-AZAD

DAMGHAN UNIVERSITY

Maresias, AofA'08

GLOSSARY

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- n-letters long words $\omega = \omega_1\omega_2\ldots\omega_n,\ \omega_i \in \mathcal{A},\ |\omega| = n,\ \omega \in \mathcal{A}^n$
- Language $\mathcal{A}^* = \{\emptyset\} \cup \mathcal{A}^1 \cup \mathcal{A}^2 \cup \mathcal{A}^3 \cup \ldots$
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- u is a Prefix of w $r = \emptyset$
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- The standard factorization of a Lyndon word is the first step in the construction of some basis of the free Lie algebra over A

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- WLOG, $\{i \mid p_i > 0\}$ has no gaps and contains 1.

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- $N = (2,0,0,2,0,0,1,0,0, \ldots)$.

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- In the uniform case ($p_i=1/q$, $1\leq i\leq q$), **Diaconis, McGrath and Pitman** (Riffle shuffles, cycles, and descents, 1995) give the exact distribution of the profile

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- in which μ is the Moebius function.

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- As n grows, $p_n(.)$ converges to the law of a sequence of independent Poisson random variables (with respective parameters $1/k$ for c_k).

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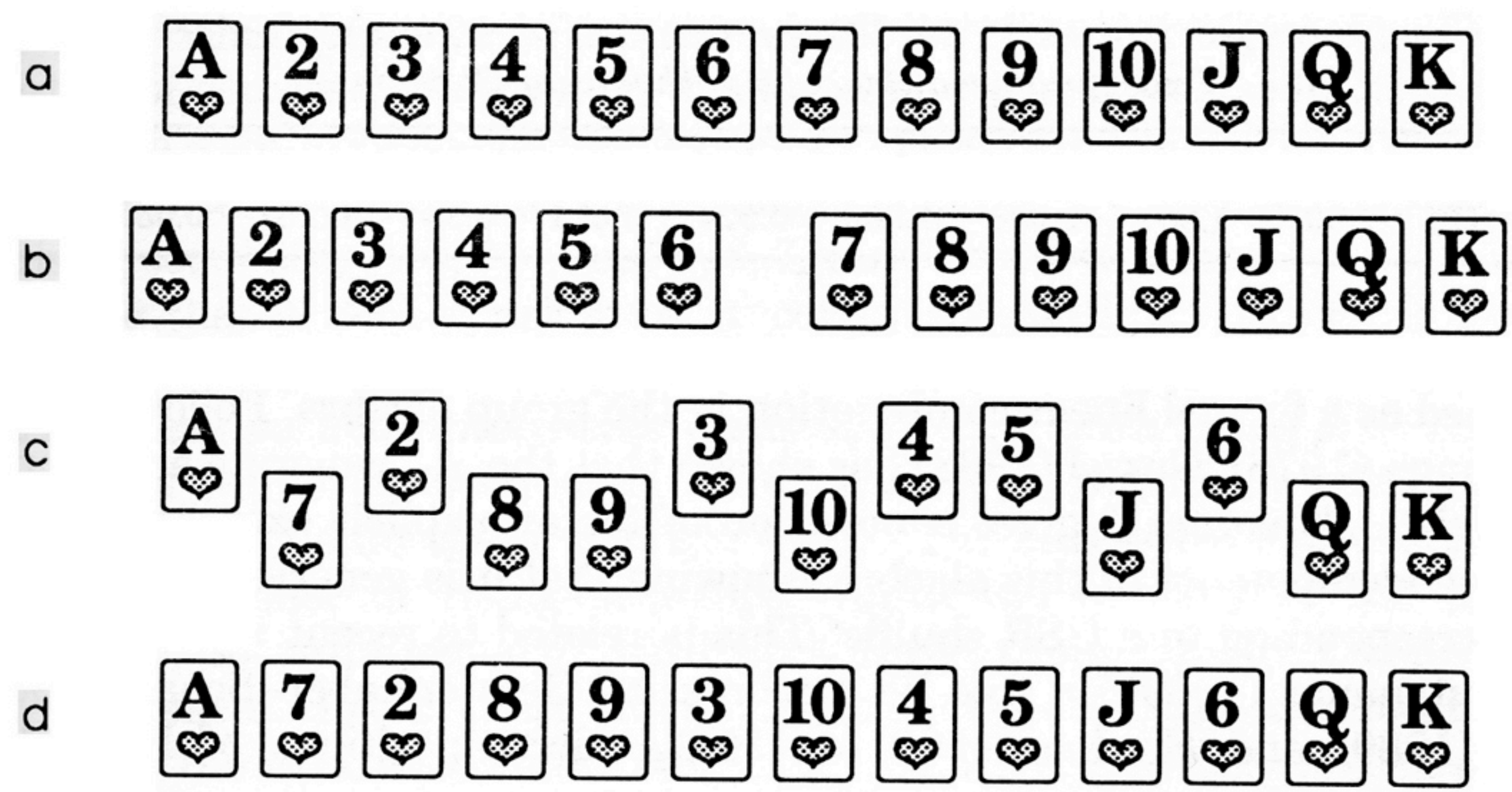
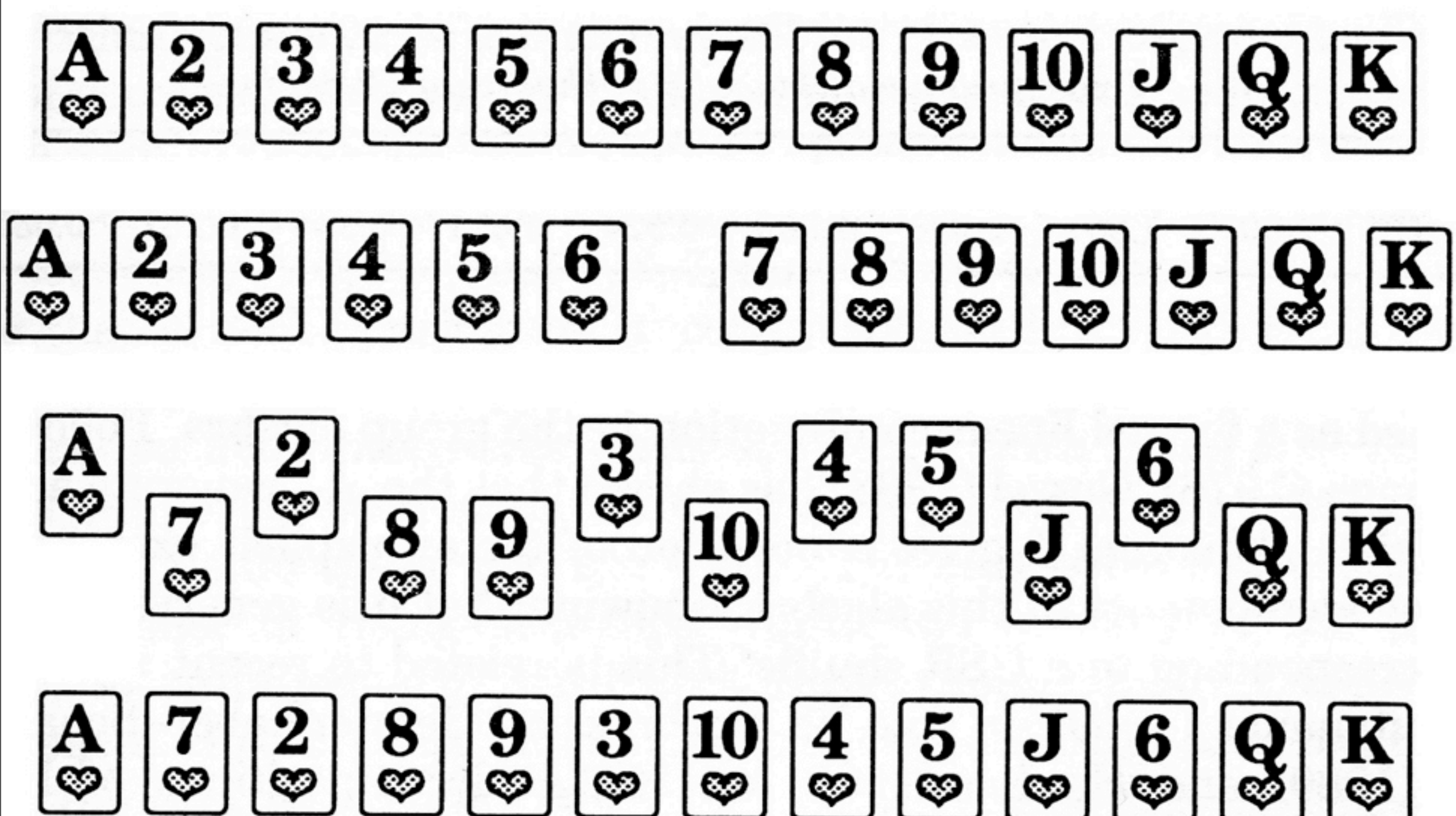


FIG. 1. *A riffle shuffle*. (a) *We begin with an ordered deck*. (b) *The deck is divided into two packets of similar size*. (c) *The two packets are riffled together*. (d) *The two packets can still be identified in the shuffled deck as two distinct "rising sequences" of face values*.



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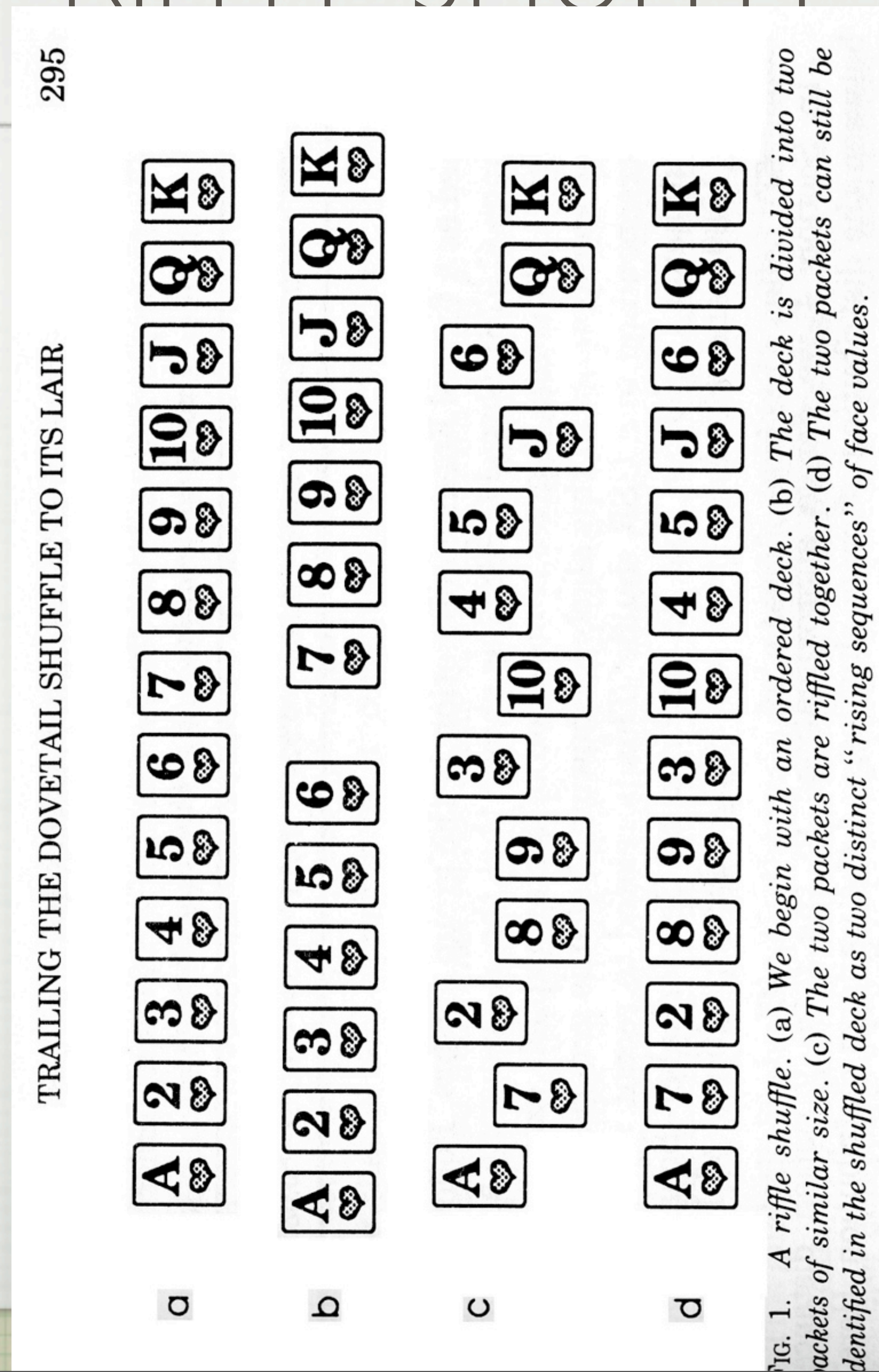

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$$RS_q \longrightarrow \text{uniform permutation,}$$

leading to the convergence of $M=(M_k)_{k\geq 1}$ to a Cauchy distribution, for

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in which $M_k(w)$ is the number of cycles with length k in the permutation w.

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- Bayer & Diaconis (1992):

$$DV(RS_q,\text{uniform}) = O(n^{3/2}/q).$$

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- And the profile of the permutation is sent on N.

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- More general distribution $p = (p_i)_{i \geq 1}$ on letters ?

MAIN RESULT

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$X_{(5)}$	$X_{(4)}$	$X_{(3)}$	$X_{(2)}$	$X_{(1)}$

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- $x_{20} = (1,1,4,9,5,0,0,\ldots)/20$
- $x_n(k)$ is the renormalised size of the k^{th} Lyndon factor, starting from the end of the word.

- For a general alphabet $A=\{a_i\}$, and a general distribution $p=(p_i)$, x_n converges to a p_1-sticky GEM(1).

GEM(1)

[redacted]

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[redacted]

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U_1

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- Terminology: Griffiths-Engen-McClosey r.v. with parameter 1, size-biased reordering of Poisson-Dirichlet(0,1) (population genetics, etc …), stickbreaking scheme …

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STICKBREAKING OCCURENCES

- $$x_k = U_1\, U_2 \ldots U_{k-1}(1-U_k).$$
- Rearranging $x=(x_k)_{k\geq 0}$ in decreasing order gives the asymptotic distributions of the normalised sizes of cycles, or of logarithms of prime factors of integers, or of degrees of prime factors of polynomials on finite fields.

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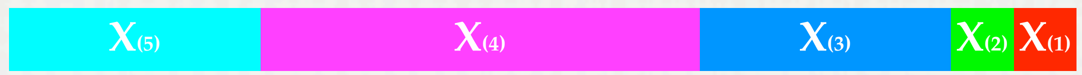
- The normalised size of the longest factor in the Lyndon decomposition converges to the Dickman distribution, regardless of $p = (p_i)$.

RELATED RESULTS

$$aabbaaababbaaabaa$$



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$aabbaaababbaaabaa$

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PROOF OF THE MAIN RESULT

EXERCISES 1 & 2 ???