Optimal stopping under mixed constraints

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Objectives

Present optimal stopping with two kinds of constraints

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Problem:

— n fixed;

— $X_1, X_2, \cdots, X_n$, i.i.d. random variables ≥ 0.

— Sequential observation (**no recall**)

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$$(1, X_1)....................(k, X_k) \bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet$$

Cost = sum of selected X_k's.

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Origin

Preview

- Probabilistic setting

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- The hierarchy of constraints

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- Recurrence

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- Precise solution for total selection cost

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- Precise solution for total selection cost

- Asymptotic behaviour of total selection cost

2. Problem formulation.

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If $I_k = 0$ then X_k is *refused.*

$\{I_k = 1\} \in \sigma$-field $\mathcal{F}_k$ generated by X_k's and I_k's together.

Selection rules $T = \{\tau := \tau_n = (I_1, I_2, \cdots, I_n)\}$.

Objective:

Find

$$v_{r,\mu}(n) = \min_{\tau \in T} \mathrm{E}\left(\sum_{k=1}^{n} I_k X_k\right), \; n \geq \mu \geq r$$

and

$$\tau^* = \arg\min_{\tau \in T} \mathrm{E}\left(\sum_{k=1}^{n} I_k X_k\right)$$

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$$\tau^* = \arg\min_{\tau \in T} \mathrm{E}\left(\sum_{k=1}^{n} I_k X_k\right)$$

subject to

$$\sum_{k=1}^{n} I_k \geq r, \quad 1 \leq r \leq n \qquad \text{(D-constraint)}$$

and

$$\mathrm{E}\left(\sum_{k=1}^{n} I_k\right) = \mu, \ \mu \in \mathbf{R}, \ \mu \geq r. \quad \text{(E-constraint)}$$

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Lemma 1 For all (stopping) times $0 \le \tau \le n$:

$$V(n|\mathcal{F}_\tau) = v_{r-N_\tau,\mu-N_\tau}(n-\tau) + \sum_{j=1}^{\tau} I_j X_j \ a.s.$$

Lemma

$$V_\delta(n) = v_{0,\mu-r}(n-\delta) + \sum_{j=1}^{\delta} I_j X_j \text{ a.s.}$$

with

$$v_{0,\mu-r}(k) = \frac{(\mu-r)^2}{2k}.$$

Sketch of Proof.

— Conditioned on $\delta = d$, ... clear.

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Statement holds unconditionally.

Remains to be shown :

$$v_{0,\mu-r}(k) = (\mu - r)^2/2k.$$

At time $\delta+$, we must design a rule which selects in expectation $\mu - r$ from $K = n - \delta$ i.i.d $U[0,1]$-random variables.

Threshold rules

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— If optimal to refuse $X_j = x$, ... optimal to refuse $X_j' > x$.

$\Longrightarrow$ Each opt. decision is based on a unique threshold!

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$$\mathrm{E}(I_{\delta+j} X_{\delta+j}) = t_j \mathrm{E}(X | X \leq t_j) = t_j^2/2.$$

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Minimize

$$\sum_{j=1}^{K} t_j^2$$

subject to

$$\sum_{j=1}^{K} \mathrm{E}(I_{\delta+j}) = \sum_{j=1}^{K} t_j = \mu - r.$$

Optimization (e.g. Lagrange multiplyer method) yields

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$$t_j \equiv (\mu - r)/K, \quad j > \delta.$$

Hence

$$v_{0,\mu-r}(K) = K\frac{\mu - r}{K} \times \frac{\mu - r}{2K} = \frac{(\mu - r)^2}{2K}.$$

□

Optimal rule.

Theorem 3.1

$$v_{r,\mu}(n) = v_{r,\mu}(n-1) - \frac{1}{2}\left[v_{r,\mu}(n-1) - v_{r-1,\mu-1}(n-1)\right]^2,$$

for $n = [\mu]^+, [\mu]^+ + 1, \cdots$, with initial conditions

$$v_{r,\mu}([\mu]^+) = \frac{\mu}{2}; \qquad v_{0,\mu-r}(n) = \frac{(\mu-r)^2}{2n},\ n = 1, 2, \cdots$$

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Proof. Suppose it is optimal to select X_1 iff $X_1 \leq t$. Then

$$\tilde{v}_{r,\mu}(n,t) = t\left[\mathrm{E}(X|X \leq t) + v_{r-1,\mu-1}(n-1)\right] + (1-t)v_{r,\mu}(n-1).$$

$\mathrm{E}(X|X \leq t) = t/2$, differentiable in t for all $t \in]0,1[$ $\partial\tilde{v}_{r,\mu}(n,t)/\partial t = 0$ with $\partial^2\tilde{v}_{r,\mu}(n,t)/\partial t^2 > 0$ minimizes $v_{r,\mu}(n,t)$.

Unique

solution

$$t^* = v_{r,\mu}(n-1) - v_{r-1,\mu-1}(n-1).$$

We must have

$$\tilde{v}_{r,\mu}(n, t^*) = v_{r,\mu}(n).$$

....insert ... elementary steps....

Initial conditions:

Suppose $\mu \in \mathbf{N}$ and $n = \mu$. The optimal policy must select all observations.... value $\mu/2$. The second initial condition stems from (4), and thus the Theorem is proved. $\square$

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For all r and μ, $v_{r,\mu}(n) \geq 0$. Hence $(v_{r,\mu}(n))$ decreases in n, whenever the sequence $(v_{r-1,\mu-1}(n))$ decreases in n.
$(v_{0,\mu-r}(n))$ decreases in n

Hence must converge (to the only possible limit 0.)

Corollary For $\mu \geq 1$ and $n \geq \mu \geq r$ $(v_{r,\mu}(n))_{n \geq \mu}$ is monotone decreasing with limit 0.

Lemma

For n fixed with $n \geq [\mu]^+$ and $\mu \geq r$

(i) $v_{r,\mu}(n) \geq v_{r-1,\mu-1}(n)$

(ii) $v_{r,\mu}(n) \geq v_{r,\mu-1}(n)$

Proof: $\tilde{v}_{\mu,r}(n)$:= minimal expected total cost of the optimal strategy for the (r,μ)-constraints under the additional hypothesis, that the rth selection for free. Then $\tilde{v}_{\mu,r}(n) \leq v_{\mu,r}(n)$. However if we play right away optimally under the weaker $(r-1,\mu-1)$-constraints, ...

$$v_{r-1,\mu-1}(n) \leq \tilde{v}_{r,\mu}(n).$$

Hence $v_{r,\mu}(n) \geq v_{r-1,\mu-1}$.

Inequality (ii) follows from $v_{0,\mu-r}(.) > v_{0,\mu-1-r}(.)$ uniformly. □

4. The optimal rule.

Definition For $s \in \{0, 1, \cdots, r\}$ and $k \in \{0, 1, \cdots, n\}$ we say we are in *state* (s, k), if s selections have been made until time $n - k$ included.

(Note that the current E-constraint is implicit for $0 \leq s \leq r$.)

Since the continuation thereafter is, by hypothesis, a fixed selection rule, it becomes irrelevant once the D-constraint is satisfied. Hence we need not list it as a separate state-coordinate.

Recall: Optyimal thresholds are all unique.

Computing optimal thresholds and values

The optimal thresholds for each state can be computed recursively.

We have to start with two independent lines of initial conditions, namely for $v_{0,\mu-r}(k)$ with $k \geq \mu - r$ and for $v_{s,k}(k)$ with $k \geq s$.

Optimal values

(A1) $v_{0,\mu-r}(k) = (\mu - r)^2/(2k)\,,\; k = \mu - r, \cdots, n - r.$

(A2) $v_{s,k}(k) = k/2, \;\; k = \mu - r, \cdots, n - r; s = 1, \cdots r.$

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(A2) $v_{s,k}(k) = k/2,\ \ k = \mu-r, \cdots, n-r; s = 1, \cdots r.$

(A3) For $s = 1, \cdots, r$ and init. cond. (A1), (A2) compute

$$v_{s,\mu-r+s}(k)$$

$$= v_{s,\mu-r+s}(k-1) - \frac{1}{2}\left[v_{s,\mu-r+s}(k-1) - v_{s-1,\mu-r+(s-1)}(k-1)\right]^2,$$

$k = \mu-r+s, \cdots, n-r;\ s = 1, \cdots r.$

Optimal thresholds

(B1) $\quad t_{r,k} = v_{0,\mu-r}(k) = (\mu - r)^2/2,\ k = \mu - r, \cdots n - r.$

(B2) $\quad t_{s,k} = v_{r-s,\mu-s}(k-1) - v_{r-s-1,\mu-s-1}(k-1),$
$s = 0, \cdots, r-1$

5.2 Bounds of $v_{r,\mu}(n)$ for general r and μ.

Motivation ...

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Lemma For all $0 \leq s \leq r, s \leq m \leq \mu$ and $\max\{s, m\} \leq k \leq n$

$$v_{r,\mu}(n) \leq v_{s,m}(k) + v_{r-s,\mu-m}(n-k).$$

Proof.

Fix indices s, m and k such that the conditions for the Lemma are fufilled. This is always possible ...at least (r, μ, n) and $(0, 0, 0)$ are possible (by definition.)

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— Leg 1 minimizes the expected total cost of accepting items until time k under the (s, m) constraint.

— Leg 2 remembers the occured cost at time k and then minimizes (independently) the additional cost of accepting further items under the constraints $r - s, mu - m$.

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Consider a two-legged strategy.

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This composed strategy is admissable since it fulfills the original constraints, and since $X_{k+1}, X_{k+2} \cdots X_n$ are independent of the past, its value is $v_{s,m}(k) + v_{r-s,\mu-m}(n-k)$. The inequality follows then by sub-optimality. □

For the special case $s = r = m$ we obtain

Corollary For $1 \leq r \leq \mu \leq n: \quad v_{r,\mu}(2n) \leq v_{r,r}(n) + \frac{1}{2n}(\mu - r)^2.$

Lemma For all $1 \leq r \leq \mu$ there exist constants $\alpha = \alpha(r, \mu)$ and $\beta = \beta(r, \mu)$ such that $\alpha/n \leq v_{r,\mu}(n) \leq \beta/n$ for all $n \geq \mu$, with n sufficiently large.

Proof. We first prove that the existence of a lower bound $\alpha(r, \mu)/n$.

By definition of the D-constraint and E-constraint we have $\mu \geq r$. Since $v_{r,\mu}(\cdot)$ is increasing in μ for fixed r and n, it suffices to show $v_{r,r}(n) \geq \alpha/n$ for some constant α.

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The expectation of the sum of these is $r(r+1)/(2(n+1)) \geq r^2/(2n)$. Hence $v_{r,\mu}(n) \geq v_{r,r}(n) \geq \alpha/n$ for $\alpha = r^2/2$.

Concerning the upper bound β we see that the statement is true, if it is true for $v_{r,r}(n)$.

Now, $v_{r,r}(rn) \leq v_{r-1,r-1}((r-1)n) + v_{1,1}(n)$, and hence by induction $v_{r,r}(rn)) \leq rv_{1,1}(n)$. The sequence $(v_{1,1}(n))$ coincides with Moser's sequence, which is known to satisfy $v_{1,1}(n) \leq c/n$ for all n.

Therefore $v_{r,r}(rn)) \leq (cr)/n$. But then, for general n we have $v_{r,r}(n) \leq v_{r,r}([n/r]r)$, where $[x]$ denotes the floor of x. Hence $v_{r,r}(n) \leq cr/[n/r] \leq (cr^2 + \epsilon)/n$ for all $\epsilon > 0$ and n sufficiently large, and the proof is complete. □

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Example:

Aldous' problem (2006). What is $v_{1,2}(n)$ and what is the behaviour of $nv_{1,2}(n)$?

We have $\mu - r = 1$, $v_{0,1}(k) = (\mu - r)^2/(2k)$. Initial condition: $v_{1,2}(2) = 1$.

Recurrence:

$$v_{1,2}(k) = v_{1,2}(k-1) - \frac{1}{2}\left(v_{1,2}(k-1) - \frac{1}{2(k-1)}\right)^2, k = 2, 3, \cdots n.$$

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What is the asymptotic behaviour of $(nv(n))$?

Answer: We will see $(nv_{1,2}(n)) \to 3/2 + \sqrt{2}$.

Asymptotic behaviour

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We rewrite for $t \in \mathbf{N}$ and $\epsilon = 1$ as

$$\frac{1}{\epsilon}\Big(v_{r,\mu}(t) - v_{r,\mu}(t-\epsilon)\Big) = -\frac{1}{2}\Big(v_{r,\mu}(t-\epsilon) - v_{r-1,\mu-1}(t-\epsilon)\Big)^2$$

with initial condition (6). We fix r and μ and can then simplify the notation by writing $v_{r-1,\mu-1}(t) =: v(t)$ and $v_{r,\mu}(t) =: w(t)$, say. Let $\tilde{v}(t)$ and $\tilde{w}(t)$ be diffentiable functions which coincide with $v(t)$ and $w(t)$ for $t \in \mathbf{N}$ with $t \geq \mu$.

It follows from Lemma 5.3 and the mean value theorem that the differential equation

$$\tilde{w}'(t) = -\frac{1}{2}\left(\tilde{w}(t) - \tilde{v}(t)\right)^2$$

defined for $t \in [\mu, \infty]$ must catch the asymptotic behaviour of $w(t)$ for $t \in \mathbf{N}$.

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Note that this is a general Riccati differential equation, and the idea is now to show that only exactly one solution of equation (12) is compatible with (11).

Theorem 6.1

If $\tilde{v}(t) = c/t$ for some constant $c \geq 2$ then the unique solution $\tilde{w}(t)$ satisfying $\lim_{t\to\infty} v_{r,\mu}(t)/\tilde{w}(t) = 1$ is the function

$$\tilde{w}(t) := \tilde{w}_1(t) = \frac{1}{t}\left(1 + c + \sqrt{1+2c}\right).$$

Proof: We first prove that $\tilde{w}_1(t) = \left(1 + c + \sqrt{1+2c}\right)/t$ is a particular solution of equation (12). Indeed, there must be a constant, c_1 say, such that c_1/t is a particular solution, because plugging in yields the equation

$$\frac{-c_1}{t^2} = \frac{-1}{2t^2}(c_1^2 - 2cc_1 + c^2)$$

with solutions in $\{1 + c + -\sqrt{1+2c}\}$.

Only the solution $c_1 = \left(1 + c + \sqrt{1+2c}\right)$ is meaningful because with $c > 0$ we would have $\left(1 + c - \sqrt{1+2c}\right) < c$ contradicting $\tilde{w}(t) \geq \tilde{v}(t)$. Hence $\tilde{w}_1(t)$ is a particular solution.

From the general theory of Riccati differential equations (see e.g. Grauert und Fischer (1967), 109-112) we know that we can generate a general solution $\{\tilde{w}_2\}$ from a particular solution by solving (substitution $u(t) = 1/(w_2(t) - w_1(t))$ the first order linear equation

$$u'(t) = -(Q(t) + 2R(t)\tilde{w}_1(t))u(t) - R(t)$$

where, in our case, $R(t) = -1/2$ and $Q(t) = c/t$. The set $\{\tilde{w}_2\}$ of solutions is then the set $\{\tilde{w}_2(t) = \tilde{w}_1(t) + u(t)^{-1}\}$ with a single undetermined constant.

Plugging our particular solution $\tilde{w}_1(t)$ into the first order equation yields, after straightforward simplification, the equation $u'(t) = u(t)\left(1+\sqrt{1+2c}\right)/t + 1/2$. We solve its associated homogeneous equation and then apply the method of the variation of constants. This yields ....

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Finally wee see that all other solutions are **incompatible** with at least one of the precedingly proved properties of $v_{\Sigma,\mu}(n)$ and $v_{0,\mu-r}(n)$

Limiting behaviour:

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Limiting behaviour:

$$n v_{r,\mu}(n) \to c_r$$

where

$$c_0 := \frac{(\mu - r)^2}{2}$$

$$c_k := c_{k-1} + 1 + \sqrt{2c_{k-1} + 1}.$$

Outlook.

***

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