Young tableaux and snakes

Yuliy Baryshnikov
joint work with Dan Romik (Hebrew University)

Motivation

- A *Young tableaux* is a filling of *Young diagram* consisting of n boxes with numbers $1, \ldots, n$ increasing top-to-down and left-to-right.

1	2	6	8	9
3	5			
4				
7				

n=9

Motivation

- A *Young tableaux* is a filling of *Young diagram* consisting of n boxes with numbers $1, \ldots, n$ increasing top-to-down and left-to-right.

1	2	6	8	9
3	5			
4				
7				

$n=9$

- The number of YTs with *given shape* λ has various interpretations (dimension of the representation λ of S_n, for example).

Motivation

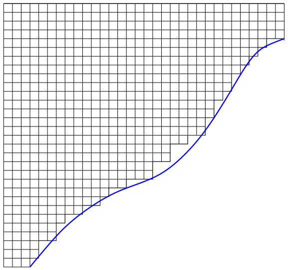
- A *Young tableaux* is a filling of *Young diagram* consisting of n boxes with numbers $1, \ldots, n$ increasing top-to-down and left-to-right.

1	2	6	8	9
3	5			
4				
7				

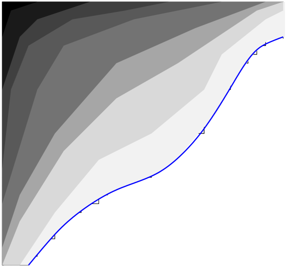
n=9

- The number of YTs with *given shape* λ has various interpretations (dimension of the representation λ of S_n, for example).
- Asymptotic regime is of interest:

- Consider a *large* shape $t\lambda$, $t \to \infty$:

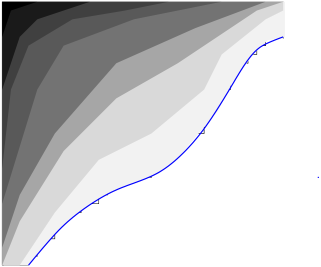


- Consider a *large* shape $t\lambda, t \to \infty$: and a typical Young tableaux filling it:



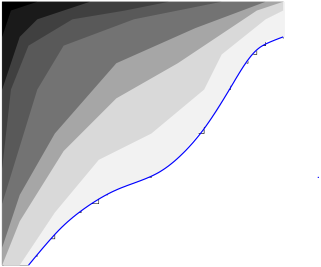
- The natural question arises:

- Consider a *large* shape $t\lambda$, $t \to \infty$: and a typical Young tableaux filling it:



- The natural question arises:
  Conjecture A *typical* YT, considered as a function on the Young diageam $t\lambda$ is close to some *deterministic* limiting function.

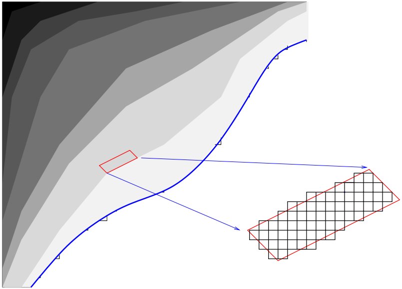
- Consider a *large* shape $t\lambda$, $t \to \infty$: and a typical Young tableaux filling it:



- The natural question arises:
Conjecture A *typical* YT, considered as a function on the Young diageam $t\lambda$ is close to some *deterministic* limiting function.
- How one would prove it?

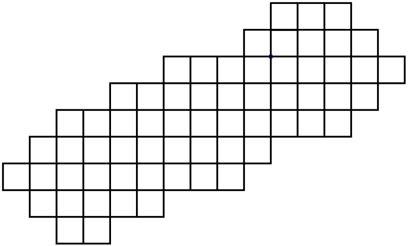
- By finding the rate function and then solving variational problem.

- By finding the rate function and then solving variational problem.
- Rate function will count the (normalized, per unit area) number of YT filling the shapes approximating a strip

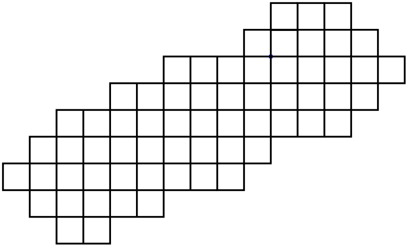


- Hence we have to compute the number of Young tableaux filling the strips like

- Hence we have to compute the number of Young tableaux filling the strips like



- Hence we have to compute the number of Young tableaux filling the strips like



- We start with the simplest task: finding the number of YT filling the strip of width 2 and slope 1.

- A permutation $\sigma \in S_n$ is called an **up-down permutation** (also zig-zag permutation, alternating permutation) if it satisfies

$$\sigma(1) < \sigma(2) > \sigma(3) < \sigma(4) > \dots$$

- A permutation $\sigma \in S_n$ is called an **up-down permutation** (also zig-zag permutation, alternating permutation) if it satisfies

$$\sigma(1) < \sigma(2) > \sigma(3) < \sigma(4) > \dots$$

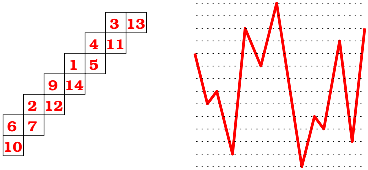
Equivalent to "2-strip" tableaux:

Up-down permutations

- A permutation $\sigma \in S_n$ is called an **up-down permutation** (also zig-zag permutation, alternating permutation) if it satisfies

$$\sigma(1) < \sigma(2) > \sigma(3) < \sigma(4) > \ldots$$

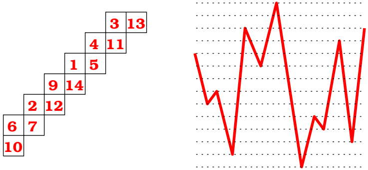
Equivalent to "2-strip" tableaux:



- A permutation $\sigma \in S_n$ is called an **up-down permutation** (also zig-zag permutation, alternating permutation) if it satisfies

$$\sigma(1) < \sigma(2) > \sigma(3) < \sigma(4) > \dots$$

Equivalent to "2-strip" tableaux:



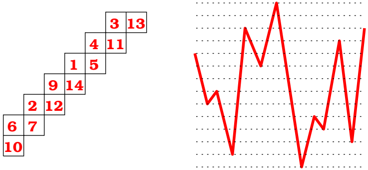
- **Theorem** (D. André 1881): Let $A_n = \#$ of n-element up-down permutations. Then

$$\sum_{n=0}^{\infty} \frac{A_n x^n}{n!} = \tan x + \sec x.$$

- A permutation $\sigma \in S_n$ is called an **up-down permutation** (also zig-zag permutation, alternating permutation) if it satisfies

$$\sigma(1) < \sigma(2) > \sigma(3) < \sigma(4) > \ldots$$

  Equivalent to "2-strip" tableaux:



- **Theorem** (D. André 1881): Let $A_n = \#$ of n-element up-down permutations. Then

$$\sum_{n=0}^{\infty} \frac{A_n x^n}{n!} = \tan x + \sec x.$$

- Up-down permutations were named *snakes* and studied by V. Arnold to enumerate *morsifications of real singularities*.

- Reminder:

$$
\begin{aligned}
\operatorname{sech} x &= \sum_{n=0}^{\infty} \frac{E_n x^n}{n!}, & (E_n)_{n\geq 0} \text{ – Euler numbers,} \\
\tan x &= \sum_{n=1}^{\infty} \frac{T_n x^{2n-1}}{(2n-1)!}, & (T_n)_{n\geq 0} \text{ – Tangent numbers,} \\
\frac{x}{e^x - 1} &= \sum_{n=0}^{\infty} \frac{B_n x^n}{n!}, & (B_n)_{n\geq 0} \text{ – Bernoulli numbers.}
\end{aligned}
$$

Up-down permutations (continued)

- Reminder:

$$\begin{aligned} \operatorname{sech} x &= \sum_{n=0}^{\infty} \frac{E_n x^n}{n!}, \qquad (E_n)_{n\geq 0} \text{ – Euler numbers,} \\ \tan x &= \sum_{n=1}^{\infty} \frac{T_n x^{2n-1}}{(2n-1)!}, \qquad (T_n)_{n\geq 0} \text{ – Tangent numbers,} \\ \frac{x}{e^x - 1} &= \sum_{n=0}^{\infty} \frac{B_n x^n}{n!}, \qquad (B_n)_{n\geq 0} \text{ – Bernoulli numbers.} \end{aligned}$$

- In this notation: $A_{2n} = |E_{2n}|, \quad A_{2n-1} = T_n = \frac{(-1)^{n-1} 4^n (4^n - 1)}{2n} B_{2n}.$

Many proofs of André's theorem, mostly algebraic. Proof using transfer operators (due to N. Elkies, 2003):

Many proofs of André's theorem, mostly algebraic. Proof using transfer operators (due to N. Elkies, 2003):

- Let $P_n = \Big\{(x_1, x_2, \ldots, x_n) \in [0,1]^n : x_1 < x_2 > x_3 < x_4 > \ldots\Big\}$.

Many proofs of André's theorem, mostly algebraic. Proof using transfer operators (due to N. Elkies, 2003):

- Let $P_n = \Big\{(x_1, x_2, \ldots, x_n) \in [0,1]^n : x_1 < x_2 > x_3 < x_4 > \ldots\Big\}$.
- Compute $\mathrm{vol}(P_n)$ in two ways:

Many proofs of André's theorem, mostly algebraic. Proof using transfer operators (due to N. Elkies, 2003):

- Let $P_n = \Big\{(x_1, x_2, \ldots, x_n) \in [0,1]^n : x_1 < x_2 > x_3 < x_4 > \ldots\Big\}$.
- Compute $\mathrm{vol}(P_n)$ in two ways: First, $\mathrm{vol}(P_n) = \frac{A_n}{n!}$;

Many proofs of André's theorem, mostly algebraic. Proof using transfer operators (due to N. Elkies, 2003):

- Let $P_n = \Big\{(x_1, x_2, \ldots, x_n) \in [0,1]^n : x_1 < x_2 > x_3 < x_4 > \ldots\Big\}$.
- Compute $\mathrm{vol}(P_n)$ in two ways: First, $\mathrm{vol}(P_n) = \frac{A_n}{n!}$;
- Second,

$$\mathrm{vol}(P_n) \quad =$$

Transfer operators

Many proofs of André's theorem, mostly algebraic. Proof using transfer operators (due to N. Elkies, 2003):

- Let $P_n = \Big\{(x_1, x_2, \ldots, x_n) \in [0,1]^n : x_1 < x_2 > x_3 < x_4 > \ldots\Big\}$.
- Compute $\mathrm{vol}(P_n)$ in two ways: First, $\mathrm{vol}(P_n) = \frac{A_n}{n!}$;
- Second,

$$\mathrm{vol}(P_n) \quad = \quad \int_0^1 dx_1$$

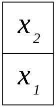
$\boxed{x_1}$

Transfer operators

Many proofs of André's theorem, mostly algebraic. Proof using transfer operators (due to N. Elkies, 2003):

- Let $P_n = \left\{ (x_1, x_2, \ldots, x_n) \in [0,1]^n : x_1 < x_2 > x_3 < x_4 > \ldots \right\}$.
- Compute $\mathrm{vol}(P_n)$ in two ways: First, $\mathrm{vol}(P_n) = \frac{A_n}{n!}$;
- Second,

$$\mathrm{vol}(P_n) \quad = \quad \int_0^1 dx_1 \int_{x_1}^1 dx_2$$



Transfer operators

Many proofs of André's theorem, mostly algebraic. Proof using transfer operators (due to N. Elkies, 2003):

- Let $P_n = \Big\{(x_1, x_2, \ldots, x_n) \in [0,1]^n : x_1 < x_2 > x_3 < x_4 > \ldots\Big\}$.
- Compute $\mathrm{vol}(P_n)$ in two ways: First, $\mathrm{vol}(P_n) = \frac{A_n}{n!}$;
- Second,

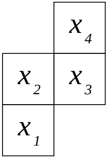
$$\begin{array}{|c|c|}\hline x_2 & x_3 \\ \hline x_1 & \\ \cline{1-1}\end{array}$$

$$\mathrm{vol}(P_n) \quad = \quad \int_0^1 dx_1 \int_{x_1}^1 dx_2 \int_0^{x_2} dx_3$$

Transfer operators

Many proofs of André's theorem, mostly algebraic. Proof using transfer operators (due to N. Elkies, 2003):

- Let $P_n = \left\{(x_1, x_2, \ldots, x_n) \in [0,1]^n : x_1 < x_2 > x_3 < x_4 > \ldots\right\}$.
- Compute $\mathrm{vol}(P_n)$ in two ways: First, $\mathrm{vol}(P_n) = \frac{A_n}{n!}$;
- Second,

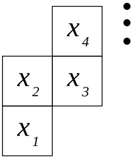


$$\mathrm{vol}(P_n) = \int_0^1 dx_1 \int_{x_1}^1 dx_2 \int_0^{x_2} dx_3 \int_{x_3}^1 dx_4$$

Transfer operators

Many proofs of André's theorem, mostly algebraic. Proof using transfer operators (due to N. Elkies, 2003):

- Let $P_n = \Big\{(x_1, x_2, \ldots, x_n) \in [0,1]^n : x_1 < x_2 > x_3 < x_4 > \ldots\Big\}$.
- Compute $\mathrm{vol}(P_n)$ in two ways: First, $\mathrm{vol}(P_n) = \frac{A_n}{n!}$;
- Second,

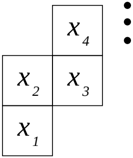


$$
\begin{aligned}
\mathrm{vol}(P_n) &= \int_0^1 dx_1 \int_{x_1}^1 dx_2 \int_0^{x_2} dx_3 \int_{x_3}^1 dx_4 \ldots \\
&=
\end{aligned}
$$

Transfer operators

Many proofs of André's theorem, mostly algebraic. Proof using transfer operators (due to N. Elkies, 2003):

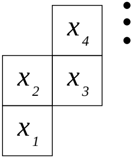
- Let $P_n = \Big\{(x_1, x_2, \ldots, x_n) \in [0,1]^n : x_1 < x_2 > x_3 < x_4 > \ldots\Big\}$.
- Compute $\mathrm{vol}(P_n)$ in two ways: First, $\mathrm{vol}(P_n) = \frac{A_n}{n!}$;
- Second,



$$\begin{aligned}\mathrm{vol}(P_n) &= \int_0^1 dx_1 \int_{x_1}^1 dx_2 \int_0^{x_2} dx_3 \int_{x_3}^1 dx_4 \ldots \\ &= \langle \ldots \circ T \circ S \circ T \circ S\mathbf{1}, \mathbf{1}\rangle_{\mathbf{L_2[0,1]}},\end{aligned}$$

Many proofs of André's theorem, mostly algebraic. Proof using transfer operators (due to N. Elkies, 2003):

- Let $P_n = \Big\{(x_1, x_2, \ldots, x_n) \in [0,1]^n : x_1 < x_2 > x_3 < x_4 > \ldots\Big\}$.
- Compute $\mathrm{vol}(P_n)$ in two ways: First, $\mathrm{vol}(P_n) = \frac{A_n}{n!}$;
- Second,



$$\begin{aligned} \mathrm{vol}(P_n) &= \int_0^1 dx_1 \int_{x_1}^1 dx_2 \int_0^{x_2} dx_3 \int_{x_3}^1 dx_4 \ldots \\ &= \langle \ldots \circ T \circ S \circ T \circ S\mathbf{1}, \mathbf{1} \rangle_{\mathbf{L_2[0,1]}}, \end{aligned}$$

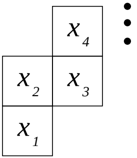
where

$$(Tf)(x) = \int_0^x f(y)dy, \qquad (Sg)(x) = \int_x^1 g(y)dy$$

Transfer operators

Many proofs of André's theorem, mostly algebraic. Proof using transfer operators (due to N. Elkies, 2003):

- Let $P_n = \Big\{(x_1, x_2, \dots, x_n) \in [0,1]^n : x_1 < x_2 > x_3 < x_4 > \dots\Big\}$.
- Compute $\mathrm{vol}(P_n)$ in two ways: First, $\mathrm{vol}(P_n) = \frac{A_n}{n!}$;
- Second,



$$
\begin{aligned}
\mathrm{vol}(P_n) &= \int_0^1 dx_1 \int_{x_1}^1 dx_2 \int_0^{x_2} dx_3 \int_{x_3}^1 dx_4 \dots \\
&= \langle \dots \circ T \circ S \circ T \circ S\mathbf{1}, \mathbf{1} \rangle_{\mathbf{L_2[0,1]}},
\end{aligned}
$$

where

$$
(Tf)(x) = \int_0^x f(y)dy, \qquad (Sg)(x) = \int_x^1 g(y)dy
$$

(continuous analogue of using adjacency matrix to count paths in graphs).

- Note that $S = C \circ T \circ C$ where $(Cg)(x) = g(1-x)$, so we have shown that $A_n = n!\langle R^{n-1}\mathbf{1}, \mathbf{1}\rangle_{L_2[0,1]}$, where

$$R = C \circ T, \qquad (Rf)(x) = \int_0^{1-x} f(y)dy.$$

Transfer operators (continued)

- Note that $S = C \circ T \circ C$ where $(Cg)(x) = g(1-x)$, so we have shown that $A_n = n! \langle R^{n-1}\mathbf{1}, \mathbf{1}\rangle_{L_2[0,1]}$, where

$$R = C \circ T, \qquad (Rf)(x) = \int_0^{1-x} f(y)dy.$$

- Therefore

$$A_n = n! \sum_{k=1}^{\infty} \lambda_k^{n-1} \langle \mathbf{1}, \phi_k\rangle_{L_2[0,1]}^2,$$

where $(\phi_k)_{k\geq 1}$ is the orthonormal system of eigenfunctions of the self-adjoint operator R, with corresponding eigenvalues $(\lambda_k)_{k\geq 1}$.

Transfer operators (continued)

- Note that $S = C \circ T \circ C$ where $(Cg)(x) = g(1-x)$, so we have shown that $A_n = n!\langle R^{n-1}\mathbf{1}, \mathbf{1}\rangle_{L_2[0,1]}$, where

$$R = C \circ T, \qquad (Rf)(x) = \int_0^{1-x} f(y)dy.$$

- Therefore

$$A_n = n! \sum_{k=1}^{\infty} \lambda_k^{n-1} \langle \mathbf{1}, \phi_k \rangle^2_{L_2[0,1]},$$

  where $(\phi_k)_{k\geq 1}$ is the orthonormal system of eigenfunctions of the self-adjoint operator R, with corresponding eigenvalues $(\lambda_k)_{k\geq 1}$.
- It remains to diagonalize the operator R.

- We can refine the enumeration A_n by splitting the number of up-down permutations according to the *last number*

- We can refine the enumeration A_n by splitting the number of up-down permutations according to the *last number*
- Then the numbers $A_{n,k}$ form the *snake triangle*:

- We can refine the enumeration A_n by splitting the number of up-down permutations according to the *last number*
- Then the numbers $A_{n,k}$ form the *snake triangle*:

$$
\begin{array}{ccccccccccc}
 & & & & & 1 & & & & & \\
 & & & & 0 & & 1 & & & & \\
 & & & 1 & & 1 & & 0 & & & \\
 & & 0 & & 1 & & 2 & & 2 & & \\
 & 5 & & 5 & & 4 & & 2 & & 0 & \\
0 & & 5 & & 10 & & 14 & & 16 & & 16 \\
 & & & & & \dots & & & & &
\end{array}
$$

- We can refine the enumeration A_n by splitting the number of up-down permutations according to the *last number*
- Then the numbers $A_{n,k}$ form the *snake triangle*:

$$
\begin{array}{ccccccccccc}
 & & & & & 1 & & & & & \\
 & & & & 0 & & 1 & & & & \\
 & & & 1 & & 1 & & 0 & & & \\
 & & 0 & & 1 & & 2 & & 2 & & \\
 & 5 & & 5 & & 4 & & 2 & & 0 & \\
0 & & 5 & & 10 & & 14 & & 16 & & 16 \\
 & & & & & \cdots & & & & &
\end{array}
$$

- Plotting the last line one already can guess the base eigenfunction!

Diagonalizing the transfer operator

- Looking for an eigenfunction:

$$\lambda f(x) \quad = \quad (Rf)(x) = \int_0^{1-x} f(y)dy,$$

- Looking for an eigenfunction:

$$
\begin{aligned}
\lambda f(x) &= (Rf)(x) = \int_0^{1-x} f(y)dy, \\
\lambda f'(x) &= -f(1-x),
\end{aligned}
$$

Diagonalizing the transfer operator

- Looking for an eigenfunction:

$$\begin{aligned}
\lambda f(x) &= (Rf)(x) = \int_0^{1-x} f(y)dy, \\
\lambda f'(x) &= -f(1-x), \\
\lambda f''(x) &= f'(1-x) = -\frac{1}{\lambda} f(x).
\end{aligned}$$

- Looking for an eigenfunction:

$$\begin{aligned}\lambda f(x) &= (Rf)(x) = \int_0^{1-x} f(y)dy, \\ \lambda f'(x) &= -f(1-x), \\ \lambda f''(x) &= f'(1-x) = -\frac{1}{\lambda} f(x).\end{aligned}$$

- So f solves the Sturm-Liouville problem:

$$f''(x) = -\frac{1}{\lambda^2} f(x),$$

- Looking for an eigenfunction:

$$\begin{aligned}\lambda f(x) &= (Rf)(x) = \int_0^{1-x} f(y)dy, \\ \lambda f'(x) &= -f(1-x), \\ \lambda f''(x) &= f'(1-x) = -\frac{1}{\lambda}f(x).\end{aligned}$$

- So f solves the Sturm-Liouville problem:

$$f''(x) = -\frac{1}{\lambda^2}f(x), \qquad f(1) = 0,$$

- Looking for an eigenfunction:

$$\begin{aligned}\lambda f(x) &= (Rf)(x) = \int_0^{1-x} f(y)dy, \\ \lambda f'(x) &= -f(1-x), \\ \lambda f''(x) &= f'(1-x) = -\frac{1}{\lambda} f(x).\end{aligned}$$

- So f solves the Sturm-Liouville problem:

$$f''(x) = -\frac{1}{\lambda^2} f(x), \qquad f(1) = 0, \qquad f'(0) = 0.$$

Diagonalizing the transfer operator

- Looking for an eigenfunction:

$$\begin{aligned}\lambda f(x) &= (Rf)(x) = \int_0^{1-x} f(y)dy, \\ \lambda f'(x) &= -f(1-x), \\ \lambda f''(x) &= f'(1-x) = -\frac{1}{\lambda} f(x).\end{aligned}$$

- So f solves the Sturm-Liouville problem:

$$f''(x) = -\frac{1}{\lambda^2} f(x), \qquad f(1) = 0, \qquad f'(0) = 0.$$

- The (normalized) solutions are

$$\phi_k(x) = \sqrt{2}\cos\left(\frac{(2k-1)\pi x}{2}\right), \qquad \lambda_k = \frac{(-1)^{k-1}}{(2k-1)}\frac{2}{\pi}, \qquad k = 1, 2, \ldots.$$

Diagonalizing the transfer operator

- Looking for an eigenfunction:

$$\begin{aligned}\lambda f(x) &= (Rf)(x) = \int_0^{1-x} f(y)dy, \\ \lambda f'(x) &= -f(1-x), \\ \lambda f''(x) &= f'(1-x) = -\frac{1}{\lambda} f(x).\end{aligned}$$

- So f solves the Sturm-Liouville problem:

$$f''(x) = -\frac{1}{\lambda^2} f(x), \qquad f(1) = 0, \qquad f'(0) = 0.$$

- The (normalized) solutions are

$$\phi_k(x) = \sqrt{2}\cos\left(\frac{(2k-1)\pi x}{2}\right), \qquad \lambda_k = \frac{(-1)^{k-1}}{(2k-1)}\frac{2}{\pi}, \qquad k = 1, 2, \ldots.$$

- So $A_n = \frac{2^{n+2} n!}{\pi^{n+1}} \sum_{k=1}^{\infty} \frac{(-1)^{(k-1)(n-1)}}{(2k-1)^{n+1}}$, which is equivalent to André's theorem. :)

The transfer operator for the $2m$-strip

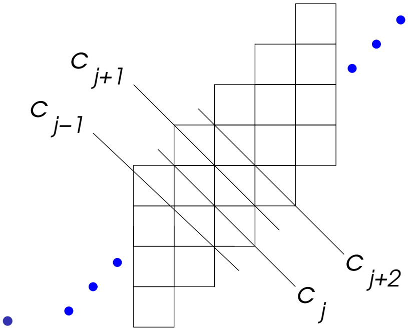


Figure: The coordinate filtration for the 4-strip.

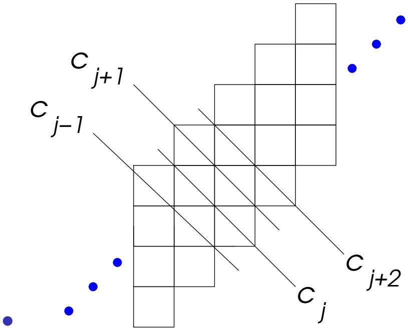


Figure: The coordinate filtration for the 4-strip.

- (main observation: better to cut tableau along diagonals!)

- The transfer operator works on the function space over the m-dimensional simplex

$$\Omega_m = \Big\{(x_1, \ldots, x_m) : 0 \leq x_1 \leq x_2 \leq \ldots \leq x_m \leq 1\Big\},$$

and is given by

The transfer operator for the $2m$-strip, continued

- The transfer operator works on the function space over the m-dimensional simplex

$$\Omega_m = \Big\{(x_1, \dots, x_m) : 0 \le x_1 \le x_2 \le \dots \le x_m \le 1\Big\},$$

and is given by

$$(Tf)(x_1, \dots, x_m) = \int_0^{1-x_m} \int_{1-x_m}^{1-x_{m-1}} \int_{1-x_{m-1}}^{1-x_{m-2}} \dots \int_{1-x_2}^{1-x_1} f(y_1, \dots, y_m) dy_m \dots dy_1.$$

The transfer operator for the $2m$-strip, continued

- Diagonalizing leads to boundary value problem:

$$\begin{aligned}
\frac{\partial^{2m} f}{\partial^2 x_1 \ldots \partial^2 x_m} &= \frac{(-1)^m}{\lambda^2} f, \\
f &\equiv 0 \quad \text{on: } x_1 = x_2, x_2 = x_3, \ldots, x_{m-1} = x_m, x_m = 1, \\
\frac{\partial f}{\partial x_1} &\equiv 0 \quad \text{on: } x_1 = 0.
\end{aligned}$$

The transfer operator for the $2m$-strip, continued

- Diagonalizing leads to boundary value problem:

$$\begin{aligned}
\frac{\partial^{2m} f}{\partial^2 x_1 \ldots \partial^2 x_m} &= \frac{(-1)^m}{\lambda^2} f, \\
f &\equiv 0 \quad \text{on: } x_1 = x_2, x_2 = x_3, \ldots, x_{m-1} = x_m, x_m = 1, \\
\frac{\partial f}{\partial x_1} &\equiv 0 \quad \text{on: } x_1 = 0.
\end{aligned}$$

- Solutions are

$$\begin{aligned}
\phi_{k_1,\ldots,k_m}(x_1,\ldots,x_m) &= 2^{m/2} \det \left(\cos\left(\frac{\pi k_i x_j}{2} \right) \right)_{i,j=1,\ldots,m}, \\
\lambda_{k_1,\ldots,k_m} &= \frac{2^m (-1)^{\frac{1}{2}\sum(k_j-1)}}{\pi^m k_1 k_2 \ldots k_m},
\end{aligned}$$

where $0 < k_1 < k_2 < \ldots < k_m$ are odd integers.

The transfer operator for the $2m$-strip, continued

- Diagonalizing leads to boundary value problem:

$$\begin{aligned}
\frac{\partial^{2m} f}{\partial^2 x_1 \dots \partial^2 x_m} &= \frac{(-1)^m}{\lambda^2} f, \\
f &\equiv 0 \quad \text{on: } x_1 = x_2, x_2 = x_3, \dots, x_{m-1} = x_m, x_m = 1, \\
\frac{\partial f}{\partial x_1} &\equiv 0 \quad \text{on: } x_1 = 0.
\end{aligned}$$

- Solutions are

$$\begin{aligned}
\phi_{k_1,\dots,k_m}(x_1,\dots,x_m) &= 2^{m/2} \det\left(\cos\left(\frac{\pi k_i x_j}{2}\right)\right)_{i,j=1,\dots,m}, \\
\lambda_{k_1,\dots,k_m} &= \frac{2^m (-1)^{\frac{1}{2}\sum(k_j-1)}}{\pi^m k_1 k_2 \dots k_m},
\end{aligned}$$

where $0 < k_1 < k_2 < \dots < k_m$ are odd integers.

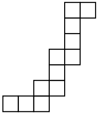
- In physicspeak: m-fermion systems

Other slopes

- The generalizations to other (rational) slopes are straightforward:

Other slopes

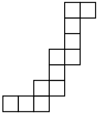
- The generalizations to other (rational) slopes are straightforward:
- One considers the "ribbon" shape



$$(SSTSTTSTTTS)^n$$

Other slopes

- The generalizations to other (rational) slopes are straightforward:
- One considers the “ribbon” shape

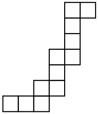


$$(SSTSTTSTTTS)^n$$

- Funds the eigenfunctions in the appropriate functional space

Other slopes

- The generalizations to other (rational) slopes are straightforward:
- One considers the "ribbon" shape



$$(SSTSTTSTTTS)^{n}$$

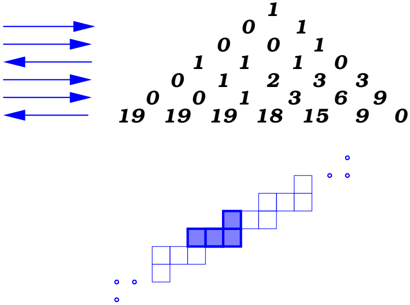
- Funds the eigenfunctions in the appropriate functional space
- For m-stack of ribbons, the eigenfunctions are m-fermionic states.

Other slopes (cont'd)

- For example, for slope $1/2$,

Other slopes (cont'd)

- For example, for slope 1/2,

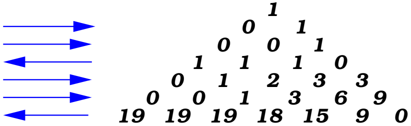


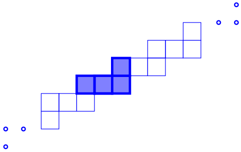
- One solves the boundary problem:

$$f'''(x) = 1/\lambda f(x),$$

Other slopes (cont'd)

- For example, for slope $1/2$,



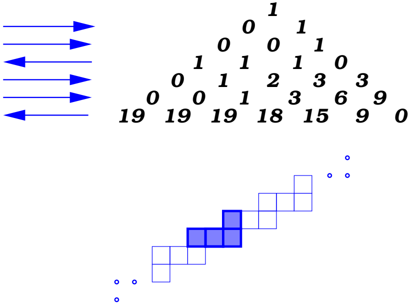


- One solves the boundary problem:

$$f'''(x) = 1/\lambda f(x), \qquad f(1) = 0,$$

Other slopes (cont'd)

- For example, for slope 1/2,

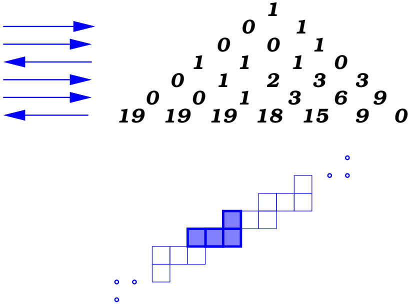


- One solves the boundary problem:

$$f'''(x) = 1/\lambda f(x), \qquad f(1) = 0, \qquad f'(0) = 0,$$

Other slopes (cont'd)

- For example, for slope 1/2,

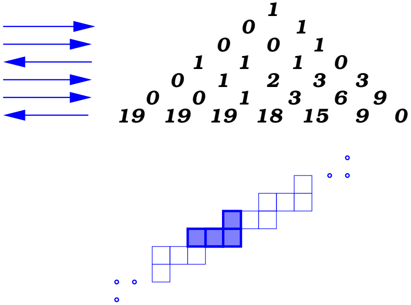


- One solves the boundary problem:

$$f'''(x) = 1/\lambda f(x), \qquad f(1) = 0, \qquad f'(0) = 0, \qquad f''(0) = 0.$$

Other slopes (cont'd)

- For example, for slope 1/2,



- One solves the boundary problem:

$$f'''(x) = 1/\lambda f(x), \qquad f(1) = 0, \qquad f'(0) = 0, \qquad f''(0) = 0.$$

- For m-stack of ribbons, the eigenfunctions are m-fermionic states.

- This leads to the a large deviation principle for Young tableaux: given a **continual Young diagram** of shape λ, the typical **growth profile** of a randomly chosen Young tableau of shape λ maximizes the functional

- This leads to the a large deviation principle for Young tableaux: given a **continual Young diagram** of shape λ, the typical **growth profile** of a randomly chosen Young tableau of shape λ maximizes the functional

$$J(g) = \int_0^1 \int_{-\infty}^{\infty} \left(\log\left(\frac{2}{\pi}\cos\left(\frac{\pi}{2}\frac{\partial g}{\partial u}\right)\right) - \log\frac{\partial g}{\partial t}\right)\frac{\partial g}{\partial t}\, du\, dt$$

subject to being a feasible growth profile for the shape λ.

- **Strip tableaux** (and to a lesser extent their generalizations with arbitrary slope) are an exactly solvable model.

- **Strip tableaux** (and to a lesser extent their generalizations with arbitrary slope) are an exactly solvable model.
- Interesting determinantal formulas - connection to determinantal point processes, random matrices?

- **Strip tableaux** (and to a lesser extent their generalizations with arbitrary slope) are an exactly solvable model.
- Interesting determinantal formulas - connection to determinantal point processes, random matrices?
- Connection to Euler and Bernoulli numbers and values of poly-zeta functions.

Conclusions

- **Strip tableaux** (and to a lesser extent their generalizations with arbitrary slope) are an exactly solvable model.
- Interesting determinantal formulas - connection to determinantal point processes, random matrices?
- Connection to Euler and Bernoulli numbers and values of poly-zeta functions.
- Connection to **square ice** model

Conclusions

- **Strip tableaux** (and to a lesser extent their generalizations with arbitrary slope) are an exactly solvable model.
- Interesting determinantal formulas - connection to determinantal point processes, random matrices?
- Connection to Euler and Bernoulli numbers and values of poly-zeta functions.
- Connection to **square ice** model

Thank you!