The Limiting Search-Cost of the Move-to-Front Strategy in a Law of Large Numbers Asymptotic Regime

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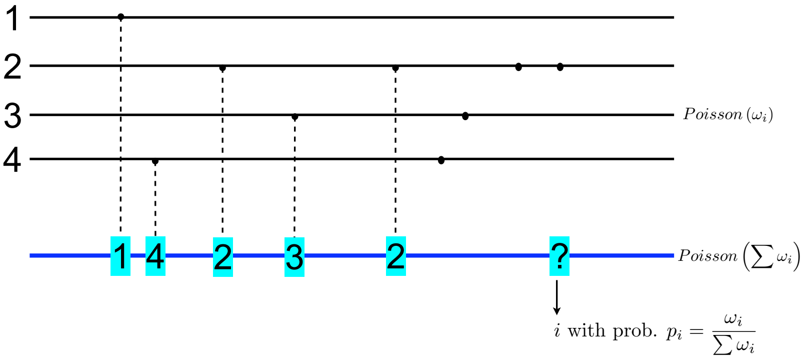
Outline

Request Process



Request Process

- Consider a list of n objects labelled $\{1, \ldots, n\}$;
- Assume that at time $t = 0$ objects are arranged in a permutation π of $\{1, \ldots, n\}$;
- Let $\mathbf{w} = (\omega_1, \ldots, \omega_n)$ be a *deterministic* nonnegative vector and consider a Poisson point process in $\mathbb{R}_+ \times \{1, \ldots, n\}$ with intensity measure $dt \otimes \mathbf{w}$.
- The request instant for an item i is given by the restriction of the point measure to $\mathbb{R}_+ \times \{i\}$ and follows a Poisson process of rate ω_i.
- For $t > 0$ each object is request independently.

Request Process

- Denote by N_t the total number of requests, which is also a standard Poisson process of rate $\sum_{i=1}^{n} \omega_i$.
- At each request the probability that object i is the one requested is
$$p_i = \frac{\omega_i}{\sum_{j=1}^{n} \omega_j},$$
we call p_i the “popularity” of object i.
- We denote by $S_i^{(n)}(t)$ the position of file i at time t and by I_k the k-th request file.
- By convention the list is update after t then $I_{N_{t-}+1}$ is the label of the first object to be requested.

The Move-to-front rule

$$4 \quad 1 \quad 2 \quad 5 \quad 7 \quad 3 \quad 8 \quad 9 \quad 6$$

$$\downarrow$$

$$4 \quad 1 \quad 2 \quad \boxed{5} \quad 7 \quad 3 \quad 8 \quad 9 \quad 6$$

$$\downarrow$$

$$\boxed{5} \quad 4 \quad 1 \quad 2 \quad 7 \quad 3 \quad 8 \quad 9 \quad 6$$

Figure: Ilustration of the MtF strategy

The MtF with independent requests

- Search-cost of the next requested item, define by:

$$S^{(n)}(t) := \sum_{i=1}^{n} S_i^{(n)}(t)\mathbf{1}_{\{I_{N_{t-}+1}=i\}}.$$

- R_t is the subset of objets which have been requested at least once in $[0, t[$.
- We decompose the search-cost into two r. v.,

$$S^{(n)}(t) = S_{eq}^{(n)}(t) + S_{oe}^{(n)}(t)$$

where

$$\begin{aligned} S_{eq}^{(n)}(t) &:= S^{(n)}(t)\mathbf{1}_{\{I_{N_{t-}+1}\in R_t\}} \\ S_{oe}^{(n)}(t) &:= S^{(n)}(t)\mathbf{1}_{\{I_{N_{t-}+1}\notin R_t\}}. \end{aligned}$$

Proposition: the search-cost for the MtF rule

We set $\mathbf{1}_{ij} = \mathbf{1}_{\pi(i)<\pi(j)}$. Let T_i be the time that has past since the last request of i or t if it has never been request

a) For all $k, i \in \{1, \dots, n\}$,

$$\boldsymbol{P}\{S_i^{(n)}(t) = k, i \in R_t\} = \boldsymbol{E}_{T_i}\left(\boldsymbol{P}\{\sum_{j=1 j\neq i}^{n} \mathbf{1}_{T_j<T_i} = k-1 \mid T_i\}\mathbf{1}_{T_i<t}\right)$$

b) For all $k, i \in \{1, \dots, n\}$,

$$\boldsymbol{P}\{S_i^{(n)}(t) = k, i \notin R_t\} = \boldsymbol{P}\Big\{\sum_{j=1, j\neq i}^{n} \mathbf{1}_{ji}+\mathbf{1}_{T_j<t}\mathbf{1}_{ij} = k-1\Big\}\boldsymbol{P}\{T_i = t\}$$

Proposition: the search-cost for the MtF rule

We set $\mathbf{1}_{ij} = \mathbf{1}_{\pi(i)<\pi(j)}$. Let $B_1(q_1)\dots, B_n(q_n)$ independent Bernoulli r. v. with given parameters $q_1,\dots,q_n$.

a) For all $k, i \in \{1,\dots,n\}$,

$$\boldsymbol{P}\{S_i^{(n)}(t) = k, i \in R_t\} = \int_0^t p_i e^{-p_i u} \boldsymbol{P}\{J_{eq}^n(u) = k-1\} du$$

where $J_{eq}^n(u) =^d \sum_{j=1, j\neq i}^n B_j(1-e^{-p_j u})$.

b) For all $k, i \in \{1,\dots,n\}$,

$$\boldsymbol{P}\{S_i^{(n)}(t) = k, i \notin R_t\} = \boldsymbol{P}\{J_{oe}^n(t) = k-1\} e^{-p_i t}$$

where $J_{oe}^n(t) =^d \sum_{j=1, j\neq i}^n B_j\left(1-e^{-p_j t}\mathbf{1}_{ij}\right)$.

c) For all $k, i \in \{1, \ldots, n\}$,

$$\boldsymbol{P}\{S_{eq}^{n}(t) = k\} = \sum_{i=1}^{n} \int_{0}^{t} p_i^2 e^{-p_i u} \boldsymbol{P}\{J_{eq}^{n}(u) = k\} du.$$

where $J_{eq}^{n}(u) =^{d} \sum_{j=1, j \neq i}^{n} B_j (1 - e^{-p_j u})$.

d) For all $k, i \in \{1, \ldots, n\}$,

$$\boldsymbol{P}\{S_{oe}^{(n)}(t) = k\} = \sum_{i=1}^{n} p_i \boldsymbol{P}\{J_{oe}^{(n)}(t) = k\} e^{-p_i t}.$$

where $J_{oe}^{(n)}(t) =^{d} \sum_{j=1, j \neq i}^{n} B_j \left(1 - e^{-p_j t} \mathbf{1}_{ij}\right)$.

The proof is direct consequence of J. Fill and L. Holst results.

Expected stationary search-cost (McCabe 65)

$$\boldsymbol{E}(S^{(n)}(\infty)) = \sum_{i \neq j} \frac{p_i p_j}{p_i + p_j}$$

Laplace transform of the transient Search-Cost

$$\phi_{S^{(n)}(t)}(z) = A_n(t,z) + B_n(t,z)$$

$$\begin{aligned} A_n(t,z) &= \int_0^t \sum_{i=1}^n p_i^2 e^{-u} \left(\prod_{j=1 j \neq i}^n ((e^{p_j u} - 1)e^{-z}) \right) du \\ B_n(t,z) &= \sum_{i=1}^n p_i e^{-t} \prod_{j=1 j \neq i}^n (\mathbf{1}_{i<j} + (e^{p_j t} - \mathbf{1}_{i<j})e^{-z}) \end{aligned}$$

Flajolet *et al.* ($t = \infty$) 92, Bodell 97 and Fill and Holst 96.

The conditional MtF

To do the asymptotic in the number of files we must do some assumption over $\mathbf{w}^{(n)}$.
For each n we consider a random or deterministic vector of intensities $\mathbf{w}^{(n)} = (\omega_1^{(n)}, \ldots, \omega_n^{(n)})$. Consider:

- The Poisson process $N_t = N_t^{(n)}$ and the search-cost $S^{(n)}(t)$, both defined *conditionally* on $\mathbf{w}^{(n)}$
- Let $\mathbf{p}^{(n)} = (p_1^{(n)}, \ldots, p_n^{(n)})$ be the vector of popularities, $p_i^{(n)} = \omega_i^{(n)} / \sum_{j=1}^{n} \omega_j^{(n)}$.

Previous Result, the limiting stationary search-cost

Proposition

JB., C. Paroissin, T. Huillet 2005 *Let ω_i be n i.i.d variable with law P. The limiting distribution of the stationary search-cost $S^{(n)}(\infty)$ satisfies*

$$\frac{S^{(n)}(\infty)}{n} \xrightarrow{d} S_\infty$$

when $n \to \infty$, where S_∞ has the density function

$$f_{S_\infty}(x) = -\frac{1}{\mu}\frac{\phi''\left(\phi^{-1}(1-x)\right)}{\phi'\left(\phi^{-1}(1-x)\right)}\mathbf{1}_{[0,1-\mathbf{p}_0]},$$

with $\mathbf{p}_0 = P(\omega_1 = 0)$, $\phi(t) = \boldsymbol{E}(e^{-\omega_1 t})$ and $\mu = \boldsymbol{E}(\omega_1)$.

Definition: The **LLN-**P Condition

We say that a sequence of (random or deterministic) vectors $\mathbf{w}^{(n)} = \left(\omega_1^{(n)}, \ldots, \omega_n^{(n)}\right)_{n \in \mathbb{N}}$ satisfies a law of large numbers with limiting law P (**LLN-**P), if there exist a probability measure $P \in \mathcal{P}(\mathbb{R}_+)$ with finite first moment $\mu \neq 0$ and positive random variables Z_n, such that the empirical measures

$$\hat{\nu}^{(n)} := \frac{1}{n} \sum_{i=1}^{n} \delta_{Z_n \omega_i^{(n)}} \to^{law} P$$

and their empirical means

$$\frac{1}{n} \sum_{i=1}^{n} Z_n \omega_i^{(n)} \to^{law} \mu.$$

Observe that, besides the i.i.d. case, **LLN-P** holds in the following examples, provided that the empirical means converge:

a) $Z_n = 1$, and $\omega_i^{(n)} = \omega_i$, with $(\omega_i)_{i \in \mathbb{N}}$ an ergodic process with invariant measure P.

b) $Z_n = 1$ and $(\omega_1^{(n)}, \ldots, \omega_n^{(n)})$, $n \in \mathbb{N}$ is exchangeable and $P-$chaotic.

c) $\omega_i^{(n)} = i^\alpha$ and some $\alpha \in \mathbb{R}$, and $Z_n = n^{-\alpha}$. By the obvious change of variable we have

$$P(dx) = \begin{cases} \frac{1}{\alpha} x^{\frac{1}{\alpha}-1} \mathbf{1}_{[0,1]}(x) dx & \text{if } \alpha > 0, \\ \delta_1(dx) & \text{if } \alpha = 0, \\ \frac{1}{|\alpha|} x^{\frac{1}{\alpha}-1} \mathbf{1}_{[1,\infty)}(x) dx & \text{if } \alpha < 0. \end{cases}$$

Thus, one can check that **LLN-P** holds if and only if $\alpha > -1$.

The initial permutation hypothesis

We study separately the random variables $S_{eq}^{(n)}(t)$ and $S_{oe}^{(n)}(t)$.
In order to observe any coherent limiting behavior of $S_{oe}^{(n)}(t)$, some assumptions on π will thus be needed.
Therefore, we assume either of the three following conditions:

LLN-P-ex: LLN-P holds, $\pi = Id$ and the vector $\mathbf{p}^{(n)}$ is exchangeable for each $n \in \mathbb{N}$.

LLN-P$^-$: LLN-P holds, $\pi = Id$ and $\mathbf{p}^{(n)}$ is decreasing a.s. for each $n \in \mathbb{N}$.

LLN-P$^+$: LLN-P holds, $\pi = Id$ and $\mathbf{p}^{(n)}$ is increasing a.s. for each $n \in \mathbb{N}$.

The limiting transient search-cost distribution

Corollary

If **LLN**-*P-ex holds, for each* $t > 0$ *we have*

$$\frac{S^{(n)}(n\mu t)}{n} \xrightarrow{d} S(t) \ ,$$

where $S(t)$ *satisfies the following relation in distribution:*

$$S(t) =^{(d)} S_\infty \mathbf{1}_{\{S_\infty \leq 1-\phi(t)\}} + U\mathbf{1}_{\{S_\infty > 1-\phi(t)\}}.$$

The random variable $S(t)$ *has density*

$$f_{S(t)}(x) = f_{S_\infty}(x)\mathbf{1}_{[0,1-\phi(t)]} + \frac{|\phi'(t)|}{\mu\phi(t)}\mathbf{1}_{[1-\phi(t),1]}$$

Continuation: the limiting transient search-cost

Corollary

If **LLN**-*P+ or* **LLN**-*P- holds, for each $t > 0$ we have*

$$\frac{S^{(n)}(n\mu t)}{n} \xrightarrow{d} S(t) \, ,$$

where $S(t)$ has the density

$$f_{S(t)}(x) = \mathbf{1}_{[0,1-\phi(t)]}(x) f_{S_\infty}(x) + \mathbf{1}_{[1-\phi(t),1]}(x) \frac{1}{\mu} \tilde{g}_t(x).$$

$$\begin{aligned} \textit{LLN-P}^- : &\quad \tilde{g}_t(y) = g_t^{-1}(1-y) &\quad ; g_t(y) = \boldsymbol{E}(e^{-\omega_1 t} \mathbf{1}_{[0,y]}), \\ \textit{LLN-P}^+ : &\quad \tilde{g}_t(y) = (1-g_t)^{-1}(y) &\quad ; g_t(y) = \boldsymbol{E}(e^{-\omega_1 t} \mathbf{1}_{(y,\infty)}). \end{aligned}$$

Examples

1) Let ω_i i.i.d. Bernoulli(p), then

$$f_{S(t)}(x) = \frac{1}{p}\mathbf{1}_{[0,p(1-e^{-t}))}(x) + \frac{e^{-t}}{1-p+pe^{-t}}\mathbf{1}_{[p(1-e^{-t}),1]}(x) \; .$$

2) Let ω_i i.i.d. Gamma($1,\alpha$), then

$$f_{S(t)}(x) = \left(1+\frac{1}{\alpha}\right)(1-x)^{1/\alpha}\mathbf{1}_{[0,u(t))}(x)+(1+t)^{-1}\mathbf{1}_{[u(t),1]}(x) \; ,$$

with $u(t) = 1-(1+t)^{-\alpha}$.

Examples

3) If ω_i i.i.d. Geometric(p), then

$$f_{S(t)}(x) = \frac{2(1-x)-p}{1-p}\mathbf{1}_{[0,u(t))}(x) + \frac{pe^{-t}}{1-(1-p)e^{-t}}\mathbf{1}_{[u(t),1]}(x) \ ,$$

where $u(t) = \frac{(1-p)(1-e^{-t})}{p+(1-p)(1-e^{-t})}$.

Examples

4) let now $\alpha \in (-1, 0)$ and define

$$\begin{aligned} P_\alpha dx &= -\frac{1}{\alpha} x^{1/\alpha - 1} \mathbf{1}_{[1\infty)} dx \text{ (Pareto distribution)} \\ \phi(s) &= \boldsymbol{E}_P(e^{\omega s}) \\ g_t(y) &= \boldsymbol{E}_P(e^{\omega s} \mathbf{1}_{[1,y]}). \end{aligned}$$

Examples

4.i) If $\omega_i = i^\alpha$ (GZipf law), the we have

$$f_{S(t)}(x) = -(\alpha+1)\frac{\phi''(\phi^{-1}(1-x))}{\phi'(\phi^{-1}(1-x))}\mathbf{1}_{[0,1-\phi(t))}(x) + (\alpha+1)g_t^{-1}(1-x)\mathbf{1}_{[1-\phi(t),1]}(x).$$

4.ii) If ω_i are i.i.d. with P_α , then

$$f_{S(t)}(x) = -(\alpha+1)\frac{\phi''(\phi^{-1}(1-x))}{\phi'(\phi^{-1}(1-x))}\mathbf{1}_{[0,1-\phi(t))}(x) + (\alpha+1)\frac{|\phi'(t)|}{\phi(t)}\mathbf{1}_{[1-\phi(t),1]}(x)\ .$$

5) Let $\alpha > 0$ and set now

$$
\begin{aligned}
P_\alpha &= \frac{1}{\alpha} x^{1/\alpha - 1} \mathbf{1}_{[0,1]} dx \text{ (Beta}(1, 1/\alpha)\text{ distribution)} \\
\phi(s) &= \boldsymbol{E}_P(e^{\omega s}) \\
g_t(y) &= \boldsymbol{E}_P(e^{\omega s} \mathbf{1}_{[y,1]}).
\end{aligned}
$$

Examples

Let $\alpha > 0$

5.i) If $\omega_i = i^\alpha$ (power law), we have

$$f_{S(t)}(x) = -(\alpha+1)\frac{\phi''(\phi^{-1}(1-x))}{\phi'(\phi^{-1}(1-x))}\mathbf{1}_{[0,1-\phi(t))}(x) + (\alpha+1)(1-g_t)^{-1}(x)\mathbf{1}_{[1-\phi(t),1]}(x)\ ,$$

5.ii) If ω_i are i.i.d. with law P_α, then

$$f_{S(t)}(x) = -(\alpha+1)\frac{\phi''(\phi^{-1}(1-x))}{\phi'(\phi^{-1}(1-x))}\mathbf{1}_{[0,1-\phi(t))}(x) + (\alpha+1)\frac{|\phi'(t)|}{\phi(t)}\mathbf{1}_{[1-\phi(t),1]}(x)\ .$$

Least-Recently-Used (LRU) Fault Probability

The probability that at time $n\mu t$ the requested document does not lay among the δn selected ones (a "fault" occurs) corresponds to the probability $\boldsymbol{P}(S^{(n)}(n\mu t) > \delta n)$. Consequently, under assumption **LLN-*P*-ex**:

$$\boldsymbol{P}(S^{(n)}(n\mu t) > \delta n) \sim \begin{cases} \dfrac{|\phi'(\eta_\delta)|}{\mu} & \text{if } \eta_\delta < t \\ \dfrac{1-\delta}{\mu} \dfrac{|\phi'(t)|}{\phi(t)} & \text{if } \eta_\delta \geq t\,, \end{cases}$$

with $\eta_\delta = \phi^{-1}(1-\delta)$.

The Fault probability for the LRU with GZipf law

Moreover, for $\omega_i = i^\alpha$, $\alpha \in (-1, 0)$, the transient asymptotic fault probability is given by $P(S(t) > \delta) =$

$$\begin{cases} -\dfrac{\alpha+1}{\alpha}\eta_\delta^{-(1+1/\alpha)}\Gamma\left(1+1/\alpha, \eta_\delta\right) & \text{if } \eta_\delta < t \\ -\dfrac{\alpha+1}{\alpha}t^{-(1+1/\alpha)}\left[\Gamma\left(1+1/\alpha, t\right) - \Gamma\left(1+1/\alpha, t\epsilon_{\delta,t}\right)\right] & \text{if } \eta_\delta \geq t\,. \end{cases}$$

where $\Gamma(z, y) := \int_y^\infty x^{z-1} e^{-x} dx$ is the incomplete Gamma function, and $\epsilon_{\delta,t} := g_t^{-1}(1-\delta)$ with g_t as in Example 4.i).

Why that the i.i.d. case works?

If $(\omega_i)_{i\in \mathbb{N}}$ i.i.d. random variables in $\mathbb{R}_+$ of law P with finite mean $\mu > 0$, and the probability vector $\mathbf{p}^{(n)} = (p_i^{(n)})$ defined by $p_i^{(n)} := \frac{\omega_i}{\sum_{j=1}^n \omega_j}$. Then, by the strong law of large numbers, we have

$$(n\mu p_1^{(n)}, \dots, n\mu p_k^{(n)}) \longrightarrow^{a.s.} (\omega_1, \dots, \omega_k).$$

In particular, for any fixed $k \leq n$ coordinates of the vector $n\mu\mathbf{p}^{(n)}$ become independent as n tends to infinity, and with limiting law equal to P. We can deduce that the empirical measures

$$\nu^{(n)} := \frac{1}{n}\sum_{i=1}^{n} \delta_{n\mu p_i^{(n)}} \rightarrow^{law} P.$$

Propagation of Chaos (H. Tanaka)

For each $n \in \mathbb{N}$, let $X^{(n)} = (X_1^{(n)}, \dots, X_n^{(n)})$ be an exchangeable random vector in $\mathbb{R}^n$ with law P_n. Then, the following assertions are equivalent:

i) There exists a probability measure P in $\mathbb{R}$ such that for all $k \in \mathbb{N}$, when $n \to \infty$,

$$law(X_1^{(n)}, \dots, X_k^{(n)}) \Longrightarrow P^{\otimes k}.$$

where $\Longrightarrow$ denotes the weak convergence of measures.

ii) The random variables $\frac{1}{n}\sum_{i=1}^{n} \delta_{X_i^{(n)}}$ (taking values in $\mathcal{P}(\mathbb{R})$) converge in law as n goes to infinity to a deterministic limit equal to P.

L.L.N. for random partitions of the interval

Theorem

Assume that $(\mathbf{w}^{(n)})_{n\in\mathbf{N}}$ *satisfy condition* **LLN-**P *and let* $(\mathbf{p}^{(n)})_{n\in\mathbf{N}}$ *be defined as the renormalized vector. Then, the empirical measure*

$$\nu^{(n)} := \frac{1}{n}\sum_{i=1}^{n}\delta_{n\mu p_i^{(n)}}$$

converges in law to the deterministic limit P.

If **LLN-**P holds for $(\omega_i^{(n)})_{i=1}^n$ and some sequence (Z_n), then it also holds for $(p_i^{(n)})_{i=1}^n$ and the sequence $(Z_n') := (n\mu)$

Remark

Assumption **LLN-P** together with previous Theorem, imply that for any bounded continuous function $\Psi : \mathcal{P}(\mathbb{R}_+) \to \mathbb{R}$, one has

$$\mathbb{E}(\Psi(\nu^{(n)})) \to \Psi(P) \text{ when } n \to \infty.$$

The proofs of our results systematically rely on this facts to compute the limits of quantities of the form

$$\mathbb{E}\left(f_n(\mathbf{p}^{(n)})\right) \to \Psi(P)$$

whenever $f_n(\mathbf{p}^{(n)}) \sim \Psi(\nu^{(n)})$, for some bounded continuous Ψ not depending on n.

Idea of the proof

Conditionally on $\mathbf{w}^{(n)}$ we have that the expected stationary search-cost is

$$\boldsymbol{E}(\tfrac{1}{n}S^{(n)}(\infty)|\,\mathbf{w}) = \frac{1}{n}\sum_{i,j=1}^{n}\frac{p_i p_j}{p_i+p_j} - \frac{1}{2n} \sim \frac{1}{n^2\mu}\sum_{i,j=1}^{n}\frac{(n\mu p_i)(n\mu p_j)}{n\mu p_i + n\mu p_j}$$

Then in terms of $\nu^{(n)}$ we have

$$\boldsymbol{E}(\tfrac{1}{n}S^{(n)}(\infty)) = \frac{1}{\mu}\left\langle \nu^{(n)}, \left\langle \nu^{(n)}, \frac{uv}{u+v}\right\rangle_u\right\rangle_v$$

Then asymptotically

$$\boldsymbol{E}(\tfrac{1}{n}S^{(n)}(\infty)) \to \frac{1}{\mu}\boldsymbol{E}_{P\otimes P}\left(\frac{uv}{u+v}\right) .$$

notice that here we don't have the bounded hypothesis!.

Remark, discussion and questions

1) Our techniques ensure some type of stability of the asymptotic behavior of random or deterministic partition, under “perturbations” that preserve the probability measure P.
2) The techniques we have introduced can be used to study other sorting algorithms, if the law of the corresponding relevant variables depend on the empirical measure of the popularities of the items. However, it is not obvious to identify the functional of the empirical measures involved.
3) A proiri our approach excludes cases of interest as for instance the Zipf laws $\omega = i^{\alpha}$ with $\alpha \leq -1$, and the Poisson-Dirichlet partition.

Thank!