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Dada $f \in \mathcal{C}^2(\mathbb{R}^N \backslash \{0\})$, seja $\varphi : (0, \infty) \to \mathbb{R}$ a função dada por:

$$\varphi(r) = \int_{\mathrm{S}_1(0)} f(r\omega) \,\mathrm{d}\sigma(\omega).$$

Temos $f_\#(x) = \varphi(|x|)$, logo (recorde a solução do exercício 2 da primeira lista):

$$\Delta(f_\#)(x) = \varphi''(|x|) + \frac{N-1}{|x|}\varphi'(|x|), \quad x \in \mathbb{R}^N \backslash \{0\}.$$

Temos:

$$\begin{aligned}
\varphi'(r) &= \int_{\mathrm{S}_1(0)} \langle \nabla f(r\omega), \omega \rangle \,\mathrm{d}\sigma(\omega) \\
&= \int_{\mathrm{S}_1(0)} \langle \nabla f(r\omega), \vec{n}(\omega) \rangle \,\mathrm{d}\sigma(\omega) \\
&= \int_{\mathrm{B}_1(0)} \operatorname{div} [\nabla f(ry)] \,\mathrm{d}y \\
&= r \int_{\mathrm{B}_1(0)} \Delta f(ry) \,\mathrm{d}y \\
&= r^{1-N} \int_{\mathrm{B}_r(0)} \Delta f(z) \,\mathrm{d}z \\
&= r^{1-N} \int_0^r \int_{\mathrm{S}_1(0)} \Delta f(\rho\omega) \rho^{N-1} \,\mathrm{d}\sigma(\omega) \,\mathrm{d}\rho,
\end{aligned}$$

logo:

$$\begin{aligned}
\varphi''(r) &= (1-N) r^{-N} \int_{\mathrm{B}_r(0)} \Delta f(z) \,\mathrm{d}z + r^{1-N} \int_{\mathrm{S}_1(0)} \Delta f(r\omega) r^{N-1} \,\mathrm{d}\sigma(\omega) \\
&= \frac{1-N}{r} \varphi'(r) + \int_{\mathrm{S}_1(0)} \Delta f(r\omega) \,\mathrm{d}\sigma(\omega),
\end{aligned}$$

e, portanto:

$$\Delta(f_\#)(x) = \frac{1-N}{|x|}\varphi'(|x|) + \int_{\mathrm{S}_1(0)} \Delta f(|x|\omega) \,\mathrm{d}\sigma(\omega) + \frac{N-1}{|x|}\varphi'(|x|) = (\Delta f)_\#(x),$$

para todo $x \in \mathbb{R}^N \backslash \{0\}$.