Research Statement

My research interests lie in differential geometry and geometric analysis. My work has concentrated according to two themes. The first is the study of submanifolds of spaces with riemannian holonomy group contained in the group G_2. The second is a question of rigidity of totally geodesic cycles in riemannian symmetric spaces. In the future, I propose to continue to work on these questions, and also to study the gauge theory on manifolds of exceptional holonomy.

Minimal Surfaces in Manifolds of G_2 Holonomy

One idiosycrasy in dimension 7 is the fact that one can define a constant 3-form on $\mathbb{R}^7$ that determines the multiplication of the octonion algebra $\mathbb{O} \cong \mathbb{R}^8$. More precisely, one can define

$$\phi(X, Y, Z) = \langle X, YZ \rangle$$

where X, Y and Z are purely imaginary octonions. ϕ is a antisymmetric and nondegenerate. Moreover, ϕ determines the metric. In the context of Riemannian geometry, this means that a manifold M that supports a form φ that is at each point isomorphic to ϕ canonically admits a metric g_φ. If the form φ is parallel with respect to the metric g_φ, (M, g_φ) is said to be a a G_2-manifold because the Riemannian holonomy group is necessarily isomorphic to a subgroup of the exceptional Lie group G_2.

By the classification theorem of Berger, there exist only seven possibilities for the holonomy group for a non-symmetric and irreducible Riemannian manifold. Among these sits the group G_2. It was only relatively recently that the first example of a manifold of holonomy precisely G_2 was discovered. Over the last fifteen years their importance has become more evident: they have appeared in M-theory in high energy physics and are closely related to the geometry of Calabi-Yau manifolds and of self-duality in 4-dimensions. We show that the submanifolds of a manifold of G_2-holonomy can be studied using the techniques of holomorphic bundles and of pseudo-holomorphic curves.

Using the techniques of calibrated geometry, the minimal submanifolds of dimensions 3 and 4 of a G_2 manifold have been much studied. For submanifolds of dimension 2, we show that strong analogies exist with the minimal surfaces of 4-dimensional spaces. Just as in that case, we see that X admits a *twistor space* $\mathcal{Z}_X$ and the minimal submanifolds of X are intimately related to the complex curves in $\mathcal{Z}_X$.

We consider a 2 dimensional submanifold $\Sigma \subseteq X$, where X is a G_2 manifold. To such a submanifold there exists canonically a normal vector field η. For example in $\mathbb{R}^7$, in which case ϕ takes the form

$$\phi = e^{123} + e^1 \wedge (e^{45} - e^{67}) + e^2 \wedge (e^{46} - e^{75}) + e^3 \wedge e^{47} - e^{56}).$$

In this case, if the tangent space to Σ, $T_\Sigma = \text{span}\{e_1, e_2\}$ is generated by e_1 and e_2 the normal vector is given by

$$\eta = -(e_1 \lrcorner\, (e_2 \lrcorner\, \phi))^{\#} = e_3$$

where $\lrcorner$ denotes the contraction of a vector in a form and $\#$ denotes the musical duality. In particular, this permits us to define a Gauss map that lifts the surface Σ to a surface $\tilde{\Sigma}$ in the unit tangent bundle $\mathcal{Z}_X = U_1(T_X)$. We can consider this bundle to be the twistor space to the manifold X.

Additionally, we can show that $U_1(T_X)$ admits a distribution $\mathcal{E}$ by hyperplanes, to which the Gauss lift $\tilde{\Sigma}$ is always tangent. $\mathcal{E}$ admits canonically a hermitian structure, defined by the form φ. We define the notion of an *adapted* submanifold $\Sigma \subseteq X$ for which we have the following result.

Theorem 1. *The Gauss lift $\tilde{\Sigma}$ of an adapted surface in a G_2-manifold is pseudo-holomorphic with respect to the hermitian structure on $\mathcal{E}$ if and only if Σ is a minimal submanifold.*

The condition that a submanifold be *adapted* in this case is that the second fundamental form, considered as a symmetric tensor with values in the normal bundle, takes values everywhere orthogonal to the vector field η.

Theorem 2. *A surface $\Sigma \subseteq X$ is adapted in this sense if and only if the Levi-Civita connection defines a holomorphic structure on the bundle $W = \eta^\perp \subseteq N_{\Sigma,X}$, which is the set of normal vectors orthogonal to η.*

We would like to generalise and extend these results to manifolds with $Spin(7)$ holonomy. In this case, the twistor space admits an almost complex structure. We would also like to use symplectic geometry and the techniques there-in of pseudo-holomorphic curves (à la Gromov and Labourie) to study adapted minimal surfaces.

This work is submitted for publication and is available at *arXiv:math.DG/1011.3084*.

Rigidity of Totally Geodesic Cycles

A simply connected symmetric space of compact type decomposes into a product of irreducible factors. We can show that the factors of rank one, with one possible exceptional case, are rigid when considered among minimal submanifolds. More precisely, let $X = X_1 \times X_2$ be a Riemannian symmetric space of compact type where X_1 is irreducible and of rank one. We exclude the example of the Cayley plane $\mathbb{O}P^2$. Let f be the standard immersion of X_1 into X as a totally geodesic factor.

Theorem 3. *There exists a $\mathcal{C}^3$ neighbourhood $\mathcal{U}$ of f in the set of immersions such that every minimal immersion contained in $\mathcal{U}$ is equivalent to f.*

The notion of equivalence that we use here is by the action of ambiant isometries and reparametrisations of the submanifold. In this sense, the non-exceptional components of rank one of the symmetric space are rigid, and isolated from minimal submanifolds of another type.

Here we note that, except for the Cayley plane, every compact symmetric space of rank one admits a riemannian submersion from a sphere with totally geodesic fibres. We can hence deduce the preceding result from the corresponding one for $X_1 = S^n$. We describe the result in that case.

In his article of 1968, Simons showed that the second fundamental form A of a minimal submanifold satisfies the non-linear second order equation

$$\nabla^2 A = -\overline{A} \circ A - A \circ \underline{A} + \bar{R}(A) + R', \tag{1}$$

if A is considered a section of a Riemannian vector bundle. Here $\overline{A}$ and $\underline{A}$ are quadratic expression in A and $R(A)$ and R' are respectively obtained by contraction of A into the ambient curvature and its covariant derivative ∇R.

We consider a closed n-dimensional submanifold M of $S^n \times X_2$ where X_2 is a compact type symmetric space and we make a hypothesis on the size of the derivative of the projection to the second factor.

Theorem 4. *There exists $C > 0$ such that for all $0 < \Lambda < 1$ and for all closed minimal submanifold M of dimension n for which $\|\pi_2\| \leq \Lambda$ the term $\bar{R}(A)$ satisfies*

$$\langle \bar{R}(A), A \rangle \geq \left(2\rho - C\Lambda^2\right)\|A\|^2.$$

Here, π_2 denotes the derivative of the map $\pi_2 : M \to X_2$, and ρ is the infimum of the Ricci curvature in directions tangent to $S^n \times X_2$. This is to say that in particular $\rho > 0$. In particular, by integration by parts and using the equation 1, we obtain the inequality

$$0 \leq \int_M \|A\|^2 \left(\|A\|^2 - \frac{\rho}{q}\right)$$

where q is a constant that depends only on the dimensions of the spaces. If M is a closed minimal submanifold in $S^n \times X_2$ such that π_2 and A are sufficiently small, then the right hand side of this inequality cannot be positive. Therefore,

Corollary 5. *There exists $\Lambda > 0$ such that if M is a closed minimal submanifold that satisfies $\|\pi_2\| \leq \Lambda$ and $\|A\|^2 < \rho/q$, then M is totally geodesic. Λ can be chosen sufficiently small so that this implies that $M = S^n \times \{pt\}$.*

This implies the Theorem 3. The question for the other factors of rank one are resolved individually, for the cases of the real, complex and quaternionic projective spaces. If a closed minimal submanifold of $\mathbb{K}P^n \times X_2$ is sufficiently close to the projective factor, then it must be a totally geodesic factor. In this case though, the proximity is with respect to the scalar curvature of the intrinsic metric on the submanifold, rather than the size of the second fundamental form.

This work is to appear in the Annales de l'Institut Fourier, volume 61 of 2011, and is currently available at *arXiv:0903.0789*.

Gauge Theory on Manifolds of G_2 Holonomy

As mentioned above, the geometry of manifolds of G_2 holonomy shows many similarities with Riemannian geometry in 3 and 4 dimensions. Motivated by these analogies, we have commenced a programme of research on gauge theory on manifolds of G_2 holonomy. Gauge theory in higher dimensions is today an active field of research; important results have already been proven by Donaldson, Segal and Thomas, Tao and Tian, Lewis, Brendle and most recently Sá Earp. One problem that we propose to study is of the existence of instantons on G_2 manifolds.

If (X, φ, g_φ) is a manifold of G_2 holonomy, one can see that the bundle of 2-forms $\Lambda^2 T^*_X$ decomposes into two factors

$$\Lambda^2 T^*_X = \Lambda^2_7 \oplus \Lambda^2_{14}$$

of dimensions 7 and 14 respectively. In particular, $\Lambda^2_{14} = \ker\{*\varphi\wedge : \Lambda^2 \to \Lambda^6\}$ where $*\varphi$ is the Hodge dual of φ.

A connection A on a vector bundle $E \to X$ is a G_2-instanton if the curvature of A satisfies

$$*\varphi \wedge F_A = 0.$$

This condition can be seen as the analogue of the equation $*F_A = -F_A$ that defines anti-self-dual connections on a 4-dimensional manifold.

The first construction of a complete and irreducible metric of holonomy G_2 was done by Bryant and Salamon. They considered the 7-dimensional manifold given as the total space of the spinor bundle on S^3

$$X = (\mathcal{F} \times \mathbb{H})/SU(2),$$

where $\mathcal{F}$ is the principal fibre bundle given by the spin structure on S^3. Bryant and Salamon showed that X admits an explicit and asymptotically conic metric with holonomy equal to G_2.

Additionally, X tautologically admits a hermitian vector bundle of complex rank 2. In work in progress we have shown that X admits two forms A_1 and A_2 such that if $A = fA_1 + gA_2$ where f and g are functions of the radius of the fibre, the connection $\nabla = d + A$ is a G_2-instanton if and only if f and g satisfy a system of ordinary differential equations.

This project is still in progress. Other interesting questions to consider include to determine the domain of definition of the connections that are obtained, and to determine the integrability of the curvature.

Gauge Theory on the Kummer Surface

Following the construction of Bryant and and Salamon, the first example of a compact Riemannian manifold woth G_2 holonomy was constructed by Joyce. He considered the orbifold $X_s = T^7/\Gamma$ where Γ is a finite group of G_2-automorphisms of the flat 7-torus. Γ was chosen so that a neighbourhood of the singular locus in the quotient was diffeomorphic to a set of copies of $T^3 \times \mathbb{C}^2/\mathbb{Z}_2$.

$\mathbb{C}^2/\mathbb{Z}_2$ has as a resolution the space Y that supports the well known metric of Eguchi-Hanson. Using this metric, Joyce constructed an approximate G_2 metric φ_ε on the desingularisation X of X_s and using non-linear analysis was able to obtain a 3-form $\varphi = \varphi_\varepsilon + d\eta$ that determined a metric of holonomy G_2.

We would like to, for $\varepsilon \ll 1$, construct via a gluing theorem a connection A on a fibre bundle $E \to X$ such that $*\varphi \wedge F_A = 0$.

The construction of Joyce has been called a *generalised Kummer construction* because of its similarity with the $K3$ surface of Kummer

$$M \to T^4/\mathbb{Z}_2.$$

For all $0 < \varepsilon \ll 1$ we can graft the Eguchi-Hanson metric on Y to a flat metric to obtain a Kähler metric ω_ε on M. Similarly, we can graft an instanton A on Y to a trivial connection to obtain a connection A_ε on a bundle over M.

We consider also an analogous question in gauge theory 4-dimensions. This is closely analogous to the 7-dimensional case and seems to exhibit many of the same properties as that more complicated geometry. We would like to construct, again by a gluing argument, anti-self-dual connections on $K3$ surfaces that degenerate towards the quotient $T^4/\mathbb{Z}_2$. We search for a solution (f, α) to a non-linear differential equation such that

1. $\omega_\varepsilon + i\partial\bar{\partial}f$ is a Ricci-flat Kähler metric,
2. $A_\varepsilon + \alpha$ is an anti-self-dual connection with respect to the metric $\omega_\varepsilon + i\partial\bar{\partial}f$.

Our method follows that of Donaldson, who chooses a good conformal model to construct a Ricci-flat Kähler metric. It is also closely related to the original gluing results of Taubes. This work is in progress.

References

[BS] R.L. Bryant, S. Salamon, *On the construction of some metrics with exceptional holonomy*, Duke Math. J. **58** (1989), 829-850.

[D] S.K. Donaldson, *Calabi-Yau metrics on Kummer surfaces as a model gluing problem*, arXiv:math.DG/1007.4218v1.

[DT] S.K. Donaldson, R.P. Thomas, *Gauge theory in higher dimensions*, The geometric universe, Oxford Univ. Press (1998) 31-47.

[ES] J. Eells, S, Salamon, *Constructions twistorielles des applications harmoniques*, C. R. Acad. Sci. Paris Sér. I Math. **296** (1983) 685-687.

[J] D.D. Joyce, *Compact riemannian 7-manifolds with holonomy G_2. I.*. J. Diff. Geom. **43** (1996) 291-328.

[Lab] F. Labourie, *Immersions isométriques elliptiques et courbes pseudo-holomorphes.* J. Diff. Geom., **30** (1989) 395-424.

[Law] H.B. Lawson Jr., *Rigidity theorems in rank-1 symmetric spaces*, J. Diff. Geom., **4** (1970) 349-357.

[N] H. Nakajima, *Moduli spaces of anti-self-dual connections on ALE gravitational instantons*, Invent. math., **102**, (1990) 267-303.

[S] J. Simons, *Minimal varieties in riemannian manifolds.* Ann. of Math. **88** (1968) 62-105.

[T] C.H. Taubes, *Self-dual connections on 4-manifolds with indefinite intersection matrix.* J. Diff. Geom. **19** (1984) 517-560.