# Nonparametric Estimation of Functional-Coefficient Autoregressive Models

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# Introduction

## Nonlinear Models:

- Exponential autoregressive model (EXPAR);

  Haggan and Ozaki (1981)

- Threshold autoregressive (TAR) model;

  Tong(1983)

- Autoregressive conditional heteroscedastic (ARCH) model;

  Engel (1982)

- Functional-coefficient autoregressive model (FAR);

  Chen and Tsay (1993)

  Cai, Z., Fan, J. and Yao, Q. (2000)

# Models

## FAR:

$$X_t = f_1(Y_{t-1})X_{t-1} + \cdots + f_p(Y_{t-1})X_{t-p} + \epsilon_t,$$

where

- $p$ is a positive integer;
- $\varepsilon_t$ is a sequence of iid random variables $(0,\sigma^2)$ such that $\varepsilon_t \perp \{x_{t-i},\ i>0\}$;
- $\{f_i(Y_{t-1})\}$ are measurable functions: $R^k \to R$;
- $Y_{t-1}=(x_{t-i1},\ x_{t-i2},\ \ldots,\ x_{t-ik})'$ with $i_j>0$ for $j=1, \ldots, k$.

# Models

$Y_{t-1}$: a threshold vector;

$i_1, \ldots, i_k$: the threshold (or delay) parameters;

$x_{t-i_j}$: the threshold variables.

Assume $\max(i_1, \ldots, i_k) \le p$.

# Models

## Special cases of the FAR model:

**The linear TAR model:**

$x_t = \theta_1^{(i)} x_{t-1} + \ldots + \theta_p^{(i)} x_{t-p} + \varepsilon_t^{(i)}$, if $x_{t-d} \subset \Omega_i$,

for $i = 1, \ldots, k$, where $\Omega_i$'s form a nonoverlapping partition of the real line.

Letting $f_j(Y_{t-1}) = \theta_j^{(i)}$, if $X_{t-d} \in \Omega_i$

# Models

**EXPAR model :**

$x_t=[a_1+(b_1 +c_1 x_{t-d})\exp(-\theta_1 x_{t-d}^2)]x_{t-1} + \ldots +$

$+ [a_p +(b_p+c_p x_{t-d})\exp(-\theta_p x_{t-d}^2)]x_{t-p} + \varepsilon_t,$

where $\theta_i \geq 0$ for $i=1, \ldots, p$.

$$f_j(Y_{t-1}) = a_j + (b_j + c_j X_{t-d}) \exp(-\theta_j X_{t-d}^2).$$

# Wavelets

From two basic functions, the

scaling function $\phi(x)$ and the wavelet $\psi(x)$ we define infinite collections of translated and scaled versions,

$$\phi_{j,k}(x) = 2^{j/2}\phi(2^j x-k), \quad \psi_{j,k}(x) = 2^{j/2}\psi(2^j x-k), \quad j,k \in Z.$$

We assume that

$$\{\phi_{l,k}(\cdot)\}_{k\in Z} \cup \{\psi_{j,k}(\cdot)\}_{j \geq l;\ k \in Z}$$

forms an orthonormal basis of $L_2(R)$, for some coarse scale $l$.

# Wavelets

for any function $f \in L_2(R)$, we can expand it in an orthogonal series

$$f(x) = \sum_{k \in Z} \alpha_{l,k}\, \phi_{l,k}(x) + \sum_{j \geq l} \sum_{k \in Z} \beta_{j,k}\, \psi_{j,k}(x),$$

for some coarse scale $l$ with the wavelet coefficients given by

$$\alpha_{l,k} = \int f(x)\phi_{l,k}(x)\,dx,$$

$$\beta_{j,k} = \int f(x)\psi_{j,k}(x)\,dx.$$

# A Review on FAR Models

There are, basically, three procedures that have been used for the estimation of these models:

a) arranged local regression; Chen and Tsay (1993);

b) kernel estimators; Cai et al.(2000);

c) spline smoothing; Huang and Shen(2004).

# Estimation

"First generation wavelets" may not be appropriete for arbitrary designs of the variable of interest.

**Three approaches:**

1) use the usual wavelet after a suitable transformation of the observations;
2) use wavelet adapted to the design. Sweldens(1997);
3) use warped wavelet. Kerkyacharian and Picard(2004).

# Estimation

The main difficulty in using the proposed FAR model is specifying the functional coefficients $f_i(\cdot)$.

For simplicity we consider only the case $Y_{t-1} = x_{t-d}$, for some $d > 0$.

# Estimation

The estimation problem consists of estimating the parameter function $f_i(\cdot)$.

We present wavelet estimators from observations

$$\{x_t,\ t=1,\ \ldots,\ T\}.$$

# Estimation(approach1)

## Estimator with the usual wavelets

We expand $f_i(\cdot)$ as

$$f_i(x_{t-d}) = \sum_{k\in Z} \alpha_{l,k}^{(i)} \phi_{l,k}(x_{t-d}) + \sum_{j\geq l} \sum_{k\in Z} \beta_{j,k}^{(i)} \psi_{j,k}(x_{t-d}),$$

where

$$\alpha_{l,k}^{(i)} = \int f_i(x_{t-d})\phi_{l,k}(x_{t-d})\,dx_{t-d},$$

$$\beta_{j,k}^{(i)} = \int f_i(x_{t-d})\psi_{j,k}(x_{t-d})\,dx_{t-d}.$$

we may let $l = 0$, $k \in I_j=\{k: k=0,1, \ldots, 2^j-1\}$ and

$j = 0, \ldots, J_T-1$ in the second term, for some maximum scale $J_T$ depending on $T$.

In general, we assume $T=2^J$ and $J_T \leq J$.

# Estimation(approache1)

We define the empirical wavelet coefficients as least square estimators, i.e., as minimizers of

$$\sum_{t=p+1}^{T}\left\{x_t-\sum_{i=1}^{p}\left[\alpha_{0,0}^{(i)}\phi(x_{t-d})+\sum_{j=0}^{J_T-1}\sum_{k\in I_j}\beta_{j,k}^{(i)}\psi_{j,k}(x_{t-d})\right]x_{t-i}\right\}^2,$$

where $J_T$-1 is the highest resolution level such that

$$2^{J_T-1}\le T^{1/2}\le 2^{J_T}.$$

# Estimation(approache1)

The solution

$$\hat{\beta} = \left(\hat{\alpha}_{0,0}^{(1)}, \cdots, \hat{\alpha}_{0,0}^{(p)}, \hat{\beta}_{0,0}^{(1)}, \cdots, \hat{\beta}_{0,0}^{(p)}, \cdots, \hat{\beta}_{J_T-1,\Delta}^{(1)}, \cdots, \hat{\beta}_{J_T-1,\Delta}^{(p)}\right)',$$

with $\Delta = 2^{J_T-1} - 1$, can be written as the least squares estimator

$$\hat{\beta} = (\Psi'\Psi)^{-1}\Psi'Y$$

in the linear model

$$Y = \Psi\beta + \gamma,$$

where

$$Y = (x_{p+1}, \cdots, x_T)',$$

$$\gamma = (\gamma_{p+1}, \cdots, \gamma_T)'$$

and

$$\gamma_t = \sum_i \sum_{j \geq J_T} \sum_{k \in I_j} \beta_{j,k}^{(i)} \psi_{j,k}(x_{t-d}) x_{t-i} + \epsilon_t.$$

# Estimation (approache1)

Write $\Psi$ as

$$\Psi = (\Psi_1, \Psi_2),$$

with

$$\Psi_1 = \begin{pmatrix} \phi(x_{p+1-d})x_p & \cdots & \phi(x_{p+1-d})x_1 \\ & \vdots & \\ \phi(x_{T-d})x_{T-1} & \cdots & \phi(x_{T-d})x_{T-p} \end{pmatrix}$$

and

$$\Psi_2 = \left(\Psi_{0,0} \vdots \cdots \vdots \Psi_{J_T-1,\Delta}\right).$$

# Estimation(approache1)

$$\Psi_{0,0} = \begin{pmatrix} \psi_{0,0}(x_{p+1-d})x_p & \cdots & \psi_{0,0}(x_{p+1-d})x_1 \\ & \vdots & \\ \psi_{0,0}(x_{T-d})x_{T-1} & \cdots & \psi_{0,0}(x_{T-d})x_{T-p} \end{pmatrix},$$

$$\Psi_{J_T-1,\Delta} = \begin{pmatrix} \psi_{J_T-1,\Delta}(x_{p+1-d})x_p & \cdots & \psi_{J_T-1,\Delta}(x_{p+1-d})x_1 \\ & \vdots & \\ \psi_{J_T-1,\Delta}(x_{T-d})x_{T-1} & \cdots & \psi_{J_T-1,\Delta}(x_{T-d})x_{T-p} \end{pmatrix}.$$

We then obtain linear estimators,

$$\hat{f}_i(x_{t-d}) = \hat{\alpha}^{(i)}_{0,0}\phi(x_{t-d}) + \sum_{j=0}^{J_T-1} \sum_{k \in I_j} \hat{\beta}^{(i)}_{j,k}\psi_{j,k}(x_{t-d}).$$

# Non-linear wavelet estimator

It is known that linear estimators can not achieve the minimax rate for some function spaces. To achieve this rate we can consider nonlinear wavelet estimators, by applying thresholds to the wavelet coefficients. For example, we can apply hard thresholding to the coefficients

$$\tilde{\beta}_{j,k}^{(i)} = \delta^{(h)}(\hat{\beta}_{j,k}^{(i)}, \lambda_{j,k}) = \hat{\beta}_{j,k}^{(i)} I(|\hat{\beta}_{j,k}^{(i)}| \geq \lambda_{j,k}),$$

with threshold parameters $\lambda_{j,k}$.

# Non-linear wavelet estimator

Finally, a non-linear threshold estimator of $f_i(x_{t-d})$ is given as

$$\tilde{f}_i(X_{t-d}) = \hat{\alpha}_{0,0}^{(i)}\phi(X_{t-d}) + \sum_{j=0}^{J_T-1}\sum_{k\in I_j}\tilde{\beta}_{j,k}^{(i)}\psi_{j,k}(X_{t-d}).$$

# Estimation(approache1)

We calculate $\phi_{j,k}(X_{t-d})$ and $\psi_{j,k}(X_{t-d})$ at the points:

$$(X_{t-d} - \min(X_t))/(\max(X_t) - \min(X_t)), \quad \text{for } t = p+1, \cdots, T.$$

# Estimation(approache2)

## Estimators with design-adapted wavelets

The adapted Haar wavelets:

. a finite sample $x_1$, ..., $x_T$;

. $T=2^J$;

define:

$$I_{jk} = [x_{((k-1)\ell_j+1)}, x_{(k\ell_j+1)}),$$

for $k = 1, \ldots, 2^j$ and $0 \le j \le J$.

Here $x_{(i)}$ denotes the $i$-th order statistics

$$l_j = T/2^j.$$

# Estimation(approache2)

$$\phi_{jk} = \tilde{\phi}_{jk} = 2^{j/2} 1\{I_{jk}\},$$

and

$$\psi_{jm}(x) = \frac{1}{\sqrt{2}}(\phi_{j+1,2m-1}(x) - \phi_{j+1,2m}(x)),$$

respectively.

See Delouille (2002) for further details.

# Estimation (approache3)

## Estimators with warped wavelets

As before we assume that $Y_{t-1} = X_{t-d}$. The idea is to "warp" the wavelet basis such that in the new basis the estimates of the coefficients seem more "natural". So in (6), instead of expanding $f_i(x)$ in terms of the basis $\{\psi_{jk}(x)\}$, we expand $f(G^{-1}(y))$ in terms of the new warped basis $\{\psi(G(x))\}$. Since $G$ is usually unknown, we use the empirical cdf $\hat{G}$ in this expansion. Also, the wavelet coeffcients $\hat{\beta}_{j,k}^{(i)}$ are thresholded, and Kerkyacharian and Picard (2004) suggest that thresholds of the form

$$\lambda_T = \left(\frac{\log T}{T}\right)^{1/2}, \quad 2^{J(T)} \approx \lambda_T^{-1}$$

be used.

# Numerical Applications

The performance of the estimators of $f_i(x_{t-d})$ are assessed via the square root of average squared errors (RASE), namely

$$RASE_j = \{\frac{1}{T-p+1}\sum_{t=p+1}^{T}[\hat{f}_j(x_{t-d}) - f_j(x_{t-d})]^2\}^{\frac{1}{2}}.$$

# Simulated Examples

**Example 1.** We consider a TAR model

$$x_t = f_1(x_{t-2})x_{t-1} + f_2(x_{t-2})x_{t-2} + \epsilon_t,$$

where

$$f_1(x_{t-2}) = 0.4I(x_{t-2} \leq 1) - 0.8I(x_{t-2} > 1),$$

$$f_2(x_{t-2}) = -0.6I(x_{t-2} \leq 1) + 0.2I(x_{t-2} > 1),$$

and $\{\epsilon_t\}$ are iid N(0,1).

T = 1026

Wavelet: Haar

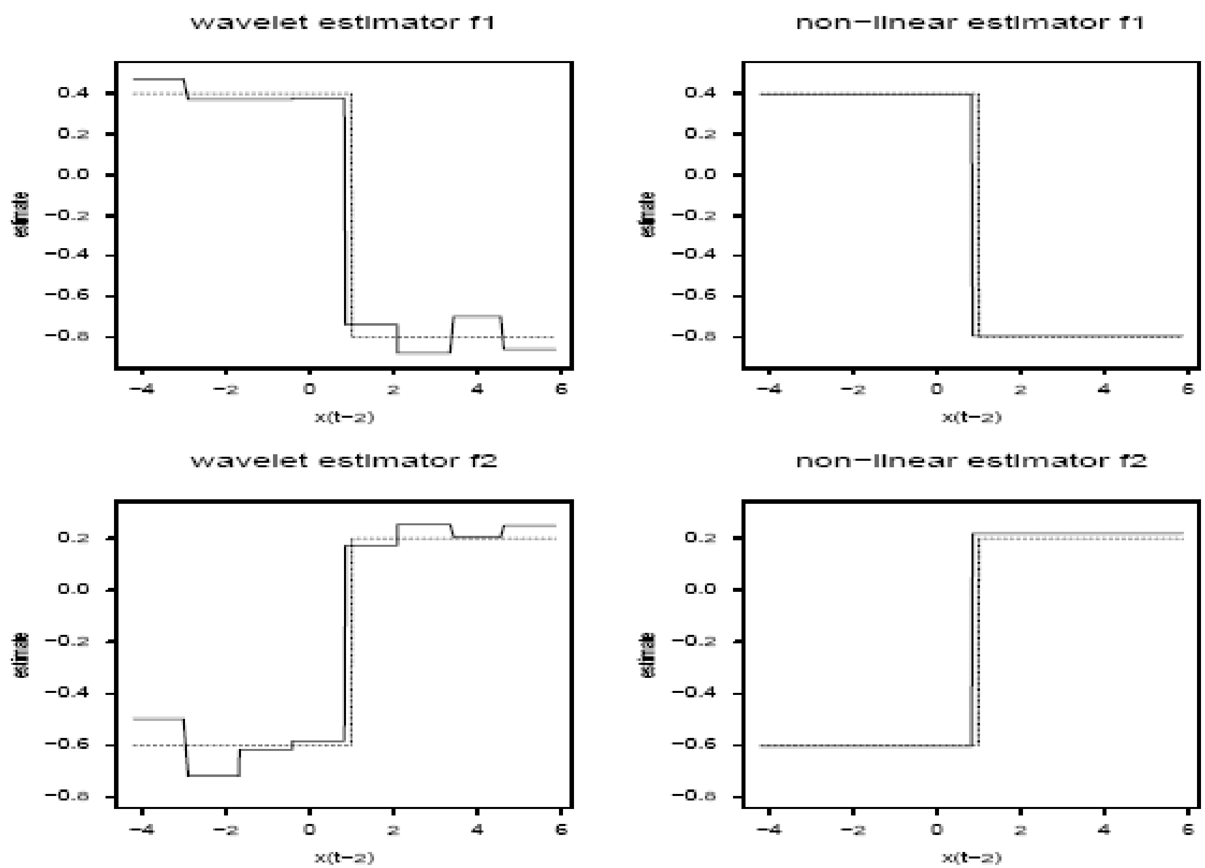


Figure 1: Example 1: Estimation with the usual wavelets for $f_1(X_{t-2})$, $f_2(X_{t-2})$; Haar wavelet. The underlying signals are represented by a dotted line and the estimators by a plain line. The corresponding RASE values are equal to 0.1618 for $\tilde{f}_1(\cdot)$ and 0.1124 for $\tilde{f}_2(\cdot)$.

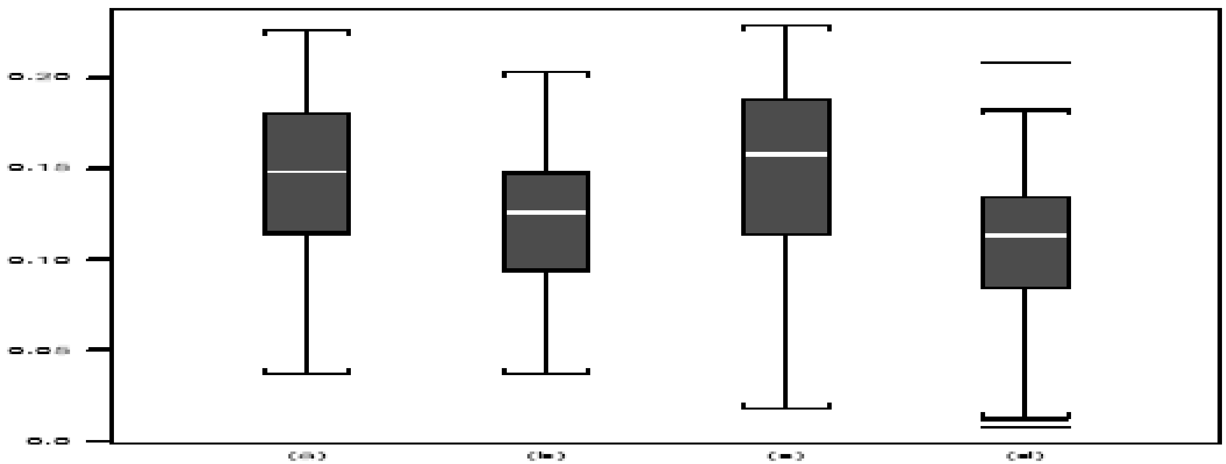

Figure 2: Boxplots of the 100 RASE-values using estimators with the usual wavelets for (a) linear estimator $\hat{f}_1(\cdot)$, (b) linear estimator $\hat{f}_2(\cdot)$, (c) non-linear estimator $\tilde{f}_1(\cdot)$, (d) non-linear estimator $\tilde{f}_2(\cdot)$.

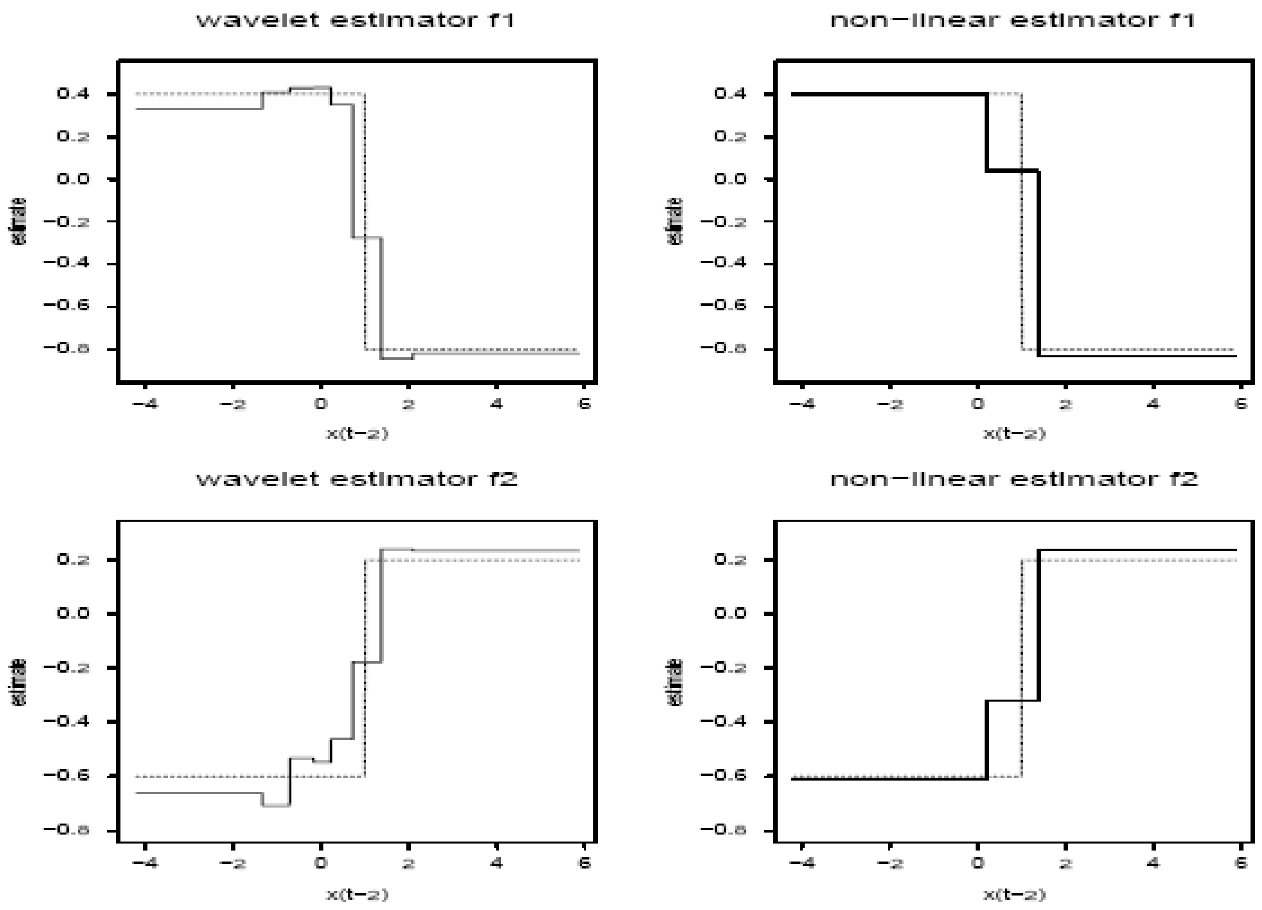


Figure 3: Example 1: Estimation with design-adapted wavelets for $f_1(X_{t-2})$, $f_2(X_{t-2})$; Haar wavelet. The underlying signals are represented by a dotted line and the estimators by a plain line. The corresponding RASE values are equal to 0.2140 for $\tilde{f}_1(\cdot)$ and 0.1591 for $\tilde{f}_2(\cdot)$.

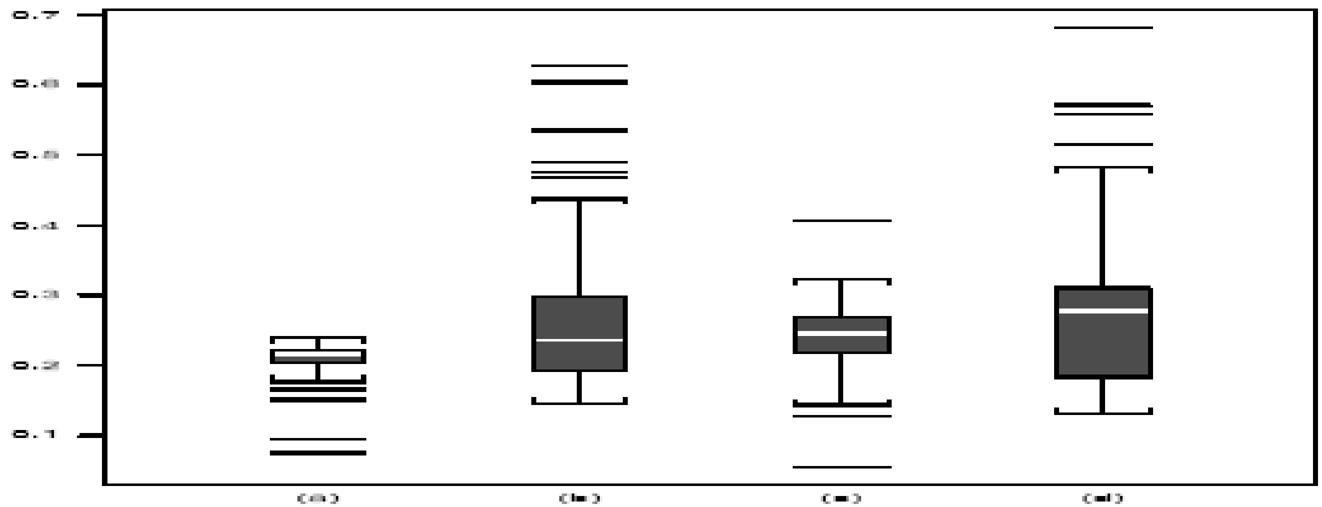

Figure 4: Boxplots of the 100 RASE-values using estimators with design-adapted wavelets for (a) linear estimator $\hat{f}_1(\cdot)$, (b) linear estimator $\hat{f}_2(\cdot)$, (c) non-linear estimator $\tilde{f}_1(\cdot)$, (d) non-linear estimator $\tilde{f}_2(\cdot)$.

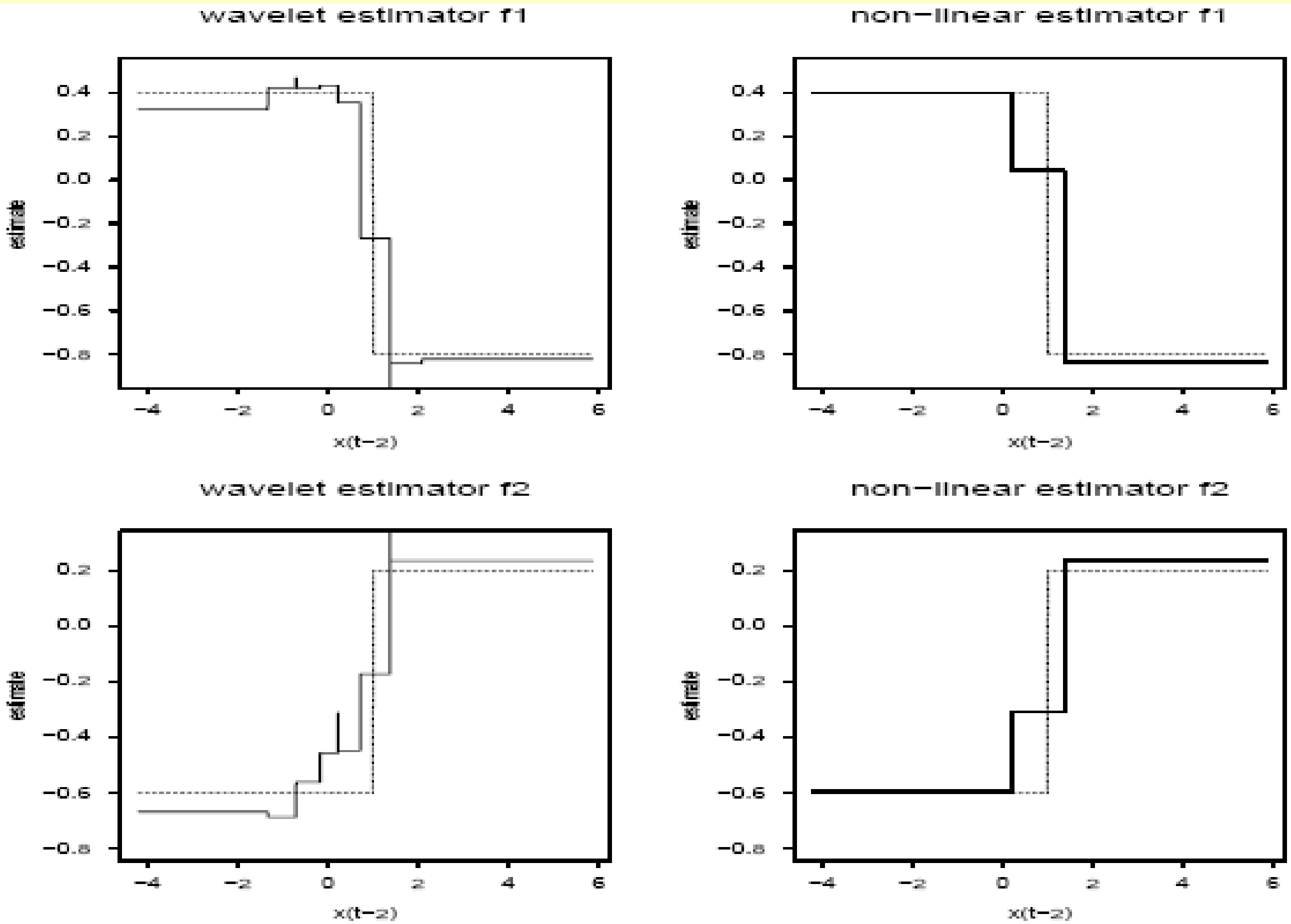


Figure 5: Example 1: Estimation with warped wavelets for $f_1(X_{t-2})$, $f_2(X_{t-2})$; Haar wavelet. The underlying signals are represented by a dotted line and the estimators by a plain line. The corresponding RASE values are equal to 0.2147 for $\tilde{f}_1(\cdot)$ and 0.1654 for $\tilde{f}_2(\cdot)$.

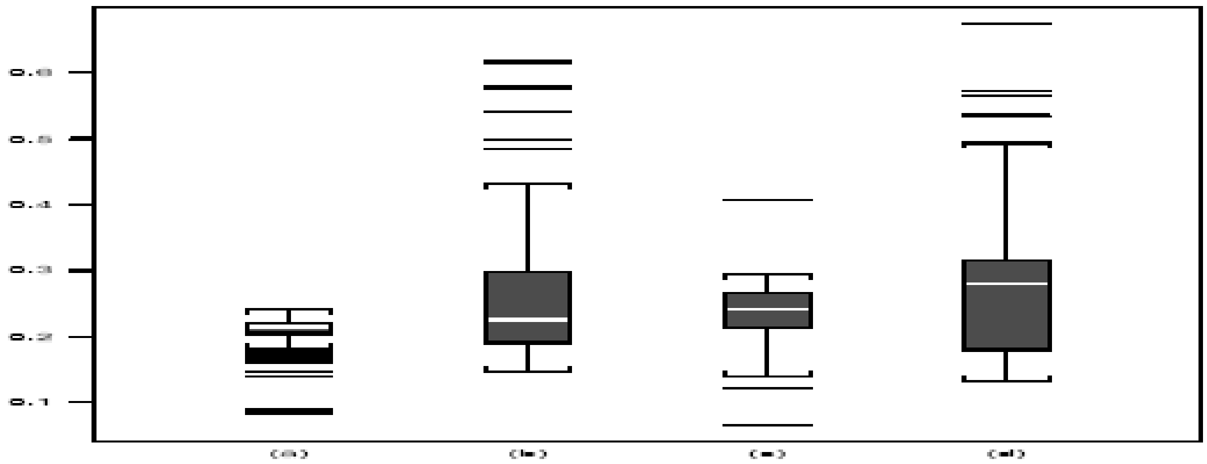


Figure 6: Boxplots of the 100 RASE-values using estimators with warped wavelets for (a) linear estimator $\hat{f}_1(\cdot)$, (b) linear estimator $\hat{f}_2(\cdot)$, (c) non-linear estimator $\tilde{f}_1(\cdot)$, (d) non-linear estimator $\tilde{f}_2(\cdot)$.

# Simulated Examples

**Example 2.** Now we consider an EXPAR model:

$$x_t = f_1(x_{t-1})x_{t-1} + f_2(x_{t-1})x_{t-2} + \epsilon_t,$$

where

$$f_1(x_{t-1}) = 0.138 + (0.316 + 0.982x_{t-1})\exp^{-3.89x_{t-1}^2},$$

$$f_2(x_{t-1}) = -0.437 - (0.659 + 1.260x_{t-1})\exp^{-3.89x_{t-1}^2},$$

and $\{\epsilon\}$ are iid $N(0, 0.2^2)$.

T = 1026

Wavelet: D8

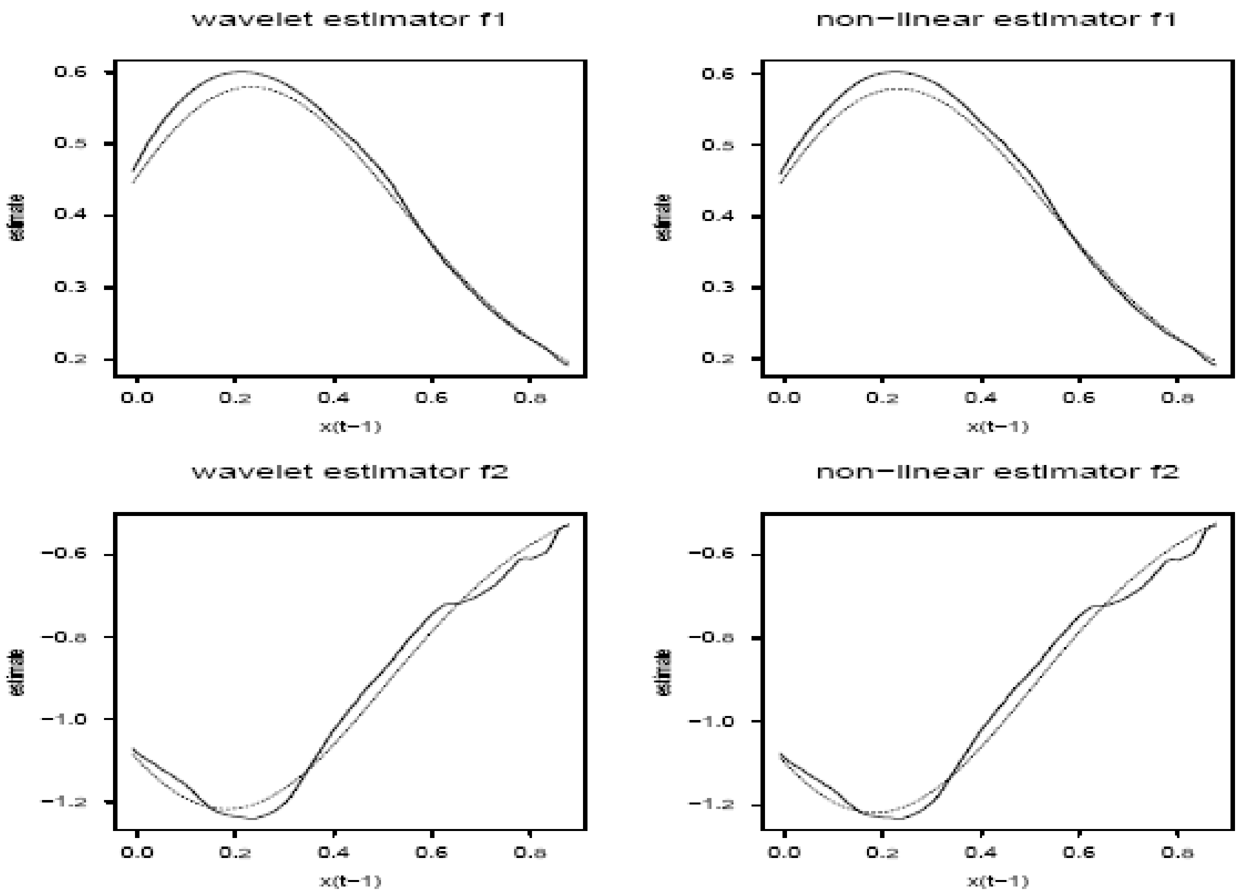


Figure 7: Example 2: Estimation with the usual wavelets for $f_1(X_{t-2})$, $f_2(X_{t-2})$; d8 wavelet. The underlying signals are represented by a dotted line and the estimators by a plain line. The corresponding RASE values are equal to 0.05614 for $\hat{f}_1(\cdot)$ and 0.0202 for $\hat{f}_2(\cdot)$.

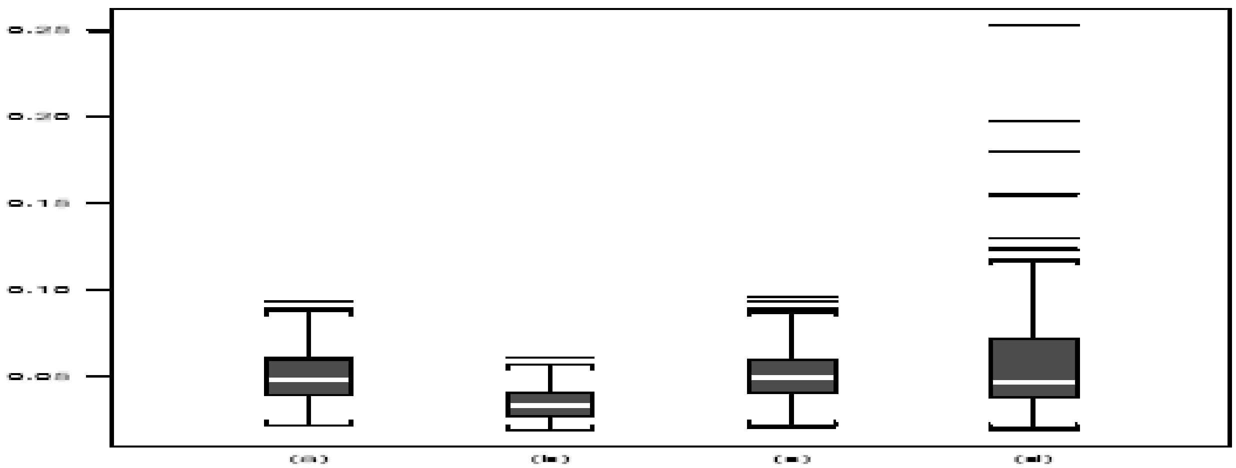


Figure 8: Boxplots of the 100 RASE-values using estimator with the usual wavelets for (a) linear estimator $\hat{f}_1(\cdot)$, (b) linear estimator $\hat{f}_2(\cdot)$, (c) non-linear estimator $\tilde{f}_1(\cdot)$, (d) non-linear estimator $\tilde{f}_2(\cdot)$.

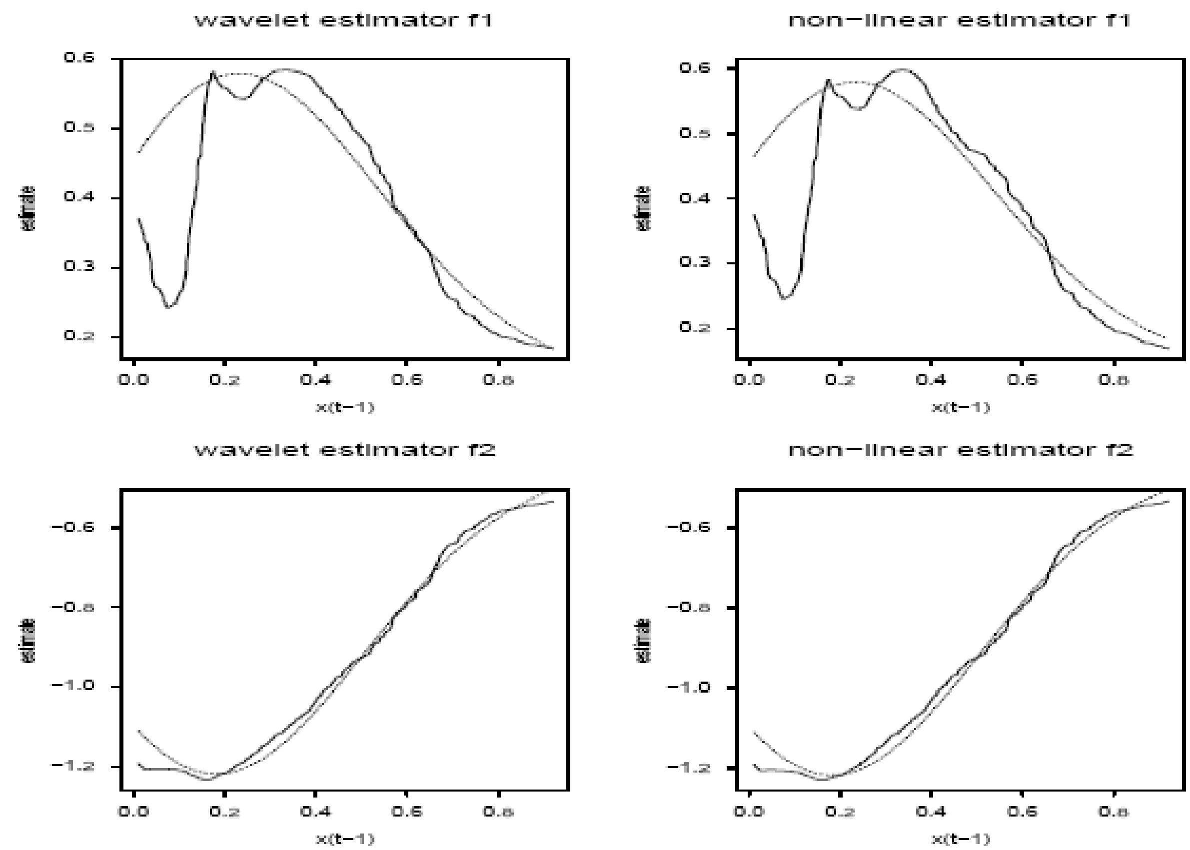


Figure 9: Example 2: Estimation with warped wavelets for $f_1(X_{t-2})$, $f_2(X_{t-2})$; d8 wavelet. The underlying signals are represented by a dotted line and the estimators by a plain line. The corresponding RASE values are equal to 0.0980 for $\hat{f}_1(\cdot)$ and 0.0306 for $\hat{f}_2(\cdot)$.

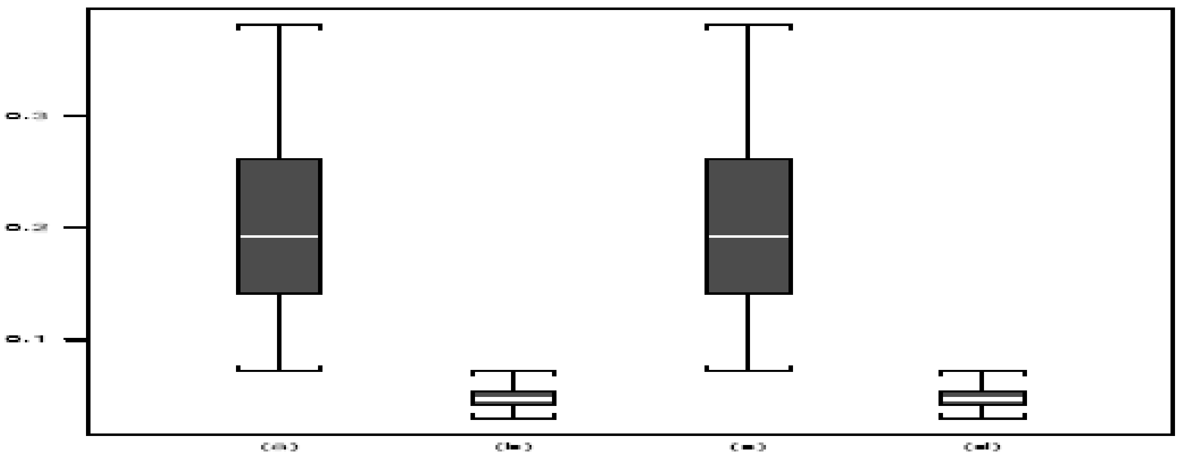

Figure 10: Boxplots of the 100 RASE-values using estimator with warped wavelets for (a) linear estimator $\hat{f}_1(\cdot)$, (b) linear estimator $\hat{f}_2(\cdot)$, (c) non-linear estimator $\tilde{f}_1(\cdot)$, (d) non-linear estimator $\tilde{f}_2(\cdot)$.

# Simulated Examples

**Example 3**. We consider an ARCH-type model:

$$X_t = \sigma_t \epsilon_t,$$

with $\epsilon_t$ iid $N(0,1)$, $\sigma_t^2 = \sigma^2(X_{t-1}) = f_0 + f_1(X_{t-1})X_{t-1}^2$, where

$$f_0 = 0.2$$

and

$$f_1(X_{t-1}) = 0.2I\{X_{t-1} < 0\} + 0.6I\{X_{t-1} \geq 0\}.$$

# Simulated Examples

Since it is desirable to have additive noise, we rewrite the ARCH model above as

$$X_t^2 = \sigma^2(X_{t-1}) + V_t = f_0 + f_1(X_{t-1})X_{t-1}^2 + V_t,$$

with $V_t = \sigma^2(X_{t-1})(\epsilon_t^2 - 1)$, $E(V_t) = 0$ and $\mathrm{Cov}(V_t, V_s) = 0$ for $s \neq t$.

Using the Approach 1, we obtain wavelet estimates $\hat{f}_0$ and $\hat{f}_1(X_{t-1})$ from the data set $\{X_t^2\}$, $t = 2, \cdots, T$. Consequently the wavelet estimator of variance is obtained by

$$\hat{\sigma}^2(X_{t-1}) = \hat{f}_0 + \hat{f}_1(X_{t-1})X_{t-1}^2$$

and its non-linear wavelet estimator by

$$\tilde{\sigma}^2(X_{t-1}) = \hat{f}_0 + \tilde{f}_1(X_{t-1})X_{t-1}^2.$$

- T = 1025 and $J_T$=3

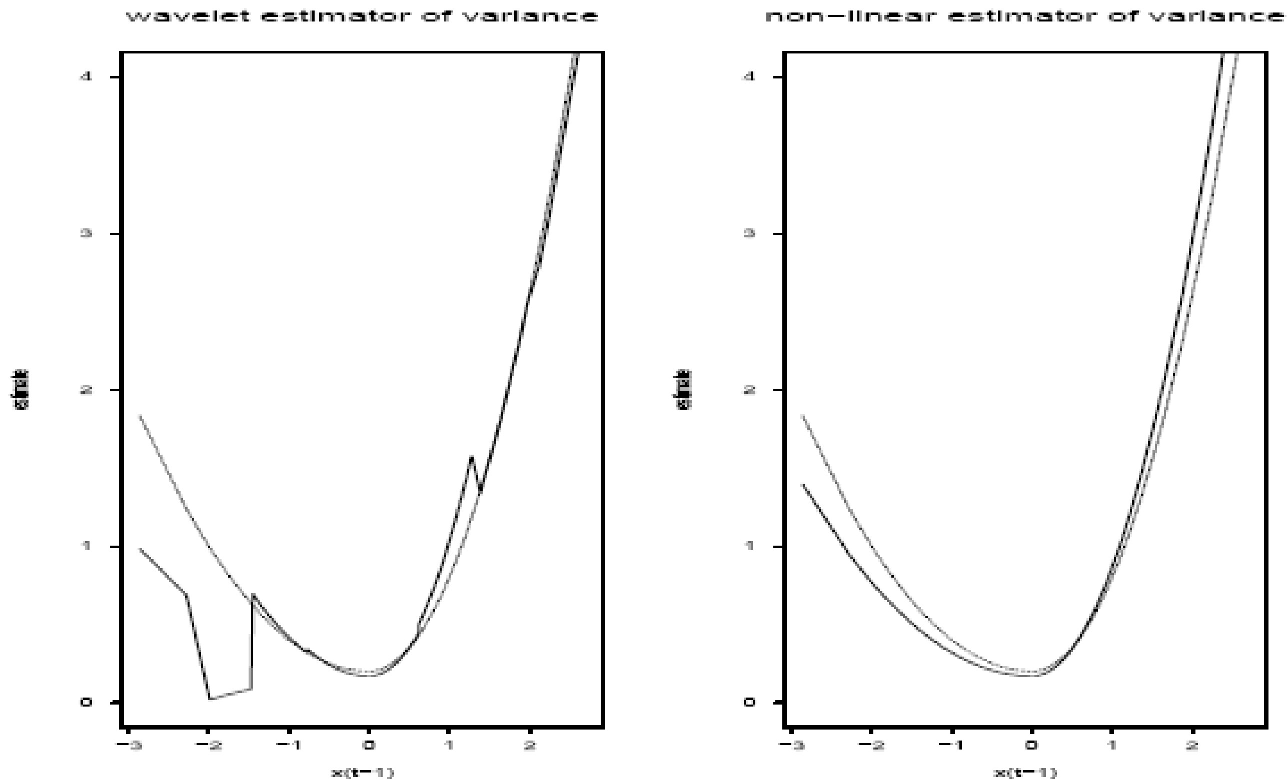


Figure 11: Example 3: Estimation with the usual wavelets for variance $\sigma^2(X_{t-1})$; Haar wavelet. The underlying signals are represented by a dotted line and the estimators by a plain line. The corresponding RASE value is equal to 0.0592 for $\tilde{\sigma}^2(\cdot)$.

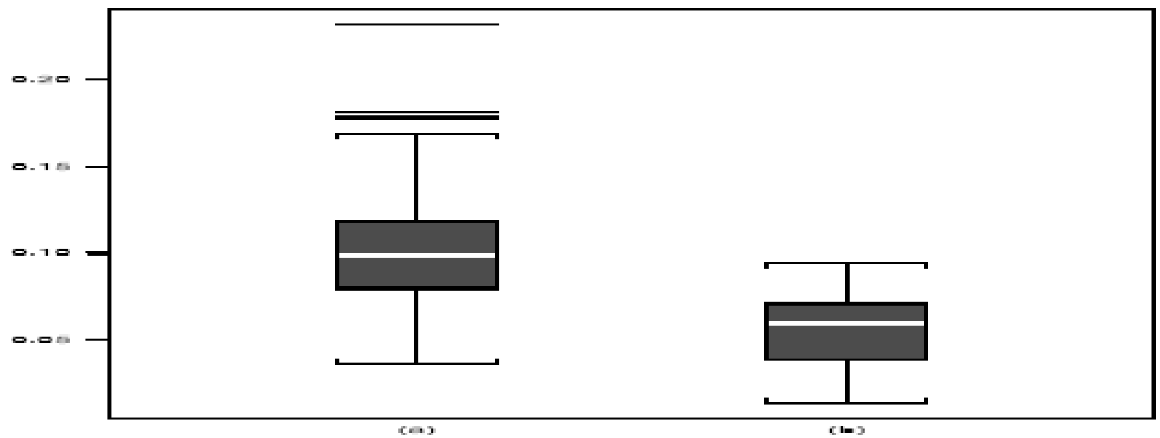


Figure 12: Boxplots of the 100 RASE-values using estimators with the usual wavelets for (a) linear estimator $\hat{\sigma}^2(\cdot)$, (b) non-linear estimator $\tilde{\sigma}^2(\cdot)$.

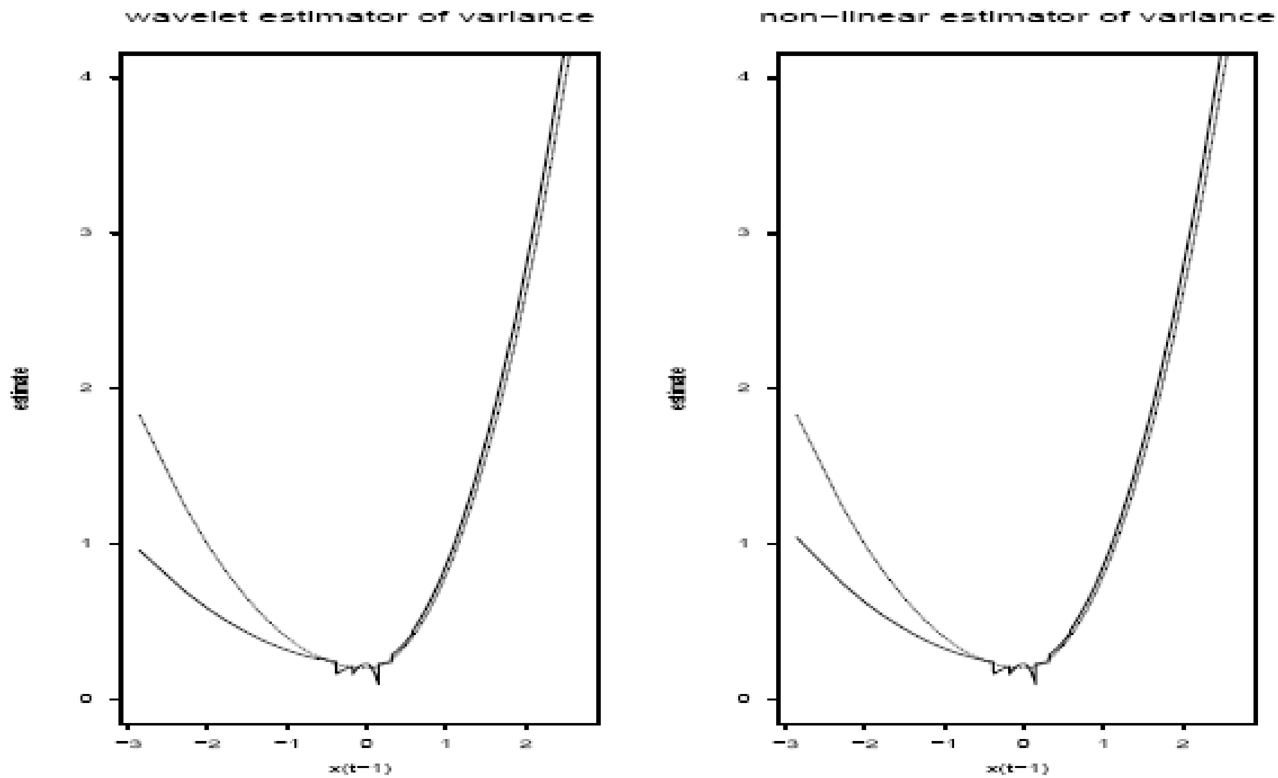


Figure 13: Example 3: Estimation with design-adapted wavelets for $\sigma^2(X_{t-1})$; Haar wavelet. The underlying signals are represented by a dotted line and the estimators by a plain line. The corresponding RASE values are equal to 0.05476 for $\tilde{\sigma}^2(\cdot)$.

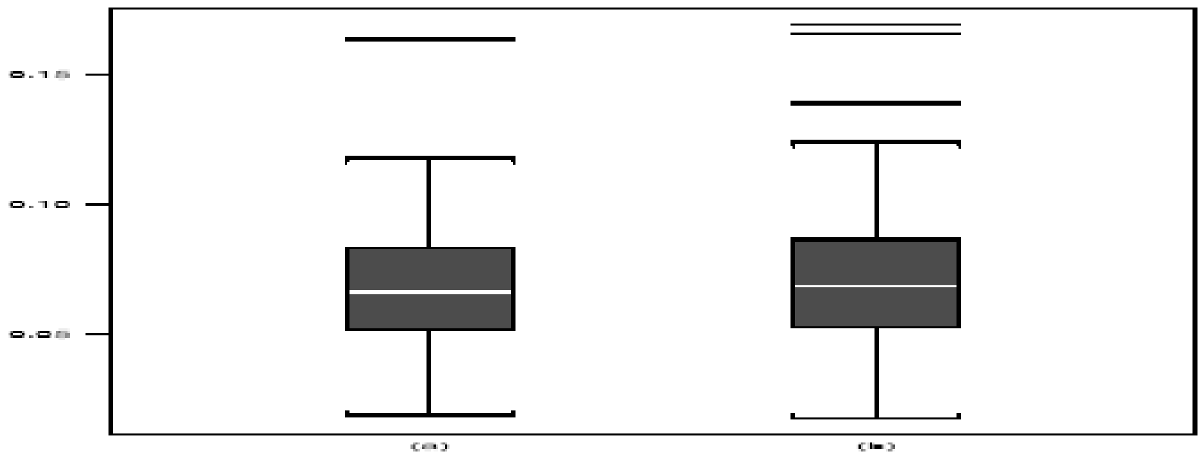


Figure 14: Boxplots of the 100 RASE-values using estimators with design-adapted wavelets for (a) linear estimator $\hat{\sigma}^2(\cdot)$, (b) non-linear estimator $\tilde{\sigma}^2(\cdot)$.

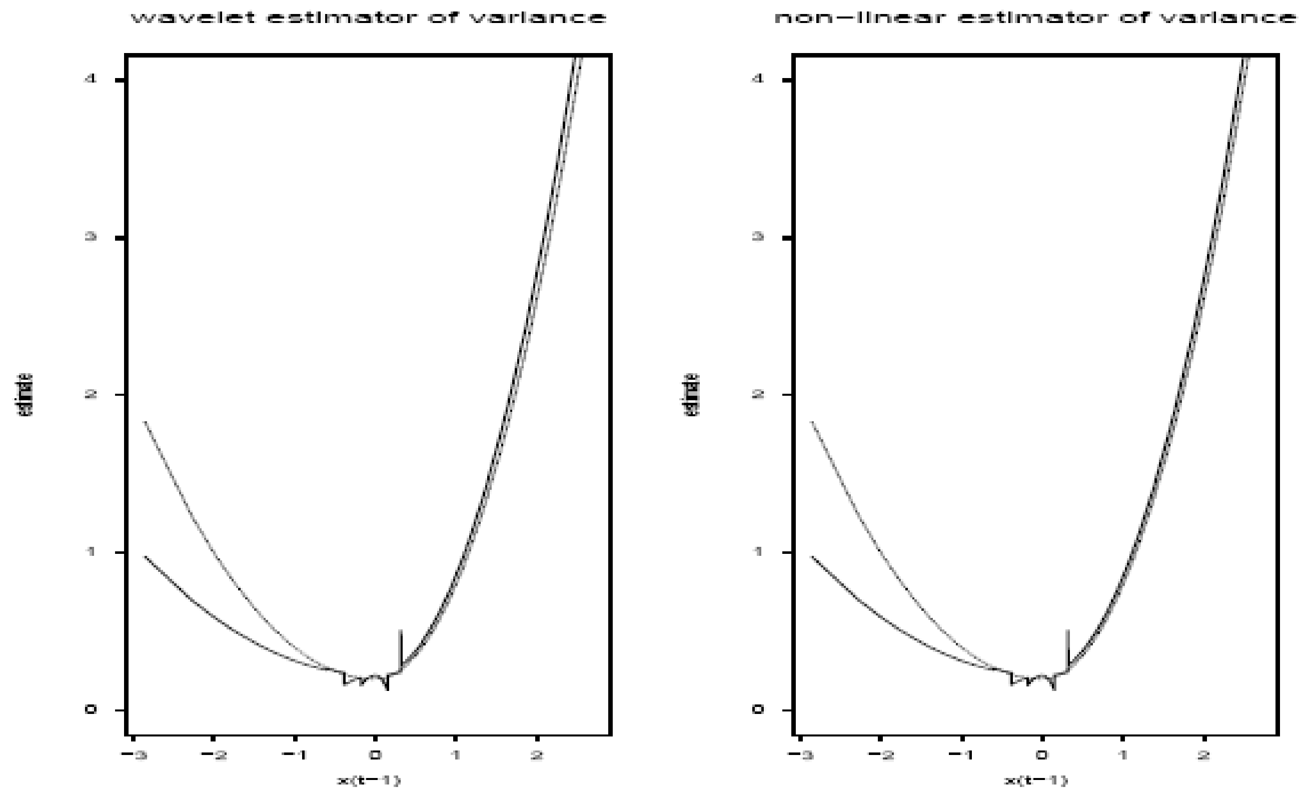


Figure 15: Example 3: Estimation with warped wavelets for $\sigma^2(X_{t-1})$; Haar wavelet. The underlying signals are represented by a dotted line and the estimators by a plain line. The corresponding RASE value is equal to 0.05616 for $\tilde{\sigma}^2$.

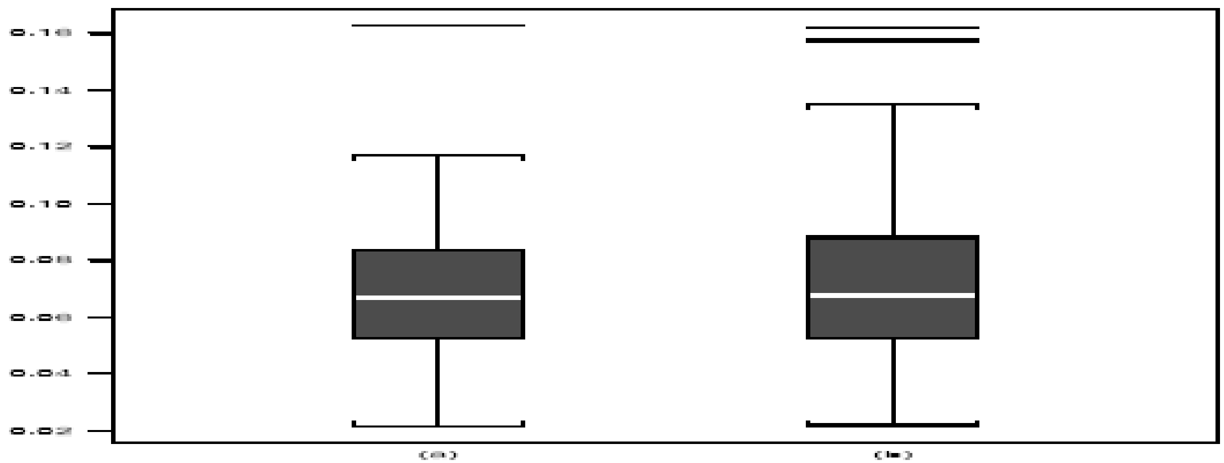

Figure 16: Boxplots of the 100 RASE-values using estimators with warped wavelets for (a) linear estimator $\hat{\sigma}^2(\cdot)$, (b) non-linear estimator $\tilde{\sigma}^2(\cdot)$.

# Real data example

**Example 4.** We fit the FAR model to the Canadian lynx data (see Stenseth et al. (1999) for further information on the data).

T = 114, $\log X_t$

Tong (1990, p.377) fitted the following TAR model with two regimes and the delay variable at lag 2 to the lynx data:

$$x_t = \begin{cases} 0.62 + 1.25x_{t-1} - 0.43x_{t-2} + \epsilon_t^{(1)}, & x_{t-2} \le 3.25 \\ 2.25 + 1.52x_{t-1} - 1.24x_{t-2} + \epsilon_t^{(2)}, & x_{t-2} > 3.25 \end{cases}$$

# Real data example

To compare with the technique proposed in this paper, we fit the lynx data with the model:

$$x_t = f_1(x_{t-2})x_{t-1} + f_2(x_{t-2})x_{t-2} + \epsilon_t.$$

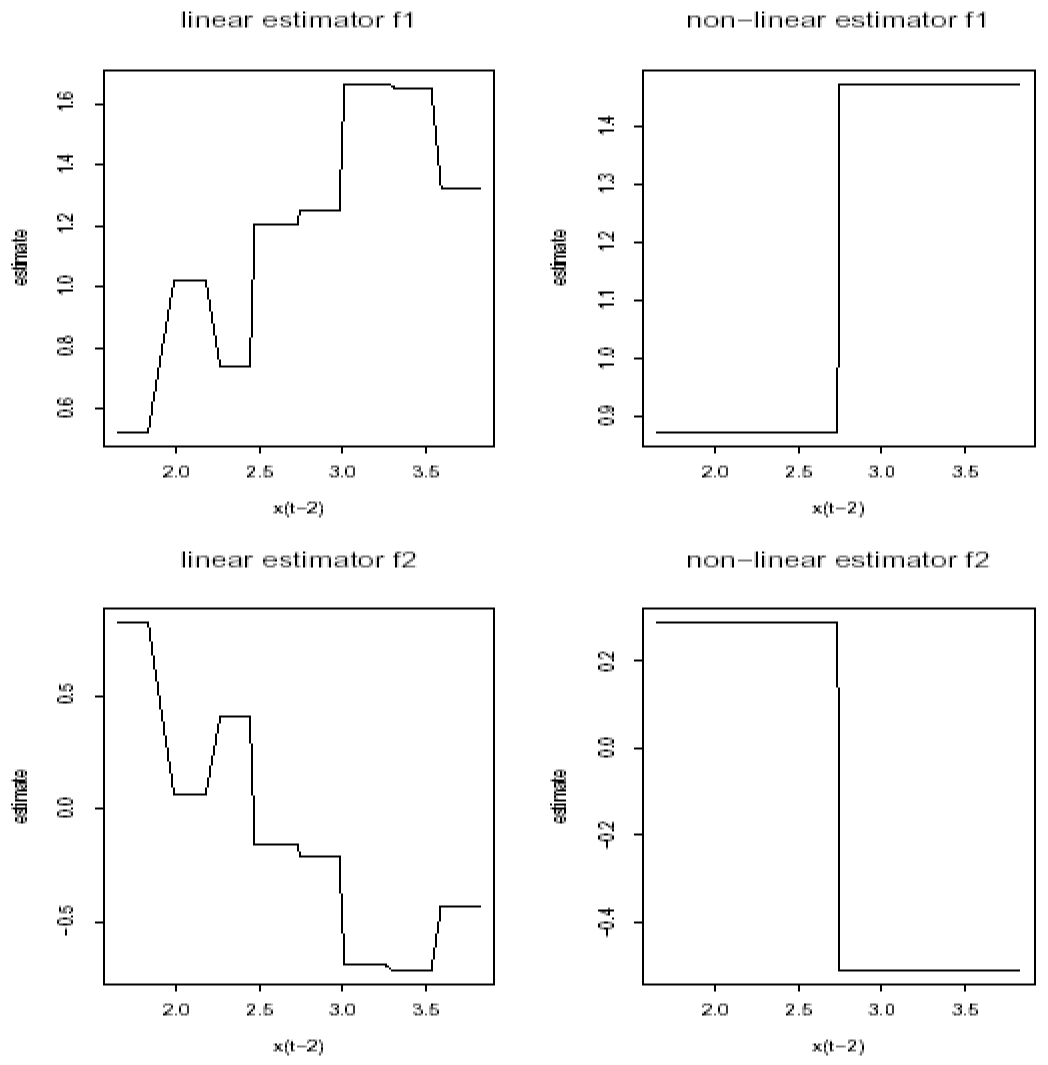


Figure 7: Example 3: linear wavelet estimate and non-linear wavelet estimate for $f_1(x_{t-2})$, $f_2(x_{t-2})$: Haar wavelet.

# Real data example

**Example 5.** We apply an AR-ARCH model to the São Paulo Stock Exchange Index(Ibovespa).

- $X_t$: 02/01/1995-06/02/1999
- T=1026

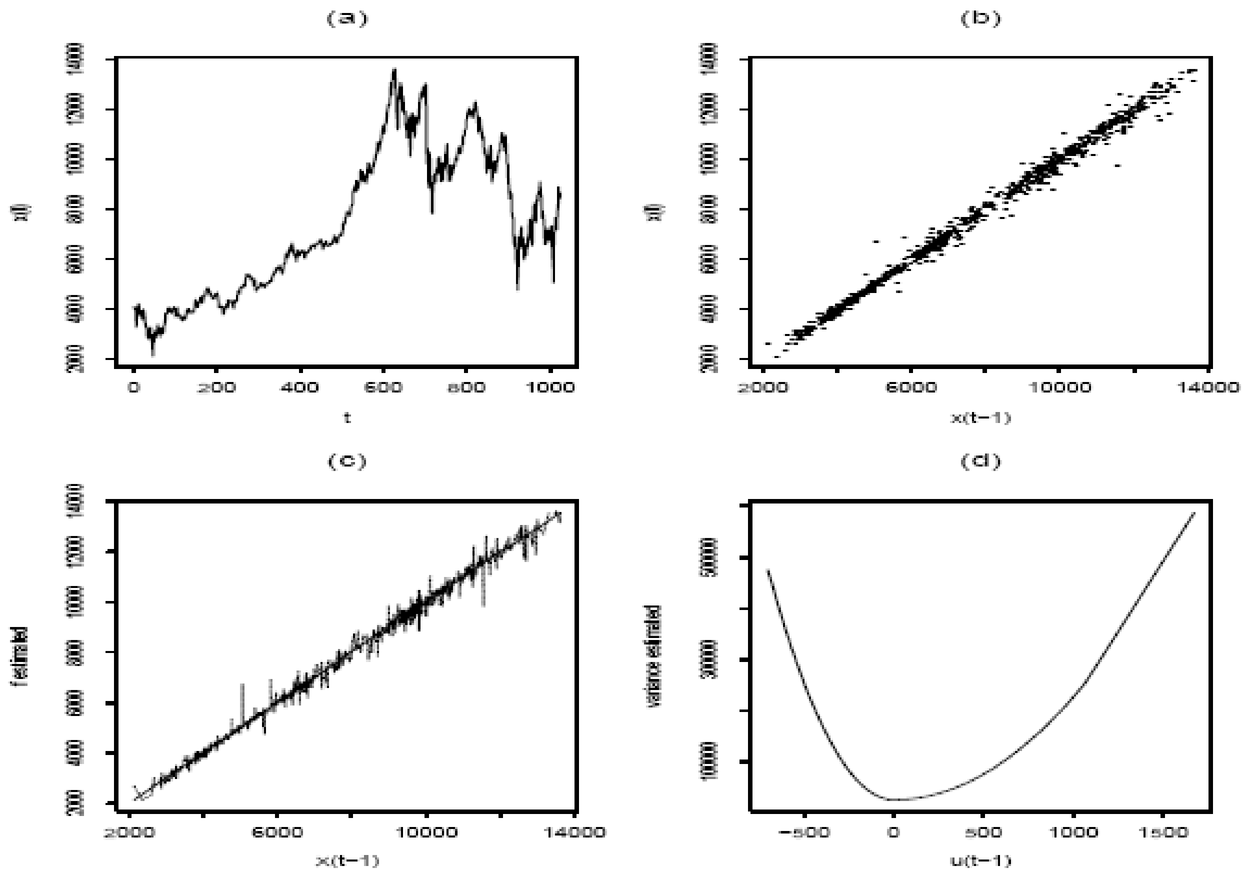


Figure 18: Example 5: (a) Iboveps index series from January 2, 1995 to February 3, 1999. (b) $X_t$ is given as a function of the lagged value $X_{t-1}$. (c) the estimation of the trend. (d) the variance function $\sigma^2(\cdot)$ is estimated as a function of the lagged estimated residuals $\hat{u}_{t-1} = X_{t-1} - \hat{m}(X_{t-2})$.

# Real data example

Figure 18(b)shows the scatter plot of $X_t$ versus $X_{t-1}$, suggesting to model $\{X_t\}$ by a nonlinear AR-process of order one. So we consider the model

$$X_t = m(X_{t-1}) + U_t,$$

with

$$m(X_{t-1}) = f(X_{t-1})X_{t-1},$$

$$U_t = \sigma_t \epsilon_t,$$

$$\sigma_t^2 = \sigma^2(U_{t-1}) = S_0 + S_1(U_{t-1})U_{t-1}^2,$$

and

$$U_{t-1} = X_{t-1} - m(X_{t-2}) = X_{t-1} - f(X_{t-2})X_{t-2}.$$

As in Example 3, we need to rewrite $U_t = \sigma_t \epsilon_t$ as $U_t^2 = \sigma_t^2 + V_t$, with $V_t = \sigma^2(U_{t-1})(\epsilon_t^2 - 1)$, $E(V_t) = 0$ and $\mathrm{Cov}(V_t, V_s) = 0$ for $s \neq t$.

The aim here is to estimate both the trend $m(\cdot)$ and the variance function $\sigma^2(\cdot)$. We estimate $f(\cdot)$, $S_0$ and $S_1(\cdot)$ in two steps:

(a) From the data set $\{X_t\}$, $t = 2, \cdots, T$, obtain an estimator $\hat{f}(\cdot)$ of $f(\cdot)$ using a linear Haar wavelet estimator with $J_T = 3$;

(b) Estimate the residuals by $\hat{U}_t = X_t - \hat{f}(X_{t-1})X_{t-1}$, for $t = 2, \cdots, T$. From the data set $\{\hat{U}_t^2\}$, $t = 3, \cdots, T$, estimate $S_0$ and $S_1(\hat{U}_{t-1})$, obtain a non-linear estimator $\tilde{\sigma}^2(\hat{U}_{t-1})$ of the variance $\sigma^2(U_{t-1})$ by $\hat{S}_0 + \tilde{S}_1(\hat{U}_{t-1})\hat{U}_{t-1}^2$.

Here we use Approach 1 to estimate the trend $m(\cdot)$ and the variance function $\sigma^2(\cdot)$. Figure 18(c) shows the estimator $\hat{m}(X_{t-1}) = \hat{f}(X_{t-1})X_{t-1}$ of the trend. Figure 18(d) shows the non-linear estimate of the variance function using hard threshold. The figure clearly shows the presence of a quadratic component in the variance function, as it is often the case for such financial time series (Gouriéroux, 1997; Hafner, 1998).

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