Spectral Analysis and Filtering

Cyclical Behavior and Periodicity

As in (1.5), we consider the periodic process

$$x_t = A\cos(2\pi\omega t + \phi) \tag{4.1}$$

for $t = 0, \pm1, \pm2, \ldots$, where ω is a frequency index, defined in cycles per unit time with A determining the height or *amplitude* of the function and ϕ, called the *phase*, determining the start point of the cosine function. We can introduce random variation in this time series by allowing the amplitude and phase to vary randomly.

$$x_t = U_1\cos(2\pi\omega t) + U_2\sin(2\pi\omega t), \tag{4.2}$$

where $U_1 = A\cos\phi$ and $U_2 = -A\sin\phi$ are often taken to be normally distributed random variables. In this case, the amplitude is $A = \sqrt{U_1^2 + U_2^2}$ and the phase is $\phi = \tan^{-1}(-U_2/U_1)$. From these facts we can show that if, and only if, in (4.1), A and ϕ are independent random variables, where A^2 is chi-squared with 2 degrees of freedom, and ϕ is uniformly distributed on $(-\pi, \pi)$, then U_1 and U_2 are independent, standard normal random variables

Consider a generalization of (4.2) that allows mixtures of periodic series with multiple frequencies and amplitudes,

$$x_t = \sum_{k=1}^{q}\left[U_{k1}\cos(2\pi\omega_k t) + U_{k2}\sin(2\pi\omega_k t)\right], \tag{4.3}$$

where U_{k1}, U_{k2}, for $k = 1, 2, \ldots, q$, are independent zero-mean random variables with variances σ_k^2, and the ω_k are distinct frequencies. Notice that (4.3) exhibits the process as a sum of independent components, with variance σ_k^2 for frequency ω_k. Using the independence of the Us and the trig identity in footnote 1, it is easy to show[2] (Problem 4.3) that the autocovariance function of the process is

$$\gamma(h) = \sum_{k=1}^{q}\sigma_k^2\cos(2\pi\omega_k h), \tag{4.4}$$

and we note the autocovariance function is the sum of periodic components with weights proportional to the variances σ_k^2. Hence, x_t is a mean-zero stationary processes with variance

[2] For example, for x_t in (4.2) we have $\operatorname{cov}(x_{t+h}, x_t) = \sigma^2\{\cos(2\pi\omega[t+h])\cos(2\pi\omega t) + \sin(2\pi\omega[t+h])\sin(2\pi\omega t)\} = \sigma^2\cos(2\pi\omega h)$, noting that $\operatorname{cov}(U_1, U_2) = 0$.

$$\gamma(0) = E(x_t^2) = \sum_{k=1}^{q} \sigma_k^2, \tag{4.5}$$

which exhibits the overall variance as a sum of variances of each of the component parts.

Example 4.1 A Periodic Series

Figure 4.1 shows an example of the mixture (4.3) with $q = 3$ constructed in the following way. First, for $t = 1, \ldots, 100$, we generated three series

$x_{t1} = 2\cos(2\pi t\, 6/100) + 3\sin(2\pi t\, 6/100)$

$x_{t2} = 4\cos(2\pi t\, 10/100) + 5\sin(2\pi t\, 10/100)$

$x_{t3} = 6\cos(2\pi t\, 40/100) + 7\sin(2\pi t\, 40/100)$

These three series are displayed in Figure 4.1 along with the corresponding frequencies and squared amplitudes. For example, the squared amplitude of x_{t1} is $A^2 = 2^2 + 3^2 = 13$. Hence, the maximum and minimum values that x_{t1} will attain are $\pm\sqrt{13} = \pm 3.61$.

Finally, we constructed

$$x_t = x_{t1} + x_{t2} + x_{t3}$$

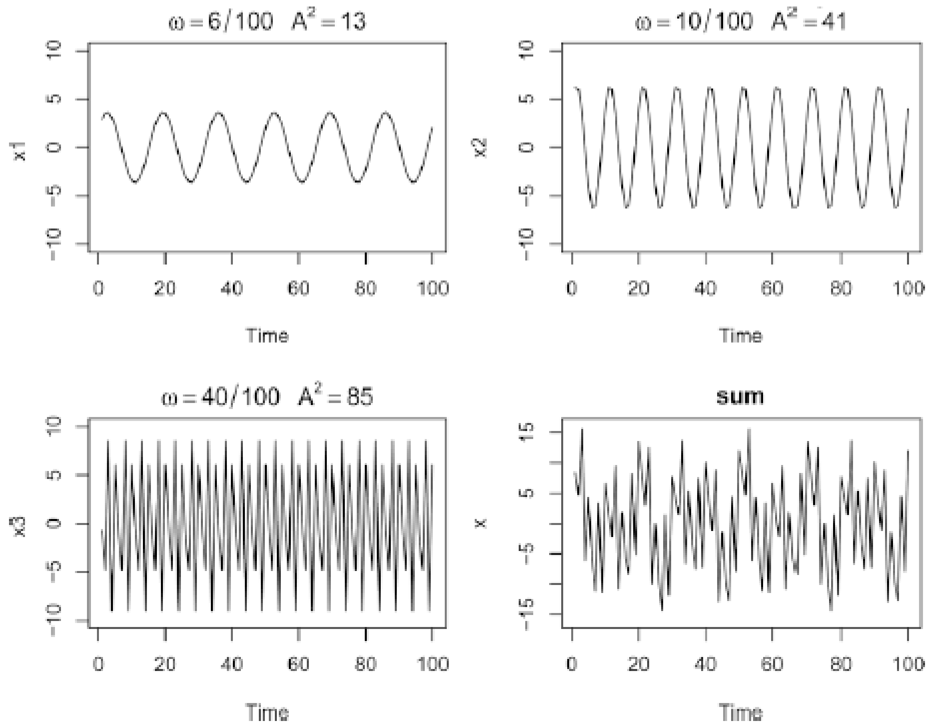


Fig. 4.1. Periodic components and their sum as described in Example 4.1.

Example 4.2 The Scaled Periodogram for Example 4.1

In §2.3, Example 2.9, we introduced the periodogram as a way to discover the periodic components of a time series. Recall that the scaled periodogram is given by

$$P(j/n) = \left(\frac{2}{n}\sum_{t=1}^{n} x_t \cos(2\pi t j/n)\right)^2 + \left(\frac{2}{n}\sum_{t=1}^{n} x_t \sin(2\pi t j/n)\right)^2, \quad (4.6)$$

and it may be regarded as a measure of the squared correlation of the data with sinusoids oscillating at a frequency of $\omega_j = j/n$, or j cycles in n time points. Recall that we are basically computing the regression of the data on the sinusoids varying at the fundamental frequencies, j/n. As discussed in Example 2.9, the periodogram may be computed quickly using the fast Fourier transform (FFT), and there is no need to run repeated regressions.

The scaled periodogram of the data, x_t, simulated in Example 4.1 is shown in Figure 4.2, and it clearly identifies the three components x_{t1}, x_{t2}, and x_{t3} of x_t. Note that

$$P(j/n) = P(1 - j/n), \quad j = 0, 1, \ldots, n-1,$$

so there is a mirroring effect at the folding frequency of 1/2; consequently, the periodogram is typically not plotted for frequencies higher than the folding frequency. In addition, note that the heights of the scaled periodogram shown in the figure are

$$P(6/100) = 13, \quad P(10/100) = 41, \quad P(40/100) = 85,$$

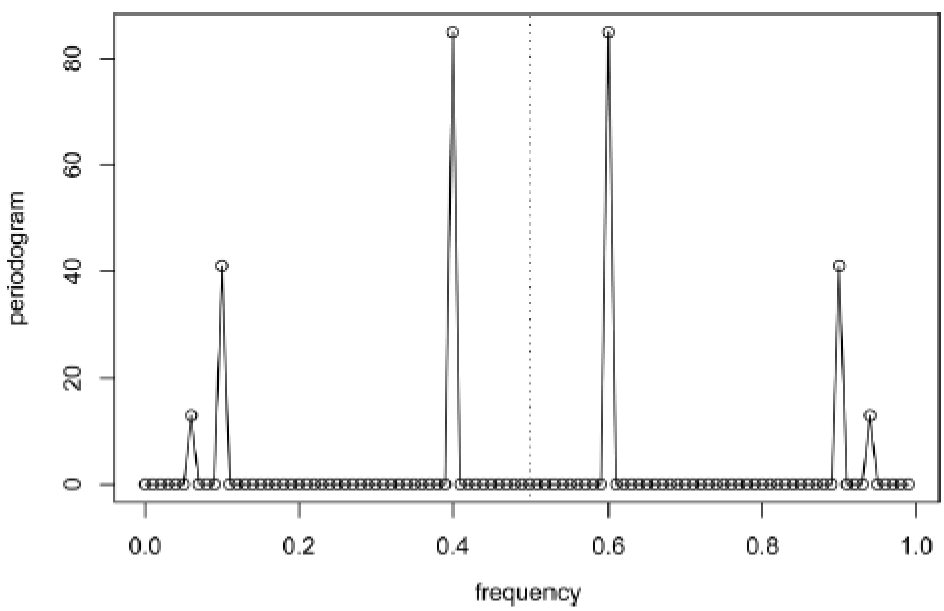


Fig. 4.2. Periodogram of the data generated in Example 4.1.

Another fact that may be of use in understanding the periodogram is that for any time series sample $x_1, \ldots, x_n$, where n is odd, we may write, *exactly*

$$x_t = a_0 + \sum_{j=1}^{(n-1)/2} \left[a_j \cos(2\pi t\, j/n) + b_j \sin(2\pi t\, j/n) \right], \tag{4.7}$$

for $t = 1, \ldots, n$ and suitably chosen coefficients. If n is even, the representation (4.7) can be modified by summing to $(n/2 - 1)$ and adding an additional component given by $a_{n/2} \cos(2\pi t\, 1/2) = a_{n/2}(-1)^t$. The crucial point here is that (4.7) is exact for any sample. Hence (4.3) may be thought of as an approximation to (4.7), the idea being that many of the coefficients in (4.7) may be close to zero. Recall from Example 2.9 that

$$P(j/n) = a_j^2 + b_j^2, \tag{4.8}$$

The Spectral Density

Example 4.3 A Periodic Stationary Process

Consider a periodic stationary random process given by (4.2), with a fixed frequency ω_0, say,

$$x_t = U_1 \cos(2\pi\omega_0 t) + U_2 \sin(2\pi\omega_0 t),$$

where U_1 and U_2 are independent zero-mean random variables with equal variance σ^2. The number of time periods needed for the above series to complete one cycle is exactly $1/\omega_0$, and the process makes exactly ω_0 cycles per point for $t = 0, \pm1, \pm2, \ldots$. It is easily shown that[3]

$$\begin{aligned}\gamma(h) = \sigma^2 \cos(2\pi\omega_0 h) &= \frac{\sigma^2}{2} e^{-2\pi i\omega_0 h} + \frac{\sigma^2}{2} e^{2\pi i\omega_0 h} \\ &= \int_{-1/2}^{1/2} e^{2\pi i\omega h} dF(\omega)\end{aligned}$$

[3] Some identities may be helpful here: $e^{i\alpha} = \cos(\alpha) + i\sin(\alpha)$ and consequently, $\cos(\alpha) = (e^{i\alpha} + e^{-i\alpha})/2$ and $\sin(\alpha) = (e^{i\alpha} - e^{-i\alpha})/2i$.

using a Riemann–Stieltjes integration, where $F(\omega)$ is the function defined by

$$F(\omega) = \begin{cases} 0 & \omega < -\omega_0, \\ \sigma^2/2 & -\omega_0 \le \omega < \omega_0, \\ \sigma^2 & \omega \ge \omega_0. \end{cases}$$

The function $F(\omega)$ behaves like a cumulative distribution function for a discrete random variable, except that $F(\infty) = \sigma^2 = \text{var}(x_t)$ instead of one. In fact, $F(\omega)$ is a cumulative distribution function, not of probabilities, but rather of variances associated with the frequency ω_0 in an analysis of variance, with $F(\infty)$ being the total variance of the process x_t. Hence, we term $F(\omega)$ the *spectral distribution function.*

Theorem C.1 in Appendix C states that a representation such as the one given in Example 4.3 always exists for a stationary process. In particular, if x_t is stationary with autocovariance $\gamma(h) = E[(x_{t+h} - \mu)(x_t - \mu)]$, then there exists a unique monotonically increasing function $F(\omega)$, called the spectral distribution function, that is bounded, with $F(-\infty) = F(-1/2) = 0$, and $F(\infty) = F(1/2) = \gamma(0)$ such that

$$\gamma(h) = \int_{-1/2}^{1/2} e^{2\pi i\omega h}\, dF(\omega). \tag{4.9}$$

A more important situation we use repeatedly is the one covered by Theorem C.3, where it is shown that, subject to absolute summability of the autocovariance, the spectral distribution function is absolutely continuous with $dF(\omega) = f(\omega)\, d\omega$, and the representation (4.9) becomes the motivation for the property given below.

Property 4.2 The Spectral Density

If the autocovariance function, $\gamma(h)$, of a stationary process satisfies

$$\sum_{h=-\infty}^{\infty} |\gamma(h)| < \infty, \tag{4.10}$$

then it has the representation

$$\gamma(h) = \int_{-1/2}^{1/2} e^{2\pi i\omega h}\, f(\omega)\, d\omega \quad h = 0, \pm 1, \pm 2, \ldots \tag{4.11}$$

as the inverse transform of the spectral density, which has the representation

$$f(\omega) = \sum_{h=-\infty}^{\infty} \gamma(h) e^{-2\pi i\omega h} \quad -1/2 \le \omega \le 1/2. \tag{4.12}$$

This spectral density is the analogue of the probability density function; the fact that $\gamma(h)$ is non-negative definite ensures

$$f(\omega) \geq 0$$

for all ω (see Appendix C, Theorem C.3 for details). It follows immediately from (4.12) that

$$f(\omega) = f(-\omega) \quad \text{and} \quad f(\omega) = f(1-\omega),$$

verifying the spectral density is an even function of period one. Because of the evenness, we will typically only plot $f(\omega)$ for $\omega \geq 0$. In addition, putting $h = 0$ in (4.11) yields

$$\gamma(0) = \text{var}(x_t) = \int_{-1/2}^{1/2} f(\omega)\, d\omega,$$

which expresses the total variance as the integrated spectral density over all of the frequencies. We show later on, that a linear filter can isolate the variance in certain frequency intervals or bands.

We note that the autocovariance function, $\gamma(h)$, in (4.11) and the spectral density, $f(\omega)$, in (4.12) are Fourier transform pairs. In particular, this means that if $f(\omega)$ and $g(\omega)$ are two spectral densities for which

$$\gamma_f(h) = \int_{-1/2}^{1/2} f(\omega) e^{2\pi i\omega h}\, d\omega = \int_{-1/2}^{1/2} g(\omega) e^{2\pi i\omega h}\, d\omega = \gamma_g(h) \qquad (4.13)$$

for all $h = 0, \pm1, \pm2, \ldots,$ then

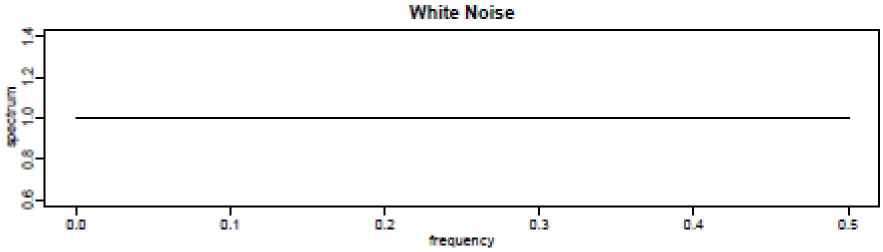
$$f(\omega) = g(\omega). \qquad (4.14)$$

Example 4.4 White Noise Series

As a simple example, consider the theoretical power spectrum of a sequence of uncorrelated random variables, w_t, with variance σ_w^2. A simulated set of data is displayed in the top of Figure 1.8. Because the autocovariance function was computed in Example 1.16 as $\gamma_w(h) = \sigma_w^2$ for $h = 0$, and zero, otherwise, it follows from (4.12), that

$$f_w(\omega) = \sigma_w^2$$

for $-1/2 \leq \omega \leq 1/2$. Hence the process contains equal power at all frequencies. This property is seen in the realization, which seems to contain all different frequencies in a roughly equal mix. In fact, the name white noise comes from the analogy to white light, which contains all frequencies in the color spectrum at the same level of intensity. Figure 4.3 shows a plot of the white noise spectrum for $\sigma_w^2 = 1$.



If x_t is ARMA, its spectral density can be obtained explicitly using the fact that it is a linear process, i.e., $x_t = \sum_{j=0}^{\infty} \psi_j w_{t-j}$, where $\sum_{j=0}^{\infty} |\psi_j| < \infty$. In the following property, we exhibit the form of the spectral density of an ARMA model. The proof of the property follows directly from the proof of a more general result, Property 4.7 given on page 222, by using the additional fact that $\psi(z) = \theta(z)/\phi(z)$; recall Property 3.1.

Property 4.3 The Spectral Density of ARMA

If x_t is ARMA(p, q), $\phi(B)x_t = \theta(B)w_t$, its spectral density is given by

$$f_x(\omega) = \sigma_w^2 \frac{|\theta(e^{-2\pi i\omega})|^2}{|\phi(e^{-2\pi i\omega})|^2} \tag{4.15}$$

where $\phi(z) = 1 - \sum_{k=1}^{p} \phi_k z^k$ and $\theta(z) = 1 + \sum_{k=1}^{q} \theta_k z^k$.

Example 4.5 Moving Average

As an example of a series that does not have an equal mix of frequencies, we consider a moving average model. Specifically, consider the MA(1) model given by

$$x_t = w_t + .5w_{t-1}.$$

A sample realization is shown in the top of Figure 3.2 and we note that the series has less of the higher or faster frequencies. The spectral density will verify this observation.

The autocovariance function is displayed in Example 3.4 on page 90, and for this particular example, we have

$$\gamma(0) = (1 + .5^2)\sigma_w^2 = 1.25\sigma_w^2; \quad \gamma(\pm 1) = .5\sigma_w^2; \quad \gamma(\pm h) = 0 \text{ for } h > 1.$$

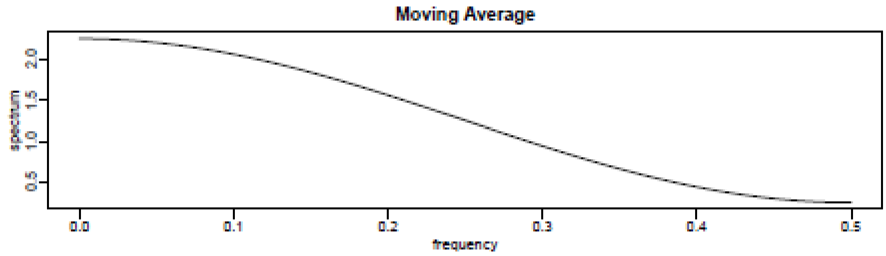
Substituting this directly into the definition given in (4.12), we have

$$\begin{aligned} f(\omega) = \sum_{h=-\infty}^{\infty} \gamma(h)\,\mathrm{e}^{-2\pi i\omega h} &= \sigma_w^2\left[1.25 + .5\left(\mathrm{e}^{-2\pi i\omega} + \mathrm{e}^{2\pi\omega}\right)\right] \\ &= \sigma_w^2\left[1.25 + \cos(2\pi\omega)\right]. \end{aligned} \tag{4.16}$$

We can also compute the spectral density using Property 4.3, which states that for an MA, $f(\omega) = \sigma_w^2|\theta(\mathrm{e}^{-2\pi i\omega})|^2$. Because $\theta(z) = 1 + .5z$, we have

$$\begin{aligned} |\theta(\mathrm{e}^{-2\pi i\omega})|^2 = |1 + .5\mathrm{e}^{-2\pi i\omega}|^2 &= (1 + .5\mathrm{e}^{-2\pi i\omega})(1 + .5\mathrm{e}^{2\pi i\omega}) \\ &= 1.25 + .5\left(\mathrm{e}^{-2\pi i\omega} + \mathrm{e}^{2\pi\omega}\right) \end{aligned}$$

which leads to agreement with (4.16).



Example 4.6 A Second-Order Autoregressive Series

We now consider the spectrum of an AR(2) series of the form

$$x_t - \phi_1 x_{t-1} - \phi_2 x_{t-2} = w_t,$$

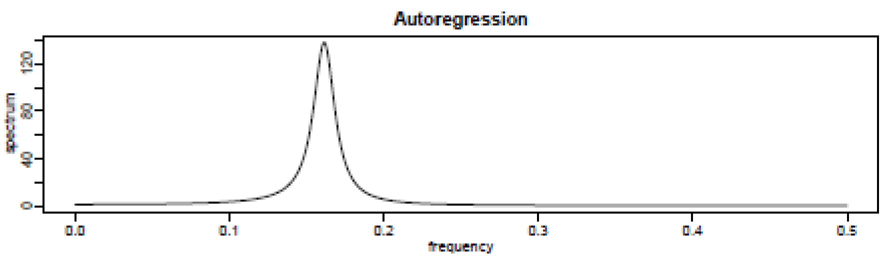
for the special case $\phi_1 = 1$ and $\phi_2 = -.9$. Figure 1.9 on page 14 shows a sample realization of such a process for $\sigma_w = 1$. We note the data exhibit a strong periodic component that makes a cycle about every six points.

To use Property 4.3, note that $\theta(z) = 1$, $\phi(z) = 1 - z + .9z^2$ and

$$\begin{aligned} |\phi(e^{-2\pi i\omega})|^2 &= (1 - \mathrm{e}^{-2\pi i\omega} + .9\mathrm{e}^{-4\pi i\omega})(1 - \mathrm{e}^{2\pi i\omega} + .9\mathrm{e}^{4\pi i\omega}) \\ &= 2.81 - 1.9(\mathrm{e}^{2\pi i\omega} + \mathrm{e}^{-2\pi i\omega}) + .9(\mathrm{e}^{4\pi i\omega} + \mathrm{e}^{-4\pi i\omega}) \\ &= 2.81 - 3.8\cos(2\pi\omega) + 1.8\cos(4\pi\omega). \end{aligned}$$

Using this result in (4.15), we have that the spectral density of x_t is

$$f_x(\omega) = \frac{\sigma_w^2}{2.81 - 3.8\cos(2\pi\omega) + 1.8\cos(4\pi\omega)}.$$



The spectral density can also be obtained from first principles, without having to use Property 4.3. Because $w_t = x_t - x_{t-1} + .9x_{t-2}$ in this example, we have

$$\begin{aligned}\gamma_w(h) &= \text{cov}(w_{t+h}, w_t)\\ &= \text{cov}(x_{t+h} - x_{t+h-1} + .9x_{t+h-2},\, x_t - x_{t-1} + .9x_{t-2})\\ &= 2.81\gamma_x(h) - 1.9[\gamma_x(h+1) + \gamma_x(h-1)] + .9[\gamma_x(h+2) + \gamma_x(h-2)]\end{aligned}$$

Now, substituting the spectral representation (4.11) for $\gamma_x(h)$ in the above equation yields

$$\begin{aligned}\gamma_w(h) &= \int_{-1/2}^{1/2} \left[2.81 - 1.9(e^{2\pi i\omega} + e^{-2\pi i\omega}) + .9(e^{4\pi i\omega} + e^{-4\pi i\omega})\right] e^{2\pi i\omega h} f_x(\omega) d\omega\\ &= \int_{-1/2}^{1/2} \left[2.81 - 3.8\cos(2\pi\omega) + 1.8\cos(4\pi\omega)\right] e^{2\pi i\omega h} f_x(\omega) d\omega.\end{aligned}$$

If the spectrum of the white noise process, w_t, is $g_w(\omega)$, the uniqueness of the Fourier transform allows us to identify

$$g_w(\omega) = [2.81 - 3.8\cos(2\pi\omega) + 1.8\cos(4\pi\omega)]\, f_x(\omega).$$

But, as we have already seen, $g_w(\omega) = \sigma_w^2$, from which we deduce that

$$f_x(\omega) = \frac{\sigma_w^2}{2.81 - 3.8\cos(2\pi\omega) + 1.8\cos(4\pi\omega)}$$

is the spectrum of the autoregressive series.