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THE BOUNDING NUMBER $\mathfrak{b}$ AND QUOTIENTS IN BANACH SPACES

Abstract. The problem whether every infinite dimensional Banach space has a separable infinite dimensional quotient is known as the separable quotient problem. In this survey, we review results connecting the bounding number $\mathfrak{b}$ to this problem and to the existence of uncountable biorthogonal systems in nonseparable Banach spaces. Our discussion highlights combinatorial methods that help differentiate the structure of Banach spaces of density equal to the bounding number $\mathfrak{b}$ from those with density smaller than $\mathfrak{b}$.

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1. Introduction

The following problem, commonly referred to as the "separable quotient problem", is one of the most significant open questions in Banach space theory.

Problem 1.1. Does every infinite dimensional Banach space have a separable infinite dimensional quotient?

It has likely been considered since the 1930s, alongside other important problems stemming from Banach's seminal work. It is attributed to Stanisław Mazur and Stefan Banach. However, there is no explicit mention of it in Banach's book [3], and I couldn't find any formal record of it. The earliest reference I am aware of, where the problem is explicitly stated, is Rosenthal's paper [31] (see Remark 2 on page 188). Over nearly a century, much research has been developed around this problem, and several partial results have been obtained. It also motivated problems related to other structures, as in the case of the article [20], which deals with topological groups.

Several other important problems from Banach's book have been solved. The basis problem, which is whether every separable Banach space admits a Schauder basis, was answered in the negative by Enflo in [11]. The basic sequence problem is whether every Banach space admits a basic sequence and a positive solution is stated in [3] with no proof. Generalizations to that solution are proved in [4]. The textbook [21] is a good reference for all these results.

Quotients of Banach spaces are useful for gaining insight into the structure of the space itself and vice versa. This is exemplified by the so called three space problems, which are of the following form: knowing that two out of the three spaces X, $Y \subseteq X$ and X/Y have a certain property, can we conclude that the third space also shares this property? There are several examples of three-space properties in the literature, see [8].

During the 1960s and the 1970s, several positive results for the separable quotient problem have been obtained. Let us mention that the following classes of Banach spaces do admit separable quotients:

- Banach spaces containing c_0 (Bessaga, Pełczyński, [5]);
- Reflexive spaces (Pełczyński, [27]);
- Weakly compactly generated spaces (Amir, Lindenstrauss, [1]);
- Spaces whose dual contains ℓ_1 (Hagler, Johnson, [14]).

Moreover, Johnson and Rosenthal proved in [16] that every separable space X contains a closed infinite dimensional linear subspace Y of X such that X/Y is infinite dimensional.

These are classical results and most of their original proofs have a structural flavour. The classical survey in [23], as well as the more recent account in [12], along with the references cited therein, are good sources for the reader interested in the separable quotient problem. In this paper, we also review results that provide partial solutions to Problem 1.1 or solve related problems. In contrast to the aforementioned works, however, the results we analyze here primarily focus on the relationship between the density of a Banach space and the cardinality of certain structures within it and their proofs often combine classical analytic methods with combinatorial approaches. The bounding number $\mathfrak{b}$ plays an important role in the results discussed here, which represent only a small sample of the diverse combinatorial constructions related to separable quotients in nonseparable Banach spaces.

The paper is organized as follows. Section 2 contains results relating the existence of separable quotients, quotients with Schauder basis and biorthogonal systems in general Banach spaces. We show that every Banach space of density smaller than $\mathfrak{b}$ admits a (separable) quotient with Schauder basis. This result has been obtained in [32], but it also follows from classical arguments from the 1970s. Section 3 focuses on the existence of biorthogonal systems in $C(K)$ spaces and its relation to properties of the corresponding compact space K. We recall constructions of nonseparable Banach $C(K)$ spaces without uncountable biorthogonal systems assuming combinatorial principles. Let us point out that there are also interesting constructions of nonseparable Banach spaces which are not $C(K)$ spaces, see e.g. [22], but they won't be addressed here. In Section 4, we detail an old consistent construction by Todorčević of a nonseparable Banach $C(K)$ space without uncountable biorthogonal systems, under the assumption that $\mathfrak{b} = \omega_1$. We chose to present Todorčević's original proof, as the outline given in his book [35] lacks detail. Moreover, his result is crucial to the brief discussion in Section 5 of recent results from [7], which combine the techniques resumed in Section 2 with the P-ideal dichotomy, introduced in [36].

The paper is intended to set theorists interested in Banach spaces. Therefore, the reader is assumed to be familiar with the classical set theory definitions, which we introduce only when necessary for the arguments presented. We also aimed to minimize the introduction of too many definitions from Banach space theory; however, readers can find these definitions in the references if needed. We follow standard notation for set theory and [15] for Banach spaces.

2. Quotients with Schauder bases

In the 1970s, Johnson and Rosenthal proved in [16, Theorem IV.1(i)] that every separable Banach space has a nontrivial separable quotient. By nontrivial we mean that it has infinite codimension. This result was later improved for Banach spaces of density strictly smaller than $\mathfrak{b}$:

Theorem 2.1 (Saxon, Sanchez-Ruiz, [32])**.** *Every Banach space of density strictly smaller than* $\mathfrak{b}$ *has a separable quotient.*

In their proof, the authors use an equivalent version of the separable quotient problem from [33]. We can also see this result as a consequence of classical results. This approach was used in [12] and a similar argument is presented below, for the convenience of the reader. Our main focus is on the use of the bounding number $\mathfrak{b}$ to refine sequences in the dual space. The bounding number $\mathfrak{b}$ is defined as the smallest cardinality of a subset of ω^ω which is unbounded with respect to the relation $\leqslant^*$, where $f \leqslant^* g$ means that $f(n) \leqslant g(n)$ fails for finitely many $n \in \omega$.

Recall that a sequence $(e_n)_n$ in a Banach space X is a Schauder basis if every vector $x \in X$ has a unique representation as a series $\sum_n \lambda_n e_n$. As noted in the introduction, it has been known since the 1970s that separable Banach spaces may not have Schauder bases, see [11]. However, it is also known that every infinite-dimensional Banach space X contains a basic sequence, meaning a sequence of vectors that forms a Schauder basis for some infinite-dimensional closed subspace of X, see [4]. In fact, Banach spaces are abundant in basic sequences, see [21], and this will be helpful in the following argument.

Sketch of the proof of Theorem 2.1. Let $(\varphi_n)_n$ be a normalized weakly* null sequence in X^* (ie. $(\varphi_n(x))_n$ converges to 0 for every $x \in X$). Its existence is guaranteed by results from Josefson–Nissenzweig theorem, see [15]. It follows also from results from [14].

Suppose that X has density strictly smaller than $\mathfrak{b}$ and let $D \subseteq X$ be norm-dense in S_X such that $|D| < \mathfrak{b}$. For each $x \in D$, since $(\varphi_n)_n$ is weakly* null, let $f_x \in \omega^\omega$ be such that

$$k \geqslant f_x(n) \Rightarrow |\varphi_k(x)| < \frac{1}{2^n}.$$

From $|D| < \mathfrak{b}$, we get that there is a $\leqslant^*$-dominating $f \in \omega^\omega$ for $\{f_x : x \in D\}$.

Without loss of generality, f can be assumed to be strictly increasing. Let $k_n = f(n)$ for each $n \in \omega$. For each $x \in D$, there is $n_0 \in \omega$ such that $n \geqslant n_0$ implies $f_x(n) \leqslant f(n)$. Hence,

$$\begin{aligned}\sum_{n\in\omega} |\varphi_{k_n}(x)| = \sum_{n\in\omega} |\varphi_{f(n)}(x)| &= \sum_{n<n_0} \varphi_{f(n)}(x)| + \sum_{n\geqslant n_0} |\varphi_{f(n)}(x)| \\ &\leqslant \sum_{n<n_0} |\varphi_{f(n)}(x)| + \sum_{n\geqslant n_0} \frac{1}{2^n},\end{aligned}$$

so that the series $\sum_{n\in\omega} |\varphi_{k_n}(x)|$ converges. From now on, we assume that for every $x \in D$, $\sum_{n\in\omega} |\varphi^*_n(x)|$ converges.

Given a subsequence $(\varphi_{n_k})_k$, consider the mapping defined by

$$Q\colon X \to (\overline{\operatorname{span}}\{\varphi_{n_k} : k \in \omega\})^* \quad Q(x)(\varphi) = \varphi(x).$$

Independently on the extra properties of the subsequence $(\varphi_{n_k})_k$, Q is a well-defined linear continuous map. Our goal is to find a subsequence $(\varphi_{n_k})_k$ which is a basic sequence such that Q is a quotient mapping onto $\overline{\operatorname{span}}\{\varphi^*_{n_k} : k \in \omega\}$, where $(\varphi^*_{n_k})_k$ are the elements of X^{**} biorthogonal to $(\varphi_{n_k})_k$. This is done similarly to the original result by Johnson and Rosenthal from [16]: given a sequence $(\varepsilon_n)_n$ in $(0,1)$, we recursively use the principle of local reflexivity to prove the following claim:

Claim. *There is an increasing sequence $(n_k)_k$ such that for every $N \in \omega$, there is a finite subset D_N of D such that given a norm-one $\varphi^* \in (\operatorname{span}\{\varphi_{n_k} : k < N\})^*$ there is a norm-one $x \in D_N$ such that*

$$\forall \varphi \in \operatorname{span}\{\varphi_{n_k} : k < N\} \quad |\varphi^*(\varphi) - \varphi(x)| < \varepsilon_N \|\varphi\|$$

and $\sum_{n \geqslant n_N} |\varphi_n(x)| < \varepsilon_N$.

Proof of the Claim. Given $(n_i)_{i<N}$, let D^*_N be a finite $\frac{\varepsilon_N}{3}$-dense in the sphere of $(\operatorname{span}\{\varphi_{n_k} : k < N\})^*$. By the local reflexivity principle, for each $\varphi^* \in D^*_N$ there is $y_{\varphi^*} \in X$ such that $\|y_{\varphi^*}\| = 1$ and

$$\forall \varphi \in \operatorname{span}\{\varphi_{n_k} : k < N\} \quad |\varphi^*(\varphi) - \varphi(y_{\varphi^*})| < \frac{\varepsilon_N}{3}\|\varphi\|.$$

Let $x_{\varphi^*} \in D$ be such that $\|y_{\varphi^*} - x_{\varphi^*}\| < \frac{\varepsilon_N}{3}$.

Since $D_N := \{x_{\varphi^*} : \varphi^* \in D^*\}$ is a subset of D, we know that for each $x \in D$, $\sum_{n\in\omega} |\varphi_n(x)|$ converges. Hence, since D_N is finite, there is $n_N > n_{N-1}$ such that

$$\forall \varphi^* \in D^*_N \quad \sum_{n \geqslant n_N} |\varphi_n(x_{\varphi^*})| < \varepsilon_N.$$

Finally, given any norm-one $\psi^* \in \operatorname{span}\{\varphi_{n_k} : k < N\}$, let $\varphi \in D^*_N$ be such that $\|\psi^* - \varphi^*\| < \frac{\varepsilon_N}{3}$. It follows that for every $\varphi \in \operatorname{span}\{\varphi_{n_k} : k < N\}$,

$$|\psi^*(\varphi) - \varphi(x_{\varphi^*})| \leqslant |\psi^*(\varphi) - \varphi^*(\varphi)| + |\varphi^*(\varphi) - \varphi(y_{\varphi^*})| + |(\varphi(y_{\varphi^*}) - \varphi(x_{\varphi^*})| < \varepsilon_N \|\varphi\|,$$

which concludes the proof of the claim. □

Now, if we impose that $\Pi_{n\in\omega}(\frac{1}{1-3\varepsilon_n}) < +\infty$, we can show that $(\varphi_{n_k})_k$ is a basic sequence in X^*, so that their biorthogonal functionals $(\varphi^*_{n_k})_k$ form a basic sequence X^{**}. Therefore, for $x \in D$, we have that $Q(x) = \sum_{k\in\omega} \varphi_{n_k}(x)\varphi^*_{n_k}$. This implies that $Q[X] \subseteq \overline{\operatorname{span}}\{\varphi^*_{n_k} : k \in \omega\}$. Finally, assuming that $\sum_{n\in\omega} \varepsilon_n < +\infty$, successive approximations guarantee that $\overline{\operatorname{span}}\{\varphi^*_{n_k} : k \in \omega\} \subseteq Q[X]$, as desired. □

In this argument, the basic sequence plays a crucial role in identifying a natural candidate for a quotient space, and it allows us to obtain a quotient with a Schauder basis at no additional cost. The following consistency result gets separable quotients with Schauder basis in spaces of large density:

Theorem 2.2 (Dodos, Lopez-Abad, Todorčević, [10])). *It is consistent with the usual axioms of ZFC that every Banach space with density at least $\aleph_\omega$ has a separable quotient with an unconditional basis.*

The proof extracts a partition property of some cardinal κ which ensures the existence of an unconditional basic sequence in the dual of every Banach space of density at least κ. A result from [14] and the fact that $\aleph_\omega$ consistently satisfies this partition property imply the previous result. Another result combining combinatorial methods and Hagler and Johnson's result to guarantee the existence of a separable quotient is the following:

Theorem 2.3 (Argyros, Dodos, Kanellopoulos, [2]). *Every dual Banach space has a separable quotient.*

Pełczyński asked in [27] whether the following problem is equivalent to the original separable quotient problem:

Problem 2.1. Does every Banach space have a nontrivial quotient with Schauder basis?

Theorem 2.1 implies that this is equivalent to the original separable quotient problem, since a separable quotient would itself have a (separable) quotient with Schauder basis. A natural version of Problem 2.1 was originally posed by Plichko in [29]:

Problem 2.2. Does every Banach space have a quotient with a long Schauder basis of the length of its density?

Long Schauder basis are natural generalizations of standard Schauder basis, but indexed in larger ordinals and useful in the nonseparable setting, see [15]. Plichko himself gave a negative answer to this question. First, he gave a negative answer to the following question, posed by Davis and Johnson in [9].

Problem 2.3. Does every Banach space have a bounded fundamental biorthogonal system?

Recall that a family of pairs $(x_\alpha, \varphi_\alpha)_{\alpha\in\kappa}$ in $X \times X^*$ is a biorthogonal system if $\varphi_\alpha(x_\alpha) = 1$ and $\varphi_\alpha(x_\beta) = 0$ if $\alpha \neq \beta$. It is a fundamental biorthogonal system if, moreover, $\mathrm{span}\{x_\alpha : \alpha \in \kappa\}$ is norm dense in X. Hence, a basic sequence, together with its biorthogonal functionals, forms a biorthogonal system, whereas a Schauder basis, together with its biorthogonal functionals, forms a fundamental biorthogonal system. A biorthogonal system $(x_\alpha, \varphi_\alpha)_{\alpha\in\kappa}$ is bounded if $\sup_{\alpha\in\kappa} \|x_\alpha\|\cdot\|\varphi_\alpha\| < +\infty$ and it easily follows from counting arguments that for any uncountable regular cardinal κ, every biorthogonal system of cardinality κ contains a bounded biorthogonal system of same cardinality. In [28], Plichko showed the following result.

Theorem 2.4. *If Γ is an index set of cardinality greater than $\mathfrak{c}$, then $\ell_\infty^c(\Gamma)$ does not admit a bounded fundamental biorthogonal system.*

$\ell_\infty^c(\Gamma)$ is the space of bounded sequences of scalars indexed in Γ with countable support. A few years later, Plichko proved in [29] that Problems 2.2 and 2.3 are equivalent. Finally, Godefroy and Louveau posed in [13] the following more general question:

Problem 2.4. Does every nonseparable Banach space have an uncountable biorthogonal system?

The next section discusses consistent negative answers to this question in the context of spaces of continuous functions.

3. $C(K)$ spaces without biorthogonal systems

Given a compact Hausdorff space K, let $C(K)$ be the space of continuous real-valued functions on K with the supremum norm. The class of $C(K)$ spaces is a great source of interesting examples, particularly in the context of nonseparable spaces, as the zoo of nonmetrizable compact spaces gave rise to interesting examples of $C(K)$ spaces, see [15]. Every Banach space X is isometrically isomorphic to a subspace of $C(B_{X^*})$, where the dual ball is equipped with the weak* topology and the structure of a $C(K)$ space can be analyzed from the properties of the topological space K. The classical Stone-Weierstrass theorem guarantees that the density of $C(K)$ equals the weight of K and Riesz Representation theorem identifies each linear continuous functional on $C(K)$ with a regular signed Borel measure on K.

There is a natural way to get biorthogonal systems in $C(K)$ from discrete subsets of K: if $\{x_\alpha : \alpha \in \Gamma\}$ is a discrete subset of K, Urysohn's Lemma guarantees the existence, for each $\alpha < \kappa$, of $\varphi_\alpha \in C(K)$ such that $\varphi_\alpha(x_\alpha) = 1$ and $\varphi_\alpha(x_\beta) = 0$ for $\beta \neq \alpha$. Taking the point-evaluating functional $\delta_\alpha \in C(K)^*$ defined by $\delta_\alpha(\varphi) = \varphi(x_\alpha)$, we get that $(\varphi_\alpha, \delta_\alpha)_{\alpha\in\Gamma}$ is a biorthogonal system in $C(K)$.

This argument can be improved to show the following result:

Theorem 3.1 (Todorčević, [37])**.** *If a compact Hausdorff space K has a nonseparable subspace, then $C(K)$ contains an uncountable biorthogonal system.*

On the other hand, the following result is some sort of contrapositive for scattered spaces:

Theorem 3.2 (Zenor, [40]; Velichko [39])**.** *Let K be a compact Hausdorff scattered space. If K^n is hereditarily separable for every $n \in \omega$, then $C(K)$ has no uncountable biorthogonal systems.*

The previous result derives from arguments from the 1980s related to pointwise convergence and the weak topology in $C(K)$, see [25]. It identifies a class of Banach spaces where one might seek counterexamples to Problem 2.4. Notably, there are several consistent constructions of nonmetrizable compact scattered spaces with hereditarily separable finite powers. The most known is likely Kunen's construction, presented in [25, Theorem 7.1] and achieved under the continuum hypothesis. Additionally, Shelah built such a space under $\diamondsuit$ (see [34]), and a variation of Ostaszewski's construction under $\clubsuit$ (see [26]) was described in [15, Theorem 4.36].

The following result by Todorčević will play a crucial role in the discussion that follows.

Theorem 3.3 (Todorčević, Theorem 2.4, [35])**.** *Assuming that $\mathfrak{b} = \aleph_1$, there exists a nonmetrizable compact scattered Hausdorff space K such that K^n is hereditarily separable for every $n \in \omega$. In particular, $C(K)$ is a nonseparable Asplund space with no uncountable biorthogonal systems.*

A complete proof of this result is provided in the next section. Several other consistent examples of nonseparable Banach spaces without uncountable biorthogonal systems have been obtained by forcing. The versatility of the forcing method has enabled the construction of examples with a wide range of properties. We highlight two constructions that illustrate this diversity by pursuing different directions.

The first construction proves the consistency of a gap between the density of a Banach space and the maximal cardinality of a biorthogonal system.

Theorem 3.4 (Brech, Koszmider, [6]). *It is consistent with the usual axioms of set theory ZFC that there exists a compact scattered Hausdorff space K of weight $\aleph_2$ such that K^n is hereditarily separable for every $n \in \omega$. In particular, $C(K)$ is a Banach space of density $\aleph_2$ with no uncountable biorthogonal systems.*

All $C(K)$ constructions discussed so far have the property of being Asplund spaces: a Banach space X is Asplund if every separable subspace has a separable dual. Namioka and Phelps proved in [24] that $C(K)$ is Asplund if and only if K is scattered. This is important in the proof of Theorem 3.2, as it ensures that the functionals on $C(K)$ are atomic measures. Indeed, Asplund spaces can be considered "small", which might explain why the nonseparable examples which do not admit uncountable biorthogonal systems were found in this class, even with density $\aleph_2$. This will be relevant in Section 5. For now, let us turn to the second construction, which demonstrates that being Asplund is consistently not a necessary condition for the nonexistence of uncountable biorthogonal systems in $C(K)$ spaces:

Theorem 3.5 (Koszmider, [17]). *It is consistent with the usual axioms of set theory ZFC that there exists a compact Hausdorff space K of weight $\aleph_1$ such that $C(K)$ is a space with no uncountable semi-biorthogonal systems, i.e. there is no sequence of pairs $(x_\alpha, \varphi_\alpha)_{\alpha\in\kappa}$ in $X \times X^*$ such that $\varphi_\alpha(x_\alpha) = 1$, $\varphi_\alpha(x_\beta) = 0$ if $\alpha > \beta$ and $\varphi_\alpha(x_\beta) \geqslant 0$ if $\alpha < \beta$.*

It follows from a result of [19] that K is not scattered, hence $C(K)$ is not Asplund.

4. A construction by Todorčević

Although an outline of the proof of Theorem 3.3 appears in [35], many details are left to the reader. Since the result gained importance in light of recent results-which will be discussed in the next section-we have chosen to include a complete and detailed proof here for the reader's convenience.

Recall that the theorem says that assuming $\mathfrak{b} = \aleph_1$, there exists a nonmetrizable compact scattered Hausdorff space K such that K^n is hereditarily separable for every $n \in \omega$. In particular, $C(K)$ is a nonseparable Asplund space with no uncountable biorthogonal systems.

Proof of Theorem 3.3. Let $(f_\alpha)_{\alpha<\omega_1}$ be an unbounded family in $(\omega^\omega, \leqslant^*)$ and without loss of generality we may assume that $f_\alpha <^* f_\beta$ for every $\alpha < \beta < \omega_1$.

Fix $e\colon [\omega_1]^2 \to \omega$ a function with the following properties:

- For every $\beta \in \omega_1$, $e_\beta := e(\{\cdot, \beta\})\colon \beta \to \omega$ is injective.
- For every $\alpha \in \omega_1$, $\{e_\beta \restriction_\alpha\colon \beta < \omega_1\}$ is a countable set.

The existence of such e is a consequence of the existence of an Aronszajn tree, see e.g. [18].

Let $\Delta(\alpha,\beta)=\min\{n\in\omega : f_\alpha(n)\neq f_\beta(n)\}$ if $\alpha\neq\beta$, $\Delta(\alpha,\alpha)=\infty$,

$$H(\beta)=\{\alpha<\beta : e(\alpha,\beta)\leqslant f_\beta(\Delta(\alpha,\beta))\}$$

and recursively define

$$V(\beta)=\{\beta\}\cup\bigcup_{\eta\in H(\beta)}\{\alpha\in V(\eta):\forall\xi\in H(\beta)\cup\{\beta\}\ (\xi\neq\eta\Rightarrow\Delta(\alpha,\xi)<\Delta(\alpha,\eta))\}.$$

Let us denote by $\varphi(\alpha,\eta,\beta)$ the sentence

$$\forall\xi\in H(\beta)\cup\{\beta\}\ (\xi\neq\eta\Rightarrow\Delta(\alpha,\xi)<\Delta(\alpha,\eta)),$$

so that

$$V(\beta)=\{\beta\}\cup\bigcup_{\eta\in H(\beta)}\{\alpha\in V(\eta):\varphi(\alpha,\eta,\beta)\text{ holds}\}.$$

Finally, let $V_n(\beta)=\{\alpha\in V(\beta):\Delta(\alpha,\beta)\geqslant n\}$ and we claim that there is a topology τ in ω_1 such that $\{V_n(\beta):n\in\omega\}$ forms a local topological basis at β. The desired space K will be the one-point compactification of $L:=(\omega_1,\tau)$.

Claim 1. *If $\alpha\in V_n(\beta)$, then there is $k\in\omega$ such that $V_k(\alpha)\subseteq V_n(\beta)$.*

Proof of Claim 1. We prove this by induction on β. Given $\alpha\in V_n(\beta)$, $\alpha\neq\beta$, let $\eta\in H(\beta)$ be such that $\alpha\in V(\eta)$ and $\varphi(\alpha,\eta,\beta)$ holds. In particular, $n\leqslant\Delta(\alpha,\beta)<\Delta(\alpha,\eta)$, so that $\alpha\in V_n(\eta)$. By the inductive hypothesis, there is $k\in\omega$ such that $V_k(\alpha)\subseteq V_n(\eta)$. We may assume without loss of generality $k\geqslant\max\{n,\Delta(\alpha,\eta)\}$ and let us check that $V_k(\alpha)\subseteq V_n(\beta)$. Fix $\alpha'\in V_k(\alpha)$. First, since $\Delta(\alpha',\alpha)\geqslant k\geqslant n$ and $\Delta(\alpha,\beta)\geqslant n$, we get that $\Delta(\alpha',\beta)\geqslant n$. Second, $\Delta(\alpha',\alpha)\geqslant\Delta(\alpha,\eta)$ implies that $\varphi(\alpha',\eta,\beta)$ holds, so that $\alpha'\in V(\beta)$. Hence, $V_k(\alpha)\subseteq V_n(\beta)$. $\square$

Let $L=(\omega_1,\tau)$ and notice that L is Hausdorff since $V_{\Delta(\alpha,\beta)}(\alpha)\cap V_{\Delta(\alpha,\beta)}(\beta)$ are disjoint for $\alpha\neq\beta$. Let us prove that it is locally compact. For each $\beta\in\omega_1$ and $n\in\omega$, let

$$F_{\beta,n}=\{\eta\in H(\beta):e(\eta,\beta)\leqslant f_\beta(n)\}$$

and notice that the first property of e guarantees that $F_{\beta,n}$ is finite.

Claim 2. *For every $\beta\in\omega_1$ and every $n\in\omega$, if $\alpha\in V_n(\beta)\setminus V_{n+1}(\beta)$, then there is $\eta\in F_{\beta,n}$ such that $\alpha\in V_{\Delta(\eta,\beta)}(\eta)$ and $\varphi(\alpha,\eta,\beta)$ holds.*

Proof of Claim 2. If $\alpha\in V_n(\beta)\setminus V_{n+1}(\beta)$, then $\Delta(\alpha,\beta)=n$ and there is $\eta\in H(\beta)$ such that $\varphi(\alpha,\eta,\beta)$ holds. In particular, $\Delta(\alpha,\beta)<\Delta(\alpha,\eta)$. Hence, $\Delta(\eta,\beta)=\Delta(\alpha,\beta)=n$, and since $\eta\in H(\beta)$, we get that $e(\eta,\beta)\leqslant f_\beta(n)$, which ensures that $\eta\in F_{\beta,n}$ and concludes the proof of the claim. $\square$

Claim 3. *For every $\beta\in\omega_1$ and every $m\in\omega$, $V_m(\beta)$ is compact.*

Proof of Claim 3. We prove it by induction on β. Let $X\subseteq V_m(\beta)$ be an infinite set and notice that one of the following alternatives holds:

(1) For every $n\geqslant m$, $X\cap V_n(\beta)$ is infinite.
(2) There exists $n\geqslant m$ such that $X\cap(V_n(\beta)\setminus V_{n+1}(\beta))$ is infinite.

If (1) holds, then β is an accumulation point of X in $V_m(\beta)$ and we are done. If (2) holds, it follows from Claim 2 that there is $\eta \in F_{\beta,n}$ such that $V_{\Delta(\eta,\beta)}(\eta) \cap X$ is infinite and $\varphi(\alpha,\eta,\beta)$ holds for every $\alpha \in V_{\Delta(\eta,\beta)}(\eta) \cap X$. By the inductive hypothesis, there is $\gamma \in V_{\Delta(\eta,\beta)}(\eta)$ an accumulation point of $V_{\Delta(\eta,\beta)}(\eta) \cap X$. Let us show that $\gamma \in V_m(\beta)$.

Given $k \geqslant \max\{n, \Delta(\gamma,\eta)\}$, there is $\alpha \in V_k(\gamma) \cap V_{\Delta(\eta,\beta)}(\eta) \cap X$. In particular, $\alpha \in V_n(\beta) \setminus V_{n+1}(\beta)$. Hence, $\Delta(\gamma,\alpha) \geqslant k \geqslant n$ and $\Delta(\alpha,\beta) = n$, so that $\Delta(\gamma,\beta) = n$. Moreover, $\Delta(\gamma,\alpha) \geqslant k \geqslant \Delta(\gamma,\eta)$, so that

$$\forall \xi \in H(\beta) \cup \{\beta\},\ \xi \neq \eta \Rightarrow \Delta(\gamma,\xi) = \Delta(\alpha,\xi) < \Delta(\alpha,\eta) = \Delta(\gamma,\eta).$$

This proves that $\varphi(\gamma,\eta,\beta)$ holds and, therefore, $\gamma \in V_n(\beta) \subseteq V_m(\beta)$. $\square$

L is clearly a scattered space, since for any nonempty $X \subset \omega_1$, $\min X$ is isolated in X. Let K be the one point compactification of L and let us denote by ω_1 the infinity point.

The proof that K^n is hereditarily separable requires some extra work. Given $\Gamma \subseteq \omega_1$, we say that a family $(\beta_1^\xi, \dots, \beta_n^\xi)_{\xi\in\Gamma} \subseteq \omega_1^n$ is cofinal if for every $\alpha \in \omega_1$, there is $\eta \in \Gamma$ such that $\alpha < \beta_i^\xi$ for every $\xi \geqslant \eta$ in Γ and every $1 \leqslant i \leqslant n$.

Claim 4 ([35, Lemma 2.0])**.** *If $(\beta_1^\xi, \dots, \beta_n^\xi)_{\xi\in\omega_1} \subseteq \omega_1^n$ is cofinal, then there are $\delta < \xi < \omega_1$ such that $\beta_i^\delta \in H(\beta_i^\xi)$ for every $1 \leqslant i \leqslant n$.*

Before proving the claim, let us finish the proof of the theorem. We want to prove that if K is the one-point compactification of L, then K^n is hereditarily separable for every $n \in \omega$. From [30, Theorem 3.1], K^n is hereditarily separable if and only if no uncountable sequence is left-separated, that is, for every uncountable $(\bar\beta_\xi)_{\xi<\omega_1} \subseteq K^n$, there is $\eta < \omega_1$ such that $\bar\beta_\eta \in \overline{\{\bar\beta_\xi : \xi < \eta\}}$. For each $\xi < \omega_1$, let $\bar\beta_\xi = (\beta_1^\xi, \dots, \beta_n^\xi) \in K^n$.

We prove this by induction on n (take $K^0 = \{\omega_1\}$). Suppose that there is $1 \leqslant j \leqslant n$ and $\Gamma \in [\omega_1]^{\omega_1}$ such that $(\beta_j^\xi)_{\xi\in\Gamma}$ is constant. Then, we can omit the jth coordinate to get an uncountable sequence in K^{n-1} which cannot be left-separated by the inductive hypothesis. This immediately yields that $(\bar\beta_\xi)_{\xi<\omega_1}$ is not left-separated either.

Otherwise, we may assume without loss of generality that each $(\beta_i^\xi)_{\xi\in\omega_1}$ is strictly increasing (and does not include ω_1). By contradiction, suppose that, for each $\xi < \omega_1$, there is $(m_1^\xi, \dots, m_n^\xi) \in \omega^n$ such that

$$\forall \xi < \omega_1 \quad \forall 1 \leqslant i \leqslant n \quad \beta_i^\xi \in V_{m_i^\xi}(\beta_i^\xi)$$

and

$$\forall \xi < \eta < \omega_1 \quad \exists 1 \leqslant i \leqslant n \quad \beta_i^\xi \notin V_{m_i^\eta}(\beta_i^\eta).$$

Passing to an uncountable subset $\Gamma \subseteq \omega_1$, we may assume that, for each $1 \leqslant i \leqslant n$, there is $m_i \in \omega$ such that $m_i^\xi = m_i$ for every $\xi \in \Gamma$. Also, refining Γ to a further uncountable subset, we may assume without loss of generality that $\Delta(\beta_i^\xi, \beta_i^\eta) \geqslant m_i$ for every $\xi < \eta$ in Γ.

Since $(\beta_1^\xi, \dots, \beta_n^\xi)_{\xi\in\Gamma} \subseteq \omega_1^n$ is cofinal, it follows from Claim 4 that there are $\xi < \eta$ in Γ such that $\beta_i^\xi \in H(\beta_i^\eta)$ for every $1 \leqslant i \leqslant n$. Since $H(\beta) \subseteq V(\beta)$ and $\Delta(\beta_i^\xi, \beta_i^\eta) \geqslant m_i$ for every $\xi < \eta$ in Γ, we conclude that $\beta_i^\xi \in V_{m_i}(\beta_i^\eta)$ for every $1 \leqslant i \leqslant n$, which contradicts our assumption and concludes the proof of the theorem.

Let us finally prove Claim 4.

Proof of Claim 4. Since ω^ω with the usual Baire topology is a second countable space, there is $I \in [\omega_1]^\omega$ such that for every $k \in \omega$ and every $\xi \in \omega_1$, there is $\delta \in I$ such that $\Delta(\beta_i^\delta, \beta_i^\xi) \geqslant k$ for every $1 \leqslant i \leqslant n$. Fix $\gamma \in \omega_1$ such that $\beta_i^\delta < \gamma$ for every $\delta \in I$ and every $1 \leqslant i \leqslant n$. Let $\Gamma \in [\omega_1]^{\omega_1}$ be such that $(\beta_1^\xi, \dots, \beta_n^\xi)_{\xi\in\Gamma}$ is still cofinal and if $\xi < \eta$ in Γ, then $\beta_i^\xi < \beta_j^\eta$ for all $1 \leqslant i, j \leqslant n$. We may assume, without loss of generality, that $\gamma < \beta_i^\xi$ for every $\xi \in \Gamma$ and every $1 \leqslant i \leqslant n$.

We will proceed by refining the cofinal family several times to some cofinal subfamily with better properties. To simplify the notation, we will keep calling Γ the uncountable subset obtained after each further refinement.

We use the second property of the function e to refine Γ to an uncountable subset such that for each $1 \leqslant i \leqslant n$, there is $e^i \colon \gamma \to \omega$ such that

$$\forall \xi \in \Gamma \quad e_{\beta_i^\xi} \restriction_\gamma = e^i.$$

We claim that we can refine Γ to some uncountable subset to ensure that for each $1 \leqslant i \leqslant n$, there is $m_i \in \omega$ such that

$$\forall \xi, \eta \in \Gamma \quad f_{\beta_i^\xi} \restriction_{m_i} = f_{\beta_i^\eta} \restriction_{m_i}$$

and

$$\forall k \in \omega \quad \exists \xi \in \Gamma \quad \forall 1 \leqslant i \leqslant n \quad f_{\beta_i^\xi}(m_i) > k.$$

We prove it for $n = 1$. Suppose by contradiction that for each $m \in \omega$ and $s \in \omega^m$ such that

$$\Gamma_s = \{\xi \in \Gamma : f_{\beta_1^\xi} \restriction_m = s\}$$

is uncountable, there is $k_s \in \omega$ such that

$$\forall \xi \in \Gamma_s \quad f_{\beta_1^\xi}(m) \leqslant k_s.$$

Let $f \in \omega^\omega$ be defined by

$$f(m) = \max\{k_s : s \in \omega^{m+1} \text{ and } \forall j \in m+1,\ s(j) \leqslant k_{s\restriction_j}\}.$$

For each $m \in \omega$, let

$$\Gamma_m = \bigcup \{\Gamma_s : s \in \omega^m \text{ and } \Gamma_s \text{ is uncountable}\}.$$

Clearly $\Gamma_{m+1} \subseteq \Gamma_m$ and by induction one proves that each Γ_m is cocountable in Γ, so that $\bigcap_{m\in\omega} \Gamma_m$ is an uncountable set. It remains to notice that

$$\forall \xi \in \bigcap_{m\in\omega} \Gamma_m \quad f_{\beta_1^\xi} \leqslant^* f,$$

which contradicts the fact that $(f_{\beta_1^\xi})_{\xi\in\Gamma}$ is unbounded since $(\beta_1^\xi)_{\xi\in\Gamma}$ is cofinal in ω_1. This holds because if $\xi \in \bigcap_{m\in\omega} \Gamma_m$, then for all $m \in \omega$, $s_m = f_{\beta_1^\xi} \restriction_m$ is such that Γ_s is uncountable. Therefore, $f_{\beta_1^\xi}(m) \leqslant k_s \leqslant f(m)$. The general case requires a multi-dimensional version of the preceding argument.

Now, we use an auxiliary and arbitrary $\xi_0 \in \Gamma$ to choose $\delta \in I$ such that $\Delta(\beta_i^\delta, \beta_i^{\xi_0}) \geqslant m_i$, so that $f_{\beta_i^\delta} \upharpoonright_{m_i} = f_{\beta_i^{\xi_0}} \upharpoonright_{m_i}$. Since $f_{\beta_i^\delta} <^* f_\gamma$, let $m_0 \in \omega$ be such that $f_{\beta_i^\delta}(k) < f_\gamma(k)$ for all $k \geqslant m_0$.

Finally, choose $\xi \in \Gamma$ such that $f_{\beta_i^\xi}(m_i) \geqslant \max\{e^i(\beta_i^\delta), f_{\beta_i^\delta}(m_i)+1\}$ for all $1 \leqslant i \leqslant n$. We have that $\delta < \xi$ are such that $e(\beta_i^\delta, \beta_i^\xi) = e_{\beta_i^\xi}(\beta_i^\delta) = e^i(\beta_i^\delta) \leqslant f_{\beta_i^\xi}(m_i)$. To conclude that $\delta \in H(\xi)$, it remains to see that $m_i = \Delta(\beta_i^\delta, \beta_i^\xi)$. From the choice of δ and the fact that $\Delta(\beta_i^\xi, \beta_i^{\xi_0}) \geqslant m_i$, we know that $m_i \leqslant \Delta(\beta_i^\delta, \beta_i^{\xi_0}) = \Delta(\beta_i^\delta, \beta_i^\xi)$. On the other hand, $f_{\beta_i^\xi}(m_i) > f_{\beta_i^\delta}(m_i)$, so that $\Delta(\beta_i^\delta, \beta_i^\xi) \leqslant m_i$, which concludes the proof. $\square$

This finishes the proof of the theorem. $\square$

5. Biorthogonal systems in nonseparable spaces

In this section we review results ensuring the consistency of the existence of uncountable biorthogonal systems in every nonseparable Banach space. We start with the following important result:

Theorem 5.1 (Todorčević, [37]). *Martin's maximum implies that every Banach space of density $\aleph_1$ has a quotient with a monotone long Schauder basis of length $\aleph_1$.*

The proof of this result involves using Martin's maximum to get an improvement of the argument presented after Theorem 2.1. We discuss below a variation of that argument, which proves the following equivalence result:

Theorem 5.2 (Brech, Todorčević, [7]). *Under the P-ideal dichotomy, the following are equivalent:*

(1) $\mathfrak{b} > \aleph_1$
(2) *All Asplund spaces of density $\aleph_1$ have a quotient with a monotone long Schauder basis of length $\aleph_1$.*
(3) *All nonseparable Asplund spaces have a biorthogonal system of length $\aleph_1$.*

Sketch of the proof. The contrapositive implication from $\neg(1) \Rightarrow \neg(3)$ follows immediately from [35, Theorem 2.4] (Theorem 3.3 above), with no use of the P-ideal dichotomy. (2)$\Rightarrow$(3) holds in ZFC because if X is a nonseparable Asplund space and Y is a subspace of X o density $\aleph_1$, (2) implies that Y has a quotient with a monotone long Schauder basis of length $\aleph_1$. The associated biorthogonal system in this quotient can be lifted to a biorthogonal system in Y using the quotient mapping. And the functionals of this biorthogonal system can be lifted to the whole space using Hahn-Banach Theorem.

The real work is to prove (1)$\Rightarrow$(2) and this is where the P-ideal dichotomy (PID, for short) comes into play. A P-ideal $\mathcal{I}$ of countable infinite subsets of some uncountable set S is an ideal of sets (ie. closed under subsets and finite unions) which satisfies, moreover, that if $(I_n)_n$ is a sequence of elements of $\mathcal{I}$, then there is $I \in \mathcal{I}$ such that $I_n \setminus I$ is finite for every $n \in \omega$. The PID was introduced by Todorčević in [36] and it says that given any P-ideal $\mathcal{I}$ of countable infinite subsets of some uncountable set S, either there is an uncountable $\Gamma \subseteq S$ such that all of its countable subsets belong to $\mathcal{I}$, or S can be partitioned into countably many pieces S_n, $n \in \omega$, in such a way that any $I \in \mathcal{I}$ has finite intersection with each piece S_n. In a certain way, when $\mathcal{I}$ imposes a property on its elements, the PID ensures that either an uncountable subset of S shares this property, or S can be split into countably many pieces whose behavior is orthogonal to that of elements of $\mathcal{I}$.

Similarly to the argument presented in Section 2, in order to prove (1)⇒(2), we construct a long basic sequence $(\varphi_\alpha)_{\alpha\in\Gamma_0}$ in X^* such that the mapping

$$Q\colon X \to (\operatorname{span}\{\varphi_\alpha : \alpha \in \Gamma_0\})^* \quad Q(x)(\varphi) = \varphi(x)$$

is a quotient mapping with range $\overline{\operatorname{span}}\{\varphi_\alpha^* : \alpha \in \Gamma_0\}$.

In the original separable setting from Johnson and Rosenthal's result, since B_{X^*} is metrizable in the weak* topology, we immediately get a normalized weakly* null sequence from the fact that B_{X^*} is weakly* dense in S_{X^*} and by recursion one can pass to a subsequence which gives the Schauder basis of a quotient of X. In the case when X has density smaller than $\mathfrak{b}$ presented in Section 2, the normalized weakly* null sequence was obtained from classical results and a similar recursion argument leads to the convenient basic sequence. The use of the fact that X has density smaller than $\mathfrak{b}$ helps in locating the range of Q within the space $\overline{\operatorname{span}}\{\varphi_{n_k}^* : k \in \omega\}$ as

$$Q(x) = \sum_{k\in\omega} \varphi_{n_k}(x)\varphi_{n_k}^*$$

for x in a dense subset of X.

Here, the argument is indeed more involved and the first and main difficulty is that we have to start the refinements from an *uncountable version* of the normalized weakly* null sequence, which is not given by classical results. In order to get it, we start from a suitable normalized sequence $(\psi_\alpha)_{\alpha\in\omega_1}$ in X^* such that for every $x \in X$, $(\psi_\alpha(x))_{\alpha\in\omega_1}$ has countable support. The PID is then used to select the uncountable version of a weakly* null convergent sequence: a sequence $(\varphi_\alpha)_{\alpha\in\omega_1}$ such that

$$\forall x \in X \quad \forall \varepsilon > 0 \quad \{\alpha \in \omega_1 : |\varphi_\alpha(x)| \geqslant \varepsilon\} \text{ is finite.}$$

Two P-ideals are used in this argument, the first one containing countable pieces of the desired uncountable sequence:

$$\mathcal{I}_1 = \{I \in [\omega_1]^\omega : (\forall x \in D)(\forall \varepsilon \in (0,1) \cap \mathbb{Q}) \quad \{\alpha \in I : |\psi_\alpha(x)| \geqslant \varepsilon\} \text{ is finite}\},$$

where D is a dense subset of X of cardinality $\aleph_1$. It is ensured to be a P-ideal using the fact that $|D| = \aleph_1 < \mathfrak{b}$ and the first alternative of the PID would lead us to the desired uncountable version of a weakly* null convergent sequence. However, we cannot guarantee that the second alternative does not hold. Hence, we have to pass to some sort of weakly* sequentially compact bidimensional version of $(\psi_\alpha)_{\alpha\in\omega_1}$ where a Fubini-like argument is used. We then define a second P-ideal and show that the second alternative of PID cannot hold for this ideal. This yields the desired uncountable sequence $(\varphi_\alpha)_{\alpha\in\omega_1}$ which is weakly* null.

From this point on, the proof follows along similar lines as the one presented in Section 2. We make use of the P-ideal formed by countable subsets of Γ satisfying the desired property:

$$\mathcal{I} = \{I \in [\omega_1]^\omega : (\forall x \in D) \quad \sum_{\alpha\in I} |f_\alpha(x)| < +\infty\}.$$

Once we show that the second alternative is impossible, we obtain an uncountable $\Gamma \subseteq \omega_1$ such that

$$\forall x \in D \quad \sum_{\alpha\in\Gamma} |\varphi_\alpha(x)| < +\infty.$$

We finish by refining the sequence to an uncountable $\Gamma_0 \subseteq \Gamma$ such that the mapping Q is indeed a quotient mapping and $(\varphi^*_\alpha)_{\alpha\in\Gamma_0}$ is a long basic sequence of its range. The local reflexivity principle is used in a similar way as in the argument presented in Section 2. $\square$

Theorem 5.1 had already been reformulated in [38], where Martin's maximum was replaced by the P-ideal dichotomy and the cardinal assumption $\mathfrak{p} > \aleph_1$. It is worth recalling that the conclusion of Theorem 5.2 holds for Asplund spaces, while both in Theorem 5.1 (and in its modification in) [38], the conclusion holds for all Banach spaces. The point is that the cardinal assumption $\mathfrak{p} > \aleph_1$ allows stronger diagonalization arguments than the weaker $\mathfrak{b} > \aleph_1$. Asplund spaces have weak* sequentially compact dual balls and this helps in finding convergent sequences and replaces the diagonalization arguments at some point. In both cases, convergent sequences are used to kill one of the alternatives of the PID.

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