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Homogeneous families on trees and subsymmetric basic sequences ☆,☆☆

C. Brech [a], J. Lopez-Abad [b,c,*], S. Todorcevic [d,e]

[a] *Departamento de Matemática, Instituto de Matemática e Estatística, Universidade de São Paulo, Caixa Postal 66281, 05314-970, São Paulo, Brazil*
[b] *Institut de Mathématiques de Jussieu–PRG, Université Paris 7, case 7012, 75205 Paris Cedex 13, France*
[c] *Departamento de Matemáticas Fundamentales, UNED, Paseo Senda del Rey 9, E-28040 Madrid, Spain*
[d] *Institut de Mathématiques de Jussieu, UMR 7586, 2 place Jussieu – Case 247, 75222 Paris Cedex 05, France*
[e] *Department of Mathematics, University of Toronto, Toronto, M5S 2E4, Canada*

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We study density requirements on a given Banach space that guarantee the existence of subsymmetric basic sequences by extending Tsirelson's well-known space to larger index sets. We prove that for every cardinal κ smaller than the first Mahlo cardinal there is a reflexive Banach space of density κ without subsymmetric basic sequences. As for Tsirelson's space, our construction is based on the existence of a rich collection of homogeneous families on large index sets for which one can estimate the complexity on any given infinite set. This is used to describe detailedly the asymptotic structure of the spaces. The collections of families are of independent interest and their existence is proved inductively. The fundamental

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* Corresponding author.
E-mail addresses: brech@ime.usp.br (C. Brech), abad@mat.uned.es (J. Lopez-Abad), stevo@math.toronto.edu (S. Todorcevic).

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stepping up argument is the analysis of such collections of families on trees.

1. Introduction

Recall that a set of *indiscernibles* for a given structure $\mathcal{M}$ is a subset X with a total ordering $<$ such that for every positive integer n every two increasing n-tuples $x_1 < x_2 < \cdots < x_n$ and $y_1 < y_2 < \cdots < y_n$ of elements of X have the same properties in $\mathcal{M}$. A simple way of finding an extended structure on κ without an infinite set of indiscernibles is as follows. Suppose that $\mathcal{F}$ is a family of finite subsets of κ that is compact, as a natural subset of the product space 2^κ, and *large*, that is, every infinite subset of κ has arbitrarily large subsets in $\mathcal{F}$. Let $\mathcal{M}_\mathcal{F}$ be the structure $(\kappa, (\mathcal{F} \cap [\kappa]^n)_n)$ that has κ as universe and that has infinitely many n-ary relations $\mathcal{F} \cap [\kappa]^n \subseteq [\kappa]^n$. It is easily seen that $\mathcal{M}_\mathcal{F}$ does not have infinite indiscernible sets.

While in set theory and model theory indiscernibility is a well-studied and unambiguous notion, in the context of the Banach space theory it has several versions, the most natural one being the notion of a *subsymmetric sequence* or a *subsymmetric set*. In a normed space $(X, \|\cdot\|)$, a sequence $(x_\alpha)_{\alpha\in I}$ indexed in an ordered set $(I, <)$ is called C-*subsymmetric* when

$$\|\sum_{j=1}^n a_j x_{\alpha_j}\| \le C \|\sum_{j=1}^n a_j x_{\beta_j}\|$$

for every sequence of scalars $(a_j)_{j=1}^n$ and every $\alpha_1 < \cdots < \alpha_n$ and $\beta_1 < \cdots < \beta_n$ in I. When $C = 1$ this corresponds exactly to the notion of an indiscernible set and it is easily seen that this can always be assumed by renorming X with an appropriate equivalent norm. Another closely related notion is *unconditionality*. Recall that a sequence $(x_i)_{i\in I}$ in some Banach space is C-*unconditional* whenever

$$\|\sum_{i\in I} \theta_i a_i x_i\| \le C \|\sum_{i\in I} a_i x_i\|$$

for every sequence of scalars $(a_i)_{i\in I}$ and every sequence $(\theta_i)_{i\in I}$ of signs. Of particular interest are the indiscernible coordinate systems, such as the Schauder basic sequences. The unit bases of the classical sequence spaces ℓ_p, $p \ge 1$ or c_0 (in any density) are subsymmetric and unconditional (in fact, symmetric, i.e. indiscernible by permutations) bases. Moreover, every basic sequence in one of these spaces has a symmetric subsequence. But this is not true in general: there are basic sequences without unconditional subsequences, the simplest example being the summing basis of c_0. However, it is more difficult to find a weakly-null basis without unconditional subsequences (B. Maurey and H.P. Rosenthal [17]). Now we know that there are Banach spaces without unconditional basic sequences. The first such example was given by W.T. Gowers and B. Maurey [9], a space which was moreover reflexive. Concerning subsymmetric sequences, we mention that the unit basis of the Schreier space [23] does not have subsymmetric subsequences, and the Tsirelson space [27] is the first example of a reflexive space without subsymmetric basic sequences.

All these are separable spaces so it is natural to ask if large spaces must contain infinite unconditional or subsymmetric sequences, since from the theory of large cardinals we know that infinite indiscernible sets exist in large structures. In general, this is a consequence of certain Ramsey principles (i.e. higher-dimensional versions of the pigeonhole principles). Indeed, it was proved by Ketonen [10] that Banach spaces with density bigger than the first ω-Erdős cardinal have subsymmetric sequences. Recall that a cardinal number κ is called ω-Erdős when every countable coloring of the collection of finite subsets of κ has an infinite subset A of κ where the color of a given finite subset F of A depends only on the cardinality of F. Such cardinals are large cardinals, and their existence cannot be proved on the basis of the standard axioms of set theory. It is therefore natural to ask what is the minimal cardinal number $\mathfrak{nc}$ ($\mathfrak{ns}$) such that every Banach space of density at least $\mathfrak{nc}$ (resp. $\mathfrak{ns}$) has an unconditional (respectively, subsymmetric) basic sequence. It is natural to consider also the relative versions of these cardinals restricted to various classes of Banach spaces like, for example, the class of reflexive spaces where we use the notations $\mathfrak{nc}_{\mathrm{refl}}$ and $\mathfrak{ns}_{\mathrm{refl}}$, respectively. Note that it follows from E. Odell's partial unconditionality result [21] that every weakly-null subsymmetric basic sequence is unconditional, hence $\mathfrak{nc}_{\mathrm{refl}} \leq \mathfrak{ns}_{\mathrm{refl}}$. Moreover, an easy application of Odell's result and Rosenthal's ℓ_1-dichotomy gives that $\mathfrak{nc} \leq \mathfrak{ns}$.

Concerning lower bounds for these cardinal numbers, it was proved by S.A. Argyros and A. Tolias [3] that $\mathfrak{nc} > 2^{\aleph_0}$, and by E. Odell [20] that $\mathfrak{ns} > 2^{\aleph_0}$. For the reflexive case, we know that $\mathfrak{nc}_{\mathrm{refl}} > \aleph_1$ ([5]), and in the recent paper [1], S.A. Argyros and P. Motakis proved that $\mathfrak{ns}_{\mathrm{refl}} > 2^{\aleph_0}$. Finally we mention that in [16] the Erdős cardinals are characterized in terms of the existence of compact and large families and the sequential version of $\mathfrak{ns}_{\mathrm{refl}}$. More precisely it is proved that the first Erdős cardinal $\mathfrak{e}_\omega$ is the minimal cardinal κ such that every long weakly-null basis of length κ has a subsymmetric basic sequence, or equivalently the minimal cardinal κ such that there is no compact and large family of finite subsets of κ.

The study of upper bounds is of different nature and seems to involve more advanced set-theoretic considerations connected to large cardinal principles. This can be seen, for example, from the aforementioned result of Ketonen or from results of P. Dodos, J. Lopez-Abad and S. Todorcevic who proved in [7] that $\mathfrak{nc} \leq \aleph_\omega$ holds consistently relative to the existence of certain large cardinals and who proved in [16], based on classical results of A. Hajnal and P. Erdős [8] and R. M. Solovay [25], that Banach's Lebesgue measure extension axiom implies that $\mathfrak{nc}_{\mathrm{refl}} \leq 2^{\aleph_0}$.

In this paper we continue the research on the existence of subsymmetric sequences in a normed space of large density, and we prove that $\mathfrak{ns}_{\mathrm{refl}}$ is rather large, distinguishing thus the cardinals $\mathfrak{ns}_{\mathrm{refl}}$ and $\mathfrak{nc}_{\mathrm{refl}}$. In contrast to the sequential version of $\mathfrak{ns}_{\mathrm{refl}}$, that is closely linked to indiscernibles of *relational* structures $\mathcal{M}_{\mathcal{F}}$ for compact and large families $\mathcal{F}$, the full version of $\mathfrak{ns}_{\mathrm{refl}}$ is more related to the existence of indiscernibles in structures that are not just relational but also have operations, suggesting that not only we need to understand families on finite sets but also "operations" with them. In the separable context, this is well-known and can be observed in the construction of the Tsirelson space, where finite products of the *Schreier family* are used in a crucial way (see [4], [6]). The natural approach in the non-separable setting would be to generalize Tsirelson's construction using analogues of the Schreier family, certain large compact families, on larger index sets. However, in the uncountable level these families cannot be spreading and therefore, if one just copies Tsirelson's construction on the basis of them, the corresponding non-separable Tsirelson-like spaces will always contain almost isometric copies of ℓ_1 ([16, Theorem 8.2]). This leads us to change our perspective and use the well-known interpolation technique [13, Example 3.b.10], an approach that appeared recently in the work of Argyros and Motakis mentioned above. In this perspective, a key tool is a suitable operation $\times$, that we call *multiplication*, of compact families of finite sets. In fact, the multiplication is an operation which associates to a family $\mathcal{F}$ on the fixed index set I and family $\mathcal{H}$ on ω, a family $\mathcal{F} \times \mathcal{H}$ on I which has, in a precise sense, many elements of the form $\bigcup_{n\in x} s_n$, where $x \in \mathcal{H}$ and $(s_n)_{n<\omega}$ is an arbitrary sequence of elements of $\mathcal{F}$. It is well known that such multiplication exists in ω and it models in some way the ordinal multiplication on uniform families. It is also the main tool to define the generalized Schreier families on ω, vastly used in modern Banach space theory (see [4]) to study ranks of compact notions (e.g. summability methods), or of asymptotic notions (e.g. spreading models). These are *uniform* families, so that any restriction of them looks like the entire family. We generalize this property to the uncountable level by defining *homogeneous* families, that despite being uncountable families on large index sets, have countable Cantor–Bendixson rank which moreover does not change substantially when passing to restrictions. In particular, if $\mathcal{F}$ is homogeneous, then the structure $\mathcal{M}_{\mathcal{F}}$ does not have infinite sets of indiscernibles, but we also get lower and upper bounds for the rank of the collection of their (finite) sets of indiscernibles.

We then introduce the notion of a *basis* of families, which is a rich collection of homogeneous families admitting a multiplication, and we prove that they exist on quite large cardinal numbers. The existence of such bases is proved inductively. For example, we prove that if κ has a basis then 2^κ has also a basis. This is done by representing 2^κ as the complete binary tree $2^{\leq\kappa}$, and observing that we can use the *height* function $\mathrm{ht} : 2^{\leq\kappa} \to \kappa{+}1$ to pull back a basis on κ to a restricted version of basis on $2^{\leq k}$, consisting of homogeneous families of finite chains of $2^{\leq\kappa}$. Actually, we prove the following more general equivalence (Theorem 3.1).

Theorem. *For an infinite rooted tree T the following are equivalent.*

(a) *There is a basis of families on T.*

(b) *There is a basis of families consisting of chains of T and there is a basis consisting of antichains of T.*

In particular, we obtain a basis on 2^ω that can be used to build a reflexive space of density 2^ω without subsymmetric basic sequences, giving another proof of the result in [1]. Also, one proves inductively that for every cardinal number κ smaller than the first inaccessible cardinal, there is a basis on κ and a corresponding Banach space of density κ with similar properties. We then use Todorcevic's method of walks on ordinals [26] to build trees on cardinals up to the first Mahlo cardinal number and find examples of reflexive Banach spaces of large densities without subsymmetric basic subsequences. Moreover, as observed above for the structure $\mathcal{M}_{\mathcal{F}}$, we can bound the complexity of the (finite) subsymmetric basic sequences and obtain the following.

Theorem. *Every cardinal κ below the first Mahlo cardinal has a basis. Consequently, for every such cardinal κ and every $\alpha < \omega_1$, there is a reflexive Banach space $\mathfrak{X}$ of density κ with a long unconditional basis and such that every bounded sequence in $\mathfrak{X}$ has an ℓ_1^α-spreading model subsequence but the space $\mathfrak{X}$ does not have ℓ_1^β-spreading model subsequences for β large enough, depending only on α. In particular, $\mathfrak{X}$ contains no infinite subsymmetric basic sequence.*

The paper is organized as follows. In Section 2 we introduce some basic topological, combinatorial and algebraic facts on families of finite chains of a given partial ordering. We then define homogeneous families and bases of them. We present some upper bounds for the topological rank of a family that uses the well-known Ramsey property of barriers on ω, a classical result by C. St. J. A. Nash-Williams ([19]), and that will be used several times along the paper. In Subsection 2.3 we give useful methods to transfer bases between partial orderings. Section 3 is the main part of this paper. The main object we study is, given a tree, the collection $\mathcal{A} \odot_T \mathcal{C}$ of all finite subtrees of T whose chains are in a fixed family $\mathcal{C}$ and such that the family of immediate successors of a given node is in another fixed family $\mathcal{A}$. We study $\mathcal{A} \odot_T \mathcal{C}$ both combinatorially and topologically. The combinatorial part is based on the canonical form of a sequence of finite subtrees, and allow us to define a natural multiplication. The topological one consists in finding upper bounds of the rank of the family $\mathcal{A} \odot_T \mathcal{C}$ in terms of the corresponding ranks of the families $\mathcal{A}$ and $\mathcal{C}$, much in the spirit of how one easily bounds the size of a finite tree from its height and splitting number. This operation allows to lift bases on chains and of immediate successors to bases on the whole tree, our main result of this work done in Theorem 3.1. We apply this in Section 4 to prove that cardinal numbers smaller than the first Mahlo cardinal have a basis. To do this, we represent such cardinals as nodes of a tree having bases on chains and on immediate successors. We achieve this last part by proving several principles of transference of basis. Finally, we use bases to build reflexive Banach spaces without subsymmetric basic sequences.

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2. Basic definitions

Let I be a set. A set $\mathcal{F}$ is called a *family on I* when the elements of $\mathcal{F}$ are finite subsets of I. Let $\mathcal{P} = (P, <)$ be a partial ordering. A *family on chains of $\mathcal{P}$* is a family on P consisting of chains of $\mathcal{P}$, i.e. totally ordered subsets of P. Let $\mathrm{Ch}_<$ be the collection of all *finite* chains of $\mathcal{P}$. Given $k \le \omega$, let

$$[I]^k := \{s \subseteq I \,:\, \#s = k\}, \qquad [P]^k_< := [P]^k \cap \mathrm{Ch}_<,$$
$$[I]^{\le k} := \{s \subseteq I \,:\, \#s \le k\}, \qquad [P]^{\le k}_< := [P]^{\le k} \cap \mathrm{Ch}_<.$$

For a family $\mathcal{F}$ on I and $A \subseteq I$, let $\mathcal{F} \restriction A := \mathcal{F} \cap \mathcal{P}(A)$. Recall that a family $\mathcal{F}$ on I is *hereditary* when it is closed under subsets and it is *compact* when it is a closed subset of $2^I := \{0,1\}^I$, after identifying each set of $\mathcal{F}$ with its characteristic function. In this case, $\mathcal{F}$ is a scattered compact space. Since each element of $\mathcal{F}$ is finite, it is not difficult to see that $\mathcal{F}$ is compact if and only if every sequence $(s_n)_{n\in\omega}$ in $\mathcal{F}$ has a subsequence $(t_n)_{n\in\omega}$ forming a Δ-system with root in $\mathcal{F}$, that is, such that

$$t_{k_0} \cap t_{k_1} = t_{l_0} \cap t_{l_1} \in \mathcal{F} \text{ for every } k_0 \ne k_1 \text{ and } l_0 \ne l_1.$$

The intersection $t_k \cap t_l$, $k \ne l$ is called the *root* of $(t_n)_n$. By weakening the notion of compactness, we say that $\mathcal{F}$ is *pre-compact* if every sequence in $\mathcal{F}$ has a Δ-subsequence (with root not necessarily in $\mathcal{F}$). It is easy to see that $\mathcal{F}$ is pre-compact if and only if its $\subseteq$-closure $\{s \subseteq I \,:\, s \subseteq t \text{ for some } t \in \mathcal{F}\}$ is compact.

Recall the *Cantor–Bendixson derivatives* of a topological space X:

$$X^{(0)} := X, \quad X^{(\alpha)} = \bigcap_{\beta<\alpha} (X^{(\beta)})'$$

where Y' denotes the collection of accumulation points of Y, that is, those points $p \in X$ such that each of its open neighborhoods has infinitely many points in Y. The minimal ordinal α such that $X^{(\alpha+1)} = X^{(\alpha)}$ is called the *Cantor–Bendixson rank* $\mathrm{rk}_{\mathrm{CB}}(X)$ of X. In the case of a compact family $\mathcal{F}$ on an index set I, being scattered, its Cantor–Bendixson index is the first α such that $\mathcal{F}^{(\alpha)} = \emptyset$, and therefore it must be a successor ordinal.

Definition 2.1 *(Rank, small rank and homogeneous families).* Given a compact family $\mathcal{F}$ on some index set I, let

$$\mathrm{rk}(\mathcal{F}) := \mathrm{rk}_{\mathrm{CB}}(\mathcal{F})^-$$

where $(\alpha+1)^- = \alpha$. We say that a compact family $\mathcal{F}$ is countably ranked when $\mathrm{rk}(\mathcal{F})$ is countable. Let $\mathcal{P}$ be a partial ordering. For a given family $\mathcal{F}$ on chains of $\mathcal{P}$, the *small rank relative to* $\mathcal{P}$ of $\mathcal{F}$ is

$$\mathrm{srk}_{\mathcal{P}}(\mathcal{F}) := \inf\{\mathrm{rk}(\mathcal{F} \upharpoonright C) \,:\, C \text{ is an infinite chain of } \mathcal{P}\}.$$

A compact and hereditary family $\mathcal{F}$ on chains of $\mathcal{P}$ is called $(\alpha,\mathcal{P})$*-homogeneous* if $\mathcal{F} = \{\emptyset\}$ if $\alpha = 0$, and if $1 \le \alpha < \omega_1$, then $[P]^{\le 1} \subseteq \mathcal{F}$ and

$$\alpha = \mathrm{srk}_{\mathcal{P}}(\mathcal{F}) \le \mathrm{rk}(\mathcal{F}) < \iota(\alpha),$$

where $\iota(\alpha)$ is defined below in Definition 2.4. $\mathcal{F}$ is $\mathcal{P}$*-homogeneous* if it is $(\alpha,\mathcal{P})$-homogeneous for some $\alpha < \omega_1$.

Observe that these notions coincide for any two total orders on the fixed index set, as being a chain with respect to a total order does not depend on the total order itself. Hence, when $\mathcal{F}$ is a family on some index set I, we use srk, α-homogeneous and homogeneous for the corresponding notions with respect to any total order on I. If $\mathcal{F}$ is a compact family on a countable index set I, then $\mathrm{rk}(\mathcal{F})$ is countable and, therefore, the small rank of $\mathcal{F}$ with respect to any partial order on I is countable. In general, $\#\mathrm{rk}(\mathcal{F}) \le \#I$ and the extreme case can be achieved.

Definition 2.2. The *normal Cantor form* of an ordinal α is the unique expression $\alpha = \omega^{\alpha[0]} \cdot n_0[\alpha] + \cdots + \omega^{\alpha[k]} \cdot n_k[\alpha]$ where $\alpha \ge \alpha[0] > \alpha[1] > \cdots > \alpha[k] \ge 0$ and $n_i[\alpha] < \omega$ for every $i \le k$.

Suppose that $*$ is an operation on countable ordinals and suppose that $\alpha > 0$ is a countable ordinal. We say that α is $*$*-indecomposable* when $\beta * \gamma < \alpha$ for every $\beta, \gamma < \alpha$.

Remark 2.3. It is well-known that

(i) α is sum-indecomposable if and only if $\alpha = \omega^\beta$.
(ii) $\alpha > 1$ is product-indecomposable if and only if $\alpha = \omega^\beta$ for some sum-indecomposable β.
(iii) For $\alpha > \omega$, α is exponential-indecomposable if and only if $\alpha = \omega^\alpha$.
(iv) Product-indecomposability imply sum-indecomposability, and exponential-indecomposability imply product and sum indecomposability.

So, $1, \omega, \omega^2$ and $1, \omega, \omega^\omega$ are the first 3 sum-indecomposable, and product-indecomposable ordinals, respectively. If we define, given $\alpha < \omega_1$, $\overline{\alpha}_0 := \alpha$, $\overline{\alpha}_{n+1} := \omega^{\overline{\alpha}_n}$ and $\overline{\alpha}_\omega := \sup_n \overline{\alpha}_n$, then $\omega, \overline{\omega}_\omega, \overline{(\overline{\omega}_\omega)}_\omega$ are the first 3 exponential-indecomposable ordinals. We will use *exp-indecomposable* to refer to exponential-indecomposable ordinals.

Definition 2.4. Given a countable ordinal α, let

$$\iota(\alpha) = \min\{\lambda > \alpha \,:\, \lambda \text{ is exp-indecomposable}\}.$$

Let $\mathrm{Fn}(\omega_1, \omega)$ be the collection of all functions $f : \omega_1 \to \omega$ such that $\operatorname{supp} f := \{\gamma < \omega_1 : f(\gamma) \neq 0\}$ is finite. Endowed with the pointwise sum $+$, $(\mathrm{Fn}(\omega_1, \omega), +)$ is an ordered commutative monoid. Let $\nu : \omega_1 \to \mathrm{Fn}(\omega_1, \omega)$ be defined by $\nu(\alpha)(\gamma) = n_i[\alpha]$ if and only if $\gamma = \alpha[i]$. Let $\sigma : \mathrm{Fn}(\omega_1, \omega) \to \omega_1$ be defined by $\sigma(f) = \sum_{i \leq n} \omega^{\alpha_i} \cdot f(\alpha_i)$, where $\{\alpha_0 > \cdots > \alpha_n \geq 0\} = \operatorname{supp} f$. In other words, σ is the inverse of ν. Given $\alpha, \beta < \omega_1$, the *Hessenberg sum* (see e.g. [24]) is defined by

$$\alpha \dot{+} \beta := \sigma(\nu(\alpha) + \nu(\beta)).$$

It is easy to see that if α is exp-indecomposable, then it is $\dot{+}$-indecomposable.

Definition 2.5. Let $\mathcal{F}$ and $\mathcal{G}$ be families on chains of a partial ordering $\mathcal{P}$. Define

$$\begin{aligned}
\mathcal{F} \cup \mathcal{G} &:= \{s \subseteq P \,:\, s \in \mathcal{F} \text{ or } s \in \mathcal{G}\}, \\
\mathcal{F} \sqcup_{\mathcal{P}} \mathcal{G} &:= \{s \cup t \,:\, s \cup t \text{ is a chain and } s \in \mathcal{F},\, t \in \mathcal{G}\}, \\
\mathcal{F} \sqcup \mathcal{G} &:= \{s \cup t \,:\, s \in \mathcal{F},\, t \in \mathcal{G}\}, \\
\mathcal{F} \boxtimes_{\mathcal{P}} (n+1) &:= (\mathcal{F} \boxtimes_{\mathcal{P}} n) \sqcup_{\mathcal{P}} \mathcal{F}; \quad \mathcal{F} \boxtimes_{\mathcal{P}} 1 := \mathcal{F}, \\
\mathcal{F} \boxtimes (n+1) &:= (\mathcal{F} \boxtimes n) \sqcup \mathcal{F}; \quad \mathcal{F} \boxtimes 1 := \mathcal{F}.
\end{aligned}$$

Observe that when $\mathcal{P}$ is a total ordering the operations $\sqcup_{\mathcal{P}}$ and $\sqcup$ are the same.

Proposition 2.6. *The operations $\cup$, $\sqcup_{\mathcal{P}}$ and $\sqcup$ preserve pre-compactness and hereditariness. Moreover, if $\mathcal{F}$ and $\mathcal{G}$ are countably ranked families on chains of $\mathcal{P}$, then*

(i) $\mathrm{rk}(\mathcal{F} \cup \mathcal{G}) = \max\{\mathrm{rk}(\mathcal{F}), \mathrm{rk}(\mathcal{G})\}$,
(ii) $\mathrm{rk}(\mathcal{F} \sqcup \mathcal{G}) = \mathrm{rk}(\mathcal{F}) \dot{+} \mathrm{rk}(\mathcal{G})$,
(iii) $\mathrm{rk}(\mathcal{F} \sqcup_{\mathcal{P}} \mathcal{G}) \leq \mathrm{rk}(\mathcal{F}) \dot{+} \mathrm{rk}(\mathcal{G})$.

Consequently, if $\mathcal{F}$ and $\mathcal{G}$ are $\mathcal{P}$-homogeneous, then $\mathcal{F} \cup \mathcal{G}$, $\mathcal{F} \sqcup_{\mathcal{P}} \mathcal{G}$ and $\mathcal{F} \sqcup \mathcal{G}$ are $(\gamma, \mathcal{P})$-homogeneous with $\gamma \geq \max\{\mathrm{srk}_{\mathcal{P}}(\mathcal{F}), \mathrm{srk}_{\mathcal{P}}(\mathcal{G})\}$.

Proof. It is easy to see that if $\mathcal{F}$ and $\mathcal{G}$ are pre-compact, hereditary, then $\mathcal{F} * \mathcal{G}$ is pre-compact, hereditary, for $* \in \{\cup, \sqcup_{\mathcal{P}}, \sqcup\}$. Let us see (i): An easy inductive argument shows that $(\mathcal{F} \cup \mathcal{G})^{(\alpha)} = \mathcal{F}^{(\alpha)} \cup \mathcal{G}^{(\alpha)}$ for every countable α. To prove (ii) and (iii), notice that by a general fact, for every compact spaces K and L and every α one has that

$$(K \times L)^{(\alpha)} = \bigcup_{\beta \dot{+} \gamma = \alpha} (K^{(\beta)} \times L^{(\gamma)}). \tag{1}$$

When $\mathrm{rk}(K)$ and $\mathrm{rk}(L)$ are countable, we have that $\mathrm{rk}(K \times L) = \mathrm{rk}(K)\dot{+}\mathrm{rk}(L)$. The proof of (1) is done by induction on α and by considering the case when α is sum-indecomposable or not. Now let $\mathcal{F}$ and $\mathcal{G}$ be countably ranked. Suppose that $\mathcal{P}$ is a total ordering, and let $\mathcal{F}\times\mathcal{G} \to \mathcal{F}\sqcup\mathcal{G}$, $(s,t) \mapsto s\cup t$. This is clearly continuous, onto and finite-to-one, so $\mathrm{rk}(\mathcal{F}\sqcup\mathcal{G}) = \mathrm{rk}(\mathcal{F}\times\mathcal{G}) = \mathrm{rk}(\mathcal{F})\dot{+}\mathrm{rk}(\mathcal{G})$, which concludes the proof of (ii). If $\mathcal{P}$ is in general a partial ordering, then it follows from this that $\mathrm{rk}(\mathcal{F}\sqcup_{\mathcal{P}}\mathcal{G}) \le \mathrm{rk}(\mathcal{F})\dot{+}\mathrm{rk}(\mathcal{G})$, proving (iii). Now suppose that $\mathcal{F}$ and $\mathcal{G}$ are $\mathcal{P}$-homogeneous. We have clearly that

$$\max\{\mathrm{srk}_{\mathcal{P}}(\mathcal{F}), \mathrm{srk}_{\mathcal{P}}(\mathcal{G})\} \le \min\{\mathrm{srk}_{\mathcal{P}}(\mathcal{F}\cup\mathcal{G}), \mathrm{srk}_{\mathcal{P}}(\mathcal{F}\sqcup_{\mathcal{P}}\mathcal{G}), \mathrm{srk}_{\mathcal{P}}(\mathcal{F}\sqcup\mathcal{G})\}. \qquad (2)$$

On the other hand, $\max\{\mathrm{rk}(\mathcal{F}\cup\mathcal{G}), \mathrm{rk}(\mathcal{F}\sqcup_{\mathcal{P}}\mathcal{G}), \mathrm{rk}(\mathcal{F}\sqcup\mathcal{G})\} \le \mathrm{rk}(\mathcal{F})\dot{+}\mathrm{rk}(\mathcal{G})$. Since $\mathrm{rk}(\mathcal{F}), \mathrm{rk}(\mathcal{G}) < \lambda := \max\{\iota(\mathrm{srk}_{\mathcal{P}}(\mathcal{F})), \iota(\mathrm{srk}_{\mathcal{P}}(\mathcal{G}))$, it follows by the indecomposability of λ that $\mathrm{rk}(\mathcal{F})\dot{+}\mathrm{rk}(\mathcal{G}) < \lambda$. This, together with (2) gives the desired result. □

2.1. Bases of homogeneous families

We recall a well-known generalization of Schreier families on ω, called uniform families. We are going to use them mainly as a tool to compute upper bounds of ranks of operations of compact families. We use the following standard notation: given $M, s, t \subseteq \omega$, we write $s < t$ to denote that $\max s < \min t$ and let $M/s := \{m \in M \,:\, s < m\}$. Notice that a family $\mathcal{F}$ on ω is pre-compact if and only if every sequence in $\mathcal{F}$ has a *block* Δ-subsequence $(s_n)_{n\in\omega}$, that is, such that $s < s_m \setminus s < s_n \setminus s$ for every $m < n$, where s is the root of $(s_n)_n$. We write $s \sqsubseteq t$ to denote that s is an initial part of t, that is, $s \subseteq t$ and $t \cap (\max s + 1) = s$, and $s \sqsubset t$ to denote that $s \sqsubseteq t$ and $s \neq t$.

Definition 2.7. Given a family $\mathcal{F}$ on ω and $n < \omega$, let

$$\mathcal{F}_{\{n\}} := \{s \subseteq \omega \,:\, n < s \text{ and } \{n\} \cup s \in \mathcal{F}\}.$$

Let α be a countable ordinal number, and let $\mathcal{F}$ be a family on an infinite subset $M \subseteq \omega$. The family $\mathcal{F}$ is called an *α-uniform family on M* when $\emptyset \in \mathcal{F}$ and

(a) $\mathcal{F} = \{\emptyset\}$ if $\alpha = 0$;
(b) $\mathcal{F}_{\{n\}}$ is β-uniform on M/n for every $n \in M$, if $\alpha = \beta + 1$;
(c) $\mathcal{F}_{\{n\}}$ is α_n-uniform on M/n for every $n \in M$ and $(\alpha_n)_{n\in M}$ is an increasing sequence such that $\sup_{n\in M} \alpha_n = \alpha$, if α is limit.

It is important to remark that uniform families are not uniform fronts, which were introduced by P. Pudlak and V. Rödl in [22] following previous works of C. Nash-Williams. Recall that a family $\mathcal{B}$ on M is called an α-uniform front on M when $\mathcal{B} = \{\emptyset\}$ if $\alpha = 0$, and if $\alpha > 0$ then $\emptyset \notin \mathcal{B}$, and $\mathcal{B}_{\{n\}}$ is a γ-uniform front on M/n for every $n \in M$, if $\beta = \gamma + 1$, and $\mathcal{B}_{\{n\}}$ is a α_n-uniform front on M/n for every $n \in M$ and $(\alpha_n)_{n\in M}$

is increasing with $\sup_{n\in M}\alpha_n$. In fact, given a uniform family $\mathcal{F}$, the collection of its $\subseteq$-*maximal* elements $\mathcal{F}^{\max}$ is a uniform front, and we can recover a uniform family out of a uniform front $\mathcal{B}$ by taking its closure under initial parts $\mathcal{B}^{\sqsubseteq}$, that is, the collection of initial parts of elements of $\mathcal{B}$:

Proposition 2.8.

(a) *Every uniform family is compact.*
(b) *The following are equivalent:*
  (b.1) *$\mathcal{F}$ is an α-uniform family on M.*
  (b.2) *$\mathcal{F}^{\max}$ is an α-uniform front on M such that $\mathcal{F}=\overline{\mathcal{F}^{\max}}=(\mathcal{F}^{\max})^{\sqsubseteq}$.*

Proof. (a) is proved by a simple inductive argument on α. To prove that (b.1) implies (b.2), one observes first that $(\mathcal{F}^{\max})^{\sqsubseteq}=\mathcal{F}$, because $\mathcal{F}$ is compact, and then again use an inductive argument. The proof of that (b.2) implies (b.1) one uses the well-known fact that if $\mathcal{B}$ is a uniform front, then $\overline{\mathcal{B}}=\mathcal{B}^{\sqsubseteq}$ (see for example [2]). □

Definition 2.9. Given two families $\mathcal{F}$ and $\mathcal{G}$ on ω their sum and product are defined by

$$\mathcal{F}\oplus\mathcal{G}:=\{s\cup t\,:\,s<t,\,s\in\mathcal{G}\text{ and }t\in\mathcal{F}\},$$
$$\mathcal{F}\otimes\mathcal{G}:=\{\bigcup_{i<n}s_i\,:\,\{s_i\}_i\subseteq\mathcal{F},\,\max s_i<\min s_{i+1},\,i<n,\text{ and }\{\min s_i\}_i\in\mathcal{G}\}.$$

The following are well-known facts of uniform fronts, and that are extended to uniform families by using the previous proposition. For more information on uniform fronts, we refer to [14], [15] and [2].

Proposition 2.10.

(a) *The rank of an α-uniform family is α.*
(b) *The unique n-uniform family on M, $n<\omega$, is $[M]^{\le n}$.*
(c) *If $\mathcal{F}$ is uniform and t is a finite set, the $\mathcal{F}_t:=\{s\subseteq\omega\,:\,t<s\text{ and }t\cup s\in\mathcal{F}\}$ is uniform on ω/t.*
(d) *If $\mathcal{F}$ is an α-uniform family on M, then $\mathcal{F}\restriction N$ is an α-uniform family on N for every $N\subseteq M$ infinite. Consequently, if $\mathcal{F}$ is an α-uniform family on ω, then $\mathcal{F}$ is α-homogeneous with* $\mathrm{srk}(\mathcal{F})=\mathrm{rk}(\mathcal{F})=\alpha$.
(e) *If $\mathcal{F}$ is an α-uniform family on M, and $\theta:M\to N$ is an order-preserving bijection, then $\{\theta''(s)\,:\,s\in\mathcal{F}\}$ is an α-uniform family on N.*
(f) *Suppose that $\mathcal{F}$ and $\mathcal{G}$ are α and β uniform families on M, respectively. Then $\mathcal{F}\cup\mathcal{G}$, $\mathcal{F}\oplus\mathcal{G}$, $\mathcal{F}\sqcup\mathcal{G}$ and $\mathcal{F}\otimes\mathcal{G}$ are* $\max\{\alpha,\beta\}$, $\alpha+\beta$, $\alpha\dot{+}\beta$ *and $\alpha\cdot\beta$-uniform families on M, respectively.*
(g) *Uniform fronts have the* Ramsey property*: if $c:\mathcal{F}\to n$ is a coloring of a uniform front on M, then there is $N\subseteq M$ infinite such that c is constant on $\mathcal{F}\restriction N$.*

(h) *Suppose that $\mathcal{F}$ and $\mathcal{G}$ are uniform families on M. Then there is some $N \subseteq M$ such that either $\mathcal{F} \upharpoonright N \subseteq \mathcal{G}$ or $\mathcal{G} \upharpoonright N \subseteq \mathcal{F}$. Moreover, when $\mathrm{rk}(\mathcal{F}) < \mathrm{rk}(\mathcal{G})$, the first alternative must hold and in addition $(\mathcal{F} \upharpoonright N)^{\max} \cap (\mathcal{G} \upharpoonright N)^{\max} = \emptyset$.*
(i) *If $\mathcal{F}$ is a uniform family on M, then there is $N \subseteq M$ infinite such that $\mathcal{F} \upharpoonright N$ is hereditary.*
(j) *If $\mathcal{F}$ is compact and $\sqsubseteq$-hereditary family on ω, then there is $M \subseteq \omega$ infinite such that $\mathcal{F} \upharpoonright M$ is a uniform family on M.* $\square$

Remark 2.11.

(i) The only new observation in the previous proposition is the fact in (f) that states that unions and square unions of uniform families is a uniform family, and that can be easily proved by induction on the maximum of the ranks using, for example, that $(\mathcal{F} \otimes \mathcal{G})_{\{n\}} = (\mathcal{F} \otimes \mathcal{G}_{\{n\}}) \oplus \mathcal{F}_{\{n\}}$.
(ii) A simple inductive argument shows that for every countable α and every infinite $M \subseteq \omega$ there is an α-uniform family $\mathcal{F}$ on M, and, although uniform families are not necessarily hereditary using (d) and (i) one can build them being hereditary.

We obtain the following consequence for families on an arbitrary partial ordering.

Corollary 2.12. *Suppose that $\mathcal{P} = (P, <_P)$ is a partial ordering, and suppose that $\mathcal{F}$ and $\mathcal{G}$ are compact and hereditary families with $\mathrm{rk}(\mathcal{F}) < \mathrm{srk}(\mathcal{G})$. Then every infinite chain C of $\mathcal{P}$ has an infinite subchain $D \subseteq C$ such that $\mathcal{F} \upharpoonright D \subseteq \mathcal{G}$.*

Proof. Fix $\mathcal{F}$, $\mathcal{G}$ and C as in the statement. By going to a subchain of C, we may assume that C is countable. Since $(C, <_P)$, so we can fix a bijection $\theta : C \to \omega$, and we set $\mathcal{F}_0 := \{\theta''(s) : s \in \mathcal{F} \upharpoonright C\}$ and $\mathcal{G}_0 := \{\theta''(s) : s \in \mathcal{G} \upharpoonright C\}$. Observe that $\mathcal{G}_0 \upharpoonright M$ has rank at least $\mathrm{srk}(\mathcal{G}) > \mathrm{rk}(\mathcal{F}_0)$ for every infinite $M \subseteq \omega$. Then by Proposition 2.10 (j), (h) there is some infinite $M \subseteq \omega$ such that $\mathcal{F}_0 \upharpoonright M \subseteq \mathcal{G}_0$, so $\mathcal{F} \upharpoonright \theta^{-1}(M) \subseteq \mathcal{G}$. $\square$

Among uniform families, the *generalized Schreier families* have been widely studied and used particularly in Banach space theory. They have an algebraic definition and have extra properties, as for example being spreading. Also, they have a sum-indecomposable rank. We recall the definition now.

Definition 2.13. The *Schreier family* is

$$\mathcal{S} := \{s \subseteq \omega : \#s \leq \min s\}.$$

A *Schreier sequence* is defined inductively for $\alpha < \omega_1$ by

(a) $\mathcal{S}_0 := [\omega]^{\leq 1}$,
(b) $\mathcal{S}_{\alpha+1} := \mathcal{S}_\alpha \otimes \mathcal{S}$, and

(c) $\mathcal{S}_\alpha := \bigcup_{n<\omega}(\mathcal{S}_{\alpha_n} \restriction \omega \setminus n)$ where $(\alpha_n)_n$ is such that $\sup_n \alpha_n = \alpha$, if α is limit.

Note that the family $\mathcal{S}_\alpha$ depends on the choice of the sequences converging to limit ordinals.

Definition 2.14 *(Spreading families).* A family $\mathcal{F}$ on ω is *spreading* when for every $s = \{m_0 < \cdots < m_k\} \in \mathcal{F}$ and $t = \{n_0 < \cdots < n_k\}$ with $m_i \leq n_i$ for every $i \leq k$ one has that $t \in \mathcal{F}$.

Proposition 2.15.

(a) *Suppose that $\mathcal{F}$ and $\mathcal{G}$ are spreading families on ω. Then $\mathcal{F} \cup \mathcal{G}$, $\mathcal{F} \oplus \mathcal{G}$ and $\mathcal{F} \otimes \mathcal{G}$ are spreading. If in addition $\mathcal{F}$ or $\mathcal{G}$ is hereditary, then $\mathcal{F} \sqcup \mathcal{G}$ is also spreading.*
(b) *If $\mathcal{F}$ is compact, hereditary and spreading, then $\mathcal{F}$ is* $\mathrm{rk}(\mathcal{F})$*-homogeneous.*
(c) *Suppose that $\mathcal{F}$ is a compact, hereditary and spreading family on ω of finite rank m. Then there is some $n < \omega$ such that $[\omega \setminus n]^{\leq m} \subseteq \mathcal{F} \subseteq [\omega]^{\leq m}$.*

Proof. (a) is easy to verify. For (b): Given an infinite set $M \subseteq \omega$, let $\theta : \omega \to M$ be the unique increasing enumeration of M. Then $s \mapsto \theta'' s$ is a 1–1 and continuous mapping from $\mathcal{F}$ into $\mathcal{F} \restriction M$, so $\mathrm{rk}(\mathcal{F} \restriction M) \geq \mathrm{rk}(\mathcal{F}) \geq \mathrm{rk}(\mathcal{F} \restriction M)$. (c): If there is some $s \in \mathcal{F}$ of cardinality at least $m+1$, then $[\omega \setminus \max s]^{\leq m+1} \subseteq \mathcal{F}$ because $\mathcal{F}$ is hereditary and spreading. And this is impossible, as it implies that the rank of $\mathcal{F}$ is at least $m+1$. On the other hand, since the rank of $\mathcal{F}$ is m, there must be some element of it of cardinality m. Pick an element s of $\mathcal{F}$ of such cardinality. Then $[\omega \setminus \max s]^{\leq m} \subseteq \mathcal{F}$. $\square$

The generalized Schreier families are uniform families and they have extra properties.

Proposition 2.16.

(a) *$\mathcal{S}_\alpha$ is hereditary, spreading and ω^α-uniform.*
(b) *For every $\alpha \leq \beta$ there is $n < \omega$ such that $\mathcal{S}_\alpha \restriction (\omega \setminus n) \subseteq \mathcal{S}_\beta$.*

Proof. (a) The first two properties are well-known. The proof of that $\mathcal{S}_\alpha$ is a ω^α-uniform family is done by induction on α. The case $\alpha = 0$ is trivial, while it is easy to verify that $\mathcal{S}$ is an ω-uniform family, so $\mathcal{S}_{\alpha+1} = \mathcal{S}_\alpha \otimes \mathcal{S}$ is a $\omega^\alpha \cdot \omega = \omega^{\alpha+1}$-uniform family by Proposition 2.10 and inductive hypothesis. Suppose that α is limit. For a given $m, n \in \omega$, let $\alpha_m^n < \omega^{\alpha_n}$ be such that $(\alpha_m^n)_m$ is increasing, $\sup_m \alpha_m^n = \alpha_n$ and $(\mathcal{S}_{\alpha_n})_{\{m\}}$ is a α_m^n uniform family. Since for every $m \in \omega$ we have that

$$(\mathcal{S}_\alpha)_{\{m\}} = \bigcup_{n \leq m} (\mathcal{S}_{\alpha_n})_{\{m\}} \restriction (\omega \setminus m)$$

it follows from Proposition 2.10 (e) that $(\mathcal{S}_\alpha)_{\{m\}}$ is a $\beta_m := \max_{n\leq m} \alpha_m^n$-uniform family on ω/m. It is easy to see that $(\beta_m)_m$ is increasing and satisfies that $\sup_m \beta_m = \omega^\alpha$. (b) is proved by a simple inductive argument. □

Definition 2.17. Let $\mathfrak{S}$ be the collection of all hereditary, spreading uniform families on ω.

Proposition 2.18. *For every $\alpha < \omega_1$ there is a hereditary, spreading α-uniform family on ω.*

Proof. Let $(\mathcal{S}_\alpha)_{\alpha<\omega}$ be a Schreier sequence, and given a countable ordinal α with normal Cantor form $\alpha = \sum_{i\leq k} \omega^{\alpha_i} \cdot n_i$ we define

$$\mathcal{F}_\alpha := (\mathcal{S}_{\alpha_0} \otimes [\omega]^{\leq n_0}) \oplus \cdots \oplus (\mathcal{S}_{\alpha_k} \otimes [\omega]^{\leq n_k}).$$

Then each $\mathcal{F}_\alpha$ is a hereditary, spreading α-uniform family on ω. □

We present now the concept of basis, which is a collection of families that intends to generalize the collection of uniform family on ω, and the multiplication $\otimes$ between the families in the basis. It seems that there is no canonical definition for the multiplication $\mathcal{F} \times \mathcal{G}$ of two families on an index set I. However, when $\mathcal{G}$ is a family on ω we can define it quite naturally as follows.

Definition 2.19. Let $\mathcal{F}$ be a homogeneous family on chains of a partial ordering $\mathcal{P}$, and let $\mathcal{H}$ be a homogeneous family on ω. We say that a family $\mathcal{G}$ on chains of $\mathcal{P}$ is a *multiplication of* $\mathcal{F}$ by $\mathcal{H}$ when

(M.1) $\mathcal{G}$ is homogeneous and $\iota(\mathrm{srk}_{\mathcal{P}}(\mathcal{G})) = \iota(\mathrm{srk}_{\mathcal{P}}(\mathcal{F}) \cdot \mathrm{srk}(\mathcal{H}))$.
(M.2) Every sequence $(s_n)_{n<\omega}$ in $\mathcal{F}$ such that $\bigcup_n s_n$ is a chain of $\mathcal{P}$ has an infinite subsequence $(t_n)_n$ such that for every $x \in \mathcal{H}$ one has that $\bigcup_{n\in x} t_n \in \mathcal{G}$.

Example 2.20.

(i) $\mathcal{F} \boxtimes_{\mathcal{P}} n$ is a multiplication of any homogeneous family $\mathcal{F}$ by $[\omega]^{\leq n}$. In general, given a family $\mathcal{H} \in \mathfrak{S}$ of finite rank n, $\mathcal{F} \boxtimes_{\mathcal{P}} n$ is a multiplication of $\mathcal{F}$ by $\mathcal{H}$ (see Proposition 2.15 (b)).
(ii) If $\mathcal{P}$ does not have any infinite chain, then any homogeneous family $\mathcal{F}$ on chains of $\mathcal{P}$ has finite rank and given any homogeneous family $\mathcal{H}$ on ω, $\mathcal{G} = \mathcal{F}$ satisfies (M.2).

Notice that always $\mathcal{F} \subseteq \mathcal{G}$ for every multiplication $\mathcal{G}$ of $\mathcal{F}$ by any family $\mathcal{H} \neq \{\emptyset\}$, and that when $\mathcal{F}, \mathcal{H} \neq \{\emptyset\}$, then (M.1) is equivalent to $\iota(\mathrm{srk}_{\mathcal{P}}(\mathcal{G})) = \max\{\iota(\mathrm{srk}_{\mathcal{P}}(\mathcal{F})), \iota(\mathrm{srk}(\mathcal{H}))\}$, because $\iota(\alpha \cdot \beta) = \max\{\iota(\alpha), \iota(\beta)\}$ if $\alpha, \beta \geq 1$. When the family $\mathcal{F} = [\kappa]^{\leq 1}$ and $\mathcal{H}$ is the Schreier family $\mathcal{S}$, then the existence of a family $\mathcal{G}$ satisfying (M.2) is equivalent to κ not being ω-Erdős (see [16], and the remarks after

Theorem 2.23). Let us use the following notation. Given a collection $\mathfrak{C}$ of families on chains of $\mathcal{P}$ and $\alpha < \omega_1$ let $\mathfrak{C}_\alpha := \{\mathcal{F} \in \mathfrak{C} : \mathrm{srk}_{\mathcal{P}}(\mathcal{F}) = \alpha\}$.

Definition 2.21 *(Basis).* Let $\mathcal{P} = (P, <)$ be a partial ordering with an infinite chain. A *basis (of homogeneous families) on chains of* $\mathcal{P}$ is a pair $(\mathfrak{B}, \times)$ such that:

(B.1) $\mathfrak{B}$ consists of homogeneous families on chains of $\mathcal{P}$, it contains all cubes $[P]^{\leq n}_{\mathcal{P}}$, and $\mathfrak{B}_\alpha \neq \emptyset$ for all $\omega \leq \alpha < \omega_1$.

(B.2) $\mathfrak{B}$ is closed under $\cup$ and $\sqcup_{\mathcal{P}}$, and if $\mathcal{F} \subseteq \mathcal{G} \in \mathfrak{B}$ is homogeneous on chains of $\mathcal{P}$ and $\iota(\mathrm{srk}_{\mathcal{P}}(\mathcal{F})) = \iota(\mathrm{srk}_{\mathcal{P}}(\mathcal{G}))$, then $\mathcal{F} \in \mathfrak{B}$.

(B.3) $\times : \mathfrak{B} \times \mathfrak{S} \to \mathfrak{B}$ is such that for every $\mathcal{F} \in \mathfrak{B}$ and every $\mathcal{H} \in \mathfrak{S}$ one has that $\mathcal{F} \times \mathcal{H}$ is a multiplication of $\mathcal{F}$ by $\mathcal{H}$.

When $\mathcal{P} = (P, <)$ is a total ordering, we simply say that $\mathfrak{B}$ is a *basis of families on* P.

Multiplications $\mathcal{F} \times \mathcal{H}$ for families $\mathcal{H}$ of finite rank always exist, so (B.3) is equivalent to the existence of multiplication by families $\mathcal{H} \in \mathfrak{S}$ of infinite rank. Also, $\{0\} \times \mathcal{H} = \{\emptyset\}$ is always a multiplication. The condition on (B.2) concerning subfamilies is a technical requirement, but is not essential (see Proposition 2.24).

Proposition 2.22. *There is basis of families on* ω.

Proof. Let $\mathfrak{B}$ be the collection of all homogeneous families $\mathcal{F}$ such that there is some uniform family $\mathcal{G}$ with $\mathcal{F} \subseteq \mathcal{G}$ and $\iota(\mathrm{srk}(\mathcal{F})) = \iota(\mathrm{srk}(\mathcal{G}))$. Given $\{\emptyset\} \subsetneq \mathcal{F} \in \mathfrak{B}$ and $\mathcal{H} \in \mathfrak{S}$, choose some uniform family $\mathcal{G}$ with $\mathcal{F} \subseteq \mathcal{G}$ and $\iota(\mathrm{srk}(\mathcal{F})) = \iota(\mathrm{srk}(\mathcal{G}))$, and define

$$\mathcal{F} \times_\omega \mathcal{H} := (\mathcal{G} \otimes \mathcal{H}) \oplus \mathcal{G}.$$

(B.1): $\mathfrak{B}_\alpha \neq \emptyset$ follows from Proposition 2.18. (B.2): $\mathfrak{B}$ is closed under $\cup$ and $\sqcup$ by Proposition 2.6 and Proposition 2.10. (B.3): Fix $\{\emptyset\} \subsetneq \mathcal{F} \in \mathfrak{B}$, and $\mathcal{H} \in \mathfrak{S}$. Choose an homogeneous family $\mathcal{G}$ defining $\mathcal{F} \times_\omega \mathcal{H} = (\mathcal{G} \otimes \mathcal{H}) \oplus \mathcal{G}$. We verify first (M.1): Let $\alpha, \beta < \omega_1$ be such that $\mathcal{G}$ and $\mathcal{H}$ are α-uniform and β-uniform, respectively. Observe that $\iota(\mathrm{skr}(\mathcal{F})) = \iota(\mathrm{srk}(\mathcal{G})) = \iota(\alpha)$ and $\mathrm{srk}(\mathcal{H}) = \beta$. Then $\mathcal{F} \times_\omega H$ is $\alpha \cdot (\beta + 1)$-uniform, $\mathrm{srk}(\mathcal{F} \times_\omega \mathcal{H}) = \alpha(\beta + 1)$, so $\iota(\mathrm{srk}(\mathcal{F} \times_\omega \mathcal{H})) = \max\{\iota(\alpha), \iota(\beta)\} = \max\{\iota(\mathrm{srk}(\mathcal{F})), \iota(\mathrm{srk}(\mathcal{H}))\} = \iota(\mathrm{srk}(\mathcal{F}) \cdot \mathrm{srk}(\mathcal{H}))$. This finishes the proof of (M.1). We check now (M.2): Suppose that $(s_k)_k$ is a sequence in $\mathcal{F}$. Let $(t_k)_{k<\omega}$ be a Δ-subsequence with root $t \in \mathcal{F}$ such that $t < t_k \setminus t < t_{k+1} \setminus t$ for every k. Suppose that $x \in \mathcal{H}$. Then $\{\min t_k \setminus t\}_{k \in x} \in \mathcal{H}$, because $\mathcal{H}$ is spreading. Hence, $\bigcup_{k \in x}(t_k \setminus t) \in \mathcal{G} \otimes \mathcal{H}$. Since $t < \bigcup_{k \in x}(t_k \setminus t)$, it follows that $\bigcup_{k \in x} t_k = t \cup \bigcup_{k \in x}(t_k \setminus t) \in (\mathcal{G} \otimes \mathcal{H}) \oplus \mathcal{G}$. $\square$

Our main result is the following:

Theorem 2.23. *Every cardinal θ strictly smaller than the first Mahlo cardinal has a basis of families on θ.*

The proof, in Section 4, is done inductively on κ and using trees with small height and levels (to be precised later) in order to step up. For example, when κ is not strong limit, there must be $\lambda < \kappa$ such that $2^\lambda \geq \kappa$, so, by the inductive hypothesis, there must be a basis on λ, and there is a rather natural way to lift it up to a basis on the nodes of the complete binary tree, via the height mapping. The case when κ is not regular is similar. When on the contrary κ is inaccessible, the tree T of cardinality κ is substantially more complicated, and in fact relies on the method of walks in ordinals. In any of these cases, the main difficulty is to pass from a basis on chains and a basis "on the antichains" of a tree T to a basis of families on the nodes of T with respect to any total ordering on them, thus getting a basis on $|T|$ (see Theorem 3.1).

Recall that κ is ω-Erdős when for every coloring $c : [\kappa]^{<\omega} \to 2$ there is an infinite c-homogeneous subset $A \subseteq \kappa$, that is, for $s, t \in A$, $c(s) = c(t)$ when $\#s = \#t$. A compact and hereditary family on κ is called *large* when $\mathrm{srk}(\mathcal{F}) \geq \omega$, or, equivalently, when $\mathcal{F}$ satisfies (M.2) for $[\kappa]^{\leq 1}$ and the Schreier family $\mathcal{S}$. It is proved in [16] that the existence of such families in κ is equivalent to κ not being ω-Erdős.

Problem 1. Characterize when κ has $\geq \omega$-homogeneous families.

Problem 2. Characterize the cardinal numbers κ such that there exists $c : [\kappa]^{<\omega} \to 2$ without infinite c-homogeneous sets, but such that there is a $\geq \omega$-homogeneous family $\mathcal{F}$ on κ such that for every sequence $(s_n)_{n<\omega}$ in $\mathcal{F}$ and every $l < \omega$ there are $n_1 < \cdots < n_l$ such that $\bigcup_{i=1}^{l} s_{n_i}$ is c-homogeneous.

The first such κ not satisfying this coloring property is at least the first Mahlo cardinal and smaller than the first ω-Erdős cardinal.

Before going into the particular case of the partial ordering being a tree, we present some operations of partial orderings and results guaranteeing that we can transfer families or bases from some partial orderings to more complex ones (Subsection 2.3). The following characterization of the existence of a basis will be useful.

Proposition 2.24. *A partial ordering $\mathcal{P}$ with an infinite chain has a basis if and only if there is a pair $(\mathfrak{B}, \times)$, called* pseudo-basis*, with the following properties:*

(B.1′) *$\mathfrak{B}$ consists of homogeneous families on chains of $\mathcal{P}$, it contains all cubes, and for every $\omega \leq \alpha < \omega_1$ there is $\mathcal{F} \in \mathfrak{B}$ such that $\alpha \leq \mathrm{srk}_{\mathcal{P}}(\mathcal{F}) \leq \iota(\alpha)$.*

(B.2′) *$\mathfrak{B}$ is closed under $\cup$ and $\sqcup_{\mathcal{P}}$.*

(B.3′) *$\times : \mathfrak{B} \times \mathfrak{S}_{\geq\omega} \to \mathfrak{B}$, $\mathfrak{S}_{\geq\omega}$ being the infinite ranked families of $\mathfrak{S}$, is such that for every $\mathcal{F} \in \mathfrak{B}$ and every $\mathcal{H} \in \mathfrak{S}$ one has that $\mathcal{F} \times \mathcal{H}$ is a multiplication of $\mathcal{F}$ by $\mathcal{H}$.*

Proof. Suppose that $(\mathfrak{B}, \times)$ satisfies (B.1′), (B.2′) and (B.3′). Let $C = \{p_n\}_n$ be an infinite chain of $\mathcal{P}$, of order type ω. Fix a basis $(\mathfrak{B}(\omega), \times_\omega)$ of families on ω. For each $\mathcal{G} \in \mathfrak{B}(\omega)$, let $\overline{\mathcal{G}} := \{\{p_n\}_{n \in x} : x \in \mathcal{G}\}$. Then $\overline{\mathcal{G}}$ is homeomorphic to $\mathcal{G}$. Given $\mathcal{F} \in \mathfrak{B}$, let $\widetilde{\mathcal{F}} := \{s \in \mathcal{F} : s \cap C = \emptyset\}$. Now let $\mathfrak{B}'$ be the collection of all unions $\widetilde{\mathcal{F}} \cup \overline{\mathcal{G}}$ such that $\mathcal{F} \in \mathfrak{B}$, $\mathcal{G} \in \mathfrak{G}$, and finally let $\mathfrak{B}''$ be the collection of all $\mathcal{P}$-homogeneous families $\mathcal{F}$ such that there is some $\mathcal{G} \in \mathfrak{B}'$ with $\mathcal{F} \subseteq \mathcal{G}$ and $\iota(\mathrm{srk}_{\mathcal{P}}(\mathcal{F})) = \iota(\mathrm{srk}_{\mathcal{P}}(\mathcal{G}))$. For each $\mathcal{F} \in \mathfrak{B}''$ we choose $\mathcal{G}_{\mathcal{F}} \in \mathfrak{B}$ and $\mathcal{H}_{\mathcal{F}} \in \mathfrak{B}(\omega)$ such that $\mathcal{F} \subseteq \widetilde{\mathcal{G}_{\mathcal{F}}} \cup \overline{\mathcal{H}_{\mathcal{F}}} \in \mathfrak{B}'$ and $\iota(\mathrm{srk}_{\mathcal{P}}(\mathcal{F})) = \iota(\mathrm{srk}_{\mathcal{P}}(\widetilde{\mathcal{G}_{\mathcal{F}}} \cup \overline{\mathcal{H}_{\mathcal{F}}}))$. If $\mathcal{H}$ has infinite rank, we define $\mathcal{F} \times' \mathcal{H} := (\widetilde{\mathcal{G}_F \times \mathcal{H}}) \cup \overline{(\mathcal{H}_{\mathcal{F}} \times_\omega \mathcal{H})}$, and when $\mathcal{H}$ has finite rank n, let $\mathcal{F} \times' \mathcal{H} := \mathcal{F} \boxtimes_{\mathcal{P}} n$. It is easy to check that $(\mathfrak{B}'', \times')$ is a basis on chains of $\mathcal{P}$. □

2.2. Ranks and uniform families

The objective of this part is to give technical tools for getting upper bounds for ranks with the use of uniform families.

Definition 2.25. Given families $\mathcal{F}$ on ω and $\mathcal{G}$ on a partial ordering $\mathcal{P}$, we say that a mapping $f : \mathcal{F} \to \mathcal{G}$ between two families is $(\sqsubseteq, \subseteq)$-increasing when $s \sqsubseteq t$ implies that $f(s) \subseteq f(t)$.

The fact that a point is in a certain derivative of a compact metrizable space K can be witnessed by a continuous and 1–1 mapping from a uniform family into K.

Proposition 2.26 *(Parametrization of ranks). Suppose that K is a compact metrizable space and let $\alpha < \omega_1$. Then a point $p \in K$ is such that $p \in K^{(\alpha)}$ if and only if for every α-uniform family $\mathcal{B}$ there is a 1–1 and continuous function $\theta : \mathcal{B} \to K$ such that $\theta(\emptyset) = p$. In case $K = \mathcal{F}$ is a compact family on I, $p \in \mathcal{F}^{(\alpha)}$ if and only if for every α-uniform family $\mathcal{B}$ there is a 1–1 and continuous mapping $\theta : \mathcal{B} \to \mathcal{F}$ such that $p = \theta(\emptyset)$ and such that θ is $(\sqsubseteq, \subseteq)$-increasing.*

Proof. Given $p \in K$ and $\varepsilon > 0$, let $B(p, \varepsilon)$ be the open ball around p and radius ε. The proof is by induction on α. Suppose that $p \in K^{(\alpha)}$ and let $\mathcal{B}$ be a α-uniform family on M and $\mathcal{C}$ the collection of $\sqsubseteq$-maximal subsets in $\mathcal{B}$. Without loss of generality we assume that $M = \omega$. Let $\alpha_n < \alpha$ be such that $\mathcal{C}_{\{n\}}$ is α_n-uniform on ω/n. Choose $(p_n)_n$ in $K^{(\alpha_n)}$ converging non-trivially to p such that there are mutually disjoint closed balls B_n around p_n with $\mathrm{diam}(B_n) \downarrow_n 0$. Since each $p_n \in K^{(\alpha_n)}$ it follows by inductive hypothesis that for each n there is a 1–1 and continuous function

$$\theta_n : \overline{\mathcal{B}_{\{n\}}} = \mathcal{C}_{\{n\}} \to B_n$$

with $\theta_n(\emptyset) = p_n$. Let $\theta : \mathcal{B} \to K$ be defined by $\theta(\emptyset) = p$, $\theta(s) := \theta_{\min s}(s \setminus \{\min s\})$, for $s \neq \emptyset$. By the choice of the balls B_n it follows that θ is 1–1. We verify now that θ is continuous: Suppose that $(s_k)_k$ tends to s. Suppose first that $s \neq \emptyset$, let $n := \min s$. Then there is k_0 such that for every $k \geq k_0$ one has that $\min s_k = n$. It follows that for every $k \geq k_0$, $\theta(s_k) = \theta_n(t_k)$ and $\theta(s) = \theta_n(t)$, where $t_k := s_k \setminus \{n\}$ and $t := s \setminus \{n\}$. Hence, $\lim_{k\to\infty} \theta(s_k) = \lim_{k\to\infty} \theta_n(t_k) = \theta_n(t) = \theta(s)$. Suppose now that $s = \emptyset$. Fix $\gamma > 0$ and suppose that $d(p, \theta(s_k)) \geq \gamma$ for every k belonging to an infinite subset $M \subseteq \omega$. Without loss of generality, we may assume that $(s_k)_{k\in M}$ is a Δ-system with empty root such that $s_k < s_l$ if $k < l$ in M. Since $\theta(s_k) \in B_{n_k}$, for $n_k := \min s_k$ for every k, and since $(n_k)_{k\in M}$ tends to infinity, $(p_{n_k})_{k\in M}$ converges to p, so that there is some k such that $d(p, \theta(s_k)) < \gamma$, a contradiction. The reverse implication is trivial.

Now, if $\mathcal{F}$ is a compact family on I and α is a countable ordinal, then $p \in \mathcal{F}^{(\alpha)}$ if and only if $p \in (\mathcal{F} \upharpoonright I_0)^\alpha$ for some countable subset I_0. With a small modification of the proof we have just exposed applied to the compact and metrizable space $K = \mathcal{F} \upharpoonright I_0$ one can find recursively on α a 1–1 and continuous $\theta : \mathcal{B} \to \mathcal{F} \upharpoonright I_0$ which in addition is $(\sqsubseteq, \subseteq)$-increasing. $\square$

Lemma 2.27. *Suppose that $\mathcal{B}$ and $\mathcal{C}$ are uniform families, $\mathcal{F}$ is a compact family on some index set I with $\mathrm{rk}(\mathcal{F}) < \mathrm{rk}(\mathcal{C})$. Suppose that $\lambda : \mathcal{B} \otimes \mathcal{C} \to \mathcal{F}$ is $(\sqsubseteq, \subseteq)$-increasing. Then there is a finite subset x of ω and some infinite set $x < M$ such that $\{x\} \sqcup \mathcal{B} \upharpoonright M \subseteq \mathcal{B} \otimes \mathcal{C}$ and such that λ is constant on $\{x\} \sqcup (\mathcal{B} \upharpoonright M)^{\max}$. If in addition λ is continuous, then λ is constant on $\{x\} \sqcup (\mathcal{B} \upharpoonright M)$.*

Proof. Let $\mathcal{B}_0 := \mathcal{B}^{\max}$, $\mathcal{C}_0 := \mathcal{C}^{\max}$, and for each $s \in \mathcal{B}_0 \otimes \mathcal{C}_0$, let $s = \bigcup_{i\leq k_s} s_i$ be the canonical decomposition of s; i.e. $s_i < s_{i+1}$ are in $\mathcal{B}_0$ and $\{\min s_i\}_{i\leq k_s} \in \mathcal{C}_0$.

Claim 2.27.1. *There is an infinite subset $M \subseteq \omega$ such that one of the following conditions hold.*

(a) *For every $s_0 < \cdots < s_{k-1}$ in $\mathcal{B}_0 \upharpoonright M$ with $\{\min s_i\}_{i<k} \in \mathcal{C} \upharpoonright M \setminus \mathcal{C}_0$ and every $s_{k-1} < x < y \in \mathcal{B}_0 \upharpoonright M$ one has that $\lambda(\bigcup_{i<k} s_i \cup x) \neq \lambda(\bigcup_{i<k} s_i \cup y)$.*
(b) *For every $s = \bigcup_{i\leq k_s} s_i \in \mathcal{B}_0 \upharpoonright M \otimes \mathcal{C}_0 \upharpoonright M$ there is $k < k_s$ such that for every $s_{k-1} < x < y \in \mathcal{B}_0 \upharpoonright M$ one has that $\lambda(\bigcup_{i<k} s_i \cup x) = \lambda(\bigcup_{i<k} s_i \cup y)$.*

Proof of Claim. We use the Ramsey property of uniform fronts and a diagonal argument. We can find a decreasing sequence $(N_j)_j$ of infinite subsets of ω, such that $n_j := \min N_j < n_{j+1} := \min N_{j+1}$ and such that for every j and every $s_0 < \cdots < s_{k-1}$ in $\{n_i\}_{i\leq j}$ with $\{\min s_i\}_{i<k} \in \mathcal{C} \setminus \mathcal{C}_0$ one has that either

(i) for every $x < y$ both in $\mathcal{B}_0 \upharpoonright N_{j+1}$ one has that $\lambda(\bigcup_{i<k} s_i \cup x) \neq \lambda(\bigcup_{i\leq j} s_i \cup y)$, or
(ii) for every $x < y$ both in $\mathcal{B}_0 \upharpoonright N_{j+1}$ one has that $\lambda(\bigcup_{i<kj} s_i \cup x) = \lambda(\bigcup_{i\leq j} s_i \cup y)$.

This is done inductively: suppose that $(N_i)_{i\leq j}$ is already defined, $n_i := \min N_i$. Given a sequence $s_0 < \cdots < s_{k-1}$ in $\{n_i\}_{i\leq j}$ with $\{\min s_i\}_{i<k} \in \mathcal{C} \setminus \mathcal{C}_0$, we can define the

coloring $c : (\mathcal{B} \oplus \mathcal{B}) \upharpoonright (N_j/s_{k-1}) \to 2$, for $x < y$ each in $\mathcal{B} \upharpoonright (N_j/s_{k-1}$ by $c(x \cup y) = 0$ if $\lambda(s_0 \cup \cdots \cup s_{k-1} \cup x) \neq \lambda(s_0 \cup \cdots \cup s_{k-1} \cup y)$ and 1 otherwise. Notice that for every $x \in \mathcal{B} \upharpoonright (N_j/s_{k-1}$ one has that $s_0 \cup \cdots \cup s_{k-1} \cup x \in \mathcal{B} \otimes \mathcal{C}$, since $\{\min s_i\}_{i<k} \in \mathcal{C} \setminus \mathcal{C}_0$ and so $C_{\{\min s_i\}_{i<k}}$ is a uniform family (Proposition 2.10) of rank > 0, so it contains all singletons $> \min s_{k-1}$.

Let now $d : \mathcal{B} \otimes \mathcal{C} \upharpoonright N \to 2$ be defined by $c(\bigcup_{i \leq k_s} s_i) = 0$ if there is $k < k_s$ such that $\lambda(\bigcup_{i \leq k} s_i) = \lambda((\bigcup_{i<k} s_i \cup s_{k+1})$ and $c(\bigcup_{i \leq k_s} s_i) = 1$ otherwise. Let $M \subseteq N$ be such that d is constant on $\mathcal{B} \otimes \mathcal{C} \upharpoonright M$ with value $\varrho \in 2$. Then if $\varrho = 1$, then (b) holds on M. If $\varrho = 0$, the (a) holds on M. □

Claim 2.27.2. *(b) above holds.*

Proof of Claim. Otherwise, (a) holds. We claim that there is some $\lambda(\emptyset) \subseteq z \in \mathcal{F}^{(\alpha)}$, $\alpha := \mathrm{rk}(\mathcal{C})$, which is impossible by hypothesis. The proof is by induction on α. Let $(s_i)_{i<\omega}$ be a sequence in $\mathcal{B}_0 \upharpoonright M$ such that $s_i < s_{i+1}$ for every i, $m_i := \min s_i$, and $N := \{m_i\}_i \subseteq M$. For each i, let $\lambda_i : \mathcal{B} \otimes \mathcal{C}_{\{m_i\}} \upharpoonright N \to \mathcal{F}$, $\lambda_i(x) := \lambda(s_i \cup x)$ for every $x \in \mathcal{B} \otimes \mathcal{C}_{\{m_i\}} \upharpoonright N$. Since λ_i also satisfies (a), we obtain that for every i there is some $\lambda(s_i) \subseteq z_i \in (\mathcal{F})^{(\alpha_i)}$, where $\alpha_i = \mathrm{rk}(\mathcal{C}_{\{m_i\}}$. By (a), $(\lambda(s_i))_i$ are pairwise different, so $(z_i)_i$ is non-trivial. Let $(z_i)_{i \in A}$ be a non-trivial Δ-sequence with root $z \in \mathcal{F}$ ($\mathcal{F}$ is compact). Since $\mathcal{C}$ is uniform, it follows that either $\alpha = \beta + 1$ and $\alpha_i = \beta$ for every i, or α is limit and $\sup_i \alpha_i = \alpha$. In any case, $z \in \mathcal{F}^{(\alpha)}$. Since λ is $(\sqsubseteq, \subseteq)$-increasing, it follows that $\lambda(\emptyset) \subseteq \lambda(s_i)$ for every i, so $\lambda(\emptyset) \subseteq z \in \mathcal{F}^{(\alpha)}$, as required. □

So, (b) above holds. Fix $s = \bigcup_{i \leq k_s} s_i \in \mathcal{B}_0 \upharpoonright M \otimes \mathcal{C}_0 \upharpoonright M$. Let $k < k_s$ be such that, setting $x := \bigcup_{i<k} s_i$, then $\lambda(x \cup y) = \lambda(x \cup z)$ for every $x < y < z$ for every $y, z \in \mathcal{B}_0 \upharpoonright M$. We claim that $\lambda(x \cup y) = \lambda(x \cup z)$ for every $x < y, z \in \mathcal{B}_0 \upharpoonright M$. Find $y, z < w \in \mathcal{B}_0 \upharpoonright M$. Then $\lambda(x \cup y) = \lambda(x \cup w) = \lambda(x \cup z)$.

If we assume that λ is in addition continuous, since $\mathcal{B} \upharpoonright M$ is scattered, the set of isolated points is dense. Hence $\{x\} \sqcup (\mathcal{B} \upharpoonright M)^{\max}$ is dense in $\{x\} \sqcup (\mathcal{B} \upharpoonright M)$. Since λ is constant on $\{x\} \sqcup (\mathcal{B} \upharpoonright M)^{\max}$, it is constant on $\{x\} \sqcup (\mathcal{B} \upharpoonright M)$. □

Our first upper estimation on ranks is the following.

Proposition 2.28. *Suppose that $\mathcal{F}$ and $\mathcal{G}$ are countably ranked families and suppose that $\lambda : \mathcal{F} \to \mathcal{G}$ is $\subseteq$-increasing. Then*

$$\mathrm{rk}(\mathcal{F}) < \sup_{t \in \mathcal{G}} (\mathrm{rk}(\{s \in \mathcal{F} : \lambda(s) \subseteq t\}) + 1) \cdot (\mathrm{rk}(\mathcal{G}) + 1).$$

If in addition λ is continuous, then we obtain that

$$\mathrm{rk}(\mathcal{F}) < \sup_{t \in \mathcal{G}} (\mathrm{rk}(\{s \in \mathcal{F} : \lambda(s) = t\}) + 1) \cdot (\mathrm{rk}(\mathcal{G}) + 1).$$

Proof. Let $\alpha := \sup_{t\in\mathcal{G}}(\mathrm{rk}(\{s \in \mathcal{F} : \lambda(s) \subseteq t\}) + 1)$, $\beta := \mathrm{rk}(\mathcal{G})$, and suppose that $\mathcal{F}^{(\alpha\cdot(\beta+1))} \neq \emptyset$. Let $\mathcal{B}$ and $\mathcal{C}$ be α and $\beta+1$ uniform families, let $f : \mathcal{B}\otimes\mathcal{C} \to \mathcal{F}$ be a 1–1, continuous and $(\sqsubseteq, \subseteq)$-increasing function, and let $\theta := \lambda \circ f$. By hypothesis, θ is $(\sqsubseteq, \subseteq)$-increasing, so it follows from Lemma 2.27 that there are $x \subset \omega$ finite and $x < M$ infinite such that $\{x\} \sqcup \mathcal{B} \upharpoonright M \subseteq \mathcal{B}\otimes\mathcal{C}$ and such that θ is constant on $\{x\} \sqcup (\mathcal{B} \upharpoonright M)^{\max}$ with value $t \in \mathcal{G}$. This implies that the mapping $\theta_0(y) := f(x \cup y)$ for every $y \in \mathcal{B} \upharpoonright M$ is a 1–1 and continuous mapping $\theta_0 : \mathcal{B} \upharpoonright M \to \{s \in \mathcal{F} : \lambda(s) \subseteq t\}$, hence $\mathrm{rk}(\{s \in \mathcal{F} : \lambda(s) \subseteq t\}) \geq \alpha$, and this is impossible.

Suppose that in addition λ is continuous. Then the desired result is proved similarly by changing the definition of α with $\alpha := \sup_{t\in\mathcal{G}}(\mathrm{rk}(\{s \in \mathcal{F} : \lambda(s) = t\}) + 1)$, and then using the particular case of continuous functions in Lemma 2.27. □

2.3. *Transferring families and bases with the use of operations*

We present several procedures to define bases from other bases, focused on transfer methods that will be used in Section 4 to step up bases on a cardinal to bases on bigger cardinal numbers. In particular we will see that if an infinite cardinal κ has a basis, then the complete binary tree $2^{\leq\kappa}$ has a basis consisting on chains of $2^{\leq\kappa}$. We will also give more involved transfer methods that will be used to understand more complicated trees (see Subsection 4.2). We start with some terminology.

Definition 2.29. Let $\mathcal{P} = (P, \leq_P)$ and $\mathcal{Q} = (Q, \leq_Q)$ be partial orderings and $\lambda : P \to Q$.

(i) λ is *chain-preserving* when $p_0 \leq_P p_1$ implies that $\lambda(p_0) \leq_Q \lambda(p_1)$ or $\lambda(p_1) \leq_Q \lambda(p_0)$.
(ii) λ is *1–1 on chains* when $\lambda \upharpoonright C$ is 1–1 for every chain C of $\mathcal{P}$.
(iii) λ is *adequate* when it is chain-preserving and 1–1 on chains.

In other words, λ is chain-preserving if it is a *graph homomorphism* between the corresponding *comparability graphs*. Observe that λ is chain-preserving if and only if $\lambda''(C)$ is a chain of $\mathcal{Q}$ for every chain C of $\mathcal{P}$. Observe also that when $\mathcal{Q}$ is a total ordering, every mapping $\lambda : P \to Q$ is chain preserving. The previous proposition can be this generalized as follows:

Theorem 2.30. *Let $\mathcal{P}$ and $\mathcal{Q}$ be partial orderings which have infinite chains, and let $\lambda : \mathcal{P} \to \mathcal{Q}$ be adequate. If $\mathcal{Q}$ has a basis of homogeneous families, then so has $\mathcal{P}$.*

In order to prove this, we introduce the following operation.

Definition 2.31 *(Preimage).* Given partial orderings $\mathcal{P}$ and $\mathcal{Q}$, $\lambda : P \to Q$ and a family $\mathcal{G}$ on chains of $\mathcal{Q}$, let

$$\lambda^{-1}(\mathcal{G}) := \{s \subseteq P : s \text{ is a chain of } \mathcal{P} \text{ and } \lambda''s \in \mathcal{G}\}.$$

Lemma 2.32. *Suppose that $\mathcal{P}$ and $\mathcal{Q}$ are two partial orderings, $\lambda : \mathcal{P} \to \mathcal{Q}$ is adequate. Suppose also that $\mathcal{G}$ is a family on chains of $\mathcal{Q}$.*

(a) *If $\mathcal{G}$ is pre-compact, hereditary, then so is $\lambda^{-1}(\mathcal{G})$.*
(b) *If $\mathcal{G}$ is countably ranked, then*

$$\mathrm{rk}(\lambda^{-1}(\mathcal{G})) < \omega \cdot (\mathrm{rk}(\mathcal{G}) + 1). \tag{3}$$

Consequently, if $\mathcal{P}$ has infinite chains, and $\mathcal{G}$ is $(\alpha, \mathcal{Q})$-homogeneous, $\alpha \geq \omega$, then $\lambda^{-1}\mathcal{G}$ is $(\beta, \mathcal{P})$-homogeneous with $\alpha \leq \beta < \iota(\alpha)$.

Proof. Set $\mathcal{F} := \lambda^{-1}(\mathcal{G})$. It is clear that $\mathcal{F}$ is hereditary when $\mathcal{G}$ is hereditary. Suppose that $\mathcal{G}$ is pre-compact. Let $(x_n)_n$ be a sequence in $\mathcal{F}$. W.l.o.g. we assume that $(x_n)_n$ converges to $A \subseteq P$. Limit of chains are chains, so A is a chain of P. The proof will be finished when we verify that A is finite. We assume that $(\lambda'' x_n)_n$ is a Δ-sequence with root y. It is easy to see that $\lambda'' A \subseteq y$. Since λ'' is 1–1 on chains, it follows that $\# A \leq \# y$, so A is finite. Suppose that $\mathcal{G}$ has countable rank. We apply Proposition 2.28 to $\lambda'' : \mathcal{F} \to \mathcal{G}$ to conclude that

$$\mathrm{rk}(\mathcal{F}) < \sup_{y \in \mathcal{G}} (\mathrm{rk}(\{x \in \mathcal{F} : \lambda''(x) \subseteq y\}) + 1) \cdot (\mathrm{rk}(\mathcal{G}) + 1). \tag{4}$$

Observe that given $y \in \mathcal{G}$, since λ is 1–1 on chains, it follows that

$$\{x \in \mathcal{F} : \lambda''(x) \subseteq y\} \subseteq [P]^{\leq \# y},$$

so from (4) we obtain the desired inequality in (3).

Suppose that $\mathcal{P}$ has infinite chains and let $\mathcal{G}$ be $(\alpha, \mathcal{Q})$-homogeneous. Let us see that $\mathcal{F}$ is $(\beta, \mathcal{P})$-homogeneous with $\beta \geq \alpha$. Let X be an infinite chain of $\mathcal{P}$ such that $\mathrm{srk}_{\mathcal{P}}(\mathcal{F}) = \mathrm{rk}(\mathcal{F} \upharpoonright X) = \beta$. Then $Y := \lambda'' X$ is an infinite chain of $\mathcal{Q}$. Since $h : \mathcal{F} \upharpoonright X \to \mathcal{G} \upharpoonright Y$, $h(s) := \lambda'' s$ is an homeomorphism, it follows that $\mathrm{rk}(\mathcal{G} \upharpoonright Y) = \beta$, hence $\mathrm{srk}_{\mathcal{Q}}(\mathcal{G}) \leq \beta$. On the other hand, it follows from (3) and the fact that $\mathcal{G}$ is homogeneous that

$$\mathrm{srk}_{\mathcal{P}}(\mathcal{F}) \leq \mathrm{rk}(\mathcal{F}) < \omega \cdot (\mathrm{rk}(\mathcal{G}) + 1) < \iota(\mathrm{srk}_{\mathcal{Q}}(\mathcal{G})) \leq \iota(\mathrm{srk}_{\mathcal{P}}(\mathcal{F})). \quad \square$$

Proof of Theorem 2.30. Let $(\mathfrak{C}, \times_{\mathcal{Q}})$ be a basis of families on chains of $\mathcal{Q}$. Let $\mathfrak{B}$ be the collection of all $\mathcal{P}$-homogeneous families $\lambda^{-1}\mathcal{G}$ with $\mathcal{G} \in \mathfrak{C}$. For each $\mathcal{F} \in \mathfrak{B}$, choose $\mathcal{G}_{\mathcal{F}}$ such that $\mathcal{F} = \lambda^{-1}(\mathcal{G}_{\mathcal{F}})$, and for $\mathcal{H} \in \mathfrak{S}$, let $\mathcal{F} \times \mathcal{H} := \lambda^{-1}(\mathcal{G}_{\mathcal{F}} \times_{\mathcal{Q}} \mathcal{H})$. We check that $(\mathfrak{B}, \times)$ satisfies (B.1′), (B.2′) and (B.3), which is enough to guarantee the existence of a basis on $\mathcal{P}$, by Proposition 2.24. Given $\mathcal{G} \in \mathfrak{C}_\alpha$, $\lambda^{-1}\mathcal{G} \in \mathfrak{B}$ and by Lemma 2.32 we know that $\lambda^{-1}\mathcal{G}$ is β-uniform with $\alpha \leq \beta < \iota(\alpha)$. We check now (B.2’). Suppose that $\mathcal{G}_0, \mathcal{G}_0 \in \mathfrak{C}$. Then $\lambda^{-1}\mathcal{G}_0 \sqcup_{\mathcal{P}} \lambda^{-1}(\mathcal{G}_1) = \lambda^{-1}(\mathcal{G}_0 \sqcup_{\mathcal{Q}} \mathcal{G}_1)$, so $\lambda^{-1}\mathcal{G}_0 \sqcup_{\mathcal{P}} \lambda^{-1}(\mathcal{G}_1) \in \mathfrak{B}$, because $\mathcal{G}_0 \sqcup_{\mathcal{Q}} \mathcal{G}_1 \in \mathfrak{C}$. Similarly one shows that $\mathfrak{B}$ is closed under $\cup$.

Finally, we verify (B.3) for $(\mathfrak{B},\times)$. Fix $\mathcal{F}\in\mathfrak{B}$ and $\mathcal{H}\in\mathfrak{S}$. Then $\mathcal{F}\times\mathcal{H}=\lambda^{-1}(\mathcal{G}_\mathcal{F}\times_\mathcal{Q}\mathcal{H})$.

$$\begin{aligned}\iota(\mathrm{srk}_\mathcal{P}(\mathcal{F}\times\mathcal{H}))&=\iota(\mathrm{srk}_\mathcal{Q}(\mathcal{G}_\mathcal{F}\times_\mathcal{Q}\mathcal{H}))=\max\{\iota(\mathrm{srk}_\mathcal{Q}(\mathcal{G}_\mathcal{F})),\iota(\mathrm{srk}(\mathcal{H}))\}\\&=\iota(\mathrm{srk}_\mathcal{P}(\mathcal{F})\cdot\mathrm{srk}(\mathcal{H})).\end{aligned}$$

Let now $(s_n)_n$ be a sequence in $\mathcal{F}$ such that $C:=\bigcup_n s_n$ is a chain. Then $\lambda''C=\bigcup_n\lambda''s_n$ is a $\mathcal{Q}$-chain, and $\lambda''s_n\in\mathcal{G}_\mathcal{F}$. By the property (M.2) of $\times_\mathcal{Q}$, we obtain that there is a subsequence $(t_n)_n$ of $(s_n)_n$ such that $\bigcup_{n\in x}\lambda''t_n\in\mathcal{G}_\mathcal{F}\times_\mathcal{Q}\mathcal{H}$ for every $x\in\mathcal{H}$. This means that $\bigcup_{n\in x}t_n\in\mathcal{F}\times\mathcal{H}$ for such $x\in\mathcal{H}$. $\square$

Recall that given a sequence of partial orderings $(\mathcal{P}_i)_{i\in I}$ we denote by $\biguplus_{i\in I}\mathcal{P}_i$ its *disjoint union*, which is the partial ordering on $\bigcup_{i\in I}P_i\times\{i\}$ defined by $(p,i)<(q,j)$ if and only if $i=j$ and $p<_{\mathcal{P}_i}q$.

Proposition 2.33. *Suppose that θ is a regular cardinal number such that $\omega_1^{\omega_1}<\theta$, and suppose that every $\xi<\theta$ has a basis on families on ξ. Suppose that $(\theta_\xi)_{\xi<\theta}$ is a sequence of infinite ordinals such that $\sup_\xi\theta_\xi=\theta$. Then the disjoint union of $\biguplus_{\xi<\theta}(\theta_\xi,<)$ has a basis of families on chains.*

Proof. The proof is a counting argument. Set $\mathcal{P}:=\biguplus_{\xi<\theta}(\theta_\xi,<)$. First of all, let $C\subseteq\theta$ be such that $\#C=\theta$ and $(\theta_\xi)_{\xi\in C}$ is strictly increasing with supremum θ. For each $\xi\in C$ let $(\mathfrak{C}^\xi,\times'_\xi)$ be a basis on $\theta_\xi\times\{\xi\}$. Let $F:C\to\omega_1^{\omega_1}$ be the mapping that to $\xi\in C$ and $\alpha<\omega_1$ assigns

$$F(\xi)(\alpha):=\min\{\mathrm{rk}(\mathcal{F})\,:\,\mathcal{F}\in\mathfrak{C}^\xi_\alpha\}<\iota(\alpha)<\omega_1.$$

Since $\omega_1^{\omega_1}<\theta$ there must be $D\subseteq C$ of cardinality θ and $f\in\omega_1^{\omega_1}$ such that $F(\xi)=f$ for every $\xi\in D$. Define now for each $\xi<\theta$, $\mu_\xi:=\min\{\gamma\in D\,:\,\theta_\xi\le\theta_\gamma\}$. Fix $\xi<\theta$. Let $\mathfrak{B}^\xi$ be equal to $\mathfrak{C}^\xi$ if $\xi\in D$, and let $\mathfrak{B}^\xi$ be the collection of families $\{x\times\{\xi\}\,:\,x\subseteq\theta_\xi$ and $x\times\{\mu_\xi\}\in\mathcal{F}\}$ for $\mathcal{F}\in\mathfrak{C}^{\mu_\xi}$. For $\xi\in D$, let $\times_\xi=\times'_\xi$. Suppose that $\xi\notin D$. For each $\mathcal{F}\in\mathfrak{B}_\xi$, let $\mathcal{G}_\mathcal{F}$ be such that $\mathcal{F}=\{x\times\{\xi\}\,:\,x\subseteq\theta_\xi$ and $x\times\{\mu_\xi\}\in\mathcal{G}_\mathcal{F}\}$, and define

$$\mathcal{F}\times_\xi\mathcal{H}:=\{x\times\{\xi\}\,:\,x\times\{\mu_\xi\}\in\mathcal{G}_\mathcal{F}\times_{\mu_\xi}\mathcal{H}\}.$$

It is easy to see that $(\mathfrak{B}^\xi,\times_\xi)$ is a basis on $\theta_\xi\times\{\xi\}$ for every $\xi<\theta$. Let $\mathfrak{B}$ be the collection of all $\mathcal{P}$-homogeneous families $\mathcal{F}$ such that $\mathcal{F}\upharpoonright\theta_\xi\times\{\xi\}\in\mathfrak{B}^\xi$. Define for $\mathcal{F}\in\mathfrak{B}$ and $\mathcal{H}\in\mathfrak{S}$

$$\mathcal{F}\times\mathcal{H}:=\bigcup_{\xi<\theta}(\mathcal{F}\upharpoonright(\theta_\xi\times\{\xi\})\times_\xi\mathcal{H}).$$

We check that $(\mathfrak{B}, \times)$ is a pseudo-basis on chains of $\mathcal{P}$. It is easy to see that $\mathfrak{B}$ contains all finite cubes. Now let $\omega \leq \alpha < \omega_1$ and we prove that $\mathfrak{B}_\alpha \neq \emptyset$. For each $\xi \in D$, let $\mathcal{F}_\xi \in \mathfrak{C}_\alpha^\xi$. Define for each $\xi < \theta$ $\mathcal{G}_\xi = \mathcal{F}_\xi$ if $\xi \in D$, and $\mathcal{G}_\xi := \{x \times \{\xi\} : x \subseteq \theta_\xi \text{ and } x \times \{\mu_\xi\} \in \mathcal{F}_{\mu_\xi}\}$. Notice that each $\mathcal{F}_\xi$ is α_ξ-uniform with $\alpha \leq \alpha_\xi < \iota(\alpha)$. Notice also that $\sup_{\xi<\theta} \mathrm{rk}(\mathcal{F}_\xi) = f(\alpha) < \iota(\alpha)$. Now let $\mathcal{G} := \bigcup_{\xi<\theta} \mathcal{G}_\xi$. Since

$$\mathrm{rk}(\mathcal{G}) \leq \sup_{\xi<\theta} \mathrm{rk}(\mathcal{G}_\xi) + 1 = f(\alpha) + 1 < \iota(\alpha),$$

and $\mathrm{srk}_{\mathcal{P}}(\mathcal{G}) = \alpha$, it follows that $\mathcal{G}$ is $(\alpha, \mathcal{P})$-homogeneous. It is easy to see that $\mathfrak{B}$ is closed under $\cup$ and $\sqcup$. Now we prove that $\times$ is a multiplication. Fix $\mathcal{F} \in \mathfrak{B}$ and $\mathcal{H} \in \mathfrak{S}$, and suppose that $\mathcal{F}$ is $(\alpha, \mathcal{P})$-homogeneous. By definition

$$\begin{aligned}
\iota(\mathrm{srk}_{\mathcal{P}}(\mathcal{F} \times \mathcal{H})) &= \iota(\min_{\xi<\theta}(\mathrm{srk}((\mathcal{F} \restriction (\theta_\xi \times \{\xi\})) \times_\xi \mathcal{H}))) \\
&= \max\{\min_{\xi<\theta} \iota(\mathrm{srk}((\mathcal{F} \restriction (\theta_\xi \times \{\xi\})), \iota(\mathrm{srk}(\mathcal{H}))\} \\
&= \max\{\iota(\alpha), \iota(\mathrm{srk}(\mathcal{H}))\} = \iota(\mathrm{srk}_{\mathcal{P}}(\mathcal{F}) \cdot \mathrm{srk}(\mathcal{H})).
\end{aligned}$$

It is easy to see that $\times$ satisfies (M.2). □

We need to analyze then the lexicographical orderings, finite or infinite. This is the content of the next part. Given I and J, let $\pi_I : I \times J \to I$, $\pi_J : I \times J \to J$ be the canonical projections. Given $i \in I$, $j \in J$, and $\Delta \subseteq I \times J$, let

$$\begin{aligned}
(\Delta)_i &:= \pi_J''(\pi_I^{-1}(i) \cap \Delta), \\
(\Delta)^j &:= \pi_I''(\pi_J^{-1}(j) \cap \Delta),
\end{aligned}$$

be the corresponding sections.

Recall that given two partial orderings $\mathcal{P} = (P, \leq_P)$ and $\mathcal{Q} = (Q, \leq_Q)$, let $\mathcal{P} \times_{\mathrm{lex}} \mathcal{Q} := (P \times Q, <_{\mathrm{lex}})$ be the lexicographical product of $\mathcal{P}$ and $\mathcal{Q}$ defined by $(p_0, q_0) <_{\mathrm{lex}} (p_1, q_1)$ if and only if $p_0 <_P p_1$, or if $p_0 = p_1$ and $q_0 <_Q q_1$. This generalizes easily to finite products $\mathcal{P}_1 \times_{\mathrm{lex}} \mathcal{P}_2 \times_{\mathrm{lex}} \cdots \times_{\mathrm{lex}} \mathcal{P}_n$, and the corresponding finite powers $\mathcal{P}_{\mathrm{lex}}^n$. One can also define infinite lexicographical products, but they are not going to be used here. Instead we use *quasi-lexicographical power* $\mathcal{P}_{\mathrm{qlex}}^{<\omega}$ on $P^{<\omega}$ defined by $(p_i)_{i<m} <_{\mathrm{qlex}} (q_i)_{i<n}$ if and only if $m < n$ or if $m = n$ and $(p_i)_{i<m} <_{\mathrm{lex}} (q_i)_{i<m}$. Finally, let $\mathrm{lh} : P^{<\omega} \to \omega$ be the length function. The main result here is the following.

Theorem 2.34. *Let $\mathcal{P}$ and $\mathcal{Q}$ be partial orderings.*

(a) *If $\mathcal{P}$ and $\mathcal{Q}$ have bases of families on the corresponding chains, then $\mathcal{P} \times_{\mathrm{lex}} \mathcal{Q}$ also has a basis of families on its chains.*

(b) *If there is a basis on chains of $\mathcal{P}$, then there is also a basis of families on chains of each finite lexicographical power $\mathcal{P}^n_{\mathrm{lex}}$ and there is a basis of families on chains of $\mathcal{P}^{<\omega}_{\mathrm{lex}}$.*

Definition 2.35 *(Fubini product of families).* Given I and J, let $\pi_I : I \times J \to I$, and $\pi_J : I \times J \to J$ be the canonical coordinate projections. Given families $\mathcal{F}$ and $\mathcal{G}$ on I and J respectively, let

$$\mathcal{F} \circledast_{\mathrm{F}} \mathcal{G} := \{x \subseteq I \times J \,:\, \pi_I'' x \in \mathcal{F} \text{ and } (x)_i \in \mathcal{G} \text{ for every } i \in I\}.$$

We call $\mathcal{F} \circledast_{\mathrm{F}} \mathcal{G}$ the Fubini product of $\mathcal{F}$ and $\mathcal{G}$. Given $n \geq 1$, let $\mathcal{F}^{n+1}_{\mathrm{F}} = \mathcal{F}^n_{\mathrm{F}} \circledast_{\mathrm{F}} \mathcal{F}$.

It is easy to see that if $\mathcal{F}$ and $\mathcal{G}$ are families on chains of $\mathcal{P}$ and $\mathcal{Q}$, respectively, then $\mathcal{F} \circledast_F \mathcal{G}$ is a family on chains of $\mathcal{P} \times_{\mathrm{lex}} \mathcal{G}$.

Definition 2.36 *(Power operation).* For each $n < \omega$, let $\mathcal{F}_n$ be a family on chains of $\mathcal{P}^n_{\mathrm{lex}}$, and let $\mathcal{G}$ be a family on ω. We define $((\mathcal{F}_n)_n)^{\mathcal{G}}$ as the collection of all $x \subseteq P^{<\omega}$ such that

(i) $x \cap [P]^n \in \mathcal{F}_n$ for every $n < \omega$.
(ii) $\mathrm{lh}'' x \in \mathcal{G}$.

Given a family $\mathcal{F}$ on chains of $\mathcal{P}$, let $\mathcal{F}^{\mathcal{G}} := ((\mathcal{F}^n_{\mathrm{F}})_n)^{\mathcal{G}}$.

Lemma 2.37. *Let $\mathcal{P}$ and $\mathcal{Q}$ be two partial orderings. For each $1 \leq n < \omega$, let $\mathcal{F}_n$ be a family on chains of $\mathcal{P}^n_{\mathrm{lex}}$, and suppose also that $\mathcal{G}$ and $\mathcal{H}$ are families on chains of $\mathcal{Q}$ and ω, respectively. Set $\mathcal{F} := \mathcal{F}_1$.*

(a) *If $\mathcal{F}_n$, $n < \omega$, $\mathcal{G}$, $\mathcal{H}$ are pre-compact, hereditary, then so are $\mathcal{F} \circledast_{\mathrm{F}} \mathcal{G}$, $((\mathcal{F}_n)_n)^{\mathcal{H}}$, and $\mathcal{F}^{\mathcal{H}}$.*
(b) *If $\mathcal{F}_n$, $n < \omega$, $\mathcal{G}$ and $\mathcal{H}$ are countably ranked families, then*
 (b.1) $\mathrm{rk}(\mathcal{F} \circledast_{\mathrm{F}} \mathcal{G}) < (\mathrm{rk}(\mathcal{G}) \cdot \omega)) \cdot (\mathrm{rk}(\mathcal{F}) + 1)$,
 (b.2) $\mathrm{rk}(((\mathcal{F}_n)_n)^{\mathcal{H}}) < \sup_{n<\omega}(\mathrm{rk}(\mathcal{F}_n) + 1) \cdot (\mathrm{rk}(\mathcal{H}) + 1)$,
 (b.3) $\mathrm{rk}(\mathcal{F}^{\mathcal{H}}) < (\mathrm{rk}(\mathcal{F}) \cdot \omega)^{\omega} \cdot (\mathrm{rk}(\mathcal{H}) + 1)$, *if* $\mathrm{rk}(\mathcal{F}) \geq 1$.
(c) *When the corresponding families are countable ranked,*
 (c.1) $\mathrm{srk}_{\mathrm{lex}}(\mathcal{F} \circledast_{\mathrm{F}} \mathcal{G}) = \min\{\mathrm{srk}_{\mathcal{P}}(\mathcal{F}), \mathrm{srk}_{\mathcal{Q}}(\mathcal{G})\}$,
 (c.2) $\mathrm{srk}_{\mathrm{qlex}}(((\mathcal{F}_n)_n)^{\mathcal{H}}) = \min\{\min_{n<\omega} \mathrm{srk}_{\mathcal{P}^n_{\mathrm{lex}}}(\mathcal{F}_n), \mathrm{srk}(\mathcal{H})\}$, *and*
 (c.3) $\mathrm{srk}_{qlex}(\mathcal{F}^{\mathcal{H}}) = \min\{\mathrm{srk}_{\mathcal{P}}(\mathcal{F}), \mathrm{srk}(\mathcal{H})\}$.
(d) *If each $\mathcal{F}_n$ is $(\alpha, \mathcal{P}^n_{\mathrm{lex}})$-homogeneous, $\mathcal{G}$ is $(\alpha, \mathcal{Q})$-homogeneous and $\mathcal{H}$ is α-homogeneous, then $\mathcal{F} \circledast_{\mathrm{F}} \mathcal{G}$ is (α, lex)-homogeneous, and both $((\mathcal{F}_n)_n)^{\mathcal{H}}$ and $\mathcal{F}^{\mathcal{H}}$ is (α, qlex)-homogeneous.*

Proof. *The operation* $\circledast_{\mathrm{F}}$: The hereditariness is easy to prove. Suppose that $\mathcal{F}$ and $\mathcal{G}$ are pre-compact. Let $(x_n)_n$ be a sequence in $\mathcal{F} \circledast_{\mathrm{F}} \mathcal{G}$. Let $M \subseteq \omega$ be infinite and such that $(\pi_I'' x_n)_{n\in M}$ is a Δ-sequence with root y. Let $N \subseteq M$ be infinite such that $((x_n)_p)_{n\in N}$ is a Δ-sequence with root z_p for every $p \in y$. Let $x := \bigcup_{p\in y} z_p$. It is easy to see that $(x_n)_{n\in N}$ is a Δ-sequence with root x. Suppose now that $\mathcal{F}$ and $\mathcal{G}$ have countable rank, and set $\mathcal{H} := \mathcal{F} \circledast_{\mathrm{F}} \mathcal{G}$. We apply Proposition 2.28 to the projection $\pi_P : P \times Q \to P$ to conclude that

$$\mathrm{rk}(\mathcal{H}) < \sup_{y\in\mathcal{F}} (\mathrm{rk}(\{x \in \mathcal{H} : \pi_P''(x) \subseteq y\}) + 1) \cdot (\mathrm{rk}(\mathcal{F}) + 1).$$

Now observe that for a given $y \in \mathcal{F}$ one has that

$$\{x \in \mathcal{H} : \pi_P'' x \subseteq y\} \subseteq \bigsqcup_{p\in y} \{\{p\} \times z \subseteq P \times Q : z \in \mathcal{G}\},$$

by definition of $\mathcal{H}$. Clearly $\{\{p\} \times z \subseteq P \times Q : z \in \mathcal{G}\}$ is homeomorphic to $\mathcal{G}$, so by Proposition 2.6 (iii.3.),

$$\mathrm{rk}(\{x \in \mathcal{H} : \pi_P'' x \subseteq y\}) < \mathrm{rk}(\mathcal{G}) \cdot \omega.$$

The power operation: The fact that this operation preserves hereditariness is trivial to prove. Suppose that each $\mathcal{F}_n$ is a pre-compact family on chains of $\mathcal{P}_{\mathrm{lex}}^n$ and that $\mathcal{H}$ is a pre-compact family on ω. Set $\mathcal{Z} := ((\mathcal{F}_n)_n)^{\mathcal{H}}$, and suppose that $(x_k)_k$ is a sequence in $\mathcal{Z}$. We assume that $(\mathrm{lh}(x_n))_n$ is a Δ-sequence in ω with root z. Now for each $n \in z$ and each $k < \omega$, let $y_k^n := \mathrm{lh}^{-1}(n) \cap x_k \in \mathcal{F}_n$. Let $(x_k)_{k\in M}$ be a subsequence of $(x_k)_k$ such that for each $n \in z$ one has that $(y_k^n)_{k\in M}$ is a Δ-sequence with root y_n. It is easy to verify that $(x_k)_{k\in M}$ is a Δ-sequence with root $\bigcup_{n\in z} y_n$. The inequality in (b.2) follows from Proposition 2.28. The properties of $\mathcal{F}^{\mathcal{H}}$ follow from the corresponding properties of the Fubini product and the power operation.

(c) follows from the fact that if $C = \{(p_n, q_n)\}_{n<\omega}$ is a chain of $\mathcal{P} \times_{\mathrm{lex}} \mathcal{Q}$ then there is an infinite $M \subseteq \omega$ such that either $p_m \neq p_m$ for every $m < n \in M$, or $p_m = p_n$ and $q_m \neq q_n$ for every $m < n$ in M. Similarly, given a chain $C = \{\overline{p}^n\}_{n<\omega}$, there is an infinite $M \subseteq \omega$ such that either $\mathrm{lh}(\overline{p}^n) = l$ for every $n \in M$ and $\{\overline{p}^n\}_{n\in M}$ is an infinite chain of $\mathcal{F}_{\mathrm{F}}^l$, or $\mathrm{lh}(\overline{p}^m) \neq \mathrm{lh}(\overline{p}^n)$ for every $m \neq n$ in M. (d) is a consequence of (a) (b) and (c). □

Proof of Theorem 2.34. (a): Suppose that $(\mathfrak{B}^{\mathcal{P}}, \times_{\mathcal{P}})$ and $(\mathfrak{B}^{\mathcal{Q}}, \times_{\mathcal{Q}})$ are bases of families on chains of $\mathcal{P}$ and $\mathcal{Q}$ respectively. Let $\mathfrak{B}$ be the collection of all $\mathcal{P} \times_{\mathrm{lex}} \mathcal{Q}$-homogeneous families $\mathcal{F}$ such that

(i) $\pi_{\mathcal{P}}''(\mathcal{F}) := \{\pi_{\mathcal{P}}''(x) : x \in \mathcal{F}\} \in \mathfrak{B}_{\mathcal{P}}$.
(ii) $(\mathcal{F})_{\mathcal{P}} := \{(x)_p : x \in \mathcal{F}, p \in P\} \in \mathfrak{B}_{\mathcal{Q}}$.

Given $\mathcal{F} \in \mathfrak{B}$ and $\mathcal{H} \in \mathfrak{S}$, let

$$\mathcal{F} \times \mathcal{H} := (\pi''_{\mathcal{P}}(\mathcal{F}) \times_{\mathcal{P}} \mathcal{H}) \circledast_{\mathrm{F}} ((\mathcal{F})_{\mathcal{P}} \times_{\mathcal{Q}} \mathcal{H}).$$

We verify that $(\mathfrak{B}, \times)$ is a pseudo-basis. First of all, each family on $\mathfrak{B}$ is $\mathcal{P} \times_{\mathrm{lex}} \mathcal{Q}$-homogeneous. Next, given $n < \omega$, $\pi''_{\mathcal{P}}([P \times Q]^{\le n}_{\mathrm{lex}}) = [P]^{\le n}_{\mathcal{P}}$, and $([P \times Q]^{\le n}_{\mathrm{lex}})_{\mathcal{P}} = [Q]^{\le n}_{\mathcal{Q}}$,so $[P \times Q]^{\le n}_{\mathrm{lex}} \in \mathfrak{B}$. Now given α infinite, we choose $\mathcal{F} \in \mathfrak{B}^{\mathcal{P}}_{\alpha}$ and $\mathcal{G} \in \mathfrak{B}^{\mathcal{Q}}_{\alpha}$. Then $\mathcal{Z} := \mathcal{F} \circledast_{\mathrm{F}} \mathcal{G} \in \mathfrak{B}_{\alpha}$: We know from Lemma 2.37 that $\mathcal{Z}$ is α-homogeneous. Since $\pi''_{\mathcal{P}}(\mathcal{Z}) = \mathcal{F}$ and $(\mathcal{Z})_{\mathcal{P}} = \mathcal{G}$, we have that $\mathcal{Z} \in \mathfrak{B}$. We check now (B.2'): Let $\mathcal{F}_0, \mathcal{F}_1 \in \mathfrak{B}$. Since $\pi''_{\mathcal{P}}(\mathcal{F}_0 \cup \mathcal{F}_1) = \pi''_{\mathcal{P}}(\mathcal{F}_0) \cup \pi''_{\mathcal{P}}(\mathcal{F}_1)$ and $(\mathcal{F}_0 \cup \mathcal{F}_1)_{\mathcal{P}} = (\mathcal{F}_0)_{\mathcal{P}} \cup (\mathcal{F}_1)_{\mathcal{P}}$, we obtain that $\mathfrak{B}$ is closed under $\cup$. Secondly, $\pi''_{\mathcal{P}}(\mathcal{F}_0 \sqcup_{\mathrm{lex}} \mathcal{F}_1) = \pi''_{\mathcal{P}}(\mathcal{F}_0) \sqcup_{\mathcal{P}} \pi''_{\mathcal{P}}(\mathcal{F}_1)$, and $(\mathcal{F}_0)_{\mathcal{P}}, (\mathcal{F}_1)_{\mathcal{P}} \subseteq (\mathcal{F}_0 \sqcup_{\mathrm{lex}} \mathcal{F}_1)_{\mathcal{P}} \subseteq (\mathcal{F}_0)_{\mathcal{P}} \sqcup_{\mathcal{Q}} (\mathcal{F}_1)_{\mathcal{P}}$. This means that

$$\begin{aligned}\iota(\mathrm{srk}_{\mathcal{Q}}((\mathcal{F}_0 \sqcup_{\mathrm{lex}} \mathcal{F}_1)_{\mathcal{P}})) &= \max\{\iota(\mathrm{srk}_{\mathcal{Q}}((\mathcal{F}_0)_{\mathcal{Q}})), \iota(\mathrm{srk}_{\mathcal{Q}}((\mathcal{F}_1)_{\mathcal{Q}}))\} \\ &= \iota(\mathrm{srk}_{\mathcal{Q}}((\mathcal{F}_0)_{\mathcal{P}} \sqcup_{\mathcal{Q}} (\mathcal{F}_1)_{\mathcal{P}})),\end{aligned}$$

so $(\mathcal{F}_0 \sqcup_{\mathrm{lex}} \mathcal{F}_1)_{\mathcal{P}} \in \mathfrak{B}_{\mathcal{Q}}$. Since in addition $\mathcal{F}_0 \sqcup_{\mathrm{lex}} \mathcal{F}_1$ is lex-homogeneous, we obtain that $\mathcal{F}_0 \sqcup_{\mathrm{lex}} \mathcal{F}_1 \in \mathfrak{B}$, by definition. Finally we check that $\times$ is a multiplication. The property (M.1) of $\times$ follows from Lemma 2.37 (c). Now suppose that $(s_n)_n$ is a sequence in $\mathcal{F} \in \mathfrak{B}$. Let $(t_n)_n$ be a subsequence of $(s_n)_n$ such that

(1) $(\pi''_{\mathcal{P}} t_n)_n$ is a Δ-sequence with root y.
(2) For every $x \in \mathcal{H}$ one has that $\bigcup_{n \in x} \pi''_{\mathcal{P}}(t_n) \in (\pi''_{\mathcal{P}}\mathcal{F}) \times_{\mathcal{P}} \mathcal{H}$.
(3) For every $x \in \mathcal{H}$ and every $p \in y$ one has that $\bigcup_{n \in x} (t_n)_p \in (\mathcal{F})_{\mathcal{P}} \times_{\mathcal{Q}} \mathcal{H}$.

Since $(\mathcal{F})_{\mathcal{P}} \subseteq (\mathcal{F})_{\mathcal{P}} \times \mathcal{H}$, the conditions above imply that given $x \in \mathcal{H}$ one has that $\bigcup_{n \in x} t_n \in \mathcal{F} \times \mathcal{H}$.

(b) Finite lexicographical powers have bases of families on chains by (a). For each $n < \omega$, let $(\mathfrak{B}_n, \times_n)$ be a basis on chains of $\mathcal{P}^n_{\mathrm{lex}}$. Let $\mathfrak{B}$ be the collection of all qlex-homogeneous families on $\mathcal{P}^{<\omega}_{\mathrm{qlex}}$ such that

(i) $\mathcal{F} \upharpoonright [P]^n \in \mathfrak{B}_n$ for every $1 \le n < \omega$.
(ii) $\mathrm{lh}''(\mathcal{F}) := \{\mathrm{lh}''(s) : s \in \mathcal{F}\} \in \mathcal{H}$.

Given $\mathcal{F} \in \mathfrak{B}$ and $\mathcal{H} \in \mathfrak{S}$, let

$$\mathcal{F} \times \mathcal{H} = (((\mathcal{F} \upharpoonright [P]^n) \times_n \mathcal{H})_n)^{\mathrm{lh}''(\mathcal{F}) \times_{\omega} \mathcal{H}}.$$

We check that $(\mathfrak{B}, \times)$ is a pseudo-basis. Given $1 \le k < \omega$, set $\mathcal{F} := [P^{<\omega}]^{\le k}_{\mathrm{qlex}}$. Then for each n one has that $\mathcal{F} \upharpoonright [P]^n = [[P]^n]^{\le k}_{\mathrm{lex}} \in \mathfrak{B}_n$, and $\mathrm{lh}''(\mathcal{F}) = [\omega]^{\le k} \in \mathfrak{B}_{\omega}$, so $\mathcal{F} \in \mathfrak{B}$. One shows as in (a) that $\mathfrak{B}$ is closed under $\cup$ and $\sqcup_{\mathrm{qlex}}$. Finally, we check that $\times$ is a multiplication. That $\times$ satisfies (M.1) it follows from Lemma 2.37 (c). Let now $\mathcal{F} \in \mathfrak{B}$, $\mathcal{H} \in \mathfrak{B}_{\omega}$, and let $(s_k)_k \in \mathcal{F}$. Let $(t_k)_k$ be a subsequence of $(s_k)_k$ such that

(1) $(\mathrm{lh}''t_k)_k$ is a Δ-sequence with root $y \subseteq \omega$.
(2) For every $x \in \mathcal{H}$ and every $n \in y$ one has that $\bigcup_{k\in x}(s_k \cap [P]^n) \in (\mathcal{F} \upharpoonright [P]^n) \times_n \mathcal{H}$.
(3) For every $x \in \mathcal{H}$ one has that $\bigcup_{k\in x} \mathrm{lh}''(y_k) \in (\mathrm{lh}''\mathcal{F} \times_\omega \mathcal{H})$.

It is easy to verify that $\bigcup_{k\in x} t_k \in \mathcal{F} \times \mathcal{H}$ for every $x \in \mathcal{H}$. $\square$

3. Bases of families on trees

The goal of this section is to give a method to step up bases to trees. The idea is very natural: A tree T is determined by its chains and antichains. Given two families $\mathcal{A}$ and $\mathcal{C}$ on antichains and on chains of a tree T, respectively, one can define a third family $\mathcal{A} \odot_T \mathcal{C}$ consisting of all subsets of T generating a subtree whose antichains are in $\mathcal{A}$ and its chains are in $\mathcal{C}$. In general, the antichains of a tree are difficult to understand; on the contrary, the particular antichains consisting of immediate successors of a node are typically simpler (e.g. in a complete binary tree), and it makes more sense to define $\mathcal{A} \odot_T \mathcal{C}$ in terms of these particular simpler antichains. This operation on families will allow us, for example, to step up from a basis of families on a cardinal number κ to a basis on its cardinal exponential 2^κ, and more.

Recall that a (set-theoretical) *tree* $T = (T, <)$ is a set of nodes T with a partial order $<$ such that $\{u < t : u \in T\}$ is well ordered for every $t \in T$. A *rooted* tree is a tree with a minimal element 0, called the *root* of T. All trees we use are rooted, so that whenever we say tree, we mean a rooted tree. Trees are a sort of lexicographical product of two orderings, the one defining the tree order $<$ and the following. Given $t \leq u$ in T, let $\mathrm{Is}_t(u)$ be the immediate successor of t which is below u, that is, $\mathrm{Is}_t(u)$ be the smallest $v \leq u$ such that $t < v$. Then, given $t \in T$ and $x \subseteq T$, let

$$\mathrm{Is}_t''x := \{\mathrm{Is}_t(u) : t < u \in x\}.$$

For simplicity, we write Is_t for $\mathrm{Is}_t''(T)$, that is, the set of all immediate successors of t in T. For every $t \in T$, fix a total ordering $<_t$ of Is_t. Let $<_a$ be the partial ordering in T defined by $t <_a u$ if and only if there is v such that $t, u \in \mathrm{Is}_v$ and $t <_v u$. Hence, a chain with respect to $<_a$ is a set of immediate successors of a fixed node. Notice that both $<$ and $<_a$ can be extended to a total ordering $\prec$ on T by defining $t_0 \prec t_1$ if and only if $t_0 < t_1$, or if t_0 and t_1 are $<$-incomparable and $\mathrm{Is}_{t_0\wedge t_1}(t_0) <_a \mathrm{Is}_{t_0\wedge t_1}(t_1)$. However, as we already observed, none of the notions involved in this work depends on the particular *total* order one can choose on some index set, since being a chain with respect to a total order does not depend on the total order itself.

We are now able to state the main result of this section.

Theorem 3.1. *The following are equivalent for an infinite tree T.*

(a) *There is a basis of families on T.*

(b) *There is a basis of families on chains of* $(T, <)$*, if there is an infinite* $<$*-chain, and there is a basis of families of families on chains of* $<_a$*, if there is an infinite* $<_a$*-chain.*

We pass now to recall well-known combinatorial principles on trees. Let $T = (T, <)$ be a complete (i.e. the meet of two nodes always exists) rooted tree with root 0. Recall that a *chain* of a tree is a totally ordered subset of it. Let

$$\mathrm{Ch}_a := \{s \in [T]^{<\omega} \, : \, s \text{ is a } <_a\text{-chain}\}$$
$$\mathrm{Ch}_c := \{s \in [T]^{<\omega} \, : \, s \text{ is a } <\text{-chain}\}.$$

Notice that it follows from König's Lemma that when T is infinite, either Ch_c is not compact, that is, there is an infinite $<$-chain, or Ch_a is not compact, that is, there is an infinite $<_a$-chain. Given $t \le u \in T$, let

$$[t, u] := \{v \in T \, : \, t \le v \le u\}.$$

Similarly, one defines the corresponding (semi) open intervals. Given $x \subseteq T$, let $x_{\le t} := x \cap [0, t]$, $x_{\ge t} := \{u \in x \, : \, t \le u\}$, and let $x_{<t}$ and $x_{>t}$ be their open analogues.

Given $t, u \in T$, let

$$t \wedge u := \max(T_{\le t} \cap T_{\le u}),$$

which is well defined by the completeness of T. We say that $s \subseteq T$ is $\wedge$*-closed* when $t \wedge u \in s$ for every $t, u \in s$. Given $s \subseteq T$, let $\langle s \rangle$ be the *subtree generated* by s, that is, the minimal $\wedge$-closed subset of T containing s. We say that a subset $\tau \subseteq T$ is a *subtree* of T when $\langle \tau \rangle = \tau$.

We introduce another operation that will be very useful. Given $t, u \in T$, let

$$t \wedge_{\mathrm{is}} u := \begin{cases} \min\{t, u\} & \text{if } t, u \text{ are comparable} \\ \mathrm{Is}_{t \wedge u}(t) & \text{if } t, u \text{ are incomparable.} \end{cases}$$

Given $s \subseteq T$, let $\langle s \rangle_{\mathrm{is}}$ be the minimal subset of T that contains s and that is closed under $\wedge$ and $\wedge_{\mathrm{is}}$. It is clear that $\langle s \rangle_{\mathrm{is}}$ is a subtree and that $\langle s \rangle \subseteq \langle s \rangle_{\mathrm{is}}$. We will say that $\tau \subseteq T$ is an is-subtree if $\langle \tau \rangle_{\mathrm{is}}$. The following is easy to prove.

Definition 3.2. Given a family $\mathcal{F}$ on T, let

$$\langle \mathcal{F} \rangle := \{x \subseteq \langle s \rangle \, : \, s \in \mathcal{F}\},$$
$$\langle \mathcal{F} \rangle^{\mathrm{is}} := \{x \subseteq \langle s \rangle_{\mathrm{is}} \, : \, s \in \mathcal{F}\},$$
$$\langle \mathcal{F} \rangle_T := \{\langle s \rangle \, : \, s \in \mathcal{F}\},$$

$$\langle\mathcal{F}\rangle_{\mathrm{is}-T} := \{\langle s\rangle_{\mathrm{is}} \,:\, s \in \mathcal{F}\},$$
$$\mathrm{Is}(\mathcal{F}) := \{\mathrm{Is}''_t s \,:\, t \in T \text{ and } s \in \mathcal{F}\}.$$

We give some basic properties. We leave the details of the proof of the firsts one to the reader.

Proposition 3.3. *For every finite set $s \subseteq T$ and every $t \in T$, we have that*

$$\langle s\rangle = \{t_0 \wedge t_1 \,:\, t_0, t_1 \in s\},$$
$$\langle s\rangle_{\mathrm{is}} = \langle s\rangle \cup \{t_0 \wedge_{\mathrm{is}} t_1 : t_0, t_1 \in s, t_0 \perp t_1\},$$
$$\mathrm{Is}''_t\langle s\rangle = \mathrm{Is}''_t s.$$

In particular, $\langle s\rangle$ is finite whenever s is finite, and $\mathrm{Is}(\mathcal{F}) = \mathrm{Is}(\langle\mathcal{F}\rangle)$ *for every hereditary family $\mathcal{F}$ on T. In general, if $(s_i)_{i\in I}$ is a family of subsets of T, then*

$$\langle\bigcup_{i\in I} s_i\rangle = \bigcup_{\{i,j\}\in[I]^2} \langle s_i \cup s_j\rangle. \quad \square$$

Proposition 3.4. *Let $\mathcal{F}$ be a family on T.*

(a) *$\mathcal{F} = \langle\mathcal{F}\rangle$ if and only if $\mathcal{F}$ is hereditary and closed under $\langle\cdot\rangle$.*
(b) *$\langle\mathcal{F}\rangle$ is compact if and only if $\langle\mathcal{F}\rangle_T$ is compact.*
(c) *$\langle\mathcal{F}\rangle^{\mathrm{is}}$ is compact if and only if $\langle\mathcal{F}\rangle_{\mathrm{is}-T}$ is compact.*
(d) *If $\langle\mathcal{F}\rangle^{\mathrm{is}}$ is compact, then so is $\langle\mathcal{F}\rangle$.*

Proof. (a) is trivial. (b): Suppose that $\langle\mathcal{F}\rangle$ is compact. Then $\langle\mathcal{F}\rangle_T$ is compact, because $\{\tau \subseteq T \,:\, \tau \text{ is a subtree}\}$ is compact. Reciprocally, suppose that $\langle\mathcal{F}\rangle_T$ is compact, and let $(s_n)_n$ be a sequence in $\langle\mathcal{F}\rangle$. If $(\langle s_n\rangle)_{n\in M}$ is a converging subsequence with limit τ, then it is easy to extract (using for example the Ramsey Theorem) a further converging subsequence $(s_n)_{n\in N}$ whose limit must belong to $\langle\mathcal{F}\rangle$ because this is always a closed set. Similarly one proves (c). (d): Since $\langle\mathcal{F}\rangle \subseteq \langle\mathcal{F}\rangle^{\mathrm{is}}$ and $\langle\mathcal{F}\rangle$ is closed, the compactness of $\langle\mathcal{F}\rangle^{\mathrm{is}}$ implies the one of $\langle\mathcal{F}\rangle$. $\square$

Given $s \subseteq T$, let

$$(s)_{\max} := \{\text{maximal elements of } s\}.$$

Definition 3.5. Let $s = (t_k)_{k\in\omega}$ be a sequence of nodes in T.

(a) s is called a *comb* if s is an antichain such that

$$t_k \wedge t_l = t_k \wedge t_m \text{ and } t_k \wedge t_l < t_l \wedge t_m \text{ for every } k < l < m.$$

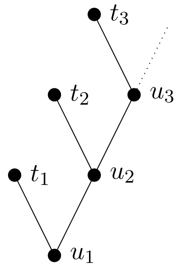

Fig. 1. A comb $(t_k)_{k\in\omega}$ and its corresponding $\wedge$-chain $(u_k)_k$.

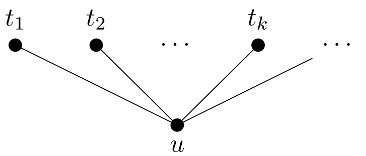

Fig. 2. A fan $(t_k)_{k\in\omega}$ and its corresponding $\wedge$-root u.

The chain $(u_k)_k$, $u_k = t_k \wedge t_l$ $(k < l)$ is called the $\wedge$-chain of the comb $(t_k)_k$ (Fig. 1).

(b) s is called a *fan* if

$$t_k \wedge t_l = t_{k'} \wedge t_{l'} \text{ for every } k \neq l \text{ and } k' \neq l'.$$

The node $u := t_k \wedge t_l$ $(k \neq l)$ is called the $\wedge$-root of the fan $(t_k)_k$ (Fig. 2).

Proposition 3.6. *Every infinite subset of T contains either an infinite chain, or an infinite comb, or an infinite fan.*

Proof. Fix a sequence $(t_k)_{k\in\omega}$ such that $t_k \neq t_l$ for $k \neq l$. By the Ramsey Theorem, there is M_0 such that either $(t_k)_{k\in M_0}$ is a chain or an antichain.

Claim 3.6.1. *If $(t_k)_{k\in M_0}$ is an antichain, then there is $M_1 \subseteq M_0$ such that $t_k \wedge t_l = t_k \wedge t_m$ for every $k < l < m$ in M_1.*

Proof of Claim. If $(t_k)_{k\in M_0}$ is an antichain and since there are no infinite decreasing chains in T, by the Ramsey Theorem, there is $M_1 \subseteq M_0$ such that

(a1) either $t_k \wedge t_l = t_k \wedge t_m$ for ever $k < l < m$ in M_1,
(b1) or else $t_k \wedge t_l < t_k \wedge t_m$ for ever $k < l < m$ in M_1.

Let us see that (b1) cannot happen: Fix $k_0 < k_1 < k_2 < k_3$ in M_1. Then

$$t_{k_0} \wedge t_{k_1} \wedge t_{k_i} = (t_{k_0} \wedge t_{k_1}) \wedge (t_{k_0} \wedge t_{k_i}) = t_{k_0} \wedge t_{k_1},$$

for $i = 2, 3$. On the other hand,

$$t_{k_0} \wedge t_{k_1} = t_{k_0} \wedge t_{k_1} \wedge t_{k_i} = (t_{k_0} \wedge t_{k_i}) \wedge (t_{k_1} \wedge t_{k_i}) \in \{t_{k_0} \wedge t_{k_i}, t_{k_1} \wedge t_{k_i}\},$$

for $i = 2, 3$; so either $t_{k_0} \wedge t_{k_1} = t_{k_0} \wedge t_{k_i}$ for some $i = 2, 3$, or else $t_{k_1} \wedge t_{k_2} = t_{k_0} \wedge t_{k_1} = t_{k_1} \wedge t_{k_3}$. Both cases are impossible since they contradict (b1). □

For each $k \in M_1$, let $u_k := t_k \wedge t_l$ for some (all) $l > k$ in M_1. Yet again, since there are no infinite decreasing chains in T, by the Ramsey Theorem, there is $M_2 \subseteq M_1$ such that

(a2) either $u_k = u_l = \overline{u}$ for every $k < l$ in M_2,
(b2) or $u_k < u_l$ for every $k < l$ in M_2.

If (a2) holds, then $(t_k)_{k \in M_2}$ is a fan with $\wedge$-root $\overline{u}$. If (b2) holds, then $(t_k)_{k \in M_2}$ is a comb with $\wedge$-chain $(u_k)_{k \in M_2}$. □

We can now use the previous result to describe better the small rank of a family on the tree T.

Corollary 3.7. *Every infinite subtree of T contains either an infinite chain or an infinite fan. Consequently,*

(a) *If $A \subseteq T$ is an infinite accumulation point of a sequence of subtrees of T, then A is a subtree of T that contains either an infinite chain or an infinite fan.*
(b) *For every family $\mathcal{F}$ on T with countable rank one has that*

$$\mathrm{srk}(\mathcal{F}) = \inf\{\mathrm{rk}(\mathcal{F} \restriction X) : X \textit{ is an infinite chain, comb or fan}\}. \quad \square$$

We can now characterize the compactness of $\langle \mathcal{F} \rangle^{\mathrm{is}}$ in terms of $<$ and $<_a$-chains.

Corollary 3.8. *For a family $\mathcal{F}$ the following are equivalent.*

(a) *$\langle \mathcal{F} \rangle^{\mathrm{is}}$ is compact.*
(b) *$\langle \mathcal{F} \rangle^{\mathrm{is}} \cap \mathrm{Ch}_c$ and $\mathrm{Is}(\langle \mathcal{F} \rangle^{\mathrm{is}})$ are compact.*

Proof. Suppose that $\langle \mathcal{F} \rangle^{\mathrm{is}}$ is compact. The family $\langle \mathcal{F} \rangle^{\mathrm{is}}$ is by definition hereditary, so $\langle \mathcal{F} \rangle^{\mathrm{is}} \cap \mathrm{Ch}_c$ is closed, and since $\langle \mathcal{F} \rangle^{\mathrm{is}} \cap \mathrm{Ch}_c \subseteq \langle \mathcal{F} \rangle^{\mathrm{is}}$, it is compact. Notice that $\mathrm{Is}(\langle \mathcal{F} \rangle^{\mathrm{is}})$ is hereditary, because $\langle \mathcal{F} \rangle^{\mathrm{is}}$ is so, by definition. Let $(s_n)_n$ be a sequence in $\mathrm{Is}(\langle \mathcal{F} \rangle^{\mathrm{is}})$. If there is an infinite subsequence $(s_n)_{n \in M}$ such that $\# s_n \leq 1$, then clearly there is a further convergent subsequence of $(s_n)_{n \in M}$ with limit ($\emptyset$ or a singleton) in $\mathrm{Is}(\langle \mathcal{F} \rangle^{\mathrm{is}})$ because it is hereditary. Otherwise, from some point on $\# s_n \geq 2$. This means that for such s_n one has that $s_n = \{t \wedge_{\mathrm{is}} u : t \neq u \in x_n\}$ for some $x_n \in \mathcal{F}$, and so $s_n \in \langle \mathcal{F} \rangle^{\mathrm{is}}$. Hence, there is a converging subsequence whose limit belongs to $\mathrm{Is}(\langle\mathcal{F}\rangle^{\mathrm{is}})$, because $\mathrm{Is}(\langle\mathcal{F}\rangle^{\mathrm{is}})$ is hereditary. Suppose now that (b) holds, and let $(s_n)_n$ be a sequence in $\langle\mathcal{F}\rangle^{\mathrm{is}}$. Going towards a contradiction, suppose that $(\tau_n)_n$ is a sequence of is-subtrees of T converging to an infinite is-subtree A of T. From Corollary 3.7 we know that one of the following holds. A contains an infinite chain C. Then $(\tau_n \cap C)_n$ is a sequence in $\langle\mathcal{F}\rangle^{\mathrm{is}} \cap \mathrm{Ch}_c$ with infinite limit C, impossible. The second option is that A contains an infinite fan X with root t. Then the sequence $(\mathrm{Is}_t''(s_n))_n$ converges to the infinite set $\mathrm{Is}_t'' X$, contradicting the fact that $\mathrm{Is}(\langle\mathcal{F}\rangle^{\mathrm{is}})$ is compact. $\square$

So, we have the following characterization of the compactness of $\langle\mathcal{F}\rangle^{\mathrm{is}}$ in terms of compact properties of $\langle\mathcal{F}\rangle$.

Corollary 3.9. *$\langle\mathcal{F}\rangle^{\mathrm{is}}$ is compact if and only if $\langle\mathcal{F}\rangle$ and $\mathrm{Is}(\langle\mathcal{F}\rangle)$ are compact.*

Proof. We know that the compactness of $\langle\mathcal{F}\rangle^{\mathrm{is}}$ implies the compactness of both $\langle\mathcal{F}\rangle$ and $\mathrm{Is}(\langle\mathcal{F}\rangle) = \mathrm{Is}(\langle\mathcal{F}\rangle^{\mathrm{is}})$. Conversely, suppose that $\langle\mathcal{F}\rangle$ and $\mathrm{Is}(\langle\mathcal{F}\rangle)$ are compact. We use the characterization of compactness of $\langle\mathcal{F}\rangle^{\mathrm{is}}$ in Corollary 3.8, so it rests to show that $\langle\mathcal{F}\rangle^{\mathrm{is}} \cap \mathrm{Ch}_c$ is compact: Set $\mathcal{C} := \langle\mathcal{F}\rangle \cap \mathrm{Ch}_c$, $\mathcal{D} := \langle\mathcal{F}\rangle^{\mathrm{is}} \cap \mathrm{Ch}_c$. Both are hereditary families, and $\mathcal{C}$ is compact, by hypothesis. Let Is be the collection of all immediate successors of T, and let $\pi : \mathrm{Is} \to T$ be the mapping assigning to each $t \in \mathrm{Is}$ its immediate predecessor $\pi(t)$. Then π is an adequate mapping, so $\pi^{-1}(\mathcal{C})$ is compact and hereditary. Since $\mathcal{D} \subseteq \mathcal{C} \sqcup \pi^{-1}(\mathcal{C})$, $\mathcal{D}$ is compact, as desired. $\square$

We analyze now the $<$ and $<_a$-chains of the tree generated by the union of two subtrees of T.

Proposition 3.10. *Suppose that τ_0, τ_1 are subtrees of T, and $t \in T$. Then*

$$\mathrm{Is}_t(\langle\tau_0 \cup \tau_1\rangle) = \mathrm{Is}_t \tau_0 \cup \mathrm{Is}_t \tau_1, \tag{5}$$

$$\mathrm{Ch}_c \upharpoonright \langle\tau_0 \cup \tau_1\rangle \subseteq (\mathrm{Ch}_c \upharpoonright \tau_0) \sqcup_c (\mathrm{Ch}_c \upharpoonright \tau_1) \sqcup_c [T]^{\leq 1}. \tag{6}$$

Proof. (5) follows from the fact that $\mathrm{Is}_t''(\langle s\rangle) = \mathrm{Is}_t''(s)$. For (6), let c be a chain of $\langle\tau_0 \cup \tau_1\rangle$, and suppose that $t_0 < t_1$ belong to $c \setminus (\tau_0 \cup \tau_1)$. Then $t_0 = u_0 \wedge v_0$ and $t_1 = u_1 \wedge v_1$ with $u_0, u_1 \in \tau_0$ and $v_0, v_1 \in \tau_1$. Then, either $u_0 \wedge u_1 = t_0$ or $v_0 \wedge v_1 = t_0$, and both are impossible. $\square$

We introduce the operation $\odot_T$, which is the crucial tool to build bases on a tree T from bases on $<$-chains and bases on $<_a$-chains.

Definition 3.11 *(The operation $\odot_T$).* Given two families $\mathcal{A}$ and $\mathcal{C}$ on T, let $\mathcal{A} \odot_T \mathcal{C}$ be the family of all finite subsets s of T such that

(a) For every $t \in T$ one has that $\mathrm{Is}_t''\langle s\rangle \in \mathcal{A}$.

(b) Every chain in $\langle s\rangle$ belongs to $\mathcal{C}$.

Remark 3.12.

(i) The family $\mathcal{A}\odot_T\mathcal{C}$ is closed under generated subtrees, that is, $\langle s\rangle\in\mathcal{A}\odot_T\mathcal{C}$ if $s\in\mathcal{A}\odot_T\mathcal{C}$.
(ii) When $[T]^{\leq 1}_a\subseteq\mathcal{A}$, the condition (a) above is equivalent to
 (a′) For every $t\in\langle s\rangle$, one has that $\mathrm{Is}''_t(s)\in\mathcal{A}$.
(iii) When $[\mathrm{Is}_t]^{\leq 1}_a\subseteq\mathcal{A}$ for all $t\in T$, $(\mathcal{A}\odot_T\mathcal{C})\cap\mathrm{Ch}_c=\mathcal{C}\cap\mathrm{Ch}_c$, and when $[T]^{\leq 2}_c\subseteq\mathcal{C}$, then $\mathrm{Is}(\mathcal{A}\odot_T\mathcal{C})=\mathcal{A}\cap\mathrm{Ch}_a$.

We introduce some notation. We are going to use a and c to refer to $<_a$ and $<$. For example,

$$\sqcup_a,\ \mathrm{srk}_a,\ \sqcup_c,\ \mathrm{srk}_c$$

denote $\sqcup_{(T,<_a)}$, $\mathrm{srk}_{(T,<_a)}$, $\sqcup_{(T,<)}$ and $\mathrm{srk}_{(T,<)}$ respectively. We now compare small ranks of $\mathcal{A}\odot_T\mathcal{C}$ and of $\mathcal{A}$ and $\mathcal{C}$.

Proposition 3.13. *Let $\mathcal{A}$ and $\mathcal{C}$ be two families on chains of $(T,<_a)$ and of $(T,<)$, respectively.*

(a) *If $\mathcal{A}$ and $\mathcal{C}$ are compact, hereditary, then so is $\mathcal{A}\odot_T\mathcal{C}$.*
(b) $\mathrm{srk}(\mathcal{A}\odot_T\mathcal{C})\leq\mathrm{srk}_a(\mathcal{A})$ *if* Ch_a *is non-compact, and* $\mathrm{srk}(\mathcal{A}\odot_T\mathcal{C})\leq\mathrm{srk}_c(\mathcal{C})$ *if* Ch_c *is non-compact.*
(c) $\mathrm{srk}(\mathcal{A}\odot_T\mathcal{C})=\mathrm{srk}_a(\mathcal{A})$ *if* Ch_c *is compact and* $[T]^{\leq 2}_c\subseteq\mathcal{C}$.
(d) $\mathrm{srk}_c(\mathcal{C})\leq\mathrm{srk}(\mathcal{A}\odot_T(\mathcal{C}\sqcup_c[T]^{\leq 1}))\leq\mathrm{srk}_a(\mathcal{C})+1$ *if* Ch_a *is compact and* $[T]^{\leq 2}_a\subseteq\mathcal{A}$.
(e) $\min\{\mathrm{srk}_a(\mathcal{A}),\mathrm{srk}_c(\mathcal{C})\}\leq\mathrm{srk}(\mathcal{A}\odot_T(\mathcal{C}\sqcup_c[T]^{\leq 1}))\leq\min\{\mathrm{srk}_a(\mathcal{A}),\mathrm{srk}_c(\mathcal{C})+1\}$ *if* Ch_a *and* Ch_c *are not compact, and* $[T]^{\leq 2}_c\subseteq\mathcal{C}$, $[T]^{\leq 2}_a\subseteq\mathcal{A}$.

Proof. Set $\mathcal{F}:=\mathcal{A}\odot_T\mathcal{C}$. (a): Hereditariness of $\mathcal{F}$ is trivial. Suppose that s is an infinite subset of T which is the limit of a sequence $(\tau_k)_k$ in $\mathcal{F}$. Since $\mathcal{F}$ is, by definition, $\wedge$-closed, we may assume without loss of generality that each τ_k is a subtree. It follows that s is a subtree of T as well. Hence, by Corollary 3.7, s contains either an infinite chain C, or an infinite fan F. In the first case, $\tau_k\cap C\in\mathcal{C}$ for every k and $\tau_k\cap C\to_k C$, which is impossible since $\mathcal{C}$ is compact. If s contains an infinite fan F with $\wedge$-root u, then $\mathrm{Is}''_u(F)$ is an accumulation point of the sequence $(\mathrm{Is}''_u(s_k))_k$ in $\mathcal{A}$, which is impossible by the compactness of $\mathcal{A}$.

(b) follows from the fact that if X is a $<_a$-chain, then $(\mathcal{A}\odot_T\mathcal{C})\restriction X\subseteq\mathcal{A}\restriction X$, and if X is a $<$-chain, then $(\mathcal{A}\odot_T\mathcal{C})\restriction X\subseteq\mathcal{C}\restriction X$. (c), (d) and (e) follow, by the means of Corollary 3.7, from the following claim.

Claim 3.13.1.

(i) *If* $[T]^{\le 2}_c \subseteq \mathcal{C}$, *then* $\mathrm{rk}((\mathcal{A} \odot_T \mathcal{C}) \upharpoonright X) \ge \mathrm{srk}_a(\mathcal{A})$ *for every infinite fan* X.
(ii) *If* $[T]^{\le 1}_a \subseteq \mathcal{A}$, *then* $\mathrm{rk}((\mathcal{A} \odot_T \mathcal{C}) \upharpoonright X) \ge \mathrm{srk}_c(\mathcal{C})$ *for every infinite chain* X.
(iii) *If* $[T]^{\le 2}_a \subseteq \mathcal{A}$, *then* $\mathrm{rk}((\mathcal{A} \odot_T (\mathcal{C} \sqcup_c [T]^{\le 1})) \upharpoonright X \ge \mathrm{srk}_c(\mathcal{C})$ *for every infinite comb* X.

Proof of Claim. (i): Set $\mathcal{F} := \mathcal{A} \odot_T \mathcal{C}$. Suppose that $X = \{t_n\}_{n<\omega}$ is an infinite fan with $\wedge$-root u. For each $n < \omega$, let $v_n := \mathrm{Is}_u(t_n)$. Since for every $x \subseteq \omega$ one has that $\langle \{t_n\}_{n\in x}\rangle = \{t_n\}_{n\in x} \cup \{u\}$, it follows that the maximal chains of $\langle \{t_n\}_{n\in x}\rangle$ have cardinality 2, so, they belong to $\mathcal{C}$, by hypothesis. This means that $\{t_n\}_{n\in x} \in \mathcal{F}$ if and only if $\{v_n\}_{n\in x} \in \mathcal{A}$, and consequently $\mathrm{rk}(\mathcal{F} \upharpoonright X) = \mathrm{rk}(\mathcal{A} \upharpoonright \{v_n\}_n) \ge \mathrm{srk}_a(\mathcal{A})$. (ii): Set again $\mathcal{F} := \mathcal{A} \odot_T \mathcal{C}$. Suppose that X is an infinite chain. Since $[T]^{\le 1}_a \subseteq \mathcal{A}$, it follows that $\mathcal{F} \upharpoonright X = \mathcal{C} \upharpoonright X$, hence $\mathrm{rk}(\mathcal{F} \upharpoonright X) \ge \mathrm{srk}_a(\mathcal{C})$. (iii): Set now $\mathcal{F} := \mathcal{A} \odot_T (\mathcal{C} \sqcup_c [T]^{\le 1}$. Suppose that $X = \{t_k\}_k$ is a comb with $\wedge$-chain $C := \{u_k\}_k$. Given $\{u_k\}_{k\in x} \in \mathcal{C}$ we know that $\{u_k\}_{k\in x} \cup \{t_p\} \in C \sqcup_c [T]^{\le 1}$, where $p := \max x$. Since

$$\langle \{t_k\}_{k\in x}\rangle = \{t_k, u_k\}_{k\in x}$$

it follows that $\langle \{t_k\}_{k\in x}\rangle \in \mathcal{A} \odot_T (\mathcal{C} \sqcup_c [T]^{\le 1})$ whenever $\{u_k\}_{k\in x} \in \mathcal{C}$. So, $\{u_k\}_{k\in x} \in \mathcal{C} \mapsto \{t_k\}_{k\in x} \in \mathcal{F}$ is continuous and 1–1. Hence $\mathrm{rk}(\mathcal{F} \upharpoonright X) \ge \mathrm{srk}_c(\mathcal{C})$. $\square$

We are ready to define a basis of T from a basis on $<$-chains and a basis on $<_a$-chains. In fact we will define a pseudobasis on T which can be modified to give a basis (Proposition 2.24).

Definition 3.14 *(The pseudo-basis on T).* Let T be an infinite tree, and suppose that for $(T, <_a)$ and $(T, <)$ either they have a basis of families on their chains or they do not have infinite chains. Let $(\mathfrak{B}^a, \times_a)$ be either a basis of families on chains of $(T, <_a)$, or $\mathfrak{B}^a := \{\mathcal{A} : \mathcal{A}$ is compact hereditary of finite rank and $\mathcal{A} \subseteq \mathrm{Ch}_a\}$ and $\mathcal{A} \times_a \mathcal{H} := \mathcal{A}$ for every $\mathcal{A} \in \mathfrak{B}^a$ and $\mathcal{H} \in \mathfrak{S}$, if there are no infinite $<_a$-chains. We define similarly $(\mathfrak{B}^c, \times_c)$. Let $\mathfrak{B}$ be the collection of all families $\mathcal{F}$ on T such that

(BT.1) $\mathcal{F}$ and $\langle \mathcal{F}\rangle$ are homogeneous, $\mathrm{Is}(\mathcal{F}) \subseteq \mathcal{F}$ and $\mathrm{rk}(\langle \mathcal{F}\rangle) < \iota(\mathrm{srk}(\mathcal{F}))$. In addition, if the rank of $\mathcal{F}$ is finite, then there is some $n < \omega$ such that $\mathcal{F} \subseteq [T]^{\le n}$.
(BT.2) $\mathrm{Is}\langle \mathcal{F}\rangle \in \mathfrak{B}^a$ and $\langle \mathcal{F}\rangle \cap \mathrm{Ch}_c \in \mathfrak{B}^c$.

Given $\mathcal{F} \in \mathfrak{B}$ and $\mathcal{H} \in \mathfrak{S}$ of infinite rank, let

(BT.3) $\mathcal{F} \times \mathcal{H} := ((\mathcal{A} \times_a \mathcal{H}) \sqcup_a [T]^{\le 1}) \odot_T ((\mathcal{C} \times_c \mathcal{H}) \boxtimes_c 5)$ where $\mathcal{A} := \mathrm{Is}(\langle \mathcal{F}\rangle)$ and $\mathcal{C} := \langle \mathcal{F}\rangle \cap \mathrm{Ch}_c$.

We are going to see that $\mathfrak{B}$ is a pseudobasis. We start by analyzing the small ranks of families of $\mathfrak{B}$.

Proposition 3.15. *Suppose that $\mathcal{F} \in \mathfrak{B}$. Then*

(a) $\iota(\mathrm{srk}(\mathcal{F})) = \iota(\mathrm{srk}(\langle\mathcal{F}\rangle))$.
(b) $\iota(\mathrm{srk}(\mathcal{F})) = \iota(\mathrm{srk}(\langle\mathcal{F}\rangle)) = \iota(\mathrm{srk}_c(\mathcal{F} \cap \mathrm{Ch}_c)) = \iota(\mathrm{srk}_c(\langle\mathcal{F}\rangle \cap \mathrm{Ch}_c))$ *if* Ch_c *is not compact.*
(c) $\iota(\mathrm{srk}(\mathcal{F})) = \iota(\mathrm{srk}(\langle\mathcal{F}\rangle)) = \iota(\mathrm{srk}_a(\mathrm{Is}(\mathcal{F}))) = \iota(\mathrm{srk}_a(\mathrm{Is}(\langle\mathcal{F}\rangle)))$ *if* Ch_a *is not compact.*

Proof. Fix $\mathcal{F} \in \mathfrak{B}$. (a):

$$\mathrm{srk}(\mathcal{F}) \le \mathrm{srk}(\langle\mathcal{F}\rangle) \le \mathrm{rk}(\langle\mathcal{F}\rangle) < \iota(\mathrm{srk}(\mathcal{F})),$$

by property (B.T.1). This means that $\iota(\mathrm{srk}(\mathcal{F})) = \iota(\mathrm{srk}(\langle\mathcal{F}\rangle))$. (b): Suppose that Ch_c is not compact. $\mathcal{F}$ is hereditary, so $\mathcal{F} \subseteq \langle\mathcal{F}\rangle$. Also, there are infinite $<$-chains of T and if C is a such chain, then $(\mathcal{F} \cap \mathrm{Ch}_c) \upharpoonright C = \mathcal{F} \upharpoonright C$. Then,

$$\mathrm{srk}(\mathcal{F}) \le \mathrm{srk}_c(\mathcal{F} \cap \mathrm{Ch}_c) \le \mathrm{srk}_c(\langle\mathcal{F}\rangle \cap \mathrm{Ch}_c) \le \mathrm{rk}(\langle\mathcal{F}\rangle) < \iota(\mathrm{srk}(\mathcal{F})).$$

This implies that $\iota(\mathrm{rk}(\mathcal{F})) = \iota(\mathrm{srk}_c(\mathcal{F} \cap \mathrm{Ch}_c)) = \iota(\mathrm{srk}_c(\langle\mathcal{F}\rangle \cap \mathrm{Ch}_c))$.

(c): Suppose that Ch_a is not compact. Similarly than for (b), now using that by hypothesis $\mathrm{Is}(\langle\mathcal{F}\rangle) = \mathrm{Is}(\mathcal{F}) \subseteq \mathcal{F} \subseteq \langle\mathcal{F}\rangle$, we obtain that

$$\mathrm{srk}(\mathcal{F}) \le \mathrm{srk}_a(\mathrm{Is}(\mathcal{F})) = \mathrm{srk}_a(\mathrm{Is}(\langle\mathcal{F}\rangle)) \le \mathrm{rk}(\langle\mathcal{F}\rangle) < \iota(\mathrm{srk}(\mathcal{F})),$$
$$\mathrm{srk}(\mathcal{F}) \le \mathrm{srk}_c(\mathcal{F} \cap \mathrm{Ch}_c) \le \mathrm{rk}(\mathcal{F}) < \iota(\mathrm{srk}(\mathcal{F})),$$

hence $\iota(\mathrm{srk}(\mathcal{F})) = \iota(\mathrm{srk}_c(\mathcal{F}\cap\mathrm{Ch}_c))$. So, $\iota(\mathrm{rk}(\mathcal{F})) = \iota(\mathrm{srk}_a(\mathrm{Is}(\mathcal{F}))) = \iota(\mathrm{srk}_a(\mathrm{Is}(\langle\mathcal{F}\rangle)))$. □

The next two results are the keys to show Theorem 3.1.

Lemma 3.16. *Suppose that $\mathcal{F}$ is a compact family such that $\mathcal{F} = \langle\mathcal{F}\rangle$. Then,*

$$\mathrm{rk}(\mathcal{F}) < \max\{\iota(\mathrm{rk}(\mathrm{Is}(\mathcal{F})) \cdot \omega)), \iota(\mathrm{rk}(\mathcal{F} \cap \mathrm{Ch}_c) \cdot \omega))\}.$$

Lemma 3.17. $\times$ *is a multiplication.*

Their proofs are involved, so we reserve specific subsections to present them. In fact, Lemma 3.16 is a consequence of an appropriate upper bound of the rank of $\mathcal{A} \odot_T \mathcal{C}$ in terms of the ranks of $\mathcal{A}$ and $\mathcal{C}$, and it is done in the next Subsection 3.1. The proof of Lemma 3.17 is mostly combinatorial; It is given in the Subsection 3.2, where we find the canonical form of a Δ-sequence of finite subtrees of T. The next gives, using Lemma 3.16, a criteria on $\mathcal{A}$ and $\mathcal{C}$ to guarantee that $\mathcal{A} \odot_T \mathcal{C} \in \mathfrak{B}$.

Proposition 3.18. *Let $\mathcal{A} \in \mathfrak{B}^a$ and $\mathcal{C} \in \mathfrak{B}^c$ be such that*

(a) $[T]_a^{\le 2} \subseteq \mathcal{A}$ *and* $[T]_c^{\le 2} \subseteq \mathcal{C}$.
(b) *If* Ch_c *is non-compact, then there is some* $\mathcal{D} \in \mathfrak{B}^c$ *such that* $\iota(\mathrm{srk}(\mathcal{D})) = \iota(\mathrm{srk}(\mathcal{C}))$ *and* $\mathcal{D} \sqcup_c [T]^{\le 1} \subseteq \mathcal{C}$.
(c) $\max\{\mathrm{srk}_a(\mathcal{A}), \mathrm{srk}_c(\mathcal{C})\} \ge \omega$.
(d) $\iota(\mathrm{srk}_a(\mathcal{A})) = \iota(\mathrm{srk}_c(\mathcal{C}))$ *if* Ch_a *and* Ch_c *are both non-compact.*

Then $\mathcal{A} \odot_T \mathcal{C} \in \mathfrak{B}$.

Proof. Fix $\mathcal{A}$ and $\mathcal{C}$ as in the statement, and set $\mathcal{F} := \mathcal{A} \odot_T \mathcal{C}$. Then $\langle \mathcal{F} \rangle = \mathcal{F}$, by definition of $\odot_T$, $\mathrm{Is}(\mathcal{F}) = \mathcal{A}$, because we are assuming $[T]_c^{\le 2} \subseteq \mathcal{C}$, and $\mathcal{F} \cap \mathrm{Ch}_c = \mathcal{C}$, because $[T]_a^{\le 1} \subseteq \mathcal{A}$. This implies that (BT.2) holds for $\mathcal{F}$. Suppose first that Ch_c is compact. Then, by the choice of $\mathfrak{B}^c$, we know that $\mathrm{rk}(\mathcal{C}) < \omega$. Also, $\mathrm{srk}(\mathcal{F}) = \mathrm{srk}_a(\mathcal{A})$. Hence, $\lambda := \iota(\mathrm{srk}_a(\mathcal{A})) = \iota(\mathrm{srk}(\mathcal{F}))$. Since $\mathcal{A}$ is homogeneous, $\mathrm{rk}(\mathcal{A}) < \lambda$. We know also that $\lambda > \omega$, hence it follows that Lemma 3.16 that $\mathrm{rk}(\mathcal{F}) < \lambda = \mathrm{srk}(\mathcal{F})$, so (BT.1) holds for $\mathcal{F}$. Suppose that Ch_a is compact. Then it follows from condition (b) and Proposition 3.13 (d) that $\mathrm{srk}(\mathcal{D}) \le \mathrm{srk}(\mathcal{A} \odot_T (\mathcal{D} \sqcup_c [T]^{\le 1})) \le \mathrm{srk}(\mathcal{F})$. Hence, by hypothesis, $\lambda = \iota(\mathrm{srk}(\mathcal{C})) \le \iota(\mathrm{srk}(\mathcal{F}))$. Since $\mathrm{srk}_a(\mathcal{A}) = 0$, it follows that $\lambda > \omega$. Since $\mathcal{A}$ has finite rank, and $\mathcal{C}$ is homogeneous, it follows from Lemma 3.16 that $\mathrm{rk}(\mathcal{F}) < \lambda = \iota(\mathrm{srk}(\mathcal{F}))$. Suppose that Ch_a and Ch_c are non-compact. Set $\lambda = \iota(\mathrm{srk}_a(\mathcal{A})) = \iota(\mathrm{srk}_c(\mathcal{C}))$. Then, as before, $\iota(\mathrm{srk}(\mathcal{F})) = \lambda > \omega$. Since both $\mathcal{A}$ and $\mathcal{C}$ are homogeneous, it follows from Lemma 3.16 that $\mathrm{rk}(\mathcal{F}) < \lambda = \iota(\mathrm{srk}(\mathcal{F}))$. □

We are now ready to prove the main result of this section.

Proof of Theorem 3.1. Let us see that $(\mathfrak{B}, \times)$ defined on Definition 3.14 is a pseudo-basis of families on T (see Proposition 2.24). (B.1′): We first see that the cubes belong to $\mathfrak{B}$: Notice that if τ is a finite tree, then

$$\#\tau \le \frac{a(\tau)^{c(\tau)+1} - 1}{a(\tau) - 1} \tag{7}$$

where $a(\tau)$ and $c(\tau)$ are the maximal cardinality of a $<_a$-chain and a $<$-chain, respectively. Set $\mathcal{F} = [T]^{\le n}$, $n \ge 1$. Then $\langle \mathcal{F} \rangle \subseteq [T]^{\le n^{n+1}}$. Hence, $\mathrm{Is}(\mathcal{F}) = [T]_a^{\le n} \subseteq [T]^{\le n} = \mathcal{F}$, $\langle \mathcal{F} \rangle$ is homogeneous and $\iota(\mathrm{srk}(\mathcal{F})) = \iota(\mathrm{srk}(\langle \mathcal{F} \rangle)) = \omega$, so $\mathcal{F}$ satisfies (BT.1). Since $\mathrm{Is}(\mathcal{F}) = [T]_a^{\le n}$, $\mathrm{Is}(\mathcal{F}) \in \mathfrak{B}^a$ whether Ch_a is compact or not. Since $\langle \mathcal{F} \rangle \subseteq [T]^{n^{n+1}}$, $\mathcal{G} := \langle \mathcal{F} \rangle \cap \mathrm{Ch}_c \subseteq [T]_c^{\le n^{n+1}}$ is a compact, hereditary family containing $[T]^{\le 1}$, so it belongs to $\mathfrak{B}^c$ when Ch_c is compact. When Ch_c is not compact, $\iota(\mathrm{srk}_c(\mathcal{G})) = \omega = \iota(\mathrm{srk}_c([T]_c^{\le n^{n+1}}))$ and $[T]_c^{\le n^{n+1}} \in \mathfrak{B}^c$. Since $\mathfrak{B}^c$ is a basis, it follows that $\mathcal{G} \in \mathfrak{B}^c$.

We fix now $\omega \le \alpha < \omega_1$, and we prove that there is $\mathcal{F} \in \mathfrak{B}$ such that $\alpha \le \mathrm{srk}(\mathcal{F}) \le \iota(\alpha)$: Suppose first that Ch_a is compact. Let $\mathcal{D} \in \mathfrak{B}_\alpha^c$, and set $\mathcal{C} := \mathcal{D} \sqcup_c [T]^{\le 1}$. It follows from Proposition 3.18 that $\mathcal{F} := [T]_a^{\le 2} \odot_T \mathcal{C} \in \mathfrak{B}$. We check that $\alpha \le \mathrm{srk}(\mathcal{F}) \le \iota(\alpha)$.

Proposition 3.13 (d) gives that $\alpha = \mathrm{srk}_c(\mathcal{D}) \leq \mathrm{srk}(\mathcal{F}) \leq \mathrm{srk}_c(\mathcal{D}) + 1 = \alpha + 1 < \iota(\alpha)$, as desired. Suppose now that Ch_c is compact. Let $\mathcal{A} \in \mathfrak{B}^a_\alpha$ and $\mathcal{C} := [T]^{\leq 2}_c$. Then, Proposition 3.18 gives that $\mathcal{F} := \mathcal{A} \odot_T \mathcal{C} \in \mathfrak{B}$, and Proposition 3.13 (c) gives that $\alpha = \mathrm{srk}_a(\mathcal{A}) = \mathrm{srk}(\mathcal{F})$. Finally, if Ch_a and Ch_c are non-compact, we choose $\mathcal{A} \in \mathfrak{B}^a_\alpha$ and $\mathcal{C} \in \mathfrak{B}^c_\alpha$. Then, Proposition 3.18 and Proposition 3.13 (e) give that $\mathcal{F} := (\mathcal{A} \sqcup_a [T]^{\leq 1}_a) \odot_T (\mathcal{C} \sqcup_c [T]^{\leq 1} \in \mathfrak{B}$ and $\alpha \leq \min\{\mathrm{srk}_a(\mathcal{A} \sqcup_a [T]^{\leq 1}), \mathrm{srk}_c(\mathcal{C})\} \leq \mathrm{srk}(\mathcal{F}) \leq \min\{\mathrm{srk}_a(\mathcal{A} \sqcup_a [T]^{\leq 1}), \mathrm{srk}_c(\mathcal{C}) + 1\} = \alpha + 1 < \iota(\alpha)$. This ends the proof of (B.1′).
(B.2′): We have to check that $\mathfrak{B}$ is closed under $\cup$ and $\sqcup$. Fix $\mathcal{F}, \mathcal{G} \in \mathfrak{B}$. Set $\mathcal{B} := \mathcal{F} \cup \mathcal{G}$. Then Proposition 2.6 gives that $\mathcal{B}$ and $\langle \mathcal{B} \rangle = \langle \mathcal{F} \rangle \cup \langle \mathcal{G} \rangle$ are homogeneous and

$$\mathrm{rk}(\langle \mathcal{B} \rangle) = \max\{\mathrm{rk}(\langle \mathcal{F} \rangle), \mathrm{rk}(\langle \mathcal{G} \rangle)\} < \max\{\iota(\mathrm{srk}(\mathcal{F})), \iota(\mathrm{srk}(\mathcal{G}))\} \leq \iota(\mathrm{srk}(\mathcal{B})).$$

Also, $\mathrm{Is}(\mathcal{B}) = \mathrm{Is}(\mathcal{F}) \cup \mathrm{Is}(\mathcal{G}) \subseteq \mathcal{B}$, so (BT.1) holds for $\mathcal{B}$. Since $\mathrm{Is}(\mathcal{B}) = \mathrm{Is}(\mathcal{F}) \cup \mathrm{Is}(\mathcal{G})$, it follows that $\mathrm{Is}(\mathcal{B}) \in \mathfrak{B}^a$ whether Ch_a is compact or not. Similarly, $\langle \mathcal{B} \rangle \cap \mathrm{Ch}_c = (\langle \mathcal{F} \rangle \cap \mathrm{Ch}_c) \cup (\langle \mathcal{G} \rangle \cap \mathrm{Ch}_c) \in \mathfrak{B}^c$. Let us see now that $\mathcal{B} := \mathcal{F} \sqcup \mathcal{G} \in \mathfrak{B}$. We suppose that $\mathcal{F}, \mathcal{G} \neq \{\emptyset\}$, so $[T]^{\leq 1} \subseteq \mathcal{F}, \mathcal{G}$ (as they are homogeneous families). We know that $\mathcal{B}$ is homogeneous and $\mathrm{rk}(\mathcal{B}) = \mathrm{rk}(\mathcal{F}) \dot{+} \mathrm{rk}(\mathcal{G})$. Suppose first that the ranks of $\mathcal{F}$ and $\mathcal{G}$ are finite. Then by (BT.1), there is some n such that $\mathcal{F}, \mathcal{G} \subseteq [T]^{\leq n}$, hence $\mathcal{B} \subseteq [T]^{\leq 2n}$. It follows from (7) that there is some m such that $\langle \mathcal{B} \rangle \subseteq [T]^{\leq m}$. This implies that $\mathrm{rk}(\langle \mathcal{B} \rangle) < \omega = \iota(\mathrm{srk}(\mathcal{B}))$, so (BT.1) holds for $\mathcal{B}$. Now, $\mathrm{Is}(\langle \mathcal{B} \rangle) = \mathrm{Is}(\mathcal{B}) = \mathrm{Is}(\mathcal{F}) \sqcup_a \mathrm{Is}(\mathcal{G}) \in \mathfrak{B}^a$ whether Ch_a is compact or not. Similarly, $\mathcal{B} \cap \mathrm{Ch}_c \in \mathfrak{B}^c$. Suppose now that $\mathcal{F}$ or $\mathcal{G}$ has infinite rank. We have, by Proposition 3.10, the following inclusions:

$$\langle \mathcal{F} \sqcup \mathcal{G} \rangle \subseteq \langle \langle \mathcal{F} \rangle \sqcup \langle \mathcal{G} \rangle \rangle \subseteq (\langle \mathcal{F} \rangle \sqcup_a \langle \mathcal{G} \rangle) \odot_T (\langle \mathcal{F} \rangle \sqcup_c \langle \mathcal{G} \rangle \sqcup_c [T]^{\leq 1}).$$

Now,

$$\begin{aligned} \mathrm{rk}(\langle F \rangle \sqcup_a \langle \mathcal{G} \rangle) &\leq \mathrm{rk}(\langle F \rangle) \dot{+} \mathrm{rk}(\langle G \rangle) < \max\{\iota(\mathrm{srk}(\mathcal{F})), \iota(\mathrm{srk}(\mathcal{G}))\} \leq \iota(\mathrm{srk}(\mathcal{B})) \\ \mathrm{rk}(\langle F \rangle \sqcup_c \langle \mathcal{G} \rangle \sqcup_c [T]^{\leq 1}) &\leq \mathrm{rk}(\langle F \rangle) \dot{+} \mathrm{rk}(\langle G \rangle) + 1 < \max\{\iota(\mathrm{srk}(\mathcal{F})), \iota(\mathrm{srk}(\mathcal{G}))\} \\ &\leq \iota(\mathrm{srk}(\mathcal{B})). \end{aligned}$$

Since $\mathcal{B}$ has infinite rank and homogeneous, $\iota(\mathrm{srk}(\mathcal{B})) > \omega$, so it follows from Lemma 3.16 that

$$\mathrm{rk}(\langle \mathcal{F} \sqcup \mathcal{G} \rangle) \leq \mathrm{rk}((\langle \mathcal{F} \rangle \sqcup_a \langle \mathcal{G} \rangle) \odot_T (\langle \mathcal{F} \rangle \sqcup_c \langle \mathcal{G} \rangle \sqcup_c [T]^{\leq 1})) < \iota(\mathrm{srk}(\mathcal{F} \sqcup \mathcal{G})).$$

On the other hand, by Proposition 3.10 $\mathrm{Is}(\langle \mathcal{B} \rangle) = \mathrm{Is}(\mathcal{B}) = \mathrm{Is}(\mathcal{F}) \sqcup_a \mathrm{Is}(\mathcal{G}) \in \mathfrak{B}_a$, whether $\mathfrak{B}^a$ is compact or not. And, $\langle \mathcal{F} \sqcup \mathcal{G} \rangle \cap \mathrm{Ch}_c \subseteq (\langle \mathcal{F} \rangle \cap \mathrm{Ch}_c) \sqcup_c (\langle \mathcal{G} \rangle \cap \mathrm{Ch}_c) \sqcup_c [T]^{\leq 1} \in \mathfrak{B}_c$. If Ch_c is compact, then $\langle \mathcal{F} \sqcup \mathcal{G} \rangle \cap \mathrm{Ch}_c \in \mathfrak{B}^c$. When Ch_c is non-compact, since

$$\begin{aligned} \iota(\mathrm{srk}(\langle \mathcal{F} \sqcup \mathcal{G} \rangle \cap \mathrm{Ch}_c)) &= \max\{\iota(\mathrm{srk}_c(\mathcal{F} \cap \mathrm{Ch}_c)), \iota(\mathrm{srk}_c(\mathcal{G} \cap \mathrm{Ch}_c))\} \\ &= \iota(\mathrm{srk}_c(((\langle \mathcal{F} \rangle \cap \mathrm{Ch}_c) \sqcup_c (\langle \mathcal{G} \rangle \cap \mathrm{Ch}_c) \sqcup_c [T]^{\leq 1})) \end{aligned}$$

it follows from the property (B.2) of $\mathfrak{B}_c$ that $\langle\mathcal{F}\sqcup\mathcal{G}\rangle\cap\mathrm{Ch}_c\in\mathfrak{B}_c$, hence $\mathcal{B}$ satisfies (BT.2). This ends the proof of property (B.2′) of $\mathfrak{B}$. (B.3) is the content of Lemma 3.17. □

3.1. The operation $\odot_T$ and ranks

We compute an upper bound of the rank of the family $\mathcal{A}\odot_T\mathcal{C}$ in terms of the ranks of $\mathcal{A}$ and $\mathcal{C}$, respectively. Fix a tree T, a compact and hereditary family $\mathcal{C}$ on chains of T and a compact and hereditary family on immediate successors of nodes of T. As we have observed in (7) for finite trees, it is natural to expect an upper bound of the rank of $\mathcal{A}\odot_T\mathcal{C}$ by an exponential-like function of the rank of $\mathcal{A}$ and the rank of $\mathcal{C}$.

Definition 3.19. Given a countable ordinal number α, we define a function $f_\alpha:\omega_1\to\omega_1$ as follows:

$$\begin{aligned} f_\alpha(0) &:= 1\\ f_\alpha(\xi+1) &:= f_\alpha(\xi)\cdot(\max\{\alpha,\xi\}\cdot\omega)\\ f_\alpha(\xi) &:= \sup_{\eta<\xi} f_\alpha(\eta),\ \text{ when } \xi \text{ is limit.}\end{aligned}$$

Remark 3.20.

(a) f_α is a continuous strictly increasing mapping such that $f_\alpha(\xi)$ is sum-indecomposable for every α and ξ.

(b) $f_\alpha(\xi)\geq(\alpha\cdot\omega)^\xi$ always, and if $\xi\leq\alpha$ then $f_\alpha(\xi)=(\alpha\cdot\omega)^\xi$.

(c) Suppose that $\lambda>\omega$. We see that $f_\alpha(\xi)<\lambda$ for every $\alpha,\xi<\lambda$ if and only if λ is exp-indecomposable: Suppose that λ is closed under $f_\cdot(\cdot)$. Let $\alpha,\xi<\lambda$. Then $\alpha^\xi\leq f_\lambda(\xi)<\alpha$. Suppose that λ is exp-indecomposable. Since $\lambda>\omega$, this is equivalent to saying that $\omega^\lambda=\lambda$. Let $\alpha,\xi<\lambda$. Since $f_\cdot(\cdot)$ is increasing in both variables, and since λ is product-indecomposable, w.l.o.g. we assume that α is sum-indecomposable, i.e. $\alpha=\omega^{\alpha_0}$, and $\xi\leq\alpha<\lambda$. Then,

$$f_\alpha(\xi)=(\alpha\cdot\omega)^\xi=\omega^{(\alpha_0+1)\cdot\xi}<\omega^\lambda=\lambda.$$

Lemma 3.16 follows from Remark 3.20 (c) and the following.

Proposition 3.21. *Suppose that $\mathcal{F}=\langle\mathcal{F}\rangle$ and $\mathrm{Is}(\mathcal{F})$ are compact. Then*

$$\mathrm{rk}(\mathcal{F})<f_{\mathrm{rk}(\mathrm{Is}(\mathcal{F}))+1}(\mathrm{rk}(\mathcal{F}\cap\mathrm{Ch}_c)\cdot 3+2).$$

The proof of this result has several parts. Recall that given a family $\mathcal{F}$, $\langle\mathcal{F}\rangle_T=\{\langle s\rangle : s\in\mathcal{F}\}$ and $\langle\mathcal{F}\rangle_{\mathrm{is}-T}=\{\langle s\rangle_{\mathrm{is}} : s\in\mathcal{F}\}$.

Lemma 3.22. *Suppose that $\langle\mathcal{F}\rangle$ is countably ranked. Then*

$$\mathrm{rk}(\langle\mathcal{F}\rangle_T) = \mathrm{rk}(\langle\mathcal{F}\rangle) \leq \mathrm{rk}(\langle\mathcal{F}\rangle^{\mathrm{is}}) = \mathrm{rk}(\langle\mathcal{F}\rangle_{\mathrm{is}-T}).$$

Proof. This follows from the following.

Claim 3.22.1. *The following are equivalent for $x \subseteq T$ and $\xi < \omega_1$:*

(i) $x \in (\langle\mathcal{F}\rangle)^{(\xi)}$.
(ii) *There is some subtree τ containing x such that $\tau \in (\langle\mathcal{F}\rangle_T)^{(\xi)}$.*
(iii) *There is some subtree τ containing x such that $\tau \in (\langle\mathcal{F}\rangle_{\mathrm{is}-T})^{(\xi)}$.*

Proof of Claim. We only give the proof of the equivalence between (i) and (ii). The equivalence between (i) and (iii) is proved in a similar way. First of all, by definition $x \in \langle\mathcal{F}\rangle$ if and only if $\langle x\rangle \in \langle\mathcal{F}\rangle_T$. Since $\langle\mathcal{F}\rangle$ is hereditary, each derivated set $(\langle\mathcal{F}\rangle)^{(\eta)}$ is also, so if there is some τ containing x such that $\tau \in (\langle\mathcal{F}\rangle_T)^{(\xi)} \subseteq (\langle\mathcal{F}\rangle)^{(\xi)}$, then $x \in (\langle\mathcal{F}\rangle)^{(\xi)}$. Now suppose that $x \in (\langle\mathcal{F}\rangle)^{(\xi)}$, and let us prove that there is a subtree τ containing x such that $\tau \in (\langle\mathcal{F}\rangle_T)^{(\xi)}$. The case $\xi = 0$ was treated above. Suppose that ξ is limit, and let $(\xi_n)_n$ be an increasing sequence with supremum ξ. By inductive hypothesis, for every n there is some subtree τ_n of T such that $x \subseteq \tau_n$ and such that $\tau_n \in (\langle\mathcal{F}\rangle_T)^{(\xi_n)}$. By compactness, there is an infinite set M such that $(\tau_n)_{n\in M}$ is a Δ-sequence with root τ. A limit of subtrees is a subtree, hence τ is a subtree that contains x and $\tau \in \bigcap_{n\in M}(\langle\mathcal{F}\rangle_T)^{(\xi_n)}$, so $\tau \in (\langle\mathcal{F}\rangle_T)^{(\xi)}$. Suppose that $x \in (\langle\mathcal{F}\rangle_T)^{(\eta+1)}$. Choose a non-trivial Δ-sequence $(x_n)_n$ in $(\langle\mathcal{F}\rangle)^{(\eta)}$ with limit x, and for each n choose a subtree τ_n of T containing x_n and in $(\langle\mathcal{F}\rangle_T)^{(\eta)}$. Now find an infinite subset $M \subseteq \omega$ such that $(\tau_n)_n$ is a Δ-sequence with root τ. Since $x_n \subseteq \tau_n$, it follows that $(\tau_n)_{n\in M}$ is non-trivial, hence $x \subseteq \tau \in (\langle\mathcal{F}\rangle_T)^{(\eta+1)}$. □

Definition 3.23. Given a subtree U of T with root t_0, let

$$\mathrm{stem}(U) := \{t \in U \,:\, \text{every } u \in U \text{ is comparable with } t\}.$$

Given a chain c and a family $\mathcal{F}$, let

$$\mathcal{F}_c := \{x \in \mathcal{F} \,:\, x \text{ is an is-subtree of } T \text{ such that } c \sqsubseteq \mathrm{stem}(x)\}.$$

In the definition above, given two chains c and d, $c \sqsubseteq d$ means that c is an initial part of d with respect to the tree ordering $<$. Notice that $\mathrm{stem}(U)$ is a non-empty chain in U, because $t_0 \in \mathrm{stem}(U)$.

Lemma 3.24. *Suppose that $\mathcal{F} = \langle\mathcal{F}\rangle^{\mathrm{is}}$ is compact, and suppose that c is a $<$-chain of T. If $\mathrm{rk}(\mathcal{F}_c) \geq f_{\mathrm{rk}(\mathrm{Is}(\mathcal{F}))+1}(\xi)$, then $c \in (\mathcal{F} \cap \mathrm{Ch}_c)^{(\xi)}$.*

Proof. Set $\mathcal{A} := \mathrm{Is}(\mathcal{F})$ and $\mathcal{C} := \mathcal{F} \cap \mathrm{Ch}_c$. For each countable ordinal ξ, set $\beta_\xi := f_{\mathrm{rk}(\mathcal{A})+1}(\xi)$. Fix a chain $c \in \mathcal{C}$, and suppose that $\mathrm{rk}(\mathcal{F}_c) \geq \beta_\xi$, and we have to prove that $c \in \mathcal{C}^{(\xi)}$. The proof is by induction on ξ. The case $\xi = 0$ or limit are trivial. Suppose that $\xi = \eta + 1$. Since $\mathcal{C}$ is compact, we can assume without loss of generality that c is maximal such that $\mathrm{rk}(\mathcal{F}_c) \geq \beta_\xi$, i.e.

$$\text{if } c \subsetneq c' \in \mathcal{C}, \text{ then } \mathrm{rk}(\mathcal{F}_{c'}) < \beta_\xi. \tag{8}$$

If $c \neq \emptyset$ we define $t_c := \max c$ and the following families.

$$\begin{aligned}\mathcal{G} &:= \{x \in \mathcal{F}_c \,:\, \mathrm{Is}''_{t_c}(x) \subseteq x\}\\ \mathcal{H} &:= \{x \in \mathcal{F}_c \,:\, x \cap \mathrm{Is}_{t_c} = \emptyset\}.\end{aligned}$$

Both $\mathcal{G}$ and $\mathcal{H}$ are compact. Now since each $x \in \mathcal{F}_c$ is a is-subtree, it follows that if $\#\mathrm{Is}''_{t_c} x \geq 2$, then $\mathrm{Is}''_{t_c} x = \{t \wedge_{\mathrm{is}} u \,:\, t \neq u \in x \setminus \{t_c\}\} \subseteq x$. Hence, for $x \in \mathcal{F}_c$, either $\mathrm{Is}''_{t_c} x \subseteq x$ or $\mathrm{Is}''_{t_c} x \cap x = \emptyset$. This means that $\mathcal{F}_c = \mathcal{G} \cup \mathcal{H}$. Since β_ξ is sum-indecomposable, it follows that when $c \neq \emptyset$, then $\max\{\mathrm{rk}(\mathcal{G}), \mathrm{rk}(\mathcal{H})\} \geq \beta_\xi$, so we have the following two cases to consider.

Case 1. $c \neq \emptyset$ and $\mathrm{rk}(\mathcal{G}) \geq \beta_\xi$. Let now

$$\begin{aligned} I &:= \{u \in \mathrm{Is}_{t_c} \,:\, \mathrm{rk}(\mathcal{F}_{c \cup \{u\}}) \geq \beta_\eta\}\\ J &:= \mathrm{Is}_{t_c} \setminus I\\ \mathcal{G}_I &:= \{x \in \mathcal{G} \,:\, \mathrm{Is}''_{t_c} x \in \mathcal{A} \upharpoonright I\}\\ \mathcal{G}_J &:= \{x \in \mathcal{G} \,:\, \mathrm{Is}''_{t_c} x \in \mathcal{A} \upharpoonright J\}.\end{aligned}$$

Clearly $\mathcal{G} \subseteq \mathcal{G}_I \sqcup \mathcal{G}_J$. So, there are two subcases to consider.

Claim 3.24.1. $\mathrm{rk}(\mathcal{G}_I) \geq \beta_\xi$ *and* I *is infinite.*

It follows from this that given $u \in I$ we know by inductive hypothesis that $c \cup \{u\} \in \mathcal{C}^{(\eta)}$, so $c \in \mathcal{C}^{(\xi)}$, as desired.

Proof of Claim 3.24.1. We argue by contradiction. Suppose first that $\mathrm{rk}(\mathcal{G}_I) < \beta_\xi$. It follows that $\mathrm{rk}(\mathcal{G}_J) \geq \beta_\xi$. Let $\lambda : \mathcal{G}_J \to \mathcal{A} \upharpoonright J$ be defined by $\lambda(x) := \mathrm{Is}_{t_c}(x)$. This mapping is $\subseteq$-increasing and since $\lambda(x) = x \cap \mathrm{Is}_{t_c}$ for every $x \in \mathcal{G}_J$, it follows that λ is continuous. By Proposition 2.28 and the properties of $f_{\mathrm{rk}(\mathcal{A})+1}(\cdot)$ we obtain that

$$\beta_\eta \cdot (\mathrm{rk}(\mathcal{A}) \cdot \omega) \leq \beta_\xi \leq \mathrm{rk}(\mathcal{G}_J) < \sup_{y \in \mathcal{A} \upharpoonright J} (\mathrm{rk}\{x \in \mathcal{G}_J \,:\, \lambda(x) = y\} + 1) \cdot (\mathrm{rk}(\mathcal{A} \upharpoonright J) + 1).$$

So there must be $y \in \mathcal{A} \upharpoonright J$ such that $\mathrm{rk}\{x \in \mathcal{G}_J \,:\, \lambda(x) = y\} \geq \beta_\eta$. We also have that

$$\{x \in \mathcal{G}_J \,:\, \lambda(x) = y\} \subseteq \bigsqcup_{u \in y} \mathcal{F}_{c\cup\{u\}}. \tag{9}$$

Observe that $y \neq \emptyset$, because $\{x \in \mathcal{G}_J \,:\, \lambda(x) = \emptyset\} = \{c\}$ has rank 0. Hence, it follows from (9) that there must be $u \in y$ such that $\mathcal{F}_{c\cup\{u\}}$ has rank at least β_η, contradicting the fact that $u \in J$. Finally, suppose that $\mathrm{rk}(\mathcal{G}_I) \geq \beta_\xi$ but I is finite. Then,

$$\mathcal{G}_I \subseteq \bigcup_{K \subseteq I} \bigsqcup_{u \in K} \mathcal{F}_{c\cup\{u\}}$$

it follows that there is some $u \in I$ such that $\mathcal{F}_{c\cup\{u\}}$ has rank at least β_ξ, contradicting (8). $\square$

Case 2. $c = \emptyset$, or $c \neq \emptyset$ and $\mathrm{rk}(\mathcal{H}) \geq \beta_\xi$. In order to unify the argument, let $\widetilde{\mathcal{F}} = \mathcal{F}$ when $c = \emptyset$, and let $\widetilde{\mathcal{F}} = \mathcal{H}$ if $c \neq \emptyset$.

Since $\mathrm{rk}(\widetilde{\mathcal{F}}) \geq \beta_\xi$, given a β_ξ-uniform family $\mathcal{B}$ we can use Proposition 2.26 to can find $f : \mathcal{B} \to \widetilde{\mathcal{F}}$ continuous, 1–1 and $(\sqsubseteq, \subseteq)$-increasing. In particular, this means that $c \subsetneq f(s)$ for every $s \neq \emptyset$. Observe that if $c \subsetneq x \in \widetilde{\mathcal{F}}$, then there is $\min(x \setminus c)$: When $c = \emptyset$, then for such x, $\min(x \setminus c)$ is the root of x; if $c \neq \emptyset$ and $c \subsetneq x \in \mathcal{H}$, then let $t \in x \setminus c$, and let $u := \mathrm{Is}_{\max c}(t)$. Then for any other $t' \in x \setminus c$ we have that $\mathrm{Is}_{\max c}(t') = u$, since otherwise $u = t \wedge_{\mathrm{is}} t' \in x$, contradicting the fact that $x \in \mathcal{H}$. Hence $\min(x \setminus c) = \bigwedge_{t \in x\setminus c} t$. So, we can define $\lambda : \mathcal{B} \to \mathcal{C}_c$ as follows. Suppose that $c \subsetneq f(\emptyset)$. We define for $s \in \mathcal{B}$

$$\lambda(s) := \{t \in f(s) \,:\, t \leq \min(f(\emptyset) \setminus c)\}.$$

Suppose that $f(\emptyset) = c$. Let $\lambda(\emptyset) = c$; and for each $\emptyset \subsetneq s$, let

$$\lambda(s) := \{t \in f(s) \,:\, t \leq \min(f(\{\min s\}) \setminus c)\}.$$

Then, λ is well-defined, that is, $\lambda(s) \in \mathcal{C}_c$ for every $s \in \mathcal{B}$. It is also clear that is $(\sqsubseteq, \subseteq)$-increasing. Moreover λ is continuous: Suppose that $(s_n)_n$ is a Δ-system with root s such that $s < s_m \setminus s < s_n \setminus s$ for every $m < n$. Suppose first that $f(s) \neq c$. It follows that $\lambda(s_n) = f(s_n) \cap [0, \min(f(s) \setminus c)]$ and $\lambda(s) = f(s) \cap [0, \min(f(s) \setminus c)]$. Since f and the intersection operation are continuous, it follows that $\lambda(s_n) \to_n \lambda(s)$. Suppose that $f(s) = c$. This implies that $s = \emptyset$, and $\lambda(s) = c$. Also, $\lambda(s_n) := f(s_n) \cap [0, \min(f(\{\min s_n\}) \setminus c)]$. It follows that $\lambda(s_n) \to_n c = \lambda(\emptyset)$.

For every $s \in \mathcal{B}^{\max}$ let $\varphi(s)$ be the maximal initial part u of s such that $\lambda(u) = \lambda(\emptyset)$, and let $M \subseteq \omega$ be infinite such that $\mathcal{B}_0 := \varphi''(\mathcal{B} \restriction M)$ is a γ-uniform family on M for some $\gamma \leq \beta_\xi$.

Claim 3.24.2. $\gamma < \beta_\xi$.

Proof of Claim. Suppose otherwise that $\gamma = \beta_\xi$. Since $c \subsetneq \lambda(s)$ for every $s \neq \emptyset$ in $\mathcal{B}$, and since $\mathcal{B}_0 \neq \{\emptyset\}$, it follows that $c \subsetneq \lambda(s) = \lambda(\emptyset) =: c'$. By the definition of $\mathcal{B}_0$ it follows that the restriction of f to $\mathcal{B}_0$ satisfies that $f(x) \in \mathcal{F}_{c'}$. Consequently, $\mathrm{rk}(\mathcal{F}_{c'}) \geq \gamma = \beta_\xi$, contradicting (8). $\square$

So, $\gamma < \beta_\xi$. Let $\mathcal{B}_1$ and $\mathcal{B}_2$ be β_η-uniform and ξ-uniform families on M, respectively. Since by definition $\beta_\xi \geq \beta_\eta \cdot (\xi \cdot \omega)$, it follows that $\gamma, \beta_\eta \cdot \xi < \beta_\xi$. Since β_ξ is sum-indecomposable, it follows that $(\beta_\eta \cdot \xi) + \gamma < \beta_\xi$. Hence, by the properties of the uniform families (Proposition 2.10) we obtain that there is $N \subseteq M$ such that

$$((\mathcal{B}_1 \otimes \mathcal{B}_2) \oplus \mathcal{B}_0) \restriction N \subseteq \mathcal{B}.$$

Fix now $s \in (\mathcal{B}_0 \restriction N)^{\max}$, set $N_s := N/s$, and $\lambda_s : (\mathcal{B}_1 \otimes \mathcal{B}_2) \restriction N_s \to \mathcal{C}_c$, $\lambda_s(r) := \lambda(s \cup r)$. Going towards a contradiction, suppose that $c \notin \mathcal{C}^{(\xi)}$. Then it follows that $\mathrm{rk}(\mathcal{C}_c) < \xi$. Since λ_s is $(\sqsubseteq, \subseteq)$-increasing and continuous, it follows from Lemma 2.27 that there is some $s < r$ finite and some $r < P$ infinite, $r \cup P \subseteq N_s$, such that $\{r\} \sqcup \mathcal{B}_1 \restriction P \subseteq (\mathcal{B}_1 \otimes \mathcal{B}_2) \restriction N$ and such that λ_s is constant on $\{r\} \sqcup \mathcal{B}_1 \restriction P$ with value $d := \lambda(s \cup r) \in \mathcal{C}_c$. Since $\mathcal{B}_1 \restriction P$ contains non-empty elements q, it follows that $s \sqsubset s \cup r \cup q$, so, since $s \in \mathcal{B}_0^{\max}$,

$$c \subseteq \lambda(\emptyset) = \lambda(s) \subsetneq \lambda(s \cup r \cup q) = \lambda(s \cup r).$$

On the other hand, the mapping $f_0 : \mathcal{B}_2 \restriction P \to \widetilde{\mathcal{F}}_d \subseteq \mathcal{F}_d$, $f_0(q) := f(s \cup r \cup q)$, witnesses that $\mathrm{rk}(\mathcal{F}_d) \geq \beta_\eta$, so by inductive hypothesis, $d \in \mathcal{C}^{(\eta)}$. In this way we can find $s < r_0 < r_1 < \cdots < r_n < \cdots$ such that $\lambda(s \cup r_n) \in \mathcal{C}^{(\eta)}$ and

$$c \subseteq \lambda(\emptyset) = \lambda(s) \subsetneq \lambda(s \cup r_n) \tag{10}$$

for every n. Since λ is continuous and $s \cup r_n \to_n s$, it follows that $\lambda(s \cup r_n) \to_n \lambda(s) = \lambda(\emptyset)$ and non-trivially, by (10). Hence, $\lambda(\emptyset) \in \mathcal{C}^{(\xi)}$, and so $c \in \mathcal{C}^{(\xi)}$, because $\mathcal{C}$ is hereditary and $c \subseteq \lambda(\emptyset)$, contradicting our hypothesis. $\square$

Proof of Proposition 3.21. Set $\mathcal{C} := \mathcal{F} \cap \mathrm{Ch}_c$, $\mathcal{D} := \langle \mathcal{F} \rangle^{\mathrm{is}} \cap \mathrm{Ch}_c$ and $\mathcal{A} := \mathrm{Is}(\mathcal{F}) = \mathrm{Is}(\langle \mathcal{F} \rangle) = \mathrm{Is}(\langle \mathcal{F} \rangle^{\mathrm{is}})$. It follows from Lemma 3.24 that $\mathrm{rk}(\langle \mathcal{F} \rangle^{\mathrm{is}}) < f_{\mathrm{rk}(\mathcal{A})+1}(\mathrm{rk}(\mathcal{D}) + 1))$, so the proof will be finished once we have the following.

Claim 3.24.3. $\mathrm{rk}(\mathcal{D}) \leq \mathrm{rk}(\mathcal{C}) \cdot 3 + 1$.

Proof of Claim. As in the proof of the Corollary 3.9, let Is be the collection of all immediate successors of T, and let $\pi : \mathrm{Is} \to T$ be the mapping assigning to each $t \in \mathrm{Is}$ its immediate predecessor $\pi(t)$. Then $\mathcal{D} \subseteq \mathcal{C} \sqcup \pi^{-1}(\mathcal{C})$. It follows from Proposition 2.28 that

$$\mathrm{rk}(\lambda^{-1}(\mathcal{C})) < \sup_{x \in \mathcal{C}} (\mathrm{rk}(\{s \in \lambda^{-1}(\mathcal{C}) \,:\, \lambda(s) \subseteq x\}) + 1) \cdot (\mathrm{rk}(\mathcal{C}) + 1). \tag{11}$$

Now, given $x \in \mathcal{C}$, let us see that $\mathrm{rk}(\{s \in \lambda^{-1}(\mathcal{C}) \,:\, \lambda(s) \subseteq x\}) \leq 1$: Set $\mathcal{E} = \{s \in \lambda^{-1}(\mathcal{C}) \,:\, \lambda(s) \subseteq x\}$. Every $s \in \mathrm{Is}$ such that $\pi(s) \subseteq x$ can be written as

$s = \{\mathrm{Is}_t(\max x)\}_{t\in\pi(s)\setminus\{\max x\}} \cup \overline{s}$ with $\overline{s} \subseteq \mathrm{Is}_{\max x}$ such that $\#\overline{s} \le 1$. We see that $\mathcal{E}' \subseteq \{\{\mathrm{Is}_t(\max x)\}_{t\in y} \,:\, y \subseteq x \setminus \{\max x\}\}$, so $\mathcal{E}'$ is finite and $\mathrm{rk}(\mathcal{E}) \le 1$: Otherwise, suppose that $s \in \mathcal{E}'$ is such that $s \cap \mathrm{Is}_{\max x} = \{u\}$. Let $(s_n)_n$ be a non-trivial Δ-sequence in $\mathcal{E}$ with root s. It follows that $s_n \cap \mathrm{Is}_{\max x} = \{u\}$ for every n. Since x is finite, we may assume that $\lambda(s_n) = y \subseteq x$ for every n. Hence $s_n = \{u\} \cup \{\mathrm{Is}_t(\max x)\}_{t\in y\setminus\{\max x\}}$, so $(s_n)_n$ is a constant sequence, a contradiction. It follows from (11) that $\mathrm{rk}(\lambda^{-1}(\mathcal{C})) < 2(\mathrm{rk}(\mathcal{C}) + 1) = 2\mathrm{rk}(\mathcal{C}) + 2$, so $\mathrm{rk}(\mathcal{D}) \le \mathrm{rk}(\mathcal{C}) \dot{+} 2\mathrm{rk}(\mathcal{C}) + 1 \le \mathrm{rk}(\mathcal{C}) \cdot 3 + 1$, as desired. □

3.2. Canonical form of sequences of finite subtrees

We prove here Lemma 3.17, that is if $\mathcal{F} \in \mathfrak{B}$ and $\mathcal{H} \in \mathfrak{S}$, then for every sequence $(s_n)_{n<\omega}$ in $\mathcal{F}$ there is an infinite subset $M \subseteq \omega$ such that $\bigcup_{n\in x} s_n \in \mathcal{F} \times \mathcal{H}$ for every $x \in \mathcal{H} \restriction M$. The proof is based on a combinatorial analysis of sequences of finite subtrees of T, done in the next Lemma 3.33, that uses crucially the Ramsey property. This relation between the Ramsey theory, uniform fronts and BQO-WQO theory of trees is well studied and has produced fundamental results like Kruskal Theorem [11] (see also Nash-Williams paper [18]) and Laver Theorem [12].

We start with some simple analysis of the tree generated by two finite subtrees τ_0 and τ_1.

Definition 3.25. Given τ_0, τ_1 two finite subtrees of T and $t \in \tau_0 \cup \tau_1$, let

$$\begin{aligned}
i(t) &:= \min\{i \in 2 \,:\, t \in \tau_i\},\\
\pi(\tau_0,\tau_1) &:= \{w \in \langle \tau_0 \cup \tau_1\rangle \,:\, (\tau_0 \setminus \tau_1)_{\ge w} \ne \emptyset \text{ and } (\tau_1 \setminus \tau_0)_{\ge w} \ne \emptyset\},\\
\sigma(\tau_0,\tau_1) &:= \pi(\tau_0,\tau_1)_{\max},\\
\overline{\sigma}(\tau_0,\tau_1) &:= \{t_0 \wedge t_1 \,:\, t_0 \perp t_1 \text{ are in } \tau_0 \cup \tau_1 \text{ and } t_0 \wedge t_1 \notin \tau_0 \cup \tau_1\}.
\end{aligned}$$

Definition 3.26. For each $w \in \overline{\sigma}(\tau_0,\tau_1)$, fix $t^0(w) \in (\tau_0 \setminus \tau_1)_{\ge w}$ and $t^1(w) \in (\tau_1 \setminus \tau_0))_{\ge w}$ such that $w = t^0(w) \wedge t^1(w) \notin \tau_0 \cup \tau_1$ and whenever $w \in \overline{\sigma}(\tau_0,\tau_1)$, then $t^0(w) \perp t^1(w)$.

Proposition 3.27. $\overline{\sigma}(\tau_0,\tau_1) \subseteq \sigma(\tau_0,\tau_1)$.

Proof. Clearly $\overline{\sigma}(\tau_0,\tau_1) \subseteq \pi(\tau_0,\tau_1)$, so given $w \in \overline{\sigma}(\tau_0,\tau_1)$, let us prove that w is maximal there, so that $w \in \sigma(\tau_0,\tau_1)$. Suppose otherwise that there is $w' \in \sigma(\tau_0,\tau_1)$ such that $w < w'$ and let us get a contradiction. If $\mathrm{Is}_w(t^0(w)) = \mathrm{Is}_w(w')$, then $w = t^0(w) \wedge t^1(w) = t^1(w') \wedge t^1(w) \in \tau_1$, a contradiction. Otherwise, $w = t^0(w) \wedge t^1(w) = t^0(w) \wedge t_0(w') \in \tau_0$, contradicting the hypothesis. □

Definition 3.28. Given two finite subtrees τ_0, τ_1, let

$$\varrho := \varrho(\tau_0,\tau_1) = \tau_0 \cap \tau_1$$

$$\begin{aligned}\overline{\varrho} &:= \overline{\varrho}(\tau_0,\tau_1) := \langle\tau_0\rangle_{\mathrm{is}} \cap \langle\tau_1\rangle_{\mathrm{is}}\\ \varrho_0 &:= \overline{\varrho}\cup\{0\}\\ (\tau_0,\tau_1)_\infty &:= \{u\in(\varrho_0)_{\max} \,:\, (\pi(\tau_0,\tau_1))_{\geq u}\neq\emptyset\}.\end{aligned}$$

Proposition 3.29. *For every $u\in(\tau_0,\tau_1)_\infty$ one has that $\#(\sigma(\tau_0,\tau_1))_{\geq u}=1$.*

Proof. If $u\in(\tau_0,\tau_1)_\infty$, then clearly $(\sigma(\tau_0,\tau_1))_{\geq u}\neq\emptyset$. Suppose there are $w_0\neq w_1\in(\sigma(\tau_0,\tau_1))_{\geq u}$ for some $u\in(\tau_0,\tau_1)_\infty$. Observe that $w_0\perp w_1$, since both of them are maximal in $\pi(\tau_0,\tau_1)$. Hence,

$$u\leq t^0(w_0)\wedge t^0(w_1)=t^1(w_0)\wedge t^1(w_1),$$

so that

$$u<t^0(w_0)\wedge_{\mathrm{is}} t^0(w_1)=t^1(w_0)\wedge_{\mathrm{is}} t^1(w_1)\in\langle\tau_0\rangle_{\mathrm{is}}\cap\langle\tau_1\rangle_{\mathrm{is}}=\overline{\varrho}\subseteq\varrho_0,$$

contradicting the maximality of u in ϱ_0. □

Definition 3.30. For every $u\in(\tau_0,\tau_1)_\infty$, let $\varpi_{\tau_0,\tau_1}(u)$ be the unique element of $(\sigma(\tau_0,\tau_1))_{\geq u}$.

Proposition 3.31. *For every $w\in\pi(\tau_0,\tau_1)$, either there is $u\in(\tau_0,\tau_1)_\infty$ such that $w\leq\varpi_{\tau_0,\tau_1}(u)$, or else there is $u\in(\varrho_0)_{\max}$ such that $w<u$. Consequently, $\overline{\sigma}(\tau_0,\tau_1)\subseteq$ ran (ϖ_{τ_0,τ_1}).*

Proof. Given $w\in\pi(\tau_0,\tau_1)$, suppose there is no $u\in(\varrho_0)_{\max}$ such that $w<u$ and let

$$u:=\max\{v\in\varrho_0 : v\leq w\}.$$

Let us prove that u is maximal in ϱ_0 so that w witnesses that $u\in(\tau_0,\tau_1)_\infty$. Suppose by contradiction that there is $v\in(\varrho_0)_{\max}$ such that $u<v$ and in particular, $v\in\overline{\varrho}$. Notice that the definition of u implies that $v\perp w$. Hence, $u\leq w\wedge v<w$, so that $u<w\wedge_{\mathrm{is}} v\leq w$. But $w\wedge_{\mathrm{is}} v=t^0(w)\wedge_{\mathrm{is}} v=t^1(w)\wedge_{\mathrm{is}} v$, so that $w\wedge_{\mathrm{is}} v\in\overline{\varrho}$ and we get a contradiction with the maximality of u below w. It follows that $u\in(\tau_0,\tau_1)_\infty$ and $w\leq\varpi_{\tau_0,\tau_1}(u)$.

Finally, suppose that $w\in\overline{\sigma}(\tau_0,\tau_1)$. Then, $w\in\sigma(\tau_0,\tau_1)$, by Proposition 3.27. It is easy to see from the definition of $\overline{\sigma}(\tau_0,\tau_1)$ that there is no $u\in(\varrho_0)_{\max}$ such that $w<u$. Hence $w\leq\varpi_{\tau_0,\tau_1}(u)$ for some $u\in(\tau_0,\tau_1)_\infty$ and it follows from the maximality of w that $w=\varpi_{\tau_0,\tau_1}(u)$. □

The following result guarantees that the new points of the tree generated by two finite subtrees τ_0 and τ_1 are given by the function ϖ_{τ_0,τ_1} and hence, they are controlled by the maximal elements of $(\tau_0,\tau_1)_\infty$.

Corollary 3.32. $\langle \tau_0 \cup \tau_1 \rangle = \tau_0 \cup \tau_1 \cup \{\varpi_{\tau_0,\tau_1}(u) \,:\, u \in (\tau_0, \tau_1)_\infty\}$.

Proof. If $w \in \langle \tau_0 \cup \tau_1 \rangle \setminus (\tau_0 \cup \tau_1)$, then there are $t_0 \in \tau_0 \setminus \tau_1$ and $t_1 \in \tau_1 \setminus \tau_0$ such that $w = t_0 \wedge t_1$ and notice that $t_0 \wedge t_1 \in \overline{\sigma}(\tau_0, \tau_1)$. Then, by Proposition 3.31, there is $u \in (\tau_0, \tau_1)_\infty$ such that $t_0 \wedge t_1 = \varpi_{\tau_0,\tau_1}(u)$. The other inclusion follows directly from the definitions. □

To prove that $\times$ is a multiplication we have to deal with the tree generated by a sequence of finite subtrees. Given a sequence $(\tau_k)_k$ of finite subtrees and $M \subseteq \omega$, let τ_M be the subtree generated by $\bigcup_{k\in M} \tau_k$. In order to be able to control the chains and the immediate successors of some τ_M, we will first find some suitable infinite M such that the subsequence $(\tau_k)_{k\in M}$ has some uniformity respective to the new points of τ_M. This is the content of the next result, that can be seen as a generalization of Proposition 3.6, which guarantees the existence of an infinite fan, chain or comb inside any infinite subset of a tree.

If we assume each τ_k to be a singleton $\{t_k\}$ and apply Proposition 3.6 to get an infinite M such that $\{t_k : k \in M\}$ is a fan, a chain or a comb, then the corresponding tree τ_M is given by $\{t_k \,:\, k \in M\} \cup \{w\}$, $\{t_k \,:\, k \in M\}$ or $\{t_k \,:\, k \in M\} \cup \{w_k \,:\, k \in M\}$, respectively. The case (2.1) corresponds to $\{t_k : k \in M\}$ being a comb, so that the new points $\{\varpi_k : k \in M\}$ form a chain; case (2.2) corresponds to $\{t_k : k \in M\}$ being a fan with root w which is the only new point; case (2.3) corresponds to $\{t_k : k \in M\}$ being a chain and no new points ($\varpi_k = t_k$); and case (2.4) corresponds to $t_k = t_{k'}$ for all $k, k' \in M$.

In the next, after refining the sequence to get a fixed τ_∞ and $\varpi_i(u) := \varpi_{i,j}(u) = \varpi_{i,k}(u)$ for $i < j < k$, each of these four cases might happen for each of the sequences of points $(\varpi_k(u))_{k\in M}$.

Theorem 3.33 *(Canonical form of sequences of subtrees). Suppose that $(\tau_k)_k$ is a sequence of finite subtrees of T forming a Δ-system with root ϱ and such that $(\langle\tau_k\rangle_{\mathrm{is}})_k$ forms a Δ-system with finite root $\overline{\varrho}$. Then there is a subsequence $(\tau_k)_{k\in M}$ such that*

(1) *For every $i \neq j$ and $k \neq l$ in M one has that*

$$\tau_\infty := (\tau_i, \tau_j)_\infty = (\tau_k, \tau_l)_\infty \tag{12}$$

(2) *Let $u \in \tau_\infty$. For each $i < j$ write $\varpi_{i,j}(u) := \varpi_{\tau_i,\tau_j}(u)$. Then $\varpi_i(u) := \varpi_{i,j}(u) = \varpi_{i,k}(u)$ for every $i < j < k$, and $\varpi_i(u) \leq \varpi_j(u)$ for every $i \leq j$.*
Moreover, one of the following holds.
(2.1) *$\varpi_i(u) < \varpi_j(u)$ for every $i < j$ and $\varpi_i(u) \notin \bigcup_k \tau_k$ for every $i < j$.*
(2.2) *$w(u) := \varpi_i(u) = \varpi_j(u) \notin \bigcup_k \tau_k$ for every i.*
(2.3) *$\varpi_i(u) < \varpi_j(u)$ and $\varpi_i(u) \in \tau_i \setminus \varrho$ for every $i < j$.*
(2.4) *$w(u) := u = \varpi_i(u) = \varpi_j(u) \in \varrho$ for every $i < j$.*

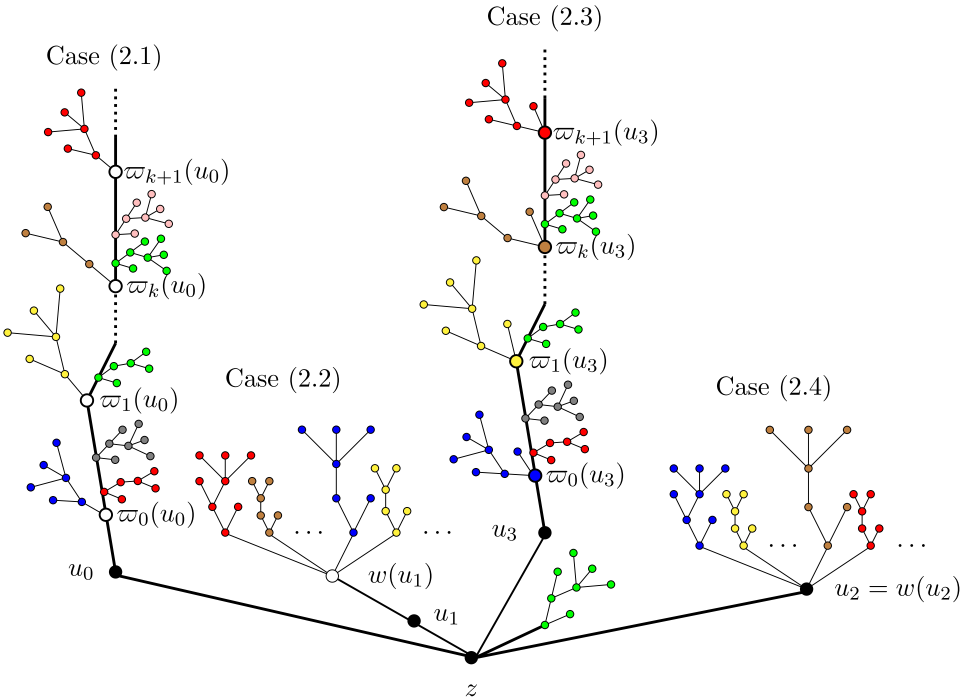

Fig. 3. A well placed sequence $(\tau_k)_k$. (For interpretation of the colors in the figure(s), the reader is referred to the web version of this article.)

Definition 3.34. We call a Δ-sequence $(\tau_k)_k$ of root ϱ such that $(\langle\tau_k\rangle_{\mathrm{is}})_k$ is a Δ-sequence of root $\overline{\varrho}$ satisfying (1) and (2) of Theorem 3.33 above a *well-placed* sequence. In this case, let τ_∞^0 be the set of those $u \in \tau_\infty$ such that $w(u) = \varpi_i(u) = \varpi_j(u)$ for every $i < j$ in ω and let $\tau_\infty^1 := \tau_\infty \setminus \tau_\infty^0$. For each $u \in \tau_\infty^1$, let $z(u) := \sup_{i\in\omega} \varpi_i(u)$.

Given $I \subseteq \omega$, we use the terminology τ_I to denote $\langle\bigcup_{k\in I} \tau_k\rangle$.

Each color in the next figure (Fig. 3) corresponds to one of the elements of the sequence: blue nodes belong to the subtree τ_0, yellow nodes to τ_1, and so on. Black is used to denote elements of the extended root $\overline{\varrho}$ and white, to nodes not belonging to any of the subtrees.

Proof of Theorem 3.33. We will apply the Ramsey Theorem and refine the sequence $(\tau_k)_{k\in\omega}$ finitely many times in order to get the desired subsequence $(\tau_k)_{k\in M}$.

First, since $\overline{\varrho}$ is finite and each $(\tau_i, \tau_j)_\infty \subseteq \varrho_0 = \overline{\varrho} \cup \{0\}$, it follows from the Ramsey Theorem that we may assume, by passing to a subsequence $(\tau_k)_{k\in M}$, that (1) holds. Now fix $u \in \tau_\infty$. Applying the Ramsey Theorem and passing again to a subsequence, we may assume that exactly one of the following holds:

(a1) $\varpi_{i,j}(u) \notin \varrho$ for every $i < j$ in M.
(b1) $\varpi_{i,j}(u) \in \varrho$ for every $i < j$ in M.

If (b1) holds, since ϱ is finite, we may pass to a further subsequence and get that $\varpi_{i,j}(u) = \varpi_{k,l}(u)$ for every $i < j$ and $k < l$ in M. In particular we get (2.4). From now on we will assume that (a1) holds and prove that we have one of the other three cases (2.1), (2.2) or (2.3).

Claim 3.34.1. *For every $i < j$ in M and $k \in M \setminus \{i, j\}$, one has that $\varpi_{i,j}(u) \notin \tau_k$.*

Proof of Claim. We color a triple $i < j < k$ by 0 if $\varpi_{i,j}(u) \in \tau_k$, by 1 if $\varpi_{i,j}(u) \notin \tau_k$ and $\varpi_{i,k}(u) \in \tau_j$, by 2 if $\varpi_{i,j}(u) \notin \tau_k$, $\varpi_{i,k}(u) \notin \tau_j$ and $\varpi_{j,k}(u) \in \tau_i$ and by 3 otherwise. By the Ramsey Theorem, we may assume that all triples in M are monochromatic. We prove that its color is 3. In the other two cases, there are $i < j$ and $k \neq l$ such that

$$\varpi_{i,j}(u) \in \tau_k \cap \tau_l = \varrho$$

which contradicts (a1). $\square$

For each $i < j$, let $t^i_{i,j}(u) \in \tau_i \setminus \varrho$ and $t^j_{i,j}(u) \in \tau_j \setminus \varrho$ be such that

$$\varpi_{i,j}(u) = t^i_{i,j}(u) \wedge t^j_{i,j}(u).$$

Since each τ_i is finite and $t^i_{i,j}(u) \in \tau_i$, we may assume by the Ramsey Theorem that

$$t_i(u) := t^i_{i,j}(u) = t^i_{i,k}(u) \text{ for every } i < j < k.$$

Hence, for each $i < j < k$ in M, $\varpi_{i,j}(u)$ and $\varpi_{i,k}(u)$ are comparable since they are both below $t_i(u)$.

Claim 3.34.2. *By passing to an infinite subset of M, we may assume that $\varpi_{i,j}(u) = \varpi_{i,k}(u)$ for every $i < j < k$ in M.*

Proof of Claim. By the Ramsey Theorem, we may assume that one of the following holds:

(a2) $\varpi_{i,j}(u) = \varpi_{i,k}(u)$ for every $i < j < k$ in M.
(b2) $\varpi_{i,j}(u) < \varpi_{i,k}(u)$ for every $i < j < k$ in M.
(c2) $\varpi_{i,j}(u) > \varpi_{i,k}(u)$ for every $i < j < k$ in M.

Notice that (c2) is impossible, since trees have no infinite strictly decreasing chains. We claim that (b2) is also impossible and therefore, (a2) holds. Given $i < j < k$, if (b2) holds, then $\varpi_{i,j}(u) \leq t^j_{i,j}(u), t^k_{i,k}(u)$, so that $\varpi_{i,j}(u) \in \pi(\tau_j, \tau_k)$. By the maximality of $\varpi_{j,k}(u)$ in $\pi(\tau_j, \tau_k)$, we get that $\varpi_{i,j}(u) \leq \varpi_{j,k}(u)$. Then, $\varpi_{i,j}(u) \leq \varpi_{i,k}(u) \wedge \varpi_{j,k}(u)$.

If $\varpi_{i,j}(u) < \varpi_{i,k}(u) \wedge \varpi_{j,k}(u)$, then the fact that $\varpi_{i,k}(u) \wedge \varpi_{j,k}(u) \in \pi(\tau_i, \tau_j)$ contradicts the fact that $\varpi_{i,j}(u)$ is maximal in $\pi(\tau_i, \tau_j)$. If $\varpi_{i,j}(u) = \varpi_{i,k}(u) \wedge \varpi_{j,k}(u)$, then we get that $\varpi_{i,j} \in \tau_k$, which is a contradiction with Claim 3.34.1. Therefore, (b2) cannot be true and we conclude that (a2) holds. □

Let now $\varpi_i(u) := \varpi_{i,j}(u)$ for every $i < j$.

Claim 3.34.3. *By passing to an infinite subset of M, we may assume that either $\varpi_i(u) = \varpi_j(u)$ for every $i < j$ in M, or $\varpi_i(u) < \varpi_j(u)$ for every $i < j$ in M.*

Proof of Claim. By the Ramsey Theorem, we may assume that one of the following holds:

(a3) $\varpi_i(u) = \varpi_j(u)$ for every $i < j$ in M.
(b3) $\varpi_i(u) < \varpi_j(u)$ for every $i < j$ in M.
(c3) $\varpi_i(u) > \varpi_j(u)$ for every $i < j$ in M.
(d3) $\varpi_i(u)$ and $\varpi_j(u)$ are incompatible for every $i < j$ in M.

Again (c3) is impossible, since trees have no infinite strictly decreasing chains. We claim that (d3) is also impossible and therefore, either (a3) or (b3) holds. If (d3) holds, then we have that for $i < j < k < l$, $u \leq \varpi_i(u) = t^k_{i,k}(u) \wedge t^k_{j,k}(u) < t^k_{i,k}(u) \wedge_{\mathrm{is}} t^k_{j,k}(u) \in \langle \tau_k \rangle_{\mathrm{is}}$ and $t^k_{i,k}(u) \wedge_{\mathrm{is}} t^k_{j,k}(u) = t^l_{i,l}(u) \wedge_{\mathrm{is}} (u) t^l_{j,l} \in \langle \tau_l \rangle_{\mathrm{is}}$, contradicting the maximality of u in ϱ_0. Hence, either (a3) or (b3) holds. □

In any case, by Claim 3.34.1 we may assume that $\varpi_i(u) \notin \tau_k$ for $i \neq k$. Hence, by the Ramsey Theorem, we may assume that one of the following holds:

(a4) $\varpi_i(u) \notin \bigcup_k \tau_k$ for every i in M.
(b4) $\varpi_i(u) \in \tau_i \setminus \varrho$ for every i in M.

Now, if (a3) holds, (b4) cannot hold and we get that u satisfies (2.2). If (b3) and (a4) hold, we get (2.1) and if (b3) and (b4) hold, then we get (2.3). This concludes the proof of the theorem. □

Corollary 3.35. *Given a well-placed sequence $(\tau_k)_{k<\omega}$, $I \subseteq \omega$ and $u \in \tau^1_\infty$, we have that:*

(i) *For every $t \in [u, z(u)]$, there is $i \in I$ such that if $t' \in (\tau_I)_{>t}$ with $\mathrm{Is}_t(t') \neq \mathrm{Is}_t(z(u))$, then $t' \in \tau_i$.*
(ii) $(\tau_I)_{<z(u)} \subseteq \bigcup_{k \in I} (\tau_k \cup \{\varpi_k(u)\})$.

Proof. (i): Let $u \leq t \leq z(u)$ and suppose there are $i_0 < i_1$ in I and $t_j \in (\tau_{i_j} \setminus \varrho)_{>t}$ such that $\mathrm{Is}_t(t_j) \neq \mathrm{Is}_t(z(u))$, $j = 0, 1$. Let $w = t_0 \wedge t_1$ and notice that $w \in \pi(\tau_{i_0}, \tau_{i_1})$. By Proposition 3.31, either there is $v \in \tau_\infty$ such that $w \leq \varpi_{i_0,i_1}(v)$ or there is $v \in (\varrho_0)_{\max}$ such that $w < v$. Since $u \in (\varrho)_{\max}$ and $u \leq t \leq w$, the second alternative cannot hold and the first alternative holds with $u = v$. Since $w \leq \varpi_{i_0,i_1}(u) = \varpi_{i_0}(u) < z(u)$, it follows that $w = t$. But then, $t = \varpi_{i_1}(u) \wedge t_1 \in \tau_{i_1}$, which is impossible both in cases (2.1) and (2.3). Finally, notice that if $t' \in (\tau_I)_{>t} \setminus \bigcup_{k\in I} \tau_k$ is such that $\mathrm{Is}_t(t') \neq \mathrm{Is}_t(z(u))$, then there are $i_0 < i_1$ in I and $t_j \in (\tau_{i_j} \setminus \varrho)_{>t'}$, so that $\mathrm{Is}_t(t_j) = \mathrm{Is}_t(t') \neq \mathrm{Is}_t(z(u))$, $j = 0, 1$, which we just proved that cannot happen.

(ii): If $t \in \tau_I \setminus \bigcup_{k\in I} \tau_k$, by Corollary 3.32, there are $i_0 < i_1$ in I such that $t = \varpi_{i_0,i_1}(v)$ for some $v \in \tau_\infty$. If $t \leq z(u)$, the maximality of u and v guarantee that $u = v$. Hence, $t = \varpi_{i_0,i_1}(u) = \varpi_{i_0}(u)$. $\square$

Corollary 3.36. *Given a well-placed sequence $(\tau_k)_{k<\omega}$, $I \subseteq \omega$ and $u \in \tau^0_\infty$, we have that:*

(i) *For every $t \in \mathrm{Is}_{w(u)}$ there is a $i \in I$ such that $(\tau_I)_{\geq t} \subseteq \tau_i$.*
(ii) *For every $t \in [u, w(u)[$ there is $i \in I$ such that if $t' \in (\tau_\omega)_{>t}$ is such that $\mathrm{Is}_t(t') \neq \mathrm{Is}_t(w(u))$, then $t' \in \tau_i$.*
(iii) *$(\tau_I)_{<w(u)} \subseteq \bigcup_{k\in I} \tau_k$.*

Proof. (i): Let $t \in \mathrm{Is}_{w(u)}$ and suppose there are $i_0 < i_1$ in I and $t_j \in (\tau_{i_j} \setminus \varrho)_{\geq t}$, $j = 0, 1$. Let $w = t_0 \wedge t_1$ and notice that $w \in \pi(\tau_{i_0}, \tau_{i_1})$. By Proposition 3.31, either there is $v \in \tau_\infty$ such that $w \leq \varpi_{i_0,i_1}(v)$ or there is $v \in (\varrho_0)_{\max}$ such that $w < v$. Since $u \in (\varrho)_{\max}$ and $u \leq w(u) < t \leq w$, the second alternative cannot hold and the first alternative holds with $u = v$, which cannot hold as well, since $\varpi_{i_0}(u) = w(u) < t \leq w$. Finally, notice that if $t' \in (\tau_I)_{>t} \setminus \bigcup_{k<\omega} \tau_k$, then there are $i_0 < i_1$ in I and $t_j \in (\tau_{i_j} \setminus \varrho)_{>t'}$, so that $t_j > t$, $j = 0, 1$, which we just proved that cannot happen.

(ii): Given $t \in [u, w(u)[$, suppose there are $i_0 < i_1$ in I and $t_j \in (\tau_{i_j} \setminus \varrho)_{>t}$ such that $\mathrm{Is}_t(t_j) \neq \mathrm{Is}_t(w(u))$, $j = 0, 1$. Let $w = t_0 \wedge t_1$ and notice that $w \in \pi(\tau_{i_0}, \tau_{i_1})$. By Proposition 3.31, either there is $v \in \tau_\infty$ such that $w \leq \varpi_{i_0,i_1}(v)$ or there is $v \in (\varrho_0)_{\max}$ such that $w < v$. Since $u \in (\varrho)_{\max}$ and $u \leq t \leq w$, the second alternative cannot hold and the first alternative holds with $u = v$. Since $w \leq \varpi_{i_0,i_1}(u) = \varpi_{i_0}(u) = w(u)$, it follows that $w = t = w(u)$, a contradiction. Finally, notice that if $t' \in (\tau_I)_{>t} \setminus \bigcup_{k\in I} \tau_k$ is such that $\mathrm{Is}_t(t') \neq \mathrm{Is}_t(w(u))$, then there are $i_0 < i_1$ in I and $t_j \in (\tau_{i_j} \setminus \varrho)_{>t'}$, so that $\mathrm{Is}_t(t_j) = \mathrm{Is}_t(t') \neq \mathrm{Is}_t(w(u))$, $j = 0, 1$, which we just proved that cannot happen.

(iii): If $t \in \tau_I \setminus \bigcup_{k\in I} \tau_k$, by Corollary 3.32, there are $i_0 < i_1$ in I such that $t = \varpi_{i_0,i_1}(v)$ for some $v \in \tau_\infty$. Hence, if $t < w(u)$, the maximality of u and v guarantee that $u = v$. Hence, $t = \varpi_{i_0,i_1}(u) = \varpi_{i_0}(u) = w(u)$, a contradiction. $\square$

Corollary 3.37. *Let $(\tau_k)_{k\in\omega}$ be a well-placed sequence of finite subtrees of T. There are finite sets $\mathfrak{a}, \mathfrak{f} \subseteq T$ and, for each $z \in \mathfrak{f}$, there is a chain $\{\varpi_i(z) : i \in \omega\}$ such that for any finite $\emptyset \neq I \subseteq \omega$, the following hold:*

(i) *For every $t \in \tau_I$, there are $z \in \mathfrak{f}$ and $i \in I$ such that*

$$\#\left([t \wedge z, t] \cap (\tau_I \setminus \tau_i)\right) \leq 1.$$

(ii) *For every* $z \in \mathfrak{f}$,

$$[0, z] \cap (\tau_I \setminus \bigcup_{i\in I} \tau_i) \subseteq \{\varpi_i(z) : i \in I\}.$$

(iii) *For every* $t \in \tau_I \setminus \mathfrak{a}$, *there is* $i \in I$ *such that*

$$\#(\mathrm{Is}_t(\tau_I) \setminus \mathrm{Is}_t(\tau_i)) \leq 1. \tag{13}$$

Proof. Let

$$\mathfrak{f} := ((\varrho_0)_{\max} \setminus \tau_\infty) \cup \{w(u) \,:\, u \in \tau_\infty^0\} \cup \{z(u) \,:\, u \in \tau_\infty^1\}$$

and let us prove (i): Given $t \in \tau_I$, if there is $u \in (\varrho_0)_{\max}$ such that $t \leq u$, then either $u \in \mathfrak{f}$, or $w(u) \in \mathfrak{f}$ or $z(u) \in \mathfrak{f}$, depending on whether $u \in (\varrho_0)_{\max} \setminus \tau_\infty$, $u \in \tau_\infty^0$ or $u \in \tau_\infty^1$, respectively. In any case, there is $z \in \mathfrak{f}$ such that $t \wedge z = t$ so that property 1 holds trivially. Otherwise, there is a unique $u \in (\varrho_0)_{\max}$ such that $u < t$. In case $u \notin \tau_\infty$, it follows from the definition of τ_∞ that $(\pi(\tau_i, \tau_j))_{\geq u} = \emptyset$ for every $i \neq j$ in I, which implies property 1. If $u \in \tau_\infty^1$, then $z(u) \in \mathfrak{f}$ and $t \wedge z(u) \in [u, z(u)]$. Then, Corollary 3.35.(i) guarantees that there is $i \in \omega$ such that $]t \wedge z(u), t] \subseteq \tau_i$, so that property 1 holds. Finally, if $u \in \tau_\infty^0$, then $w(u) \in \mathfrak{f}$ and $u \leq t \wedge w(u) \leq w(u)$. If $t \wedge w(u) < w(u)$, then Corollary 3.36.(ii) guarantees that there is $i \in \omega$ such that $]t \wedge w(u), t] \subseteq \tau_i$, so that condition 1 holds. If $t \wedge w(u) = w(u)$, it follows that $w(u) \leq t$ and the case when $t = w(u)$ is trivial. If $w(u) < t$, Corollary 3.36.(i) applied to $\mathrm{Is}_{w(u)}(t)$ guarantees that there is $i \in \omega$ such that $]t \wedge w(u), t] \subseteq \tau_i$, so that property 1 holds and this concludes the proof of (i).

To prove (ii), given $z \in \mathfrak{f}$, let us consider three different cases. If $z = z(u)$ for some $u \in \tau_\infty^1$, let $(\varpi_i(z))_{i\in\omega}$ be the sequence $(\varpi_i(u))_{i\in\omega}$ and if $z = w(u)$ for some $u \in \tau_\infty^0$, let $(\varpi_i(z))_{i\in\omega}$ be the constant sequence equal to $\varpi_i(u) = w(u)$. Finally, if $z \in (\varrho_0)_{\max} \setminus \tau_\infty$, let $(\varpi_i(z))_{i\in\omega}$ be the constant sequence equal to z.

Now, given $t \in \tau_I \setminus \bigcup_{i\in I} \tau_i$, by Corollary 3.32 there are $i_0 < i_1$ in I such that $t = \varpi_{i_0,i_1}(v) = \varpi_{i_0}(v)$ for some $v \in \tau_\infty$ since the sequence is well-placed. Then, if $t \leq z$, the maximality of v implies that $z \in \tau_\infty$ and since u is also maximal, $u = v$ and $t = \varpi_{i_0}(v) = \varpi_{i_0}(u)$, which concludes the proof of (ii).

It remains to prove (iii). Let

$$\mathfrak{a} := \varrho_0 \cup \{w(u) \,:\, u \in \tau_\infty^0\}$$

and fix $t \in \tau_I \setminus \mathfrak{a}$. Let $u \in (\varrho_0)_{\max}$ be such that $u < t$ or $t < u$ (the equality cannot hold since $\varrho_0 \subseteq \mathfrak{a}$ and $t \notin \mathfrak{a}$).

Suppose that $t < u$. We see that there is at most one index i such that $\mathrm{Is}_t''(\tau_i) \setminus \{\mathrm{Is}_t(u)\} \neq \emptyset$, that proves (13). We suppose otherwise that there are $i_0 < i_1$ such that $\mathrm{Is}_t''(\tau_{i_j}) \setminus \{\mathrm{Is}_t(u)\} \neq \emptyset$ for $j = 0, 1$. Choose $t_j \in \mathrm{Is}_t(\tau_{i_j}) \setminus \{\mathrm{Is}_t(u)\}$, $j = 0, 1$. Then, $t_0, t_1 \perp u$, hence $\tau_{i_0} \ni t_0 \wedge u = t = t_1 \wedge u \in \tau_{i_1}$, and so $t \in \varrho$, which is impossible.

Suppose now that $u < t$. If $u \notin \tau_\infty$, then (13) holds trivially. If $u < t$ and $u \in \tau_\infty^0$, we have to consider that cases $t < w(u)$ and $w(u) < t$ (again the equality cannot hold since $w(u) \in \mathfrak{a}$ and $t \notin \mathfrak{a}$). If $t < w(u)$, then Corollary 3.36.(ii) implies that $\mathrm{Is}_t(\tau_I) = \mathrm{Is}_t''(\tau_i) \cup \{\mathrm{Is}_t(w(u))\}$ for some $i \in I$, so (13) holds. If $w(u) < t$, then Corollary 3.36.(i) applied to $\mathrm{Is}_{w(u)}(t)$ implies that $\mathrm{Is}_t''(\tau_I) = \mathrm{Is}_t(\tau_i)$ for some $i \in I$, so (13) holds. If $u < t$ and $u \in \tau_\infty^1$, then Corollary 3.35.(i) implies that $\mathrm{Is}_t''(\tau_I) = \mathrm{Is}_t(\tau_i) \cup \{\mathrm{Is}_t(z(u))\}$ for some $i \in I$, so (13) holds, including in case $z(u) < t$. $\square$

Proof of Lemma 3.17. Fix $\mathcal{F} \in \mathfrak{B}$ and $\mathcal{H} \in \mathfrak{S}$ of infinite rank. We recall that

$$\mathcal{G} := \mathcal{F} \times \mathcal{H} = ((\mathcal{A} \times_a \mathcal{H}) \sqcup_a [T]^{\le 1}) \odot_T ((\mathcal{C} \times_c \mathcal{H}) \boxtimes_c 5)$$

where $\mathcal{A} := \mathrm{Is}(\langle \mathcal{F} \rangle)$ and $\mathcal{C} := \langle \mathcal{F} \rangle \cap \mathrm{Ch}_c$. We know from Proposition 3.15 that $\iota(\mathrm{srk}(\mathcal{F})) = \iota(\mathrm{srk}_a(\mathcal{A}))$ when Ch_a is not compact and $\iota(\mathrm{srk}(\mathcal{F})) = \iota(\mathrm{srk}_c(\mathcal{C}))$, when Ch_c is not compact. We check first that $\mathcal{G} \in \mathfrak{B}$, and for this purpose we use Proposition 3.18. Set

$$\overline{\mathcal{A}} := (\mathcal{A} \times_a \mathcal{H}) \sqcup_a [T]^{\le 1} \in \mathfrak{B}^a, \, \overline{\mathcal{C}} := (\mathcal{C} \times_c \mathcal{H}) \boxtimes_c 5 \in \mathfrak{B}^c.$$

Then when Ch_a or Ch_c is not compact, then $\overline{\mathcal{A}}$ and $\overline{\mathcal{C}}$ fulfill the conditions (a), (b) and (c) of that Proposition, so $\mathcal{G} \in \mathfrak{B}$. Suppose that Ch_a and Ch_c are both non-compact. Then (d) of Proposition 3.18, i.e., $\iota(\mathrm{srk}_a(\overline{\mathcal{A}})) = \iota(\mathrm{srk}_c(\overline{\mathcal{C}}))$ holds because $\iota(\mathrm{srk}_a(\overline{\mathcal{A}})) = \iota(\mathrm{srk}(\mathcal{F})) = \iota(\mathrm{srk}_c(\overline{\mathcal{C}}))$. Now we verify that $\mathcal{G}$ satisfies (M.1): Since $\mathcal{G} \in \mathfrak{B}$, it follows from Proposition 3.15 that if Ch_a is non-compact then

$$\begin{aligned}\iota(\mathrm{srk}(\mathcal{G})) &= \iota(\mathrm{srk}_a(\overline{\mathcal{A}})) = \iota(\mathrm{srk}_a(\mathcal{A} \times_a \mathcal{H})) = \max\{\iota(\mathrm{srk}_a(\mathcal{A})), \iota(\mathrm{srk}(\mathcal{H}))\} \\ &= \max\{\iota(\mathrm{srk}(\mathcal{F})), \iota(\mathrm{srk}(\mathcal{H}))\},\end{aligned}$$

and if Ch_c is non-compact then

$$\begin{aligned}\iota(\mathrm{srk}(\mathcal{G})) &= \iota(\mathrm{srk}_c(\overline{\mathcal{C}})) = \iota(\mathrm{srk}_c(\mathcal{C} \times_c \mathcal{H})) = \max\{\iota(\mathrm{srk}_c(\mathcal{C})), \iota(\mathrm{srk}(\mathcal{H}))\} \\ &= \max\{\iota(\mathrm{srk}(\mathcal{F})), \iota(\mathrm{srk}(\mathcal{H}))\}.\end{aligned}$$

This ends the proof of property (M.1) for $\times$. Let us prove now (M.2) for $\times$. Let $(\tau_k)_k$ be a sequence in $\langle \mathcal{F} \rangle$. Since $\langle \mathcal{F} \rangle \in \mathfrak{B}$, it follows that $\mathrm{Is}(\langle \mathcal{F} \rangle) \subseteq \langle \mathcal{F} \rangle$, so $\langle \mathcal{F} \rangle^{\mathrm{is}}$ is compact, by Corollary 3.9. Hence we may assume that each τ_k is a tree and that the sequence $(\langle \tau_k \rangle_{\mathrm{is}})_k$ is a Δ-sequence. By Corollary 3.37 there is a subsequence $(\sigma_k)_{k<\omega}$ of $(\tau_k)_k$ and there are $\mathfrak{a}, \mathfrak{f}$ finite subsets of T such that 1., 2. and 3. there hold. By refining the subsequence $(\nu_k)_{k<\omega}$ finitely many times, we may assume that for every $I \in \mathcal{H}$, we have that:

(i) For all $z \in \mathfrak{f}$, $\{w_i(z) : i \in I\} \in \mathcal{C} \times_c \mathcal{H}$.
(ii) For all $z \in \mathfrak{f}$, $\bigcup_{i \in I} (\nu_i \cap [0, z]) \in \mathcal{C} \times_c \mathcal{H}$.

(iii) For all $t \in \mathfrak{a}$, $\bigcup_{i\in I} \mathrm{Is}''_t \nu_i \in \mathcal{A} \times_a \mathcal{H}$.

For (i) we use that $[T]^{\leq 1} \subseteq \mathcal{F}$. Fix $I \in \mathcal{H} \upharpoonright M$ and let us prove that $\nu_I \in \mathcal{F} \times \mathcal{H}$, which is enough to guarantee that $\bigcup_{i\in I} \nu_i \in \mathcal{F} \times \mathcal{H}$, since this family is hereditary.

Claim 3.37.1. *For every $t \in \nu_I$ one has that $(\nu_I)_{\leq t} \in \overline{\mathcal{C}}$.*

Proof of Claim. Given $t \in \nu_I$, by property 1. of the sequence $(\nu_k)_{k\in M}$, there are $z \in \mathfrak{f}$, $\bar{t} \in \nu_M$ and $i \in M$ such that

$$(\nu_M) \cap [t \wedge z, t] \subseteq \nu_i \cup \{\bar{t}\}.$$

Then, the property 2. of $(\nu_k)_{k\in M}$ implies that

$$\nu_I \cap [0, z] \subseteq (\bigcup_{i\in I} (\nu_i \cap [0, z])) \cup \{w_i(z) : i \in I\} \cup \{z\}.$$

Hence,

$$\begin{aligned}\nu_I \cap [0, t] &\subseteq \nu_I \cap [t \wedge z, t] \cup (\nu_I \cap [0, z])\\ &\subseteq (\nu_i \cap [t \wedge z, t]) \cup (\bigcup_{i\in I} (\nu_i \cap [0, z])) \cup \{w_i(z) : i \in I\} \cup \{\bar{t}, z\}.\end{aligned}$$

Now notice that

- $\nu_i \cap [t \wedge z, t] \in \mathcal{C} \subseteq \mathcal{C} \times_c \mathcal{H}$;
- $\bigcup_{i\in I} (\nu_i \cap [0, z]) \in \mathcal{C} \times_c \mathcal{H}$ by (ii) above;
- $\{w_i(z) : i \in I\} \in \mathcal{C} \times_c \mathcal{H}$ by (i) above;
- $\{\bar{t}, z\} \in [T]^{\leq 2} \subseteq \mathcal{C} \boxtimes 2$.

Putting all together, we obtain that

$$\nu_I \cap [0, t] \in (\mathcal{C} \times_c \mathcal{H}) \boxtimes_c 5 = \overline{\mathcal{C}}. \quad \square$$

Claim 3.37.2. *For every $t \in \nu_I$ one has that $\mathrm{Is}''_t(\nu_I) \in \overline{\mathcal{A}}$.*

Proof of Claim. Given $t \in \nu_I$, if $t \notin \mathfrak{a}$, the property 2. of $(\nu_k)_{k\in M}$ implies that there are $j \in I$ and $\bar{t} \in \mathrm{Is}_t$ such that

$$\mathrm{Is}''_t(\nu_I) \subseteq \mathrm{Is}''_t(\nu_j) \cup \{\bar{t}\} \in \mathcal{A} \sqcup_a [T]^{\leq 1}.$$

If $t \in \mathfrak{a}$, it follows from (iii) above that

$$\mathrm{Is}''_t(\nu_I) \subseteq \bigcup_{i\in I} \mathrm{Is}''(\nu_i) \in \mathcal{A} \times_a \mathcal{H}$$

In any case, we have that $\mathrm{Is}_t''(\nu_I) \in \overline{\mathcal{A}}$. $\square$

These two claims imply that $\nu_I \in \mathcal{G}$ for every $I \in \mathcal{H}$. $\square$

4. Bases of families on (not so) large cardinals

The purpose of the section is to use Theorem 3.1 to prove Theorem 2.23, that is, to show the existence of bases of families on every cardinal below the first Mahlo cardinal.

4.1. Binary trees

We start by analyzing the case of binary trees.

Theorem 4.1. *Suppose that κ has a basis. Then 2^κ also has a basis.*

Proof. Suppose that κ has a basis. Let T be the complete binary tree $2^{\leq\kappa}$. The height mapping $\mathrm{ht} : T \to \kappa+1$ is strictly monotone, so it follows from Theorem 2.30 that there is a basis of families on chains of T. Each set Is_t has size 2, so it follows from Theorem 3.1 that T has a basis. Since T has cardinality 2^κ, we are done. $\square$

Remark 4.2. Let ε be the first exp-indecomposable ordinal $> \omega$, that, is ε is the first ordinal such that $\omega^\varepsilon = \varepsilon$. Then it is easy to see that $f_1(\alpha) = (\alpha \cdot \omega)^\alpha$ for every $\alpha < \varepsilon$. Using this, and the construction of the basis on 2^κ from the one in κ we can give upper bounds of the ranks of ω^α-homogeneous families in small exponential cardinals. Let $\mathrm{ht} : 2^{\leq\aleph_0} \to \aleph_0 + 1$ be the height function. We know that ht is adequate. For each $1 \leq \alpha < \varepsilon$, let $\mathcal{H}_\alpha$ be a ω^α-homogeneous family of rank exactly ω^α (e.g. the Schreier families on the index set $\aleph_0 + 1 \sim \omega$). Let $\mathcal{C}_\alpha := \mathrm{ht}^{-1}(\mathcal{H}_\alpha)$. It follows from Lemma 2.32 that $\mathcal{C}_\alpha$ is a β-homogeneous family on chains of T such that $\omega^\alpha \leq \beta < \omega(\omega^\alpha+1)$ and such that $\mathrm{rk}(\mathcal{C}_\alpha) < \omega(\omega^\alpha+1)$. In fact, we can have a better estimation. From Proposition 2.28 it follows that

$$\mathrm{rk}(\mathcal{C}_\alpha) < \sup_{x\in\mathcal{H}_\alpha} (\mathrm{rk}(\{s \in \mathcal{C}_\alpha \,:\, \mathrm{ht}(s) \subseteq x\}) + 1) \cdot (\mathrm{rk}(\mathcal{H}_\alpha) + 1).$$

Let $x \in \mathcal{H}_\alpha$, and set $\mathcal{E} := \{s \in \mathcal{C}_\alpha \,:\, \mathrm{ht}(s) \subseteq x\}$. We claim that $\mathrm{rk}(\mathcal{E}) \leq 1$: Suppose that $s \in \mathcal{E}'$ and let $(s_n)_n$ be a non-trivial Δ-sequence in $\mathcal{E}$ with root s. We claim that $\aleph_0 \notin \mathrm{ht}''(s)$ (recall that $\aleph_0+1 = \aleph_0\cup\{\aleph_0\}$). Otherwise, let $f \in s$ be the unique node such that $\mathrm{ht}(f) = \aleph_0$. Since each s_n is a chain, it follows that $\{f \in s_n \,:\, \mathrm{ht}(f) = \aleph_0\} = \{f\}$, so

$$s_n \subseteq \{f \upharpoonright m\}_{m\in x\setminus\{\aleph_0\}} \cup \{f\}.$$

Then there is an infinite set $M \subseteq \omega$ such that $s_n = s$ for every $n \in M$, a contradiction. Let us see that $\mathcal{E}'' = \emptyset$. Suppose otherwise, and let $s \in \mathcal{E}''$ and suppose that $(s_n)_n$ is a non-trivial Δ-sequence in $\mathcal{E}'$ with root s. Since $\{r : \mathrm{ht}''(r) \subseteq x \setminus \{\aleph_0\}\}$ is a finite set, it follows that $M = \{n \in \omega : \aleph_0 \in \mathrm{ht}''(s_n)\}$ is co-finite, in particular non-empty. Let $n \in M$. Then $s_n \in \mathcal{E}'$ and $\aleph_0 \in \mathrm{ht}'' s_n$, contradicting the previous fact. We have just proved that

$$\mathrm{rk}(\mathcal{C}_\alpha) < 2(\omega^\alpha + 1) = \omega^\alpha + 2.$$

Let $\mathcal{F}_\alpha := [2^{\leq \aleph_0}]^{\leq 2}_{a} \odot_T (\mathcal{C}_\alpha \sqcup_c [2^{\leq \aleph_0}]^{\leq 1})$. We know that $\mathcal{F}_\alpha$ is an homogeneous family and $\omega^\alpha + 1 \leq \mathrm{srk}(\mathcal{F}_\alpha) \leq \omega^\alpha + 2$. On the other hand, it follows from Proposition 3.21 that

$$\omega^\alpha + 1 \leq \mathrm{srk}(\mathcal{F}_\alpha) \leq \mathrm{rk}(\mathcal{F}_\alpha) < f_1((\omega^\alpha + 2) \cdot 3 + 2) = f_1(\omega^\alpha \cdot 3 + 8) \leq \omega^{\omega^\alpha \cdot 3 + \alpha \cdot 8 + 8}.$$

One step up further, we use $\mathcal{F}_\alpha$ (on $2^{\aleph_0} \sim 2^{\leq \aleph_0}$) and the height function to find homogeneous families $\mathcal{D}_\alpha$ on chains of $2^{\leq 2^{\aleph_0}}$ with

$$\omega^\alpha + 1 \leq \mathrm{srk}_c(\mathcal{D}_\alpha) \leq \mathrm{rk}(\mathcal{D}_\alpha) < \omega \cdot (\omega^{\omega^\alpha \cdot 3 + \alpha \cdot 8 + 8} + 1) = \omega^{\omega^\alpha \cdot 3 + \alpha \cdot 8 + 8} + \omega.$$

Set $T := 2^{\leq 2^{\aleph_0}}$, and let $\mathcal{G}_\alpha := [T]^{\leq 2}_{a} \odot_T \mathcal{D}_\alpha$. We know that this is an homogeneous family with

$$\omega^\alpha + 1 \leq \mathrm{srk}(\mathcal{G}_\alpha) \leq \mathrm{rk}(\mathcal{G}_\alpha) < \omega^{\omega^{\omega^\alpha \cdot 3 + \alpha \cdot 8 + 8} + \omega^{\alpha + 1}}$$

4.2. Trees from walks on ordinals

We pass now to study certain trees on inaccessible cardinal numbers. They are produced using the method of walks on ordinals. We introduce some basic notions of this. For more details we refer the reader to the monograph [26].

Definition 4.3. A *C-sequence* $\overline{C} := (C_\alpha)_{\alpha<\theta}$ is a sequence such that $C_\alpha \subseteq \alpha$ is a closed and unbounded subset of α with $\mathrm{otp}(C_\alpha) = \mathrm{cof}(\alpha)$. The $\overline{C}$*-walk* from β to $\alpha < \beta$ is the finite sequence of ordinals defined recursively by

$$\mathrm{Tr}(\alpha, \beta) := (\beta)^\frown \mathrm{Tr}(\alpha, \min(C_\beta \setminus \alpha))$$
$$\mathrm{Tr}(\alpha, \alpha) := (\alpha).$$

We write then the $\overline{C}$-walk as $\beta = \pi_0(\alpha, \beta) > \cdots > \pi_l(\alpha, \beta) = \alpha$, where $l+1 = \mathrm{ht}(\pi(\alpha, \beta)$, and for each $i \leq l$, $\pi_i(\alpha, \beta)$ is the i^{th} term of $\mathrm{Tr}(\alpha, \beta)$. Let

$$\varrho_2(\alpha, \beta) := \mathrm{ht}(\mathrm{Tr}(\alpha, \beta)) - 1.$$

We now define the mapping $\varrho_0 : [\theta]^2 \to (\mathcal{P}(\theta))^{<\omega}$ for $\alpha \leq \beta$ recursively by

$$\varrho_0(\alpha,\beta) := (C_\beta \cap \alpha)^\frown \varrho_0(\alpha, \min(C_\beta \setminus \alpha))$$
$$\varrho_0(\alpha,\alpha) := \emptyset.$$

Let $T = T(\varrho_0)$ be the tree whose nodes are $\varrho_0(\cdot,\beta) \restriction \alpha$, $\alpha \leq \beta$, ordered by end-extension as functions. Given $t \in T(\varrho_0)$, let $\alpha_t \leq \beta_t$ be such that $t = \varrho_0(\cdot,\beta_t) \restriction \alpha_t$. We say that (α_t, β_t) represents t.

Proposition 4.4. *T has size θ, and if θ is strong limit, then for every $t \in T$ one has that $\#\mathrm{Is}_t(T) < \theta$.*

Proof. This is a tree on a quotient of $[\theta]^{\leq 2}$, so it has cardinality θ. Now observe that the immediate successors of $t = \varrho_0(\cdot,\beta) \restriction \alpha$ are extensions u of t whose support is $\alpha + 1$. It follows that the number of them is at most $(2^\alpha)^{<\omega} < \theta$, when we assume that θ is strong limit. $\square$

In other words, the partial ordering $<_a$ is the disjoint union of small partial orderings. Observe that if $t < u$ in $T(\varrho_0)$, then we can take $(\alpha_t,\beta_t), (\alpha_u,\beta_u)$ representing t and u respectively such that $\alpha_t < \alpha_u$ and $\beta_t \leq \beta_u$: Take representatives (α_t,β_t), (α_u,β_u) of t and u respectively. Then $\alpha_t < \alpha_u$ and $\varrho_0(\cdot,\beta_t) \restriction \alpha_t = \varrho_0(\cdot,\beta_u) \restriction \alpha_t$, hence $(\alpha_t, \min\{\beta_t,\beta_u\})$ is a representative of t and satisfies the required condition together with (α_u,β_u). The following is well-known.

Proposition 4.5. *$t < u$ if and only if $\varrho_0(\alpha_t,\beta_t) = \varrho_0(\alpha_t,\beta_u)$.* $\square$

Definition 4.6. Given a C-sequence $\overline{C}$ on θ, let

$$\mathcal{I}(\overline{C}) := \{C \subseteq \theta \,:\, C \sqsubseteq C_\alpha \text{ for some } \alpha < \theta\}.$$

We consider $\mathcal{I}(\overline{C})$ ordered by $\sqsubset$.

Proposition 4.7. *$\varrho_0 : (T,<) \to \mathcal{I}(\overline{C})^{<\omega}_{\mathrm{lex}}$, $\varrho_0(t) := \varrho_0(\alpha_t,\beta_t)$, is strictly monotone, and consequently $\varrho_0 : (T,<) \to \mathcal{I}(\overline{C})^{<\omega}_{\mathrm{qlex}}$ is adequate.*

Proof. Suppose that $t < u$ in $T(\varrho_0)$.

Claim 4.7.1. *Suppose that $\varrho_0^i(t) = \varrho_0^i(u)$ for every $i \leq k$. Then $\pi_i(\alpha_t,\beta_u) = \pi_i(\alpha_u,\beta_u)$ for every $i \leq k+1$ and $\varrho_0^{k+1}(t) \sqsubseteq \varrho_0^{k+1}(u)$.*

Proof of Claim. Induction on $k \geq 0$. Suppose is true for $k-1$. Then $\pi_i(\alpha_t,\beta_u) = \pi_i(\alpha_u,\beta_u)$ for every $i \leq k$. It follows that

$$\pi_{k+1}(\alpha_t,\beta_u) = \min(C_{\pi_k(\alpha_t,\beta_u)} \setminus \alpha_t) = \min(C_{\pi_k(\alpha_u,\beta_u)} \setminus \alpha_t)$$
$$C_{\pi_k(\alpha_t,\beta_t)} \cap \alpha_t = \varrho_0^k(t) = \varrho_0^k(u) = C_{\pi_k(\alpha_u,\beta_u)} \cap \alpha_u.$$

In particular, $C_{\pi_k(\alpha_u,\beta_u)} \cap [\alpha_t, \alpha_u[= \emptyset$ hence $\min(C_{\pi_k(\alpha_u,\beta_u)} \setminus \alpha_t) = \min(C_{\pi_k(\alpha_u,\beta_u)} \setminus \alpha_u)$, so

$$\pi_{k+1}(\alpha_t, \beta_u) = \pi_{k+1}(\alpha_u, \beta_u).$$

Finally,

$$\begin{aligned} \varrho_0^{k+1}(t) = \varrho_0^{k+1}(\alpha_t, \beta_t) = \varrho_0^{k+1}(\alpha_t, \beta_u) &= C_{\pi_{k+1}(\alpha_t,\beta_u)} \cap \alpha_t \\ &= C_{\pi_{k+1}(\alpha_u,\beta_u)} \cap \alpha_t \sqsubseteq \varrho_0^{k+1}(u). \quad \square \end{aligned}$$

It follows that $\varrho_0^0(t) \sqsubseteq \varrho_0^0(u)$, so there must be $k < \varrho_2(\alpha_u, \beta_u)$ such that $\varrho_0^k(t) \sqsubset \varrho_0^k(u)$, since otherwise for every k $\pi_k(\alpha_t, \beta_u) = \pi_k(\alpha_u, \beta_u)$ would imply that $\alpha_t = \alpha_u$. $\square$

So, the mapping ϱ_0 is adequate, hence if $\mathcal{I}(\overline{C})^{<\omega}_{\mathrm{lex}}$ has a basis of families on chains, then ϱ_0 will transfer it to a basis on $<$-chains.

4.3. Cardinals smaller than the first Mahlo have a basis

Definition 4.8. A C-sequence on θ is *small* when there is a function $f : \theta \to \theta$ such that $\mathrm{otp}(C_\alpha) < f(\min C_\alpha)$ for every $\alpha < \theta$.

Proposition 4.9. *A strong limit cardinal θ has a small C-sequence if and only if θ is smaller than the first Mahlo cardinal.*

Proof. Suppose that θ is smaller than the first Mahlo cardinal. Choose a closed and unbounded set $D \subseteq \theta$ consisting of non-inaccessible cardinals. For each $\alpha < \theta$ let $\lambda(\alpha) \in D$ be the maximal element of D smaller or equal than α, and for each λ in D let λ_D^+ be the first element of D bigger than λ. Let $f : \theta \to \theta$, $f(\alpha) = 2^{\lambda(\alpha)_D^+} + 1$. $f(\alpha) < \theta$ because we are assuming that θ is strong limit. Observe also that $2^\alpha < f(\alpha)$. We define now the $\overline{C}$ sequence. Fix $\alpha \in \theta$. Suppose first that $\alpha \notin D$. Write $\alpha = \lambda(\alpha) + \beta$. Since $\mathrm{cof}(\alpha) = \mathrm{cof}(\beta)$, we can choose a club $C_\alpha \subseteq [\lambda(\alpha), \alpha[$ with $\mathrm{otp}(C_\alpha) = \mathrm{cof}(\beta)$. It follows that $\mathrm{otp}(C_\alpha) = \mathrm{cof}(\beta) < \lambda(\alpha)_D^+ < f(\lambda(\alpha)) = f(\min C_\alpha)$. Suppose that $\alpha \in D$. If α is singular, then we choose C_α in a way that $\mathrm{otp}(C_\alpha) = \mathrm{cof}(\alpha) < \min C_\alpha$. Observe that then $\mathrm{otp}(C_\alpha) < \min C_\alpha < f(\min C_\alpha)$. Finally, if α is not strong limit, then let $\beta < \alpha$ be such that $2^\beta \geq \alpha$. Let now $C_\alpha \subseteq [\beta, \alpha[$. It follows that $\mathrm{otp}(C_\alpha) = \mathrm{cof}(\alpha) \leq \alpha \leq 2^\beta \leq 2^{\min C_\alpha} < f(\min C_\alpha)$.

Suppose now that θ is bigger or equal to a Mahlo cardinal κ. By pressing down Lemma there is a stationary set S of inaccessible cardinals of κ and $\gamma < \kappa$ such that $\min C_\alpha = \gamma$ for every $\alpha \in S$. Hence $\alpha = \mathrm{cof}(\alpha) = \mathrm{otp}(C_\alpha) < f(\gamma)$ for every α, and this is of course impossible. $\square$

Proof of Theorem 2.23. The proof is by induction on $\theta < \mu_0$. We see that there is a tree $T = (T, <)$ on θ that has bases of families on chains of $(T, <)$ and of $(T, <_a)$. Suppose that θ is not strong limit. Then there is $\kappa < \theta$ such that $\theta \leq 2^\kappa$. By Theorem 4.1, 2^κ has a basis, so θ does.

Suppose that θ is strong limit. Let $T := T(\varrho_0)$ on θ. Let $\overline{C}$ be a small C-sequence on θ, and let $f : \theta \to \theta$ be a witness of it. By inductive hypothesis, and by Proposition 2.33 we know that the disjoint union $\biguplus_{\xi<\theta} f(\xi)$ of $(f(\xi))_{\xi<\theta}$ has a basis of families on chains. Let now $\lambda : \mathcal{I}(\overline{C}) \to \biguplus_{\xi<\theta} f(\xi) = \bigcup_{\xi<\theta} f(\xi) \times \{\xi\}$, $\lambda(C) := (\mathrm{otp}(C), \min C)$. Then $\lambda : (\mathcal{I}(\overline{C}), \sqsubset) \to (\biguplus_{\xi<\theta} f(\xi), <)$ is chain preserving and 1–1 on chains. Hence, by Theorem 2.30, there is a basis of families on chains of $\mathcal{I}(\overline{C})^{<\omega}_{\mathrm{qlex}}$. Since T has infinite chains and Proposition 4.7 tells that ϱ_0 is strictly monotone, it follows again by Theorem 2.30 that there is a basis of families on chains of $(T, <)$.

Observe that $\bigcup_{t\in T} \mathrm{Is}_t$ is a disjoint union, $\#T = \theta$. Since we are assuming that θ is inaccessible, it follows that T is $< \theta$-branching, hence for every $t \in T$ there is a basis of families on Is_t. Hence, by Proposition 2.33, there is a basis of families on chains of $(T, <_a)$. $\square$

5. Subsymmetric sequences and ℓ_1^α-spreading models

We present now new examples of Banach spaces without subsymmetric basic sequences of density κ. Their construction uses bases of families on κ. On one side, the multiplication of families will imply the non-existence of subsymmetric basic sequences. On the other side, the fact that the families are homogeneous will allow to bound the complexity of finite subsymmetric basic sequences. In fact, we will give examples of spaces such that every non-trivial sequence on it has a subsequence such that a large family of finite further subsequences behave like ℓ_1^n. Since in addition the spaces are reflexive, we will have, as for the Tsirelson space, that there are no subsymmetric basic sequences.

Definition 5.1. Recall that a non-constant sequence $(x_n)_n$ in a Banach space $X = (X, \|\cdot\|)$ is called *subsymmetric* when there is a constant $C \geq 1$ such that

$$\|\sum_{i=1}^n a_i x_{l_i}\| \leq C \|\sum_{i=1}^n a_i x_{k_i}\| \tag{14}$$

for every n, scalars $(a_i)_{i=1}^n$, $l_1 < l_2 < \cdots < l_n$ and $k_1 < k_2 < \cdots < k_n$.

Sometimes it is assumed, not in here, that subsymmetric sequences are unconditional basic sequences. Notice that by Rosenthal's ℓ_1-Theorem and Odell's partial unconditionality, it follows that if $(x_n)_n$ is a subsymmetric basic sequence, then either is equivalent to the ℓ_1 unit basis, or its difference sequence $(x_{2n} - x_{2n-1})_n$ is subsymmetric and unconditional. This is sharp, as it is shown by the summing basis of c_0.

Definition 5.2. Let $\mathcal{S}_\alpha$ be an α-Schreier family on ω. A bounded sequence $(x_n)_n$ in a normed space $\mathfrak{X}$ is called an ℓ_1^α-*spreading model* when there is a constant $C > 0$ such that

$$\|\sum_{n\in s} a_n x_n\| \geq C \sum_{n\in s} |a_n| \text{ for every } s \in \mathcal{S}_\alpha.$$

Let us say that a sequence in a Banach space is *non-trivial* when it does not have norm-convergent subsequences.

Remark 5.3.

(a) A subsequence of a ℓ_1^α-spreading model is again a ℓ_1^α-spreading model: Suppose that $(x_n)_n$ is a ℓ_1^α-spreading model with corresponding constant C, and suppose that $(y_n)_n$ is a subsequence of $(x_n)_n$, that is $y_n = x_{k_n}$, where $(k_n)_n$ is strictly increasing; since $\mathcal{S}_\alpha$ is spreading, given $(a_n)_{n\in s}$ supported in $s \in \mathcal{S}_\alpha$, we have that $t := \{k_n\}_{n\in s} \in \mathcal{S}_\alpha$, so setting $b_{k_n} := a_n$ for $n \in s$, $\|\sum_{n\in s} a_n y_n\| = \|\sum_{k\in t} b_k x_k\| \geq C\sum_{n\in s}|a_n|$.

(b) Suppose that $(\mathcal{S}_\alpha)_{\alpha<\omega_1}$ is a generalized Schreier sequence. If $(x_n)_n$ is a ℓ_1^α-spreading model, and $\beta \leq \alpha$, then $(x_n)_n$ is a ℓ_1^β-spreading model: This is a consequence of the fact that for every $\beta < \alpha$ there is some integer n such that $\mathcal{S}_\beta \upharpoonright \omega \setminus n \subseteq \mathcal{S}_\alpha$.

(c) Suppose that a space X does not contain ℓ_1 and it is such that every non-trivial sequence has a ℓ_1-spreading model subsequence. Then X does not have subsymmetric sequences. If in addition X has an unconditional basis, then X is in addition reflexive: Suppose otherwise that $(x_n)_n$ is a subsymmetric sequence $(x_n)_n$. It follows that $(x_n)_n$ is bounded and that $(x_n)_n$ does not have norm-convergent subsequences. So, by hypothesis, there is a ℓ_1-spreading model subsequence $(y_n)_n$. This implies that $(y_n)_n$ is equivalent to the unit basis of ℓ_1, and this is impossible. The latter condition follows from the James criterion of reflexivity.

(d) Suppose that $(x_n)_n$ is a non-trivial weakly-null sequence in some Banach space X, α is a countable ordinal and $\mathcal{F}$ is a spreading and ω^α-uniform family. Then $(x_n)_n$ has a ℓ_1^α-spreading model subsequence if and only if $(x_n)_n$ has a subsequence $(y_n)_n$ such that for some $C > 0$ one has that $\|\sum_{n\in s} a_n y_n\| \geq C\sum_{n\in s}|a_n|$ for every $(a_n)_{n\in s}$ supported in $s \in \mathcal{F}$: Suppose that $(y_n)_n$ is a subsequence of $(x_n)_n$ satisfying that last condition with respect to the family $\mathcal{F}$ and with constant $C > 0$. By the classical result fo Mazur, we can find a further subsequence $(z_n)_n$ of $(y_n)_n$ which is a basic sequence with basic constant ≤ 2. We consider the auxiliar family $\mathcal{G} := \mathcal{F} \oplus [\omega]^{\leq 1}$. This is a spreading and $\omega^\alpha + 1$-uniform family, so we can find $M = \{m_n\}_n \uparrow$ such that $\mathcal{S}_\alpha \upharpoonright M \subseteq \mathcal{G}$. Then the subsequence $(w_n)_n$, $w_n := z_{m_n}$, is a ℓ_1^α-spreading model: Let $s \in \mathcal{S}_\alpha$ and set $t := \{m_n\}_{n\in s}$; since $t \in \mathcal{S}_\alpha \upharpoonright M$ it follows that $t \in \mathcal{G}$, i.e. $u := t \setminus \{\min t\} \in \mathcal{F}$. Then, using that $(w_n)_n$ is a 2-basic sequence,

$$\|\sum_{n\in s} a_n w_n\| \geq \frac{1}{3}\|\sum_{n\in s\setminus\{\min s\}} a_n w_n\| = \frac{1}{3}\|\sum_{n\in s\setminus\{\min s\}} a_n z_{m_n}\| \geq \frac{C}{3}\sum_{n\in s\setminus\{\min s\}} |a_n|.$$

Again, since $(w_n)_n$ is 2-basic, $\|\sum_{n\in s} a_n w_n\| \geq K|a_{\min s}|$, where $K := (4 \cdot \sup_n \|w_n\|)^{-1}$. Putting all this together,

$$\|\sum_{n\in s} a_n w_n\| \geq \frac{1}{2}\min\{\frac{C}{3}, K\}\sum_{n\in s}|a_n|.$$

The reverse implication has exactly the same proof.

Definition 5.4. Recall that given a family $\mathcal{F}$ on κ, we define the corresponding *generalized Schreier space* $X_{\mathcal{F}}$ as the completion of $c_{00}(\kappa)$ with respect to the norm

$$\|x\|_{\mathcal{F}} := \max\{\|x\|_\infty, \max_{s\in\mathcal{F}}\sum_{\xi\in s}|(x)_\xi|\}.$$

It is easy to see that the unit basis of $c_{00}(\kappa)$ is a 1-unconditional basis of $X_{\mathcal{F}}$, and that $X_{\mathcal{F}}$ is c_0-saturated if $\mathcal{F}$ is compact, and contains a copy of ℓ_1 otherwise. When the family $\mathcal{F}$ is compact, hereditary and α-homogeneous with α infinite, then every subsequence of the unit basis of $X_{\mathcal{F}}$ has a ℓ_1-spreading model subsequence, consequently, no subsequence of the unit basis is subsymmetric. These families $\mathcal{F}$ exist on cardinal numbers not being ω-Erdős (see [16]).

Theorem 5.5. *Suppose that θ is smaller than the first Mahlo cardinal number. Then for every $\alpha < \omega_1$ there is a Banach space X of density θ with a long 1-unconditional basis $(u_\xi)_{\xi<\theta}$ such that every subsequence of $(u_\xi)_{\xi<\theta}$ has a further ℓ_1^α-spreading model subsequence, and no subsequence of $(u_\xi)_{\xi<\theta}$ is a $\ell_1^{\iota(\omega^\alpha)}$-spreading model.*

Proof. Fix a basis $(\mathfrak{B}, \times)$ on θ, let $\mathcal{F}$ be an $\omega^\alpha + 1$-homogeneous family in $\mathfrak{B}$ and let $X := X_{\mathcal{F}}$. Let $(u_\xi)_{\xi\in M}$ be an infinite subsequence of the unit basis $(u_\xi)_{\xi<\theta}$ of $X_{\mathcal{F}}$. Since $\mathrm{rk}(\mathcal{F} \upharpoonright M) > \omega^\alpha$, and $\mathrm{rk}(\mathcal{S}_\alpha) = \omega^\alpha$ there is some infinite subset N of M such that $\{\xi_n\}_{n\in x} \in \mathcal{F}$ for every $x \in \mathcal{S}_\alpha \upharpoonright N$. Let $N = \{n_k\}_k$ be the increasing enumeration of N, and set $x_k := u_{\xi_{n_k}}$ for every $k < \omega$. We claim that

$$\|\sum_{k\in x} a_k x_k\|_{\mathcal{F}} = \sum_{k\in x}|a_k|$$

for every $x \in \mathcal{S}_\alpha$: Fix $x \in \mathcal{S}_\alpha$. Then $\{n_k\}_{k\in x} \in \mathcal{S}_\alpha \upharpoonright N$, because $\mathcal{S}_\alpha$ is spreading. This means that $\{\xi_{n_k}\}_{k\in x} \in \mathcal{F}$, so

$$\|\sum_{k\in x} a_k x_k\|_{\mathcal{F}} = \|\sum_{k\in x} a_k u_{\xi_{n_k}}\|_{\mathcal{F}} \geq \sum_{k\in x}|a_k|.$$

On the other hand, let given a subsequence $(x_n)_{n<\omega}$ be a subsequence of $(u_\xi)_{\xi<\theta}$, $x_n := u_{\xi_n}$. We assume that $\xi_n < \xi_{n+1}$ for every n. Let $\mathcal{G} := \{x \subseteq \omega : \{\xi_n\}_{n\in x} \in \mathcal{F}\}$. Then the mapping $x \in \mathcal{G} \mapsto \{\xi_n\}_{n\in x} \in \mathcal{F}$ is continuous and 1–1, hence $\mathrm{rk}(\mathcal{G}) \leq \mathrm{rk}(\mathcal{F}) < \iota(\omega^\alpha)$. Since $\iota(\omega^\alpha)$ is exp-indecomposable, $\mathrm{rk}(\mathcal{S}_{\iota(\omega^\alpha)}) = \omega^{\iota(\omega^\alpha)} = \iota(\omega^\alpha)$. It follows by the quantitative version of Ptak's Lemma (see for example [15, Corollary 4.8]) that for every $\varepsilon > 0$ there is some convex combination $(a_n)_{n\in x}$ supported in $x \in \mathcal{S}_{\iota(\omega^\alpha)}$ such that

$$\sup_{x\in\mathcal{G}}\sum_{n\in x}|a_n|<\varepsilon.$$

This means that $\|\sum_{n\in x}a_nx_n\|_{\mathcal{F}}<\varepsilon\sum_{n\in x}|a_n|$. $\square$

Definition 5.6. Recall that given an α-Schreier family $\mathcal{S}_\alpha$, let $T_\alpha:=T_{\mathcal{S}_\alpha}$ be the α-Tsirelson space defined as the completion of c_{00} under the norm

$$\|x\|_\alpha:=\max\{\|x\|_\infty,\sup_{(E_i)_i}\frac{1}{2}\sum_i\|E_ix\|_\alpha\}$$

where the sup above runs over all sequences of sets $(E_i)_i$ such that $E_i<E_{i+1}$ and $\{\min E_i\}_i\in\mathcal{S}_\alpha$, and where $Ex=\sum_{n\in E}(x)_n$.

An equivalent way of defining $\|\cdot\|_\alpha$ is as follows. Let $K_0:=\{\pm u_n\}_{n<\omega}$ and let

$$K_{n+1}:=K_n\cup\{\frac{1}{2}\sum_{i<k}\varphi_i\ :\ \{\varphi_i\}_{i<k}\subseteq K_n,\ \varphi_i<\varphi_{i+1},\ i<k-1,$$
$$\text{and }\{\min\operatorname{supp}\varphi_i\}_{i<k}\in\mathcal{S}_\alpha\}.$$

Let $K:=\bigcup_n K_n$. Then $\|x\|_\alpha=\sup_{\varphi\in K}\langle\varphi,x\rangle$. It is easy to see that each $\varphi\in K$ has a decomposition

$$\varphi=\sum_i\frac{1}{2^i}\varphi_i$$

where φ_i is a vector with coordinates -1 or 1, supported in ($\mathcal{S}^i_\alpha:=\overbrace{\mathcal{S}_\alpha\otimes\cdots\otimes\mathcal{S}_\alpha}^{(i)}$) and $(\varphi_i)_i$ pairwise disjointly supported. It is well known that every normalized block subsequence of the unit basis $(u_n)_n$ of T_α is equivalent to a subsequence of the unit basis. Since clearly from the definition every subsequence of $(u_n)_n$ is a ℓ_1^α-spreading model, it follows that every non-trivial sequence in T_α has a ℓ_1^α-spreading model subsequence. On the other hand, let us see that there is no $\ell_1^{\alpha\cdot\omega}$-spreading model sequence: Suppose that $(x_n)_n$ is a non-trivial sequence in T_α. Since T_α is reflexive, there is a non-trivial weakly-convergent subsequence $(x_{n_k})_{k\in\mathbb{N}}$ of $(x_n)_n$ with limit x; since $(x_{n_k}-x)_{k\in\mathbb{N}}$ is a non-trivial weakly-null sequence, it has a subsequence equivalent to a subsequence of the basis, so without loss of generality we may assume that $x_k=u_{m_k}+x$ for all k. Fix $\varepsilon>0$, and let n be such that $\varepsilon 2^{n-1}>1$. Let $N\subseteq\mathbb{N}$ be such that $(\mathcal{S}_{\alpha(n+1)}\oplus\mathcal{S}_{\alpha(n+1)})\restriction\{m_k\}_{k\in N}\subseteq\mathcal{S}_{\alpha\cdot\omega}$; this is possible since $\mathcal{S}_{\alpha(n+1)}\oplus\mathcal{S}_{\alpha(n+1)}$ is a $\omega^{\alpha(n+1)}2$ uniform family and $\mathcal{S}_{\alpha\cdot\omega}$ is $\omega^{\alpha\cdot\omega}$ uniform. By the quantitative version of Ptak's Lemma ([15, Corollary 4.8]) there are two convex combination $(a_k)_{k\in s}$, $(b_k)_{k\in t}$ such that $s,t\subseteq N$, $s<t$, $\{m_k\}_{k\in s},\{m_k\}_{k\in t}\in\mathcal{S}_{\alpha\cdot(n+1)}$, and such that $\sup_{\{m_k\}_{k\in z}\in\bigcup_{i<n}\mathcal{S}_{\alpha\cdot n}}\sum_{k\in z}|a_k|+|b_k|<\varepsilon$. We claim that

$$\|\sum_{k\in s} a_k x_k - \sum_{k\in t} b_k x_k\|_\alpha \le 3\varepsilon.$$

Fix $\varphi \in K$, $\varphi = \sum_i 2^{-i}\varphi_i$ decomposed as above. Then,

$$\begin{aligned}
&|\langle \varphi, \sum_{k\in s} a_k x_k - \sum_{k\in t} b_k x_k\rangle| \\
&= |\langle \varphi, \sum_{k\in s} a_k u_{m_k} - \sum_{k\in t} b_k u_{m_k}\rangle| \le |\langle \varphi, \sum_{k\in s} a_k u_{m_k}\rangle| + |\langle \varphi, \sum_{k\in t} b_k u_{m_k}\rangle| \\
&\le \sum_{i<n} \frac{1}{2^i}\left(|\langle \varphi_i, \sum_{k\in s} a_k u_{m_k}\rangle| + |\langle \varphi_i, \sum_{k\in t} b_k u_{m_k}\rangle|\right) + \frac{1}{2^n}\left(\sum_{k\in s}|a_k| + \sum_{k\in t}|b_k|\right) \\
&\le \sum_{i<n} \frac{1}{2^i}\left(\sum_{\{m_k\}_k\in \operatorname{supp}\varphi_i} |a_k| + \sum_{\{m_k\}_k\in \operatorname{supp}\varphi_i} |b_k|\right) + \frac{1}{2^n}\left(\sum_{k\in s}|a_k| + \sum_{k\in t}|b_k|\right) \\
&\le 3\varepsilon.
\end{aligned}$$

We have just proved the following.

Theorem 5.7. *T_α is a reflexive Banach space whose unit basis is 1-unconditional and such that every non-trivial sequence has a ℓ_1^α-spreading model subsequence but it does not have $\ell_1^{\alpha\cdot\omega}$-spreading models. Consequently, T_α does not have subsymmetric basic sequences.* □

5.1. The interpolation method

We recall the following well-known construction, presented in a general, not necessarily separable, context: fix an infinite cardinal number κ, let $(\|\cdot\|_n)_{n\in\omega}$ be a sequence of norms in $c_{00}(\kappa)$ and $\|\cdot\|_X$ be a norm on $c_{00}(\mathbb{N})$ such that $(e_n)_n$ is a 1-unconditional basic sequence of the completion X of $(c_{00}(\mathbb{N}), \|\cdot\|_X)$. Let X_n, $n\in\mathbb{N}$, be the completion of $(c_{00}(\kappa), \|\cdot\|_n)$. For $x \in c_{00}(\kappa)$, define

$$|||x||| := \|\sum_n \frac{\|x\|_n}{2^{n+1}} e_n\|_X.$$

It is not difficult to see that $|||\cdot|||$ is a norm on $c_{00}(\kappa)$ (the fact that $(u_n)_n$ is a 1-unconditional basic sequence of $(c_{00}(\mathbb{N}), \|\cdot\|_X)$ is crucial to prove the triangle inequality). Let $\mathfrak{X}$ be the completion of $(c_{00}, |||\cdot|||)$.

Remark 5.8. Observe that the dual unit ball of $\mathfrak{X}$ is closed under the following operation. Given $x_i^* \in B_{X_i^*}$ for $i = 1, \dots, n$ and $\sum_{i=1}^n b_i e_i^* \in B_{X^*}$, then

$$\sum_{i=1}^n \frac{b_i}{2^{i+1}} x_i^* \in B_{\mathfrak{X}^*}.$$

To see this, there is a simple computation. Given $x \in c_{00}$ we have that

$$|(\sum_{i=1}^{n} \frac{b_i}{2^{i+1}} x_i^*)(x)| \le \sum_{i=1}^{n} \frac{|b_i|}{2^{i+1}} \|x\|_i = |(\sum_{i=1}^{n} |b_i| e_i^*)(\sum_{i=1}^{n} \frac{1}{2^{i+1}} \|x\|_i e_i)|$$
$$\le \left\| \sum_{i=1}^{n} |b_i| e_i^* \right\|_{X^*} \left\| \sum_{i=1}^{n} \frac{1}{2^{i+1}} \|x\|_i e_i \right\|_{X} \le \|x\|_{\mathfrak{X}}$$

The following follows easily from the definition.

Proposition 5.9. *Suppose that $(x_\xi)_{\xi<\lambda}$ is a C-unconditional basic sequence of each X_n. Then $(x_\xi)_{\xi<\lambda}$ is a C-unconditional basic sequence of $\mathfrak{X}$.* □

In our construction, this will be the case, so that we will be able to apply the following result.

Proposition 5.10. *Suppose that X is a space with an unconditional basis and without isomorphic copies of ℓ_1. Then the following are equivalent.*

(a) *Every non-trivial bounded sequence in X has an ℓ_1^α-spreading model subsequence.*
(b) *Every non-trivial weakly-convergent sequence in X has ℓ_1^α-spreading model subsequence.*
(c) *Every non-trivial weakly-null sequence in X has ℓ_1^α-spreading model subsequence.*

Proof. Suppose that (b) holds. It follows that c_0 does not embed into X. Hence, by James' criteria of reflexivity for spaces with an unconditional basis, X is reflexive, whence every bounded sequence has a weakly-convergent subsequence and now (a) follows directly from (b). Now suppose that (c) holds and let us prove (b): Suppose that $(x_k)_k$ is a non-trivial weakly-convergent sequence with limit x. Let $x^* \in S_{X^*}$ be such that $x^*(x) = \|x\|$. Let $y_k := x_k - x$ for every k. By hypothesis, we can find $\varepsilon > 0$ and a subsequence $(z_n)_n$ of $(y_n)_n$ such that $\|\sum_n a_n y_n\| \ge \varepsilon \sum_n |a_n|$ for every sequence of scalars $(a_n)_n$ supported in $\mathcal{S}_\alpha$. Let $(v_n)_n$ be a further subsequence of $(z_n)_n$ such that $|x^*(v_n)| \le \varepsilon/2$ for every n. We claim that $(v_n + x)_n$ is a subsequence of $(x_n)_n$ which is a ℓ_1^α-spreading model: Fix a sequence $(a_n)_n$ supported in $\mathcal{S}_\alpha$, and let $y^* \in B_{X^*}$ be such that $y^*(\sum_n a_n z_n) \ge \varepsilon \sum_n |a_n|$. Let $z^* := y^* - \lambda x^* \in \mathrm{Ker}(x) \cap 2B_{X^*}$, where $\lambda := y^*(x)/\|x\|$. Then,

$$\|\sum_n a_n(v_n + x)\| \ge \frac{1}{2} z^* \left(\sum_n a_n(v_n + x) \right) = \frac{1}{2} z^* \left(\sum_n a_n v_n \right) \ge \frac{1}{2} y^* \left(\sum_n a_n v_n \right)$$
$$- \lambda x^* \left(\sum_n a_n v_n \right) \ge \frac{\varepsilon}{4} \sum_n |a_n|. \quad \square$$

Finally, one of the interesting features of the resulting space of the interpolation is that subspaces of it essentially come from subspaces of X and of X_n's. We present it in a more general way that will be used later.

Proposition 5.11. *Let $\mathcal{F}$ be a spreading family of finite sets, and suppose that either $\mathcal{F} \sqcup \mathcal{F} = \mathcal{F}$, that is $\mathcal{F}$ is closed under unions, or else $\|x\|_n \leq \|x\|_{n+1}$ for every n and every $x \in \mathfrak{X}$. Then for every sequence $(x_n)_n$ in $\mathfrak{X}$ one of the following holds:*

(a) *there is a subsequence $(y_k)_k$ of $(x_k)_k$, some $n \in \mathbb{N}$ and some $K > 0$ such that*

$$\frac{1}{2^{n+1}} \|\sum_{j\in s} a_j y_j\|_n \leq \|\sum_{j\in s} a_j y_j\|_{\mathfrak{X}} \leq K \|\sum_{j\in s} a_j y_j\|_n;$$

(b) *there is a normalized block subsequence $(y_k)_k$ of $(x_k)_k$ such that the support s_k of each y_k with respect to $(x_j)_j$, that is $y_k = \sum_{j\in s_k} a_j x_j$, is in $\mathcal{F}$, and $(y_k)_k$ is 3-equivalent to a block subsequence of the basis $(e_n)_n$ of X.*

In particular, if Y can be isomorphically embedded into $\mathfrak{X}$, then a subspace Z of Y can be isomorphically embedded into some X_n or into X.

Proof. The latter part follows from the dichotomy in this way: take $\mathcal{F} := [\omega]^{<\infty}$ the collection of all finite subsets of ω, and let $(x_n)_n$ be a basic sequence in $\mathfrak{X}$ that is equivalent to a sequence in Y. If (a) holds, then there is a subsequence of $(x_n)_n$ equivalent to a sequence in some X_n; if (b) holds, then $(x_n)_n$ has a block subsequence equivalent to a block subsequence of the basis $(e_n)_n$ of X.

Let us prove the dichotomy. Fix all parameters, without loss of generality we assume that each x_n is normalized, and suppose that (a) does not hold. We are going to build the appropriate block subsequence $(y_n)_n$ of $(x_n)_n$ as follows. Fix a strictly positive summable sequence $(\varepsilon_n)_n$, $\sum_n \varepsilon_n < 1/4$. Let $y_0 := x_0$. Let $n_0 \in \mathbb{N}$ be such that

$$\|\sum_{n>n_0} \frac{\|y_0\|_n}{2^{n+1}} e_n\|_X \leq \varepsilon_0.$$

Let $y_1 = \sum_{j\in s_1} a_j x_j$ be such that $0 < s_1$, $s_1 \in \mathcal{F}$ and

$$\max_{n\leq n_0} \|y_1\|_n \leq \varepsilon_1.$$

Note that this is possible by the hypothesis on $\mathcal{F}$ and of $(\|\cdot\|_n)_n$ and since we are assuming that (a) does not hold. Let $n_1 > n_0$ be such that

$$\|\sum_{n>n_1} \frac{\|y_1\|_n}{2^{n+1}} e_n\|_X \leq \varepsilon_1.$$

Let now $y_2 = \sum_{j\in s_s} a_j x_j$, $s_1 < s_2$, $s_2 \in \mathcal{F}$, be such that

$$\max_{n\le n_1} \|y_2\|_n \le \varepsilon_2.$$

In this way, we can find an strictly increasing sequence $(n_k)_{k\in\mathbb{N}}$ of integers and a normalized block subsequence $(y_k)_k$ of $(x_k)_k$ such that y_k is supported (w.r.t. $(x_k)_k$) in $\mathcal{F}$ and such that for every k one has that

$$\|\sum_{n>n_k} \frac{\|y_k\|_n}{2^{n+1}} e_n\|_X \le \varepsilon_k$$

$$\max_{n\le n_{k-1}} \|y_k\|_n \le \varepsilon_k.$$

Set $w_0 := \sum_{n\le n_0}(\|y_0\|_n/2^{n+1})e_n \in X$, and for each $k\ge 1$, let

$$w_k := \sum_{n=n_{k-1}+1}^{n_k} \frac{\|y_k\|_n}{2^{n+1}} e_n \in X.$$

Let $(a_k)_k$ be a sequence of scalars with $\max_k |a_k| = 1$. Then, after some computations, one can show that

$$\begin{aligned}
&\left| \|\sum_k a_k y_k\|_{\mathfrak{X}} - \|\sum_k a_k w_k\|_X \right| \\
&\le |a_0| \|\sum_{n\le n_0} \frac{1}{2^{n+1}} \|y_0\|_n e_n\|_X \\
&\quad + \sum_{k\ge 1} |a_k| \left(\|\sum_{n\le n_{k-1}} \frac{1}{2^{n+1}} \|y_k\|_n e_n\|_X + \|\sum_{n>n_k} \frac{1}{2^{n+1}} \|y_k\|_n e_n\|_X \right) \\
&< \frac{1}{2}.
\end{aligned}$$

Since $\|w_k\|_X \ge 3/4$, and $(w_k)_k$ is a block subsequence of the basis $(e_n)_n$, it follows that $\|\sum_k a_k w_k\|_X \ge 3/4 \max_k |a_k|$, we obtain that

$$\left| \|\sum_k a_k y_k\|_{\mathfrak{X}} - \|\sum_k a_k w_k\|_X \right| < \frac{2}{3} \|\sum_k a_k w_k\|_X.$$

Hence,

$$\frac{1}{3} \|\sum_k a_k w_k\|_X \le \|\sum_k a_k y_k\|_{\mathfrak{X}} \le \frac{4}{3} \|\sum_k a_k w_k\|_X. \quad \square$$

5.2. The Banach space $\mathfrak{X}$

The next result gives the existence of the desired Banach space $\mathfrak{X}$ of density κ, subject to the existence of bases of families on κ.

Theorem 5.12. *Suppose that κ has a basis of families. Then for every $1 \le \alpha < \omega_1$ there is a reflexive Banach space $\mathfrak{X}_\alpha$ of density κ with a long unconditional basis such that*

(a) *every bounded sequence without norm convergent subsequences has a ℓ_1^α-spreading model subsequence, and*
(b) *$\mathfrak{X}_\alpha$ does not have $\ell_1^{\iota(\omega^\alpha)+\alpha\cdot\omega}$-spreading models.*

Consequently,

(c) *$\mathfrak{X}_\alpha$ does not have subsymmetric sequences;*
(d) *if $\iota(\omega^\alpha)+\alpha\cdot\omega \le \beta$, then $\mathfrak{X}_\alpha$ and $\mathfrak{X}_\beta$ are totally incomparable, i.e. there is no infinite dimensional subspace of $\mathfrak{X}_\alpha$ isomorphic to a subspace of $\mathfrak{X}_\beta$.*

Proof. Let $(\mathfrak{B}, \times)$ be a basis of families on κ. Let $\mathcal{F}_0 := [\kappa]^{\le 1}$, and for $n < \omega$ let $\mathcal{F}_{n+1} := \mathcal{F}_n \times \mathcal{S}_\alpha$. Notice that $\mathrm{rk}(\mathcal{F}_n) < \iota(\omega^\alpha)$ for every n. Let $\mathfrak{X}$ be the interpolation space from an α-Tsirelson space T_α and the sequence of generalized Schreier spaces $X_{\mathcal{F}_n}$, $n \in \mathbb{N}$. Since each $X_{\mathcal{F}_n}$ is c_0-saturated, and T_α is reflexive, it follows from Proposition 5.11 that $\mathfrak{X}$ does not have isomorphic copies of ℓ_1.

Claim 5.12.1. *Every non-trivial bounded sequence has an ℓ_1^α-spreading model.*

From this claim, and the fact that the unit basis $(u_\gamma)_{\gamma<\kappa}$ is unconditional, we obtain that $\mathfrak{X}$ is reflexive. We pass now to prove that previous claim.

Proof of Claim. Fix such sequence $(x_k)_k$. Since $\mathfrak{X}$ does not have isomorphic copies of ℓ_1, we may assume, by Proposition 5.10, that $(x_k)_k$ is a non-trivial weakly-null sequence. Since $(u_\xi)_{\xi<\kappa}$ is a Schauder basis of $\mathfrak{X}$, by going to a subsequence if needed, we assume that $(x_k)_k$ is disjointly supported, and

$$\|x_k\| \ge \gamma > 0 \text{ for every } k.$$

We now apply Proposition 5.11 for the family $\mathcal{F} := [\mathbb{N}]^1 := \{\{n\} : n \in \mathbb{N}\}$.
Case 1. There is $\varepsilon > 0$, $n \in \mathbb{N}$ and an infinite subsequence $(y_k)_k$ of $(x_k)_k$ such that

$$\|y_k\|_n \ge \varepsilon \text{ for every } k.$$

For each k choose $s_k \in \mathcal{F}_n \restriction \operatorname{supp} y_k$ such that

$$\sum_{\xi \in s_k} |u_\xi^*(y_k)| \geq \varepsilon.$$

By hypothesis, $\mathcal{F}_{n+1} = \mathcal{F}_n \times \mathcal{S}_\alpha$, so there is a subsequence (t_k) of $(s_k)_k$ such that $\bigcup_{k \in v} t_k \in \mathcal{F}_{n+1}$ for every $v \in \mathcal{S}_\alpha$. Let $(z_k)_k$ be the subsequence of $(y_k)_k$ such that $\sum_{\xi \in t_k} |u_\xi^*(z_k)| \geq \varepsilon$ for every k. We claim that $(z_k)_k$ is a ℓ_1^α-spreading model. So, fix a sequence of scalars $(a_k)_{k \in s}$ indexed by $s \in \mathcal{S}_\alpha$. Then $t := \bigcup_{k \in s} t_k \in \mathcal{F}_{n+1}$, hence,

$$\begin{aligned}
\| \sum_{k \in s} a_k z_k \| &\geq \frac{1}{2^{n+2}} \| \sum_{k \in s} a_k z_k \|_{n+1} \geq \frac{1}{2^{n+2}} \sum_{\xi \in t} \left| u_\xi^*(\sum_{k \in s} a_k z_k) \right| \\
&= \frac{1}{2^{n+2}} \sum_{k \in s} |a_k| \sum_{\xi \in t_k} \left| u_\xi^*(z_k) \right| \geq \frac{\varepsilon}{2^{n+2}} \sum_{k \in s} |a_k|.
\end{aligned}$$

Case 2. There is a subsequence $(y_k)_k$ of $(x_k)_k$ that is equivalent to a block subsequence of the unit basis $(t_k)_k$ of T_α. In this case Theorem 5.7 gives us a ℓ_1^α-spreading model subsequence of $(y_k)_k$, so we are done. $\square$

(b): Suppose otherwise that $(x_k)_k$ is a weakly-null ℓ_1^β-spreading model sequence where $\beta := \iota(\omega^\alpha) + \alpha \cdot \omega$. By Remark 5.3 (d) we may assume that there is some constant $\varepsilon > 0$ such that

$$\| \sum_{k \in s} a_k x_k \|_{\mathfrak{X}} \geq \varepsilon \sum_{k \in s} |a_k| \text{ for every } s \in \mathcal{S}_{\iota(\omega^\alpha)} \otimes \mathcal{S}_{\alpha \cdot \omega} \tag{15}$$

We apply now Proposition 5.11 to $(x_k)_k$ and the family $\mathcal{S}_{\alpha \cdot \omega}$.
Case 1. There is some subsequence $(y_k)_k$ of $(x_k)_k$ some n and some $C > 0$ such that

$$\frac{1}{2^{n+1}} \| \sum_{k \in s} a_k y_k \|_n \leq \| \sum_{k \in s} a_k y_k \|_{\mathfrak{X}} \leq C \| \sum_{k \in s} a_k y_k \|_n$$

for every $(a_k)_{k \in s}$ supported in $s \in \mathcal{S}_{\iota(\omega^\alpha)}$. Let $\theta : \mathcal{F}_n \to \mathrm{FIN}$, $\theta(s) := \{k \in \omega : \operatorname{supp} y_k \cap s \neq \emptyset\}$. This is a continuous mapping, so $\mathcal{C} := \theta''\mathcal{F}_n$ is compact, and $\gamma := \mathrm{rk}(\mathcal{C}) \leq \mathrm{rk}(\mathcal{F}_n) < \iota(\omega^\alpha)$, by the choice of $\mathcal{F}_n$. By the quantitative version of Ptak's Lemma we can find a convex combination $(a_k)_{k \in s}$ supported in $\mathcal{S}_{\iota(\omega^\alpha)}$ such that $\sum_{k \in t} |a_k| < \varepsilon/(C \sup_k \|y_k\|_n)$ for every $t \in \mathcal{C}$. Let $v \in \mathcal{F}_n$ be such that

$$\| \sum_{k \in s} a_k y_k \|_n = \sum_{\xi \in v} |u_\xi^*(\sum_{k \in s} a_k y_k)|.$$

Then

$$\| \sum_{k \in s} a_k y_k \|_{\mathfrak{X}} \leq C \| \sum_{k \in s} a_k y_k \|_n \leq \sum_{k \in \theta(s)} |a_k| \|y_k\|_n < \varepsilon,$$

and this is impossible.

CASE 2. There is a normalized block subsequence $(y_k)_k$ of $(x_k)_k$ such that each y_k is supported, w.r.t. $(x_k)_k$, in an element of $\mathcal{S}_{\alpha\cdot\omega}$, and that it is 3-equivalent to a block subsequence $(w_k)_k$ of the unit basis $(t_n)_n$ of T_α. We write each $y_k := \sum_{j\in s_k} b_j x_j$ with $s_k \in \mathcal{S}_{\iota(\omega^\alpha)}$.

Let $K := \sup_k \|x_k\|_{\mathfrak{X}}$. We know from Theorem 5.7 that T_α does not have $\ell_1^{\alpha\cdot\omega}$-spreading models, so we can find scalars $(a_k)_{k\in v}$ supported in $v \in \mathcal{S}_{\alpha\cdot\omega}$ such that $\|\sum_{k\in v} a_k w_k\|_{T_\alpha} < (\varepsilon/(3K))\sum_{k\in v}|a_k|$. Hence,

$$\|\sum_{k\in v} a_k \sum_{j\in s_k} b_j x_j\|_{\mathfrak{X}} \le 3\|\sum_{k\in v} a_k w_k\|_{T_\alpha} < \frac{\varepsilon}{K}\sum_{k\in v}|a_k| \le \varepsilon \sum_{k\in v}|a_k|\sum_{j\in s_k}|b_j|,$$

contradicting (15), because $\bigcup_{k\in v} s_k \in \mathcal{S}_{\iota(\omega^\alpha)} \otimes \mathcal{S}_{\alpha\cdot\omega}$. $\square$

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