

## AN ISOMETRICALLY UNIVERSAL BANACH SPACE INDUCED BY A NON-UNIVERSAL BOOLEAN ALGEBRA

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ABSTRACT. Given a Boolean algebra  $A$ , we construct another Boolean algebra  $B$  with no uncountable well-ordered chains such that the Banach space of real-valued continuous functions  $C(K_A)$  embeds isometrically into  $C(K_B)$ , where  $K_A$  and  $K_B$  are the Stone spaces of  $A$  and  $B$ , respectively. As a consequence we obtain the following: If there exists an isometrically universal Banach space for the class of Banach spaces of a given uncountable density  $\kappa$ , then there is another such space which is induced by a Boolean algebra which is not universal for Boolean algebras of cardinality  $\kappa$ . Such a phenomenon cannot happen on the level of separable Banach spaces and countable Boolean algebras. This is related to the open question of whether the existence of an isometrically universal Banach space and of a universal Boolean algebra are equivalent on the nonseparable level (both are true on the separable level).

### 1. INTRODUCTION

If  $A$  is a Boolean algebra, we denote by  $K_A$  the Stone space of  $A$ ; that is, a compact Hausdorff totally disconnected space such that  $A$  is isomorphic to the algebra of all clopen subsets of  $K_A$  (see [14]).  $C(K)$  denotes the Banach space of real-valued continuous functions on a compact Hausdorff space  $K$  with the supremum norm.

Given an infinite cardinal  $\kappa$ , let  $\mathcal{B}_\kappa$  denote the class of Banach spaces of density at most  $\kappa$ ,  $\mathcal{C}_\kappa$  denote the class of compact spaces of weight at most  $\kappa$  and  $\mathcal{A}_\kappa$  denote the class of Boolean algebras of cardinality at most  $\kappa$ . We say that  $X \in \mathcal{B}_\kappa$  is isometrically universal for  $\mathcal{B}_\kappa$  if for every  $Y \in \mathcal{B}_\kappa$  there is a linear isometry  $T : Y \rightarrow X$  onto its range. We say that  $K \in \mathcal{C}_\kappa$  is universal for  $\mathcal{C}_\kappa$  if for every  $L \in \mathcal{C}_\kappa$  there is a continuous surjection  $\phi : K \rightarrow L$  and we say that  $A \in \mathcal{A}_\kappa$  is universal for  $\mathcal{A}_\kappa$  if for every  $B \in \mathcal{A}_\kappa$  there is a Boolean isomorphism  $h : B \rightarrow A$  onto its range.

Classical functorial arguments involving the dual ball of a Banach space imply (see, e.g., the introduction to [2] or [15]) that for any infinite cardinal  $\kappa$  the first

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two statements below are equivalent (by the Stone duality) and that they imply the third one:

- There is a universal Boolean algebra for  $\mathcal{A}_\kappa$ .
- There is a universal compact Hausdorff space for  $\mathcal{C}_\kappa$ .
- There is an isometrically universal Banach space for  $\mathcal{B}_\kappa$ .

By classical results (e.g., the Banach-Mazur Theorem) the three statements are true for  $\kappa = \omega$  (for a survey on related topics see [15]). Moreover, by the results of I. I. Parovichenko [16] and A. S. Esenin-Volpin [10], GCH implies that all the above statements are true for any infinite cardinal  $\kappa$ . Concerning the failure of the above statements for uncountable cardinals, since classical objects like  $\wp(\mathbb{N})/Fin$  and  $\ell_\infty/c_0$  fall into the case where  $\kappa = 2^\omega$ , this is the case which has been mostly addressed in the literature ([2, 3, 7]) and seems to us to be the most interesting. Also, it can be noted (e.g., in [3, 7]) that the cases of cardinals which are fixed on the scale of alephs, like  $\omega_1$ , are easier to deal with because they can be consistently made strictly smaller than  $2^\omega$ .

The consistency of the failure of the statements for  $\kappa = 2^\omega$  has been proved by A. Dow and K. P. Hart for Boolean algebras and compact spaces (see [7]) and by S. Shelah and A. Usvyatsov for isometric embeddings (see [17]). Actually, one can even prove the nonexistence of a universal Banach space in  $\mathcal{B}_{2^\omega}$  for isomorphic embeddings (see [2, 3]). Whether the equivalence of all the three statements above for  $\kappa = 2^\omega$  can be proved without additional assumptions is the main question that motivates our research in this paper. The reader may look at the discussions in [2], [3], [15] and Conjectures 2 and 3 in [9] for a wider view of this and similar questions.

Not knowing how to attack the problem itself, an intermediate question is whether an isometrically universal Banach space of the form  $C(K_A)$  must be induced by a universal Boolean algebra  $A$ , as it is known that the existence of an isometrically universal Banach space implies the existence of one of the form  $C(K_A)$  (Fact 1.1. of [2]). This is the case on the countable level: one can see that if  $C(K_A)$  is separable and isometrically universal<sup>1</sup> for separable Banach spaces, then  $K_A$  cannot be scattered and so  $A$  must contain an infinite free algebra, meaning that  $A$  is a universal algebra among countable algebras and  $K_A$  is a universal compact space among metrizable compact spaces. The main purpose of this paper is to prove that the situation is different on the uncountable level:

**Theorem 1.1.** *For any uncountable  $\kappa$ , if there exists an isometrically universal Banach space for  $\mathcal{B}_\kappa$ , then there is also such a space of the form  $C(K_A)$  for some Boolean algebra  $A$  where  $A$  does not contain well-ordered uncountable chains. In particular,  $A$  is not a universal Boolean algebra for  $\mathcal{A}_\kappa$  nor is  $K_A$  universal for  $\mathcal{C}_\kappa$ .*

*Proof.* Suppose that  $X$  is an isometrically universal Banach space for  $\mathcal{B}_\kappa$ . By the above discussion, by isometrically embedding  $X$  into  $C(B_{X^*})$  and taking a totally disconnected continuous preimage of  $B_{X^*}$ , we may assume that  $X$  is of the form  $C(K_A)$  for some Boolean algebra  $A$ . Then, Proposition 3.5 produces a Boolean algebra  $B$  of the same cardinality  $\kappa$  such that  $C(K_A)$  isometrically embeds into  $C(K_B)$  and hence  $C(K_B)$  is also isometrically universal for the same class of Banach spaces  $\mathcal{B}_\kappa$ . However, still by Proposition 3.5, the Boolean algebra generated by a well-ordered chain of type  $\omega_1$  cannot be embedded into  $B$ . By the Stone duality,

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<sup>1</sup>Actually,  $C(K_A)$  being isomorphically universal suffices.

$K_B$  cannot be continuously mapped onto  $[0, \omega_1]$  with the order topology. Therefore,  $K_B$  is not universal for  $\mathcal{A}_\kappa$  and  $K_B$  is not universal for  $\mathcal{C}_\kappa$ .  $\square$

In the literature there are some results assuming the axiom OCA and concerning similar behaviour of the Boolean algebra  $\wp(\mathbb{N})/Fin$  and the Banach space  $\ell_\infty/c_0 \equiv C(K_{\wp(\mathbb{N})/Fin})$ . However, there is at present no known example showing that  $\ell_\infty/c_0$  is not isometrically universal under OCA while there are many examples showing that  $\wp(\mathbb{N})/Fin$  is not universal ([5, 6, 18]). For example, A. Dow and K. P. Hart have proved in [6] that OCA implies that the measure algebra  $M$  does not embed into  $\wp(\mathbb{N})/Fin$ , while it is well known that  $C(K_M) \equiv L_\infty$  isometrically embeds into  $C(K_{\wp(\mathbb{N})/Fin}) \equiv \ell_\infty/c_0$  (see, e.g., p. 304 of [1]). Moreover, while the algebra of clopen subsets of the closure of a cozero set in  $\mathbb{N}^*$  does not embed into  $\wp(\mathbb{N})/Fin$  under OCA (see [18]), it is still open if the corresponding Banach space  $\ell_\infty(\ell_\infty/c_0)$  can be embedded under OCA into  $\ell_\infty/c_0$  (cf. [4, 8]).

Our paper contains basically one construction: given a Boolean algebra  $A$ , we construct another Boolean algebra  $B$  such that  $C(K_A)$  embeds isometrically into  $C(K_B)$  but  $B$  does not contain uncountable well-ordered chains (Proposition 3.5). Thus, when  $A$  does contain uncountable well-ordered chains as, for example, in the cases of Boolean algebras of the clopen subsets of  $[0, \omega_1]$  or  $\mathbb{N}^*$ , we cannot have a Boolean embedding of  $A$  into  $B$ .  $B$  is the algebra of clopen sets of some (standard) totally disconnected preimage of the dual ball  $B_{C(K)^*}$  of  $C(K_A)$  with the weak\* topology. In Section 2 we show that if it had an uncountable well-ordered chain of clopen sets we would have a chain of some sets in the ball  $B_{C(K)^*}$  and this cannot occur as is shown in Section 3.

It would be interesting to know if objects other than uncountable well-ordered chains can be used in the above argument. Or conversely, for which Boolean algebras  $A$ , the existence of an isometric embedding  $C(K_A)$  into  $C(K_B)$  implies the existence of a Boolean embedding of  $A$  into  $B$ . By Theorem 12.30 (ii) of [11] these cannot be uncountable antichains, i.e., the algebra  $A = FinCofin(\kappa)$  for any  $\kappa$  has the above property for any algebra  $B$ . Also, if  $A$  has an independent family of cardinality  $\kappa$  and  $C(K_A)$  isometrically embeds into  $C(K_B)$ , by the Holsztyński Theorem ([12]) we have a closed set  $F \subseteq K_B$  which maps onto  $\{0, 1\}^\kappa$ . This map can be always extended to a totally disconnected superspace and in particular  $K_B$  maps onto  $\{0, 1\}^\kappa$ , and so  $B$  contains an independent family of cardinality  $\kappa$ . But we do not know if an isometric embedding of  $C(K_{\wp(\mathbb{N})}) \equiv \ell_\infty$  into  $C(K_B)$  implies the existence of an isomorphic copy of  $\wp(\mathbb{N})$  in  $B$ .

Terminology should be standard; concerning the Banach spaces we follow [11] and for Boolean algebras we follow [14].  $f[X]$ ,  $f^{-1}[X]$  denote the image and the preimage of  $X$  under  $f$ , respectively.  $f|X$  denotes the restriction of  $f$  to  $X$ ,  $\{0, 1\}^{<\mathbb{N}} = \bigcup_{n \in \mathbb{N}} \{0, 1\}^n$ .

## 2. CHAINS IN TOTALLY DISCONNECTED PREIMAGES

**Lemma 2.1.** *Let  $K$  and  $L$  be compact spaces and  $\psi : K \rightarrow L$  be a surjective continuous mapping. Suppose  $U \subseteq K$  is clopen. For every  $x \in U$ , if  $\psi^{-1}[\{\psi(x)\}] \subseteq U$ , then  $\psi(x) \in \text{int}(\psi[U])$ . Hence,*

$$\psi[\{x \in U : \psi^{-1}[\{\psi(x)\}] \subseteq U\}] \subseteq \text{int}(\psi[U]).$$

*Proof.* Note that given  $x \in U$ , if  $\psi^{-1}[\{\psi(x)\}] \subseteq U$ , then  $\psi(x) \notin \psi[K \setminus U]$ . But  $L = \psi[U] \cup \psi[K \setminus U]$ , and so  $L \setminus \psi[K \setminus U]$  is an open set included in  $\psi[U]$ , so  $\psi(x) \in \text{int}(\psi[U])$ .  $\square$

The previous lemma will be applied to a restriction of the function  $\phi_I$  defined as follows. Let  $\phi : \{0, 1\}^{\mathbb{N}} \rightarrow [0, 1]$  be given by

$$\phi(x) = \sum_{n=1}^{\infty} \frac{x(n)}{2^n}.$$

Given any set of indices  $I$ , we consider  $\phi_I : (\{0, 1\}^{\mathbb{N}})^I \rightarrow [0, 1]^I$  which is defined coordinatewise by

$$\phi_I(x)(i) = \phi(x(i))$$

for  $x \in (\{0, 1\}^{\mathbb{N}})^I$  and  $i \in I$ . Since  $\phi_I$  is defined coordinatewise, we immediately obtain the following:

**Lemma 2.2.** *Suppose  $x, x', x'' \in (\{0, 1\}^{\mathbb{N}})^I$  are such that  $\phi_I(x) = \phi_I(x')$  and that for each  $i \in I$  either  $x''(i) = x(i)$  or  $x''(i) = x'(i)$ . Then  $\phi_I(x'') = \phi_I(x) = \phi_I(x')$ .*

We will consider the standard basis of clopen sets of  $\{0, 1\}^{\mathbb{N}}$ , i.e., the sets of the form

$$[s] = \{x \in \{0, 1\}^{\mathbb{N}} : s \subseteq x\}$$

for  $s \in \{0, 1\}^{<\mathbb{N}}$ . By the definition of the product topology, the sets of the form

$$U(i, [s]) = \{x \in (\{0, 1\}^{\mathbb{N}})^I : x(i) \in [s]\}$$

for  $i \in I$  and  $s \in \{0, 1\}^{<\mathbb{N}}$  form a topological subbasis for  $(\{0, 1\}^{\mathbb{N}})^I$  which consists of clopen sets. Note that

$$(*) \quad \phi[[s]] = \left[ \sum_{1 \leq n \leq |s|} \frac{x(n)}{2^n}, \sum_{1 \leq n \leq |s|} \frac{x(n)}{2^n} + \frac{1}{2^{|s|}} \right]$$

for any  $s \in \{0, 1\}^{<\mathbb{N}}$ . Hence,  $\phi$  sends the standard basic clopen sets onto closed subintervals of  $[0, 1]$ , in particular onto convex sets.

**Definition 2.3.** For a subspace  $X$  of  $(\{0, 1\}^{\mathbb{N}})^I$ ,  $n \in \mathbb{N}$  and a subset  $J \subseteq I$  we will say that  $Y \subseteq X$   $n$ -depends on  $J \subseteq I$  in  $X$  if and only if whenever  $x, y \in X$  and  $x(i)|n = y(i)|n$  for each  $i \in J$ , then

$$x \in Y \Leftrightarrow y \in Y.$$

It is immediate that  $U(i, [s])$   $n$ -depends on  $\{i\}$  in  $(\{0, 1\}^{\mathbb{N}})^I$  where  $n = |s|$  and so,  $U(i, [s]) \cap L$   $n$ -depends on  $\{i\}$  in any  $L \subseteq (\{0, 1\}^{\mathbb{N}})^I$ . Since clopen sets of compact Hausdorff spaces are Boolean combinations of finitely many sets from any clopen subbasis, it follows that any clopen subset of any closed  $L \subseteq (\{0, 1\}^{\mathbb{N}})^I$   $n$ -depends on  $J \subseteq I$  in  $L$ , for  $n$  being any natural number larger than  $|s|$  for all  $s$  which appear in the finite Boolean combination of sets of the form  $U(i, [s])$  and  $J$  is the set of all  $i$  which appear in such a Boolean combination. We are now ready for the main result of this section.

**Proposition 2.4.** *Let  $I$  be a set and let  $K$  be a closed convex subspace of  $[0, 1]^I$  such that no closed convex subspace  $F$  of  $K$  has an uncountable well-ordered chain of open (in  $F$ ) sets  $(V_\alpha)_{\alpha < \omega_1}$  satisfying  $\overline{V_\alpha} \subseteq V_\beta$  for  $\alpha < \beta < \omega_1$ . Then  $L = \phi_I^{-1}[K]$  has no uncountable well-ordered chain of clopen sets  $(U_\alpha)_{\alpha < \omega_1}$ .*

*Proof.* Suppose that  $(U_\alpha \cap L)_{\alpha \in \omega_1}$  is a sequence of clopen subsets of  $L$  where the  $U_\alpha$ 's are clopen subsets of  $(\{0, 1\}^{\mathbb{N}})^I$ . As we noticed, it follows that each  $U_\alpha \cap L$   $n_\alpha$ -depends on some finite set  $J_\alpha \subseteq I$  in  $(\{0, 1\}^{\mathbb{N}})^I$ , for some  $n_\alpha$ . Using the  $\Delta$ -system lemma (see [13]) and the fact that  $\{0, 1\}^{<\mathbb{N}}$  and  $\mathbb{N}$  are countable, we may assume that  $(J_\alpha)_{\alpha < \omega_1}$  is a  $\Delta$ -system with root  $\Delta$  and each  $U_\alpha$   $n$ -depends on  $J_\alpha$  in  $(\{0, 1\}^{\mathbb{N}})^I$  for a fixed  $n \in \mathbb{N}$  and all  $\alpha < \omega_1$ .

For each  $f \in (\{0, 1\}^n)^\Delta$ , consider

$$U(f) = \bigcap_{i \in \Delta} U(i, [f(i)]).$$

$(\{0, 1\}^{\mathbb{N}})^I$  is the disjoint union of the family of clopen sets  $\{U(f) : f \in (\{0, 1\}^n)^\Delta\}$ . It follows that there is  $f_0 \in (\{0, 1\}^n)^\Delta$  such that  $(U_\alpha \cap L \cap U(f_0))_{\alpha < \omega_1}$  forms an uncountable sequence. By going to a subsequence we may assume that all elements  $U_\alpha \cap L \cap U(f_0)$  are distinct. Consider  $L' = L \cap U(f_0)$ ,  $F = \phi_I[L']$  and  $\psi = \phi_I|_{L'}$  from  $L'$  onto  $F$  and put

$$V_\alpha = \text{int}_F(\psi[U_\alpha \cap L']).$$

Note that  $F = \phi_I[\phi_I^{-1}[K] \cap U(f_0)] = \phi_I[U(f_0)] \cap K$ , which is convex as the intersection of two convex sets. Secondly, note that  $U_\alpha \cap L'$   $n$ -depends on  $J_\alpha \setminus \Delta$  in  $L'$ , for each  $\alpha < \omega_1$ . Indeed, whenever  $x, y \in L'$ , we have that  $x(i)|n = f_0(i) = y(i)|n$  for all  $i \in \Delta$  and so, whenever we have additionally that  $x(i)|n = y(i)|n$  for all  $i \in J_\alpha \setminus \Delta$ , we may use the fact that  $U_\alpha$   $n$ -depends on  $J_\alpha$  in  $(\{0, 1\}^{\mathbb{N}})^I$ .

By the hypothesis on convex sets in  $K$  applied to  $F$ , there are  $\alpha < \beta < \omega_1$  such that  $\overline{V_\alpha} \not\subseteq V_\beta$ . Aiming at a contradiction, let us assume that  $(U_\alpha \cap L)_{\alpha < \omega_1}$  is a well-ordered chain and hence,  $U_\alpha \cap L' \subseteq U_\beta \cap L'$ . Then, since  $\psi[U_\alpha \cap L']$  is a closed set,  $\overline{V_\alpha} \subseteq \psi[U_\alpha \cap L']$  and we conclude that there is

$$y \in \psi[U_\alpha \cap L'] \setminus \text{int}_F(\psi[U_\beta \cap L']) \subseteq \psi[U_\beta \cap L'] \setminus \text{int}_F(\psi[U_\beta \cap L']).$$

Let  $x \in U_\alpha \cap L'$  be such that  $\psi(x) = y$ . Lemma 2.1 gives that  $\psi^{-1}[\{y\}] \not\subseteq U_\beta \cap L'$  and so there is  $x' \in L'$  such that  $x' \notin U_\beta \cap L'$  but  $\psi(x') = y$ . Now we will combine  $x$  and  $x'$  following Lemma 2.2 and will obtain a contradiction with the hypothesis that  $U_\alpha \cap L' \subseteq U_\beta \cap L'$ . Define

$$x''(\xi) = \begin{cases} x'(\xi) & \text{if } \xi \notin J_\beta \setminus \Delta, \\ x(\xi) & \text{otherwise.} \end{cases}$$

By Lemma 2.2 the point  $x''$  is in  $L$ . Note that  $x'' \in U(f_0)$  as  $x''|_\Delta = x|_\Delta$  i.e.,  $x'' \in L \cap U(f_0) = L'$ . Also,  $x''|_{J_\alpha} = x|_{J_\alpha}$  and so  $x'' \in U_\alpha$  since  $U_\alpha \cap L'$   $n$ -depends on  $J_\alpha$  in  $L'$  and  $x \in U_\alpha$ . On the other hand,  $x''|(J_\beta \setminus \Delta) = x'|(J_\beta \setminus \Delta)$ , so  $x'' \notin U_\beta$  as  $U_\beta \cap L'$   $n$ -depends on  $J_\beta \setminus \Delta$  in  $L'$  and  $x' \notin U_\beta$ . This shows that  $U_\alpha \cap L' \not\subseteq U_\beta \cap L'$ , contradicting our hypothesis. Hence,  $(U_\alpha \cap L)_{\alpha < \omega_1}$  is not a well-ordered chain and this completes the proof of the proposition.  $\square$

### 3. WELL-ORDERED CHAINS IN THE DUAL BALL

**Proposition 3.1.** *If  $F$  is any closed convex subspace of the dual unit ball of a Banach space endowed with the weak\* topology, then  $F$  does not have an uncountable well-ordered chain of open sets  $(V_\alpha)_{\alpha < \omega_1}$  such that  $\overline{V_\alpha} \subseteq V_\beta$  for any  $\alpha < \beta < \omega_1$ .*

*Proof.* Let  $X$  be a Banach space,  $B_{X^*}$  its dual unit ball endowed with the weak\* topology and  $F$  a closed convex subspace of  $B_{X^*}$ . Suppose  $(V_\alpha)_{\alpha \in \omega_1}$  is a well-ordered chain of open sets of  $F$  such that  $\overline{V_\alpha} \subseteq V_\beta$  for any  $\alpha < \beta < \omega_1$ . Put

$V = \bigcup_{\alpha \in \omega_1} V_\alpha$ . Since  $F$  is a closed subspace of  $B_{X^*}$  and  $B_{X^*}$  is weakly\* compact, then  $F \setminus \overline{V}$  is a nonempty closed set of  $F$ . It is as well a weakly closed set, since the weak topology is finer than the weak\* topology. It cannot be a weakly open set in  $F$ , because  $F$  is convex, and since  $X$  endowed with the weak topology is a topological vector space,  $F$  is weakly connected. Then, there is  $x \in \overline{B_{X^*} \setminus V^w} \cap \overline{V^w}$ .

Now, by Kaplansky's Theorem (e.g., 4.49 of [11]), every Banach space has countable tightness in its weak topology. So, there is a countable set  $D \subseteq V$  such that  $x \in \overline{D^w}$ . Since  $D$  is countable, there is  $\gamma \in \omega_1$  such that  $D \subseteq V_\gamma$ , which implies that  $x \in \overline{V_\gamma^w} \subseteq \overline{V_\gamma^{w^*}} \subseteq V_{\gamma+1} \subseteq V$ , contradicting the fact that  $x \in \overline{F \setminus V^w}$ .  $\square$

The hypothesis of  $F$  being convex is crucial, as any compact Hausdorff space  $K$  can be homeomorphically embedded in the dual ball of  $C(K)$  with the weak\* topology by associating the Dirac  $\delta_x$  to each  $x \in K$ . In the context of the above result it is also worthy to mention the following:

**Proposition 3.2.** *Note that if  $X$  is a Banach space of density  $\kappa$ , then there is a well-ordered increasing chain  $(U_\xi)_{\xi < \kappa}$  of open sets in the dual unit ball  $B_{X^*}$  of  $X$  with the weak\* topology.*

*Proof.* Using the Hahn-Banach Theorem, construct by transfinite induction a sequence  $(x_\xi, x_\xi^*)_{\xi < \kappa} \subseteq X \times B_{X^*}$  such that  $x_\xi^*(x_\eta) = 0$  for  $\eta < \xi < \kappa$  and  $x_\xi^*(x_\xi) = 1$  for  $\xi < \kappa$ . Then

$$U_\xi = \bigcup_{\eta < \xi} \{x^* \in B_{X^*} : x^*(x_\eta) \neq 0\}$$

is as required.  $\square$

**Lemma 3.3.** *Given a Banach space  $X$  and a dense subset  $D$  of its unit ball, the natural restriction mapping  $f : B_{X^*} \rightarrow [-1, 1]^D$  defined by  $f(x^*) = x^*|_D$  is a homeomorphism onto its image with respect to the weak\* and the product topologies, with the property that  $F \subseteq B_{X^*}$  is convex if and only if  $f(F)$  is convex.*

*Proof.* The preimages of standard basic open sets in the product are weakly\* open, so that  $f$  is continuous. As two distinct functionals must differ on an element of the unit ball we see that  $f$  is a homeomorphism onto its image.  $\Phi : X^* \rightarrow \mathbb{R}^D$  given by  $\Phi(x^*)(d) = x^*(d)$  is linear and one-to-one and hence its inverse is linear and one-to-one. Both mappings preserve convexity. The lemma follows as  $f = \Phi|_{B_{X^*}}$ .  $\square$

**Proposition 3.4.** *The dual unit ball of every Banach space endowed with the weak\* topology has a continuous preimage which is compact, totally disconnected of same weight and with no uncountable well-ordered chain of clopen sets.*

*Proof.* Let  $X$  be a Banach space of density  $\kappa$ . Let  $D \subseteq X$  be a dense subset of the unit ball of cardinality  $\kappa$ . Let  $f : B_{X^*} \rightarrow [-1, 1]^D$  be defined by  $f(x^*) = x^*|_{B_X}$  as in Lemma 3.3, let  $g$  be the linear order-preserving homeomorphism from  $[-1, 1]$  onto  $[0, 1]$  and let  $g^D : [-1, 1]^D \rightarrow [0, 1]^D$  be defined coordinatewise by  $g^D(x)(d) = g(x(d))$  for any  $d \in D$  and  $x \in [-1, 1]^D$ . Clearly  $g^D$  is a homeomorphism such that  $F \subseteq [-1, 1]^D$  is convex if and only if  $g^D[F]$  is convex. Then,  $h = g \circ f : B_{X^*} \rightarrow [0, 1]^D$  is a homeomorphism onto its image such that  $F \subseteq B_{X^*}$  is convex if and only if  $h[F]$  is convex.

Let  $K = h[B_{X^*}] \subseteq [0, 1]^D$ . If  $F$  is a convex subset of  $K$ , then  $h^{-1}[F]$  is a convex subset of  $B_{X^*}$  and Proposition 3.1 guarantees that it contains no uncountable well-ordered chain of open sets  $(V_\alpha)_{\alpha < \omega_1}$  such that  $\overline{V_\alpha}^F \subseteq V_\beta$  for  $\alpha < \beta < \omega_1$ . Since  $h$  is a homeomorphism onto  $K$ , we get that  $F$  has no such chain of open sets either.

Finally, since  $K$  satisfies the hypotheses of Proposition 2.4, we get that  $L = (\phi_D)^{-1}[K]$  has no uncountable well-ordered chain of clopen sets. Since the weight of  $K$  cannot be bigger than  $D$ , this concludes the proof.  $\square$

**Proposition 3.5.** *Suppose that  $A$  is a Boolean algebra. There is a Boolean algebra  $B$  of same cardinality as  $A$  but without uncountable well-ordered chains such that the Banach space  $C(K_B)$  contains an isometric copy of  $C(K_A)$ .*

*Proof.* Let  $X = C(K_A)$  and by Proposition 3.4, the dual unit ball  $B_{X^*}$  has a continuous preimage  $L$  which is compact, totally disconnected, of the same weight as  $B_{C(K_A)^*}$  and with no uncountable well-ordered chain of clopen sets. Hence,  $B = \text{Clop}(L)$  is a Boolean algebra of the same cardinality as  $A$  which has no uncountable well-ordered chains and, therefore, has no isomorphic copy of  $A$ . But  $K_B$  is homeomorphic to  $L$ , so that  $C(K_B)$  is isometric to  $C(L)$ , which contains an isometric copy of  $C(B_{X^*})$ , which in turn contains an isometric copy of  $X = C(K_A)$ .  $\square$

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