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ISOMETRIES OF COMBINATORIAL BANACH SPACES

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ABSTRACT. We prove that every isometry between two combinatorial spaces is determined by a permutation of the canonical unit basis combined with a change of signs. As a consequence, we show that in the case of Schreier spaces, all the isometries are given by a change of signs of the elements of the basis. Our results hold for both the real and the complex cases.

1. INTRODUCTION

For a full survey of isometry groups on Banach spaces we refer to the books of Fleming and Jamison [5, 6]. In this paper we concentrate on the study of isometry groups of certain sequence spaces called combinatorial.

Classical results guarantee that (surjective) isometries of the sequence spaces c_0 or ℓ_p, $1 \leq p < \infty$, $p \neq 2$, are determined by a permutation of the elements of the canonical unit basis and a change of sign of these vectors (see, e.g., [10, Theorem 9.8.3 and Theorem 9.8.5] or [9, Proposition 2.f.14]). So the isometry groups of these spaces are rich in some sense. At the other side of the spectrum, it was proved in [3] and [4] that the isometry groups on James space J_2, and the generalized James space J_p, are trivial, i.e., the only isometries are plus or minus the identity. Recently, it has been shown by Antunes, Beanland, and Viet Chu [1] that the real Schreier spaces of finite order have a structure which is nontrivial but more rigid than the ones on c_0 or ℓ_p: isometries of these spaces correspond to a change of signs of the elements of the canonical unit basis. In this paper we generalize this result to higher-order Schreier spaces and more general combinatorial spaces, in both the real and the complex cases, in some cases obtaining forms of rigidity which are intermediate between the c_0 or ℓ_p example and the Schreier example. This answers a question posed by K. Beanland in a private conversation. We also characterize the isometries that may arise between two different combinatorial spaces, determining when two combinatorial spaces are or are not isometric.

In what follows, we consider spaces with either real or complex scalars. In this context, we call a scalar of modulus 1 a *sign* (so simply ± 1 in the real case). Recall that for a given regular family $\mathcal{F}$ (i.e., hereditary, compact, and spreading; see Definition 1) of finite subsets of $\mathbb{N}$, the combinatorial Banach space $X_{\mathcal{F}}$ is the

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completion of c_{00}, the vector space of finitely supported scalar sequences, with respect to the norm:

$$\|x\| = \sup\left\{\sum_{i\in F} |x(i)| : F \in \mathcal{F}\right\}.$$

The sequence of unit vectors $(e_n)_n$ forms an unconditional Schauder basis, and $X_\mathcal{F}$ is c_0-saturated (see [7]), so, in particular, it contains no copies of ℓ_1. Therefore the basis $(e_n)_n$ is shrinking (see Theorem 1.c.9 in [9]), and hence $(e_n^*)_n$ is a Schauder basis of the dual space $X_\mathcal{F}^*$.

The simplest examples of regular families are the families $[\mathbb{N}]^{\leq n}$ of all subsets of $\mathbb{N}$ of cardinality at most n for some fixed $n \in \mathbb{N}$. More interesting examples are the Schreier family

$$\mathcal{S} := \{F \in [\mathbb{N}]^{<\omega} : |F| \leq \min F\} \cup \{\emptyset\}$$

and its versions of higher order, which will be considered in Section 4.

Given two combinatorial spaces $X_\mathcal{F}$ and $X_\mathcal{G}$ and a surjective isometry $T : X_\mathcal{F}^* \to X_\mathcal{G}^*$ between the corresponding dual spaces, we can use the classical fact that extreme points of the unit balls are preserved by T to analyze the expansion of each $Te_i^* = \sum_j \alpha_j^i e_j^*$. This is the analysis we make in Section 3 to prove our main result (Theorem 10), which states that if $\mathcal{F}$ and $\mathcal{G}$ are regular families, then T is induced by what we call a *signed permutation*, i.e., for every $i \in \mathbb{N}$, $Te_i^* = \theta_i e_{\pi(i)}^*$ for some permutation $\pi : \mathbb{N} \to \mathbb{N}$ and some sequence of signs $(\theta_i)_i$. Since the adjoint operator of an isometry is an isometry, it follows, in particular, that any isometry $T : X_\mathcal{F} \to X_\mathcal{G}$ is also induced by a signed permutation.

Together with the fact that surjective isometries between Banach spaces preserve extreme points of the unit balls, we are going to use extensively in our arguments the following description of the extreme points of the dual ball of a combinatorial space:

$$\mathrm{Ext}(X_\mathcal{F}^*) = \left\{\sum_{i\in F} \theta_i e_i^* : F \in \mathcal{F}^{MAX} \text{ and } (\theta_i)_{i\in F} \text{ is a sequence of signs}\right\},$$

where $\mathcal{F}^{MAX}$ denotes the family of maximal elements of $\mathcal{F}$ with respect to inclusion. Since this characterization of extreme points holds in the real case (see [1,8]) but was not known in the complex case, we shall give a proof for complex combinatorial spaces; see Proposition 5. Our proof includes the real case and seems to be simpler than the original proof.

Classical examples of combinatorial spaces are the spaces $X_{\mathcal{S}_\alpha}$ associated to the so-called generalized Schreier families $\mathcal{S}_\alpha$, for $\alpha < \omega_1$. As a consequence of our main result and specific properties of these families, we prove in Section 4 that any isometry T of $X_{\mathcal{S}_\alpha}^*$ acts on the canonical unit basis as a change of signs, that is, $Te_i^* = \theta_i e_i^*$ for some sequence of signs $(\theta_i)_i$.

2. Preliminaries

We start with the combinatorial background for our results.

2.1. **Regular families.** Let $[\mathbb{N}]^{<\omega}$ denote the family of all finite subsets of $\mathbb{N}$, and by a family we always mean a family of finite subsets of $\mathbb{N}$ which contains all singletons. We denote by $\mathcal{F}^{MAX}$ the family of maximal elements of a family $\mathcal{F}$ with respect to inclusion.

Definition 1. We say that a given family $\mathcal{F}$ is regular if it satisfies the following three conditions:

- $\mathcal{F}$ is hereditary (closed under subsets);
- $\mathcal{F}$ is compact as a subset of $2^{\mathbb{N}}$, where each element of $\mathcal{F}$ is identified with its characteristic function;
- $\mathcal{F}$ is spreading, that is, if $F \in \mathcal{F}$ and $\sigma : F \to \mathbb{N}$ is such that $\sigma(n) \geq n$ for every $n \in F$, then $\sigma(F) \in \mathcal{F}$.

An easy property shared by regular families and which will be frequently used is the fact that any element can be extended "to the right" to a maximal one, as in the following lemma.

Lemma 2. *If $\mathcal{F}$ is a regular family and $F \in \mathcal{F}$, then for every infinite set $N \subseteq \mathbb{N}$, there is $F \subseteq E \in \mathcal{F}^{MAX}$ such that $E \setminus F \subseteq N$.*

Proof. Given $F \in \mathcal{F} \setminus \mathcal{F}^{MAX}$ and $N \subseteq \mathbb{N}$ infinite let $F \subsetneq E_1 \in \mathcal{F}$ and spread E_1 to some F_1 in such a way that $F \subsetneq F_1 \in \mathcal{F}$ and $F_1 \setminus F \subseteq N$. If $F_1 \in \mathcal{F}^{MAX}$ we are done. If not, let $F_1 \subsetneq E_2 \in \mathcal{F}$ and spread E_2 to some F_2 in such a way that $F_1 \subsetneq F_2 \in \mathcal{F}$ and $F_2 \setminus F_1 \subseteq N$, so that $F_2 \setminus F \subseteq N$. Repeat this process until achieving some $F \subseteq F_n \in \mathcal{F}^{MAX}$ such that $F_n \setminus F \subseteq N$. This will necessarily happen, as if not, $(F_n)_n$ will be a strictly increasing chain of elements of $\mathcal{F}$ converging to the infinite set $Y = \bigcup_{n \in \mathbb{N}} F_n \notin \mathcal{F}$, contradicting the compactness of $\mathcal{F}$. $\square$

Lemma 3. *Suppose $\mathcal{F}$ is a regular family, and let $n \in \mathbb{N}$ with $\{n\} \notin \mathcal{F}^{MAX}$. Then we can find a sequence of finite sets $n < G_1 < G_2 < \ldots$ such that for any $i \in \mathbb{N}$, $|G_i| \leq |G_{i+1}|$ and $G_i \cup \{n\} \in \mathcal{F}^{MAX}$.*

Proof. By Lemma 2, we can find $F_1 \in \mathcal{F}^{MAX}$ such that $n = \min F_1$. Clearly, $|F_1| \geq 2$, and let $G_1 := F_1 \setminus \{n\}$. Using that $\mathcal{F}$ is spreading, we can find $F_2' \in \mathcal{F}$ such that $|F_1| = |F_2'|$, $n = \min F_2'$, and $G_1 < F_2' \setminus \{n\}$. Next, from Lemma 2, it follows that we can "fill in" F_2' to the right, if necessary, to obtain a set $F_2 \in \mathcal{F}^{MAX}$. Let $G_2 := F_2 \setminus \{n\}$, and clearly $|G_1| \leq |G_2|$. Continuing in this manner we obtain the conclusion of the lemma. $\square$

2.2. **Extreme points in the dual space.** Denote by $\mathbb{K}$ the field of scalars $\mathbb{R}$ or $\mathbb{C}$. We say that a subset N of a Banach space X is *sign invariant* if for any sign $\theta \in \mathbb{K}$, we have $\theta N = N$. We recall a very classical lemma in its real/complex version.

Lemma 4. *Let X be a Banach space over $\mathbb{K}$, and let $N \subseteq B_{X^*}$ be a sign invariant norming set for X. Then*

$$B_{X^*} = \overline{conv(N)}^{w*}.$$

Proof. Denote $S := \overline{conv(N)}^{w*}$ and note that S is sign invariant. Assume by contradiction that the conclusion is false and pick $f \in B_{X^*} \setminus S$. S is convex, w^*-compact, and disjoint from $\{f\}$ (which is convex and w^*-closed). From the Hahn–Banach separation theorem we have that there exists $x \in X$ and a real number t such that

$$\mathrm{Re}(g(x)) < t < \mathrm{Re}(f(x)), \text{ for all } g \in S.$$

Multiplying a given $g \in S$ by the appropriate sign we may assume that $\mathrm{Re}(g(x)) = |g(x)|$. Since S is sign invariant, we have

$$|g(x)| < t < \mathrm{Re}(f(x)), \text{ for all } g \in S.$$

Taking the supremum over all $g \in N$, we obtain the contradiction

$$\|x\| < \mathrm{Re}(f(x)) \leq |f(x)| \leq \|f\|\|x\| \leq \|x\|,$$

which finishes the proof. $\square$

Proposition 5. *If $\mathcal{F}$ is a regular family, then*

$$\mathrm{Ext}(X_{\mathcal{F}}^*) = \left\{ \sum_{i \in F} \theta_i e_i^* : F \in \mathcal{F}^{MAX} \textit{ and } (\theta_i)_{i \in F} \textit{ is a sequence of signs} \right\}.$$

Proof. Let $N := \{\sum_{i \in F} \theta_i e_i^* : F \in \mathcal{F}^{MAX}, |\theta_i| = 1\}$, and let $M := \{\sum_{i \in F} \theta_i e_i^* : F \in \mathcal{F}, |\theta_i| = 1\}$. Note that M is norming for $X_{\mathcal{F}}$ and is sign invariant. From Lemma 4 it follows that $B_{X_{\mathcal{F}}^*} = \overline{conv(M)}^{w^*}$. We claim that M is w^*-closed. Then both M and $B_{X_{\mathcal{F}}^*} = \overline{conv(M)}^{w*}$ are compact in the locally convex space $(X_{\mathcal{F}}^*, w^*)$, so by Milman's theorem (see [11, Theorem 3.25]), every extreme point of $B_{X_{\mathcal{F}}^*}$ lies in M.

Since $N \subseteq \mathrm{Ext}(B_{X_{\mathcal{F}}^*})$ and no $x \in M \setminus N$ is an extreme point (any such x is easily written as the middle point of two different points of N), it follows that $\mathrm{Ext}(B_{X_{\mathcal{F}}^*}) = N$.

To prove the claim we note that if a sequence of points of M converges w^* to some y, then the compactness of $\mathcal{F}$ implies that the support of y belongs to $\mathcal{F}$. The w^* convergence implies coordinatewise convergence, and so each nonzero coordinate of y must be a sign, which concludes the proof of the claim. $\square$

3. Isometries between combinatorial spaces

Let $T : X_{\mathcal{F}}^* \to X_{\mathcal{G}}^*$ be an operator between duals of combinatorial spaces $X_{\mathcal{F}}$ and $X_{\mathcal{G}}$, sending extreme points of the unit ball of $X_{\mathcal{F}}^*$ to extreme points of the unit ball of $X_{\mathcal{G}}^*$. Our goal is to show that for any $i \in \mathbb{N}$, Te_i^* is of the form $\sum_{j \in A_i} \theta_j^i e_j^*$, for finite subsets A_i of $\mathbb{N}$ and sequences of signs $(\theta_j^i)_{j \in A_i}$ (Proposition 9). Then, assuming T is a surjective isometry, we prove that T is induced by a signed permutation.

Since

$$\mathrm{Ext}(X_{\mathcal{F}}^*) = \left\{ \sum_{i \in F} \theta_i e_i^* : F \in \mathcal{F}^{MAX} \text{ and } (\theta_i)_{i \in F} \text{ is a sequence of signs} \right\},$$

it follows that if $x^* \in \mathrm{Ext}(X_{\mathcal{F}}^*)$, then for any $i \in \mathbb{N}$, $|x^*(e_i)| \in \{0, 1\}$.

Lemma 6. *Suppose $n \in \mathbb{N}$, $\{n\} \notin \mathcal{F}^{MAX}$, and let $k \in Supp(Te_n^*)$ such that $|\alpha_k^n| \neq 1$. Then $|\alpha_k^n| = \frac{1}{2}$, and for any $F \in \mathcal{F}^{MAX}$ such that $n \in F$, there exists a unique $m \in F \setminus \{n\}$ such that $|\alpha_k^m| = \frac{1}{2}$. Moreover, $\alpha_k^j = 0$ for all $j \in F \setminus \{n, m\}$.*

Proof. Let $F \in \mathcal{F}^{MAX}$ such that $n \in F$. If for all $j \in F \setminus \{n\}$, $k \notin Supp(Te_j^*)$, then we have

$$T(\sum_{j \in F} e_j^*)(e_k) = \sum_{j \in F} Te_j^*(e_k) = \alpha_k^n.$$

Since $|\alpha_k^n| \notin \{0, 1\}$, it follows that $T(\sum_{j \in F} e_j^*) \notin \mathrm{Ext}(X_{\mathcal{G}}^*)$, contradicting the fact that $\sum_{j \in F} e_j^* \in \mathrm{Ext}(X_{\mathcal{F}}^*)$ and T preserves extreme points. Therefore there exists $m \in F$, $m \neq n$, such that $k \in Supp(Te_m^*)$. Consider $(\theta_j)_{j \in F}$ as a sequence of signs such that for any $j \in F$ we have $\theta_j \alpha_k^j \geq 0$. Since $\sum_{j \in F} \theta_j e_j^* \in \mathrm{Ext}(X_{\mathcal{F}}^*)$, it follows that $T(\sum_{j\in F} \theta_j e_j^*) \in \mathrm{Ext}(X_{\mathcal{G}}^*)$, and hence

$$T(\sum_{j\in F} \theta_j e_j^*)(e_k) = \sum_{j\in F} \theta_j Te_j^*(e_k) = \sum_{j\in F} \theta_j \alpha_k^j = 1 \tag{1}$$

as all $\theta_j \alpha_k^j$ are nonnegative and at least two, namely $\theta_n \alpha_k^n$ and $\theta_m \alpha_k^m$, are positive.

On the other hand $\sum_{j\in F\setminus\{n\}} \theta_j e_j^* - \theta_n e_n^*$ is also an extreme point whose image by T has real value in e_k, and hence

$$\sum_{j\in F\setminus\{n\}} \theta_j Te_j^*(e_k) - \theta_n Te_n^*(e_k) = \sum_{j\in F\setminus\{n\}} \theta_j \alpha_k^j - \theta_n \alpha_k^n \in \{-1, 0, 1\}. \tag{2}$$

From (1), (2), and the fact that $\theta_n \alpha_k^n$ and $\theta_m \alpha_k^m$ are positive, it follows easily that $\sum_{j\in F\setminus\{n\}} \theta_j \alpha_k^j - \theta_n \alpha_k^n = 0$, and solving for $\theta_n \alpha_k^n$ we obtain that $\theta_n \alpha_k^n = \frac{1}{2}$. In a similar manner, reversing the roles of n and m we obtain that $\theta_m \alpha_k^m = \frac{1}{2}$ as well. Plugging these values into (1), and taking into account that all $\theta_j \alpha_k^j$ are nonnegative, it follows that $\alpha_k^j = 0$ for all $j \in F \setminus \{n, m\}$. $\square$

Lemma 7. *Suppose $n \in \mathbb{N}$, $\{n\} \notin \mathcal{F}^{MAX}$, and let $k \in Supp(Te_n^*)$ such that $|\alpha_k^n| = 1$. Then for any $F \in \mathcal{F}^{MAX}$ such that $n \in F$, and for any $j \in F \setminus \{n\}$, $\alpha_k^j = 0$.*

Proof. Pick $F \in \mathcal{F}^{MAX}$ such that $n \in F$, and consider the extreme points $\sum_{j\in F} \theta_j e_j^*$, where $(\theta_j)_{j\in F}$ are choices of signs such that $\theta_j \alpha_k^j$ is nonnegative for all $j \in F$. Since $T(\sum_{j\in F} \theta_j e_j^*)$ is also an extreme point, it follows that

$$\theta_n \alpha_k^n + \sum_{j\in F\setminus\{n\}} \theta_j \alpha_k^j \in \{-1, 0, 1\},$$

for all signs $(\theta_j)_{j\in F}$. Clearly this is only possible if $\alpha_k^j = 0$ for all $j \in F \setminus \{n\}$. $\square$

Lemma 8. *For any finitely supported $x^* = \sum_{i\in A} \theta_i e_i^* \in X_{\mathcal{F}}^*$ with $|\theta_i| = 1$, we have that $\|x^*\| = 1$ if and only if $A \in \mathcal{F}$ (and $\|x^*\| > 1$ otherwise).*

Proof. Since each $e_i \in X_{\mathcal{F}}$ has norm 1 and $|x^*(e_i)| = |\theta_i| = 1$ for $i \in A$, clearly $\|x^*\| \geq 1$.

If $A \in \mathcal{F}$, then given $x = \sum_j \alpha_j e_j \in X_{\mathcal{F}}$ such that $\|x\|_{\mathcal{F}} = 1$, we have that $\sum_{j\in A} |\alpha_j| \leq 1$, so that

$$|x^*(x)| \leq \sum_{i\in A} |\theta_i| \cdot |\alpha_i| = \sum_{i\in A} |\alpha_i| \leq 1.$$

Therefore, $\|x^*\| \leq 1$.

Conversely, if $A \notin \mathcal{F}$, then $|A| \geq 2$, so let $x = \frac{1}{|A|-1} \sum_{i\in A} \bar{\theta}_i e_i$ and notice that $\|x\| \leq 1$ and

$$x^*(x) = \frac{1}{|A|-1} \sum_{i\in A} \theta_i \bar{\theta}_i = \frac{1}{|A|-1} \sum_{i\in A} |\theta_i|^2 = \frac{|A|}{|A|-1} > 1,$$

so that $\|x^*\| > 1$. $\square$

Proposition 9. *Let $T : X_{\mathcal{F}}^* \to X_{\mathcal{G}}^*$ preserve extreme points, where $\mathcal{F}, \mathcal{G}$ are regular families. Then the vectors Te_i^*, $i \in \mathbb{N}$, are of the form $\sum_{j\in A_i} \theta_j^i e_j^*$ for sets $A_i \in \mathcal{G}$ of $\mathbb{N}$ and signs θ_j^i.*

Proof. With the previous notation, we are going to show first that for any $n \in \mathbb{N}$ and any $k \in \mathbb{N}$ we have $|\alpha_k^n| \neq \frac{1}{2}$. Fix $n \in \mathbb{N}$ arbitrary and note first that if $\{n\} \in \mathcal{F}^{MAX}$, then e_n^* is an extreme point, and hence so is Te_n^*, and it follows that for any $k \in \mathbb{N}$ we have that $|Te_n^*(e_k)| = |\alpha_k^n| \in \{0, 1\}$. When $\{n\} \notin \mathcal{F}^{MAX}$, assume towards a contradiction that there exists $k \in \mathbb{N}$ such that $|\alpha_k^n| = \frac{1}{2}$. We are going to consider separately two cases: when n belongs to a maximal set of size at least three, and when n belongs only to maximal sets of size two.

Case 1 (There exists $F \in \mathcal{F}^{MAX}$ such that $n \in F$ and $|F| \geq 3$). In this case, construct a sequence as in Lemma 3, starting with $G_1 := F \setminus \{n\}$. It follows that for each $i \in \mathbb{N}$, $|G_i| \geq 2$. Since $|\alpha_k^n| = \frac{1}{2}$, and for any $i \in \mathbb{N}$ we have that $G_i \cup \{n\} \in \mathcal{F}^{MAX}$ and $|G_i| \geq 2$, from Lemma 6 we conclude that there exists a sequence $p_i \in G_i$, $i \in \mathbb{N}$, such that $\alpha_k^{p_i} = 0$. From Lemma 2, there is $E \in \mathcal{F}^{MAX}$ such that $E \subseteq \{n, p_1, p_2, \ldots\}$ and $n = \min E$. However,

$$|T(\sum_{j \in E} e_j^*)(e_k)| = |\alpha_k^n + \sum_{p_i \in E \setminus \{n\}} \alpha_k^{p_i}| = |\alpha_k^n| = \frac{1}{2},$$

contradicting the fact that $T(\sum_{j \in E} e_j^*)$ is an extreme point.

Case 2 (For any $F \in \mathcal{F}^{MAX}$ such that $n \in F$, $|F| = 2$). Assume there exists $m > n$ such that $\{n, m\} \in \mathcal{F}^{MAX}$ and m belongs to a maximal set of size at least 3. Then it follows from Lemma 6 that $|\alpha_k^m| = \frac{1}{2}$, and from Case 1, applied to m, we obtain a contradiction. Hence, we may also assume that for any $m > n$ such that $\{n, m\} \in \mathcal{F}^{MAX}$, only m belongs to maximal sets of size 2. Construct a sequence of sets $n < G_1 < G_2 < \ldots$ as in Lemma 3. Then we must have that each G_i is a singleton, so we obtain a sequence $n < q_1 < q_2 < \ldots$ such that $\{n, q_i\} \in \mathcal{F}^{MAX}$ for all $i \in \mathbb{N}$. Also, from Lemma 6, we conclude that $|\alpha_k^{q_i}| = \frac{1}{2}$ for all $i \in \mathbb{N}$. From spreading we have that $\{q_i, q_j\} \in \mathcal{F}$ for all $i < j$, and since no q_i belongs to a maximal set of size at least 3, it follows that actually $\{q_i, q_j\} \in \mathcal{F}^{MAX}$ for all $i < j$.

For each i write $T(e_{q_i}^*) = \frac{1}{2}\epsilon_i e_k^* + \frac{1}{2} y_i^* + z_i^*$, where ϵ_i is a sign, the three vectors are disjointly supported (possibly y_i^* or z_i^* is 0), and y_i^* and z_i^* only have coordinates of modulus 1 on their support. In the complex case, we note that $T(\theta_i e_{q_i}^* + \theta_j e_{q_j}^*)$, being an extreme point for all signs θ_i, θ_j, contradicts the fact that its k-coordinate is $\frac{1}{2}(\theta_i \epsilon_i + \theta_j \epsilon_j)$, which can assume values of modulus different from 0 and 1. In the real case, passing to a subsequence we may assume that ϵ_i is constant, and without loss of generality, equal to 1. From the fact that for $i \neq j$, $T(e_{q_i}^* \pm e_{q_j}^*)$ must be an extreme point and therefore does not have $\pm\frac{1}{2}$ coordinates, we deduce that the support of y_i^* is some finite set C independent of i and that z_i^* is disjointly supported from k, from y_i^*, and from all other z_j^*. Since C is finite we find $i \neq j$ such that $y_i^* = y_j^*$, and we compute

$$T(e_{p_i}^* + e_{p_j}^*) = e_k^* + y_i^* + z_i^* + z_j^*$$

and

$$T(e_{p_i}^* - e_{p_j}^*) = +z_i^* - z_j^*,$$

and so the second vector has support strictly included in the support of the first one. But this contradicts that both must belong to $\mathcal{G}^{MAX}$.

The fact that $A_i \in \mathcal{G}$ follows from Lemma 8. $\square$

Theorem 10. *Let $T : X_{\mathcal{F}}^* \to X_{\mathcal{G}}^*$ be an isometry, where $\mathcal{F}, \mathcal{G}$ are regular families. Then there exists a permutation $\pi : \mathbb{N} \to \mathbb{N}$ and a sequence of signs $(\theta_i)_i$ such that $Te_i^* = \theta_i e_{\pi(i)}^*$ for all $i \in \mathbb{N}$.*

Proof. Note first that if $\{i, k\} \in \mathcal{F}$ and $i \neq k$, then $SuppTe_i^* \cap SuppTe_k^* = \emptyset$. Indeed, from Proposition 9, $|(Te_i^*)(e_k)|, |(Te_k^*)(e_k)| \in \{0,1\}$. Hence, for a given j, let θ_i and θ_k be signs such that $\theta_i(Te_i^*)(e_j) = |(Te_i^*)(e_j)|$ and $\theta_k(Te_k^*)(e_j) = |(Te_k^*)(e_j)|$. Since $\|\theta_i e_i^* + \theta_k e_k^*\| = 1$, we have that $\theta_i(Te_i^*)(e_j) + \theta_k(Te_k^*)(e_j) \leq \|\theta_i Te_i^* + \theta_k Te_k^*\| = 1$, so that $|(Te_i^*)(e_k)|, |(Te_k^*)(e_k)|$ cannot both be 1. This guarantees that the supports are disjoint. Of course, a similar fact holds true for a pair $\{j, l\} \in \mathcal{G}$ and T^{-1}.

We are going to show that for any $n \in \mathbb{N}$, the support of Te_n^* is a singleton. From the fact that T is a bijection, the conclusion of the theorem follows immediately.

Fix $n \in \mathbb{N}$ arbitrary, and from Proposition 9 it follows that we can write

$$Te_n^* = \sum_{i \in A} \theta_i e_i^*,$$

where θ_i are signs and $A \in \mathcal{G}$. From the hereditary property we have that for any $j, l \in A$, $j \neq l$, the set $\{j, l\} \in \mathcal{G}$. Therefore, from the remark at the beginning of the proof, we conclude that $T^{-1}e_j^*$ and $T^{-1}e_l^*$ have disjoint support. Hence

$$\bigcup_{i \in A} Supp(T^{-1}e_i^*) = Supp\left(\sum_{i \in A} \theta_i T^{-1} e_i^*\right) = Supp\left(T^{-1} \sum_{i \in A} \theta_i e_i^*\right) = \{n\}.$$

Therefore $|A| \leq |\bigcup_{i \in A} Supp(T^{-1}e_i^*)| = 1$, and from this it follows immediately that A is a singleton, as claimed. This finishes the proof. $\square$

The following example shows that a bounded operator that sends extreme points to extreme points and vectors of disjoint support to vectors of disjoint support is not necessarily given by a permutation of the basis.

Example 11. The map T defined on the dual of the Schreier space by $T(e_n^*) = e_{2n}^* + e_{2n+1}^*$ sends extreme points to extreme points, sends disjoint supports to disjoint supports, but is not induced by a signed permutation.

Proof. For $n \geq 1$ any sum of e_i^* supported on some F such that $|F| = \min F = n$ has image supported on some F' such that $|F'| = 2n = \min |F'|$. $\square$

Corollary 12. *Assume $\mathcal{F}, \mathcal{G}$ are regular families. Then TFAE:*

(i) *$X_{\mathcal{F}}$ and $X_{\mathcal{G}}$ are isometric;*
(ii) *$X_{\mathcal{F}}^*$ and $X_{\mathcal{G}}^*$ are isometric;*
(iii) *there is a permutation π of $\mathbb{N}$ such that $\mathcal{G}^{MAX} = \{\pi(F) : F \in \mathcal{F}^{MAX}\}$;*
(iv) *there is a permutation π of $\mathbb{N}$ such that $\mathcal{G} = \{\pi(F) : F \in \mathcal{F}\}$.*

Proof.
(i) implies (ii) for any two Banach spaces.

It follows from Theorem 10 that if $T : X_{\mathcal{F}}^* \to X_{\mathcal{G}}^*$ is an isometry, then it is induced by a signed permutation. Since T takes extreme points to extreme points, in particular, we get that $F \in \mathcal{F}^{MAX}$ if and only if $\pi(F) \in \mathcal{G}^{MAX}$.

(iii) trivially implies (iv).

Finally, if π is a permutation of $\mathbb{N}$ such that $\mathcal{G} = \{\pi(F) : F \in \mathcal{F}\}$, it is easy to see that $T : X_{\mathcal{F}} \to X_{\mathcal{G}}$ defined by $T(\sum_i \lambda_i e_i) = \sum_i \lambda_i e_{\pi(i)}$ is an onto isometry. $\square$

The following remark is immediate from the corollary above.

Remark 13. Assume $\mathcal{F}$ is a regular family. Then $T : X_{\mathcal{F}} \to X_{\mathcal{F}}$ is an onto isometry if and only if $Te_i = \theta_i e_{\pi(i)}$ for some permutation π of $\mathbb{N}$ such that $\mathcal{F} = \{\pi(F) : F \in \mathcal{F}\}$.

4. Isometries of Schreier spaces

Definition 14. Given a countable ordinal α, we define the Schreier family of order α inductively as follows:

- $\mathcal{S}_1 = \mathcal{S}$;
- $\mathcal{S}_{\alpha+1} = \{\bigcup_{j=1}^k E_j : E_j \in \mathcal{S}_\alpha \text{ and } \{\min E_j : 1 \le j \le k\} \in \mathcal{S}\} \cup \{\emptyset\}$;
- $\mathcal{S}_\alpha = \{F \in [\mathbb{N}]^{<\omega} : F \in \mathcal{S}_{\alpha_n} \text{ for some } n \le \min F\} \cup \{\emptyset\}$, if α is a limit ordinal and $(\alpha_n)_n$ is a fixed increasing sequence of ordinals converging to α.

Note that the sequence of Schreier families $(\mathcal{S}_\alpha)_{\alpha<\omega_1}$ depends on the choice of the sequences $(\alpha_n)_n$ converging to each limit ordinal α. It is a well-known fact [2] that Schreier families are regular families, so that we may apply the results from the previous section to these families.

Lemma 15. *Let E and F be two maximal sets in $\mathcal{S}_\alpha$, where $\alpha < \omega_1$. If F is a spreading of E, then* $\min E = \min F$.

Proof. We are going to prove the statement by transfinite induction. It clearly holds true for $\mathcal{S}_1$, and assuming it holds for $\mathcal{S}_\beta$, for all $\beta < \alpha$, we will prove it for $\mathcal{S}_\alpha$.

Case 1 (α is a successor ordinal, hence $\alpha = \beta+1$ for some $\beta < \omega_1$). Let $E = \bigcup_{j=1}^k E_j$ for some $E_j \in \mathcal{S}_\beta$, $E_j < E_{j+1}$, and $\{\min E_j : 1 \le j \le k\} \in \mathcal{S}$. Since E is maximal, $k = \min E_1 = \min E$. Let $\sigma : E \to F$ be the order-preserving bijection, and, since F is a spreading of E, then $\sigma(n) \ge n$ for every $n \in E$. In particular, $F_j := \sigma(E_j) \in \mathcal{S}_\beta$, as $\mathcal{S}_\beta$ is spreading, and $\{\min F_j : 1 \le j \le k\} \in \mathcal{S}$, as $\mathcal{S}$ is spreading. Since F is maximal, we get that $\min F = \min F_1 = k = \min E$.

Case 2 (α is a limit ordinal). Let $n < \min E$ be such that $E \in \mathcal{S}_{\alpha_n}$. Since F is a spreading of E, we get that $F \in \mathcal{S}_{\alpha_n}$. The maximality of E and F in $\mathcal{S}_\alpha$ implies, by Lemma 2 of $\mathcal{S}_{\alpha_n}$, that they are both also maximal in $\mathcal{S}_{\alpha_n}$. By the inductive hypothesis, we get that $\min E = \min F$. □

Theorem 16. *Let $T : X^*_{\mathcal{S}_\alpha} \to X^*_{\mathcal{S}_\alpha}$ be an isometry. Then there exists a sequence of signs $(\theta_i)_i$ such that for any $i \in \mathbb{N}$, $Te^*_i = \theta_i e^*_i$.*

Proof. Fix $T : X^*_{\mathcal{S}_\alpha} \to X^*_{\mathcal{S}_\alpha}$, and from Theorem 10 we have that there exists a permutation $\pi : \mathbb{N} \to \mathbb{N}$ and a sequence of signs $(\theta_i)_i$ such that $Te^*_i = \theta_i e^*_{\pi(i)}$. Assume towards a contradiction that the conclusion is not true, and let k_0 be the smallest integer such that $p_0 := \pi(k_0) \ne k_0$. Note that from the proof of Theorem 10 follows that π sends maximal singletons to maximal singletons, and since $\{1\}$ is the only maximal singleton in $\mathcal{S}_\alpha$ we have that $k_0 > 1$ and $\{k_0\} \notin \mathcal{S}^{MAX}_\alpha$. From the minimality of k_0 we also have that $p_0 > k_0$.

Pick $k_1 > k_0$ such that $p_1 := \pi(k_1) \ge k_1$ and $p_1 > p_0$. Note that we can always do that, since any permutation will contain an increasing sequence, and we can go far enough along that sequence to pick a suitable k_1. Continuing in this manner we construct infinite sequences $\{k_0, k_1, k_2, \dots\}$ and $\{p_0, p_1, p_2, \dots\}$ such that $p_i := \pi(k_i)$ and $\{p_0, p_1, p_2, \dots\}$ is a spreading of $\{k_0, k_1, k_2, \dots\}$. From the barrier property, we can find an initial segment $E \subset \{k_0, k_1, k_2, \dots\}$ such that $E \in \mathcal{S}_\alpha^{MAX}$. Since T sends extreme points to extreme points, and $E \in \mathcal{S}_\alpha^{MAX}$, it follows that $\pi(E) \in \mathcal{S}_\alpha^{MAX}$ as well. Hence, from Lemma 15 we must have that $\min E = \min \pi(E)$. That is, $k_0 = p_0$, which contradicts the initial assumption. This finishes the proof. □

5. Final remarks

Theorem 10 guarantees that all the isometries of a combinatorial space or its dual are determined by a permutation of the elements of the basis and a change of signs. A natural and general question which remains open is the following.

Question 17. For which combinatorial spaces can we explicitly describe the group of its isometries?

In Section 4 we described the group of the isometries on Schreier spaces, showing that the identity is the only permutation allowed. The following example illustrates an intermediate situation where more permutations are allowed, though not all of them.

Example 18. Given an increasing sequence $(k_n)_n$ such that $k_0 = 0$, let

$$\mathcal{F} = \{F \in [\mathbb{N}]^{<\omega} : |F| \leq n, \text{ where } k_{n-1} \leq \min F < k_n\}.$$

Then $T : X_\mathcal{F}^* \to X_\mathcal{F}^*$ is an isometry if and only if there is a permutation π of $\mathbb{N}$ such that for all $n \in \mathbb{N}$, $\pi(I_n) = I_n$ and $T(e_n^*) = \pm e_{\pi(n)}^*$, where $I_n = [k_n, k_{n+1}[$.

Proof. It is easy to see that $\mathcal{F}$ is hereditary and spreading, and to prove compactness one should follow similar arguments to the Schreier families. Moreover, one easily sees that $\mathcal{F}^{MAX}$ is a barrier.

Given an isometry $T : X_\mathcal{F}^* \to X_\mathcal{F}^*$, by Theorem 10, there exists a permutation $\pi : \mathbb{N} \to \mathbb{N}$ such that $T(e_n^*) = \pm e_{\pi(n)}^*$ for every $n \in \mathbb{N}$. Note that from the proof of Theorem 10 it follows that $\pi(F) \in \mathcal{F}^{MAX}$ iff $F \in \mathcal{F}^{MAX}$. On the other hand, $F \in \mathcal{F}^{MAX}$ iff $|F| = n$ for the unique n such that $k_{n-1} \leq \min F < k_n$. It follows easily that $\pi(I_n) = I_n$.

Conversely, given a permutation π of $\mathbb{N}$ such that $\forall n \in \mathbb{N}\ \pi(I_n) = I_n$, we have that $\mathcal{F} = \{\pi(F) : F \in \mathcal{F}\}$. Hence, if $T(e_n^*) = \pm e_{\pi(n)}^*$ for every $n \in \mathbb{N}$, one can take T to be the linear operator that takes e_n^* to $e_{\pi(n)}^*$, and it is easy to see that T is an isometry. □

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