AN EXAMPLE DISTINGUISHING TWO CONVEX SEQUENTIAL PROPERTIES

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Dedicated to István Juhász, on the occasion of his 80th birthday, celebrating a lifetime of dancing feet and topological mind.

Abstract. Corson and Efremov introduced convex notions of countable tightness and the Fréchet-Urysohn property in the context of Banach spaces. We present an old unpublished example which consistently distinguishes these properties. Together with a recent result from [12], it yields that it is independent from ZFC whether these properties are equivalent or not.

1. Introduction

Recall the following sequential properties of a given topological space X:

- X has countable tightness if every point in the closure of a set $F \subseteq X$ is in the closure of a countable subset of F.
- X is Fréchet-Urysohn if every point in the closure of a set $F \subseteq X$ is the limit of a sequence in F.

It is clear that every Fréchet-Urysohn space has countable tightness and results by Balogh [2] and Fedorchuk [8] established the independence of the converse implication for compact spaces.

The main purpose of this note is to present a consistent example of a Banach space which distinguishes convex counterparts introduced by Corson and Efremov in [5] and [6] of the aforementioned topological properties. In fact, it distinguishes between the property of Corson and an intermediate property recently introduced by Martínez-Cervantes in [11].

Definition 1. *Let X be a Banach space. Let us consider the following properties:*

- *X has the property (C) of Corson if every family of closed convex subsets of X whose intersection is empty has a countable subfamily with empty intersection.*
- *X has the property (E') if every weak* sequentially closed convex set $C \subseteq X^*$ is weak* closed.*
- *X has the property (E) of Efremov if every point in a bounded weak* closed convex set $C \subseteq X^*$ is the weak* limit of a sequence in C.*

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Notice that (E) is an immediate convex analogue of Fréchet-Urysohn in the context of dual spaces and Pol proved in [16] the following characterization of (C), turning it into a convex analogue of tightness: a Banach space has property (C) if and only if every point of a bounded weak* closed convex set $C \subseteq X^*$ is in the weak* closure of a countable subset of C. (E) clearly implies (E'), which in turn implies (C), see [12, Lemma 2]. Plichko and Yost ([15], pg. 352) asked whether (C) implies (E) and the main result of this note gives a consistent negative answer to this question:

Theorem 2. *It is consistent with ZFC that there is a compact Hausdorff scattered space K such that:*

(i) every finite power K^n of K is hereditarily separable;
(ii) $C(K)$ does not have property (E').

Compact spaces are an important source of counterexamples for questions about the topology and the structure of Banach spaces. Given a compact Hausdorff space K, let $C(K)$ be the Banach space of continuous scalar-valued functions defined on K, with the supremum norm. It is well-known that K is scattered if and only if $C(K)$ is an Asplund space (i.e. every separable subspace of $C(K)$ has separable dual), see [13]. Moreover, if all finite powers of K are hereditarily separable, then $C(K)$ is weakly hereditarily Lindelöf (see e.g. [9, Theorem 4.38]). Any weakly Lindelöf Banach space has property (C), since closed convex sets are weakly closed (see [7, Theorem 3.19]). Hence, we get the following corollary:

Corollary 3. *It is consistent with ZFC that there is an Asplund space with property (C) and which does not have property (E').*

Martínez-Cervantes and Poveda proved in [12] that, under the Proper Forcing Axiom, every Banach space which has property (C) also has property (E'), establishing the independence of this statement. Another unpublished example of a space that has (C) and does not have (E) has been constructed by Justin T. Moore as a modification of Ostaszewski's space from [14] assuming the principle $\diamondsuit$. An example of a Banach space with property (E') which fails property (E) has been given under the Continuum Hypothesis in [1], but the question whether the implication fails in ZFC remains open. We refer to [12] for a complete account on these and related problems.

Our construction appears originally in the author's PhD thesis [3] and was never published. It is a simplification of the construction made in [4], inspired by [10] and [17] of a locally compact Fréchet-Urysohn space of weight ω_2. The construction in [4] gives a consistent example of a compact space of weight ω_2 with hereditarily separable finite powers, which yields a consistent example of an Asplund space of density ω_2 with interesting structural properties. In the case of the present work, we run a similar construction replacing ω_2 by ω_1 as the underlying set of the topological space K. This makes the arguments simpler and allows us to analyse the sequences in the space $C(K)^*$ and prove the main result.

In the next section we introduce the partial order used to force the existence of the space K. We recall some of its properties and how they provide some of the desired properties of the space K. In Section 3, we show Theorem 11, which contains the relevant information about weak* convergence of sequences in $C(K)^*$ and implies that $C(K)$ fails to have property (E'). It is a modification of Rabus's Lemma 5.4 [17], where the convergence of points in the space K is analysed.

2. Preliminary lemmas and the construction of K

Let us fix the following notation:

Definition 4 (Juhász, Soukup [10]). *Given finite nonempty sets of ordinals x and y such that $\max x < \max y$, we define*

$$x * y = \begin{cases} x \setminus y & \text{if } \max x \in y, \\ x \cap y & \text{if } \max x \notin y. \end{cases}$$

The following definition is a simplification of [10, Definition 2.1] replacing ω_2 by ω_1:

Definition 5. *Let $\mathbb{P}$ be the forcing formed by conditions $p = (D_p, h_p, i_p)$ where:*
1. *$D_p \in [\omega_1]^{<\omega}$;*
2. *$h_p : D_p \to \wp(D_p)$ and for all $\xi \in D_p$, $\max h_p(\xi) = \xi$;*
3. *$i_p : [D_p]^2 \to [D_p]^{<\omega}$ and for all $\xi, \eta \in D_p$, $\xi < \eta$, we have that:*
   (a) $h_p(\xi) * h_p(\eta) \subseteq \bigcup_{\gamma \in i_p(\{\xi,\eta\})} h_p(\gamma)$,
   (b) $i_p(\{\xi, \eta\}) \subseteq \xi$;

ordered by $p \leq q$ if $D_p \supseteq D_q$, for all $\xi \in D_q$, $h_p(\xi) \cap D_q = h_q(\xi)$ and $i_p|_{[D_q]^2} = i_q$.

The underlying set in [10, Definition 2.1] is ω_2 and the partial order depends on a function $f : [\omega_2]^2 \to [\omega_2]^{\leq\omega}$ with the so called strong property Δ. In our case, if we take $f : [\omega_1]^2 \to [\omega_1]^{\leq\omega}$ to be defined by $f(\{\xi, \eta\}) = \min\{\xi, \eta\}$, then we have an exact analogue of Definition 2.1 in [10], where ω_2 is replaced by ω_1. The role of the function f is to put some control on the image of the functions i_p in order to prove, for instance, that $\mathbb{P}$ has the countable chain condition (ccc). Since in the case of the present work the underlying set is ω_1, condition 3.(b) already limits the image of a pair $\{\xi, \eta\}$ to a countable set. In particular, the following lemma follows from similar arguments as in [17, 10]:

Lemma 6 (Rabus [17], Lemma 4.1; Juhász, Soukup [10], Lemma 2.8). *$\mathbb{P}$ satisfies ccc.*

We will also need the following technical lemmas, whose versions in [10] consider the forcing with ω_2 as the underlying set, but still hold in our case:

Lemma 7 (Lemma 2.2, [10]). *For each $\alpha < \omega_1$, the set $D = \{p \in \mathbb{P} : \alpha \in D_p\}$ is dense in $\mathbb{P}$.*

Lemma 8 (Lemma 2.16, [10]; see also [17]). *Let $t = (D_t, h_t, i_t) \in \mathbb{P}$, $D_t = T \cup E \cup F$, where $T < E < F$, $E = \{\alpha_1 < \cdots < \alpha_k\}$, $F = \{\alpha_i^1, \alpha_i^2 : 1 \leq i \leq k\}$, $H \subseteq T$ and*

$$\forall 1 \leq i \leq k \quad h_t(\alpha_i^1) \cap h_t(\alpha_i^2) = \bigcup_{\xi \in H \cup E} h_t(\xi).$$

Then there is $u = (D_u, h_u, i_u) \in \mathbb{P}$ such that $D_u = T \cup E$ and:

(a) $u \leq t|_T$;
(b) $u \leq t|_{H \cup E}$;
(c) $T \setminus \bigcup_{\xi \in H \cup E} h_t(\xi) \subseteq h_u(\alpha_1)$.

Let us finally define the space K. Fix the ground model V and a generic filter G.

Definition 9 (Juhász, Soukup [10], Definition 2.3)**.** *For each $\xi < \eta < \omega_1$, working in $V[G]$, let*

$$h(\xi) = \bigcup_{p \in G} h_p(\xi) \quad \textit{and} \quad i(\{\xi, \eta\}) = \bigcup_{p \in G} i_p(\{\xi, \eta\}),$$

and let L be the topological space (ω_1, τ), where τ is the topology on ω_1 which has the family of sets

$$\{h(\xi) : \xi < \omega_1\} \cup \{\omega_1 \setminus h(\xi) : \xi < \omega_1\}$$

as a topological subbasis.

It follows from [10, Theorem 1.5] that for all $\xi < \omega_1$, $h(\xi)$ is a compact subset of L and it easy to check that

$$\{h(\xi) \setminus \bigcup_{\eta \in F} h(\eta) : F \in [\xi]^{<\omega}\}$$

forms a local topological basis at ξ. Therefore L is a locally compact scattered zero-dimensional space. In $V^{\mathbb{P}}$, let K be the one-point compactification of L and let us denote the point of compactification by ω_1, ie. $K \setminus L = \{\omega_1\}$. Then, we get the following result:

Theorem 10 (Theorem 3.2, [4])**.** *In $V[G]$, K is a compact scattered zero-dimensional space such that K^n is hereditarily separable for every $n \in \mathbb{N}$.*

The proof that K^n is hereditarily separable for every $n \in \mathbb{N}$ in the forcing extension is done similarly to [4, Theorem 3.2]. Again there is a use of the function f with strong property Δ, which guarantees that any uncountable family of conditions has a pair of conditions having three properties (i), (ii) and (iii) (see page 510 of [4]), which can therefore be amalgamated using [4, Lemma 2.7]. The existence of such a pair in our case follows with no use of the function f, since conditions (i), (ii) and (iii) get trivial when $f(\{\xi, \eta\}) = \min\{\xi, \eta\}$.

Theorem 10 guarantees the first part of Theorem 2. The next section is devoted to prove that the space K constructed in this section also satisfies assertion (ii) of Theorem 2, that is, $C(K)$ fails to have property (E').

3. $C(K)$ does not have property (E')

Let us now prove the main result about the weak* convergence of sequences in $C(K)^*$. Recall that by the Riesz Representation Theorem, $C(K)^*$ can be seen as the space of bounded regular Borel measures on K. Given $x \in K$, we denote by δ_x the point-evaluation functional, i.e. $\delta_x(f) = f(x)$.

It is well-known that bounded regular Borel measures on a compact scattered space K are atomic, see e.g. [18, Theorem 19.7.6]. This means that each $\mu \in C(K)^*$ is of the form $\sum_{x \in S} a_x \delta_x$ for some countable $S \subseteq K$ and a sequence of nonzero scalars $(a_x)_{x \in S}$ such that the series $\sum_{x \in S} a_x$ converges absolutely. The variation of a measure $\mu = \sum_{x \in S} a_x \delta_x \in C(K)^*$ on some set $X \subseteq K$ is defined by $|\mu|(X) = \sum_{x \in S \cap X} |a_x|$. This characterization of the elements $C(K)^*$ will be helpful in our proof.

Theorem 11. *In $V[G]$, if $(\mu_n)_{n \in \mathbb{N}} \subseteq B_{C(K)^*}$ is a sequence weak* convergent to δ_{ω_1}, then there is $n_0 \in \mathbb{N}$ such that, for all $n \geq n_0$, we have that $\mu_n(\{\omega_1\}) \neq 0$.*

Proof. Suppose by contradiction that there is, in $V[G]$, a sequence $(\mu_n)_{n\in\mathbb{N}} \subseteq B_{C(K)^*}$ that converges weakly* to δ_{ω_1} and such that for all $n \in \mathbb{N}$, $\mu_n(\{\omega_1\}) = 0$.

In V, let $0 < \varepsilon < \frac{1}{5}$ and let $\dot{\delta}_{\omega_1}$ be a $\mathbb{P}$-name for δ_{ω_1} and $(\dot{\mu}_n)_{n\in\mathbb{N}}$ a sequence of names for elements of $B_{C(K)^*}$ such that

$$\mathbb{P} \Vdash \forall n \in \mathbb{N}\ \dot{\mu}_n(\{\omega_1\}) = 0 \text{ and } (\dot{\mu}_n)_{n\in\mathbb{N}} \text{ converges weakly}^* \text{ to } \dot{\delta}_{\omega_1}.$$

Since K is scattered, it follows that each μ_n is atomic. Therefore:

$$\mathbb{P} \Vdash \forall n \in \mathbb{N}\ \exists F_n \subseteq L \text{ finite such that } |\dot{\mu}_n|(K \setminus F_n) < \check{\varepsilon}.$$

For each $n \in \mathbb{N}$, let A_n be a maximal antichain in $\mathbb{P}$ such that for every $p \in A_n$ decides F_n, i.e. there exists a finite subset F_n^p of ω_1 such that p forces $|\dot{\mu}_n|(K \setminus \check{F}_n^p) < \check{\varepsilon}$ and for every $\alpha \in F_n^p$, there is $a_\alpha \in \mathbb{R}$ such that p forces that $\dot{\mu}_n(\{\check{\alpha}\}) = \check{a}_{\check{\alpha}}$. By Lemma 7, we can assume, without loss of generality, that for every $n \in \mathbb{N}$ and every $p \in A_n$, $F_n^p \subseteq D_p$.

From the fact that $\mathbb{P}$ is ccc, it follows that there exists $\gamma < \omega_1$ such that

$$\bigcup\{D_p : p \in A_n,\ n \in \mathbb{N}\} \subseteq \gamma.$$

Given $q \in \mathbb{P}$, since $h_q(\gamma) \subseteq \gamma \subseteq \omega_1$ and, therefore, $q \Vdash \omega_1 \notin h(\gamma)$, it follows that $q \Vdash \dot{\delta}_{\omega_1}(h(\gamma)) = 0$. Since $\mathbb{P}$ forces $(\dot{\mu}_n)_{n\in\mathbb{N}}$ to converge weakly* to $\dot{\delta}_{\omega_1}$, there are $r \leq q$ and $m \in \mathbb{N}$ such that

$$r \Vdash \forall n \geq m\ |\dot{\mu}_n(h(\gamma))| < \check{\varepsilon}.$$

Once again by Lemma 7, we can assume, without loss of generality, that $\gamma \in D_r$. Let $H = D_r \cap \gamma$ and $E = D_r \setminus \gamma = \{\gamma = \alpha_1 < \cdots < \alpha_k\}$. Let $F \subseteq \omega_1$ be such that $E < F$ and $|F| = 2|E|$ and denote $F = \{\alpha_i^1, \alpha_i^2 : 1 \leq i \leq k\}$.

We will obtain, after 3 steps, $u \in \mathbb{P}$ and $n \in \mathbb{N}$ such that $u \leq r$, $n \geq m$ and $u \Vdash |\dot{\mu}_n(h(\gamma))| > \check{\varepsilon}$, contradicting the fact that $r \Vdash \forall n \geq m\ |\dot{\mu}_n(h(\gamma))| < \check{\varepsilon}$. In Step 1, we extend r to a condition s such that $D_s = D_r \cup F$ and for every $1 \leq i \leq k$, we have $h_s(\alpha_i^1) \cap h_s(\alpha_i^2) = \bigcup_{\xi\in D_r} h_s(\xi)$; in Step 2, we extend s to a condition t such that $D_t \subseteq \gamma \cup E \cup F$ and for which there exist $n \geq m$ and $p \in A_n$ such that $t \leq p$ and

$$t \Vdash |\dot{\mu}_n(\bigcup_{\xi\in\check{D}_r} h(\xi))| < \check{\varepsilon};$$

finally, in Step 3 we will obtain u such that $D_u = (D_t \cap \gamma) \cup E$, $u \leq r$ and

$$u \Vdash |\dot{\mu}_n(h(\gamma))| > \check{\varepsilon},$$

as desired.

Step 1. Define $s = (D_s, h_s, i_s)$ by $D_s = D_r \cup F$;

$$h_s(\xi) = \begin{cases} h_r(\xi) & \text{if } \xi \in D_r, \\ D_r \cup \{\xi\} & \text{if } \xi \in F; \end{cases}$$

and

$$i_s(\{\xi,\eta\}) = \begin{cases} i_r(\{\xi,\eta\}) & \text{if } \xi,\eta \in D_r, \\ \min\{\xi,\eta\} \cap D_s & \text{otherwise.} \end{cases}$$

Clearly s satisfies conditions 1 and 2 of Definition 5 and condition 3.(a) for $\xi, \eta \in D_r$ follow from the fact that $r \in \mathbb{P}$.

Let $\xi \in D_r$ and $\eta \in F$. Hence, $\xi \in h_s(\eta)$ and $h_s(\xi) \subseteq h_s(\eta)$ so that $h_s(\xi) * h_s(\eta) = h_s(\xi) \setminus h_s(\eta) = \emptyset$. Therefore, s satisfies condition 3.(a) of Definition 5 for these pairs.

Let now be $\xi, \eta \in F$ and $\xi < \eta$. In this case, $\xi \notin h_s(\eta)$ and hence, $h_s(\xi) * h_s(\eta) = h_s(\xi) \cap h_s(\eta) = D_r$. But $D_r = H \cup E \subseteq \xi \cap D_s$, so that s satisfies condition 3.(a) for these pairs.

We get that $s \in \mathbb{P}$ and it is easy to see that $s \leq r$.

Step 2. Note that $\mathbb{P}$ forces that $\bigcup_{\xi \in \check{D}_r} h(\xi)$ is a clopen set and $\omega_1 \notin \bigcup_{\xi \in \check{D}_r} h(\xi)$. Hence, $\mathbb{P}$ forces that $\dot{\delta}_{\omega_1}(K \setminus \bigcup_{\xi \in \check{D}_r} h(\xi)) = 1$.

Since $\mathbb{P}$ forces that $(\dot{\mu}_n)_{n\in\mathbb{N}}$ converges weakly* to $\dot{\delta}_{\omega_1}$, there are $t \leq s$ and $n \geq m$ such that

$$t \Vdash \dot{\mu}_n(K \setminus \bigcup_{\xi \in \check{D}_r} h(\xi)) > 1 - \check{\varepsilon}.$$

But A_n is a maximal antichain and hence, we can assume, without loss of generality, that there exists $p \in A_n$ such that $t \leq p$. Since $t \leq p, r$, we have that

$$t \Vdash \dot{\mu}_n(\check{F}_n^p \setminus \bigcup_{\xi \in \check{D}_r} h(\xi)) \geq \dot{\mu}_n(K \setminus \bigcup_{\xi \in \check{D}_r} h(\xi)) - |\dot{\mu}_n|(K \setminus \check{F}_n^p) > 1 - 2\check{\varepsilon},$$

i.e,

$$\sum \{a_\alpha : \alpha \in F_n^p \setminus \bigcup_{\xi \in D_r} h_t(\xi)\} > 1 - 2\varepsilon.$$

Step 3. Let $T = D_t \cap \gamma$ and observe that t, T, E, F and H satisfy the assumptions of Lemma 8. Hence, there exists $u = (D_u, h_u, i_u) \in \mathbb{P}$ such that $D_u = T \cup E$, $u \leq t|_T$, $u \leq t|_{H \cup E}$ and $T \setminus \bigcup_{\xi \in H \cup E} h_t(\xi) \subseteq h_u(\alpha_1)$ and notice that $t|_T \leq p$, $H \cup E = D_r$ and $t|_{H\cup E} = r$.

It remains to show the statement below and we have a contradiction with the fact that $u \leq r$ and that $r \Vdash |\dot{\mu}_n(h(\gamma))| < \check{\varepsilon}$:

Claim. $u \Vdash \dot{\mu}_n(h(\gamma)) > \check{\varepsilon}$.

Proof of the claim. Consider

$$I = \{\alpha \in F_n^p : t \Vdash \check{\alpha} \notin \bigcup_{\xi \in \check{D}_r} h(\xi)\} = F_n^p \setminus \bigcup_{\xi \in D_r} h_t(\xi)$$

and note that, since $D_r = H \cup E$, $\alpha_1 = \gamma$ and $F_n^p \subseteq D_p \subseteq D_t \cap \gamma = T$, we have that

$$I \subseteq T \setminus \bigcup_{\xi \in D_r} h_t(\xi) \subseteq h_u(\gamma).$$

As $u \leq t|_T \leq p$ and p forces that $\dot{\mu}_n(\{\check{\alpha}\}) = \check{a}_{\check{\alpha}}$ for every $\alpha \in F_n^p$, we have that

$$u \Vdash \dot{\mu}_n(\check{I}) = \sum_{\alpha \in \check{I}} \check{a}_\alpha > 1 - 2\check{\varepsilon},$$

and, since $\mathbb{P}$ forces $\|\dot{\mu}_n\| \leq 1$, we have that

$$u \Vdash |\dot{\mu}_n|(h(\gamma) \setminus \check{I}) \leq |\dot{\mu}_n|(K \setminus \check{I}) \leq \|\dot{\mu}_n\| - |\dot{\mu}_n|(\check{I}) < 1 - (1 - 2\check{\varepsilon}) = 2\check{\varepsilon}.$$

Therefore,

$$u \Vdash \dot{\mu}_n(h(\gamma)) \geq \dot{\mu}_n(\check{I}) - |\dot{\mu}_n|(h(\gamma) \setminus \check{I}) > 1 - 4\check{\varepsilon} > \check{\varepsilon},$$

completing the proof of the claim and the theorem. □

In particular, it follows from the previous result that there is no sequence of points from L converging to ω_1 in K.

Corollary 12. *In $V[G]$, $C = \{\mu \in C(K)^* : \mu(\{\omega_1\}) = 0\}$ is a weak* sequentially closed convex subset of $C(K)^*$ which is not weak* closed. Therefore, $C(K)$ does not have property (E').*

Proof. C is clearly convex and it follows from Theorem 11 that C is weak* sequentially closed. Indeed, since K is scattered, given a sequence $(\mu_n)_{n\in\mathbb{N}}$ in C, it follows that each μ_n is atomic, that is $\mu_n = \sum_{k=1}^{\infty} a_k^n \delta_{\alpha_k^n}$ for some sequence $(a_k^n)_{k\in\mathbb{N}}$ of scalars and some sequence $(\alpha_k^n)_{k\in\mathbb{N}}$ of distinct elements of K. Since $\mu_n \in C$, we get moreover that $(\alpha_k^n)_{k\in\mathbb{N}}$ is indeed a sequence in $K \setminus \{\omega_1\} = L$.

Given $\mu \in C(K)^*$, we can write $\mu = \sum_{\alpha\in S} a_\alpha \delta_\alpha$ for some countable $S \subseteq K$ and a sequence of scalars $(a_\alpha)_{\alpha\in S}$. If $\mu \notin C$, then $\omega_1 \in S$ and $a_{\omega_1} \neq 0$. Now, if $(\mu_n)_{n\in\mathbb{N}}$ converges to μ, let $\nu_n = \mu_n - \sum_{\alpha\in S\setminus\{\omega_1\}} a_\alpha \delta_\alpha$ and notice that $(\nu_n)_{n\in\mathbb{N}}$ is a bounded sequence in $C(K)^*$ weak* convergent to $a_{\omega_1}\delta_{\omega_1}$, contradicting Theorem 11. This concludes the proof that C is weak* sequentially closed.

Finally, let us show that $\delta_{\omega_1} \in \overline{C}^{w^*}$. Given $\varepsilon > 0$, $f_1, \ldots, f_n \in C(K)$, we have to

$$U = \bigcap_{i=1}^{n} f_i^{-1}[(f_i(\omega_1) - \varepsilon, f_i(\omega_1) + \varepsilon)]$$

is an open neighborhood of ω_1. Since ω_1 is an accumulation point of K, there is $x \in K \setminus \{\omega_1\} \cap U$ and it follows that

$$\delta_x \in \{\mu \in C(K)^* : \forall 1 \leq i \leq n,\ |\mu(f_i) - \delta_{\omega_1}(f_i)| < \varepsilon\} \cap C.$$

Therefore $\delta_{\omega_1} \in \overline{C}^{w^*}$. □

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