Amenability test spaces for Polish groups

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History

Lebesgue, 1904

The Lebesgue measure μ is the unique complete translation invariant measure on the Lebesgue measurable subsets of $\mathbb{R}$.

Question, Lebesgue, 1904

What if countable additivity is replaced by finite additivity? That is, is Lebesgue measure the unique function μ on the Lebesgue measurable subsets of $\mathbb{R}$ such that:

1. μ is non-negative
2. μ is complete
3. μ is translation invariant
4. μ is finitely additive

Banach, 1923

No:
There is a non-negative, complete, translation-invariant, finitely but not countably additive measure μ on $\mathbb{R}$ defined on all subsets of $\mathbb{R}$ (not just the Lebesgue measurable ones). Moreover $\mu(\mathbb{R}) = 1$, contrasting with the fact that the Lebesgue measur of $\mathbb{R}$ is ∞. In modern language, μ is an invariant mean on $\mathbb{R}$ and $\mathbb{R}$ is amenable as a discrete group.

History: The Banach-Tarski Paradox

In the 1920s and 30s, the question of the existence of an invariant mean for a group G acting on a set X was investigated by Banach and Tarski. In 1938 Tarski showed that such a mean exists if and only if X does not admit a "G-paradoxical decomposition".

History: The Banach-Tarski Paradox

Definition

Let G be a group acting on a set X. Subsets $A, B \subseteq X$ are said to be G-equidecomposable if A and B can each be partitioned into the same finite number of respectively G-congruent pieces.

Formally, $A = \bigcup_{i=1}^{n} A_i$ and $B = \bigcup_{i=1}^{n} B_i$ where for $1 \leq i < j \leq n$, $A_i \cap A_j = B_i \cap B_j = \emptyset$; and there are $g_1, ..., g_n \in G$ such that for each $1 \leq i \leq n$, $g_i(A_i) = B_i$

Notation

If A and B are G-equidecomposable write $A \sim B$.

Definition

Let G be a group acting on a set X. The set X is G-equidecomposable if There are two proper disjoint subsets A and B of X such that $X = A \cup B$, $A \sim X$ and $B \sim X$.

Roughly speaking, the set X can be cut up into finitely many pieces from which two copies of X can be put together using elements of G.

Notation

If $X = G$ we say that the group G is paradoxical.

Tarski, 1938

Suppose a group G acts on a set X. Then there is a finitely additive, G-invariant measure μ on X such that $\mu(X) = 1$ and μ is defined on all subsets of X if and only if X is not G-paradoxical.

History: The Banach-Tarski Paradox

Example

1. A free group of rank 2 is paradoxical.
2. For all $n \geq 3$, the groups $O(n, \mathbb{R})$ and $SO(n, \mathbb{R})$ are paradoxical. (Hausdorff, 1914)
3. The sphere S^n is $SO(n+1, \mathbb{R})$-paradoxical for every $n \geq 2$ (Banach-Tarski Paradox).

History: Amenability

In 1929 Von Neumann introduced and studied the class of groups having an invariant mean, a finitely additive measure on the group. He used this class to explain why the Banach-Tarski Paradox occurs only for dimension greater than or equal to tree. He showed that the deep reason for this difference lies in the group of isometries of R^n (viewed as a discrete group) which is amenable for $n \leq 2$ and which is not so for $n \geq 3$.

History: Amenability

The term amenable for groups which admit an invariant mean was introduced by Day in 1950.

German: mittelbar

French: moyennable

English: amenable

Portuguese: mediavel? or amenavel?

History: Amenability

There are many equivalent formulations of amenability, involving, for example:

1. fixed point properties
2. representation theory
3. random walks
4. operator algebras

N.P. Brown and N. Ozawa., 2008

Amenability of a group admits the largest known number of equivalent definitions: $10^{10^{10}}$.

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Warning

Many authors use the phrase amenable group to mean a group which is amenable in its discrete topology. The danger of this is that many theorems concerning amenable discrete groups do not generalize in the ways one might expect.

Mean

1. G is a discrete group
2. $L^\infty(G)$=Bounded functions $f : G \longrightarrow \mathbb{C}$.
3. The group G acts on $L^\infty(G)$ on the left: for all $g \in G$ and $f \in L^\infty(G)$, $g.f = {}^g f$ where ${}^g f(x) = f(g^{-1}x)$ for all $x \in G$. This is called the left-regular representation.
4. An invariant mean is a linear functional $m : L^\infty(G) \longrightarrow \mathbb{R}$ such that:
    1. $m(f) \geq 0$ if $f \geq 0$
    2. $m(\chi_G) = 1$
    3. $m({}^g f) = f$ for all $g \in G$ and $f \in L^\infty(G)$.

Mean

1. $\mathcal{M}$ denote the set of mean on G
2. $\mathcal{N}$ the set of finitely additive measure on all subsets of G
3. The map $\mathcal{M} \ni m \longmapsto \mu_m \in \mathcal{N}$ where $\mu_m(A) = m(\chi_A)$ for all $A \subseteq G$ is a bijection.

Mean

In general:

1. If G is a topological group
2. $RUCB(G)$=Right Uniformly Continuous Bounded functions $f : G \longrightarrow \mathbb{C}$.
3. $RUCB(G)$ is invariant by the left-regular representation.
4. An invariant mean is a linear functional $m : RUCB(G) \longrightarrow \mathbb{R}$ such that:
    1. $m(f) \geq 0$ if $f \geq 0$
    2. $m(\chi_G) = 1$
    3. $m({}^g f) = f$ for all $g \in G$ and $f \in RUCB(G)$.

Amenability

A topological group G is amenable if there exist an invariant invariant mean on $RUCB(G)$.

Amenability

For a topological group G, the following are equivalents:

1. G is amenable.

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4. There is an invariant probability measure on the greatest ambit $S(G)$.

Some amenable polish groups

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- Compact groups are amenable (Haar measure)
  1. The orthogonal group
  $$O(n,\mathbb{R}) := \{A \in M_n(\mathbb{R}) | < Ax, Ay > = < x, y >, \text{ for all } x, y \in \mathbb{R}^n\}$$
  2. The special orthogonal group
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- The infinite symetric group S_∞, i.e the group of all self-bijections of $\mathbb{N}$ equipped with the topology of simple convergence. $(S_\infty = \bigcup_{n\in\mathbb{N}} S_n)$

Some non-amenable groups

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- For all $n \leq 2$, the groups $O(n, \mathbb{R})$ and $SO(n, \mathbb{R})$ are amenable for discrete topology
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- The group $Aut\,(X, \mu)$ of all measure-preserving automorphisms of a standard Borel measure space (X, μ), equipped with the uniform topology $(d(\tau, \sigma) = \mu\{x \in X : \tau(x) \neq \sigma(x)\})$ is non-amenable. (Giordano and Pestov 2002)
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Some properties

1. Let G and H be topological groups such that there is a continuous surjective homomorphism from G onto H. If G is amenable, so is H.
2. Let G be a topological group and suppose that there is a dense subset A of G such that every finite subset of A is included in an amenable subgroup of G. Then G is amenable.
3. Let G be a topological group and H a normal subgroup of G. If H and G/H are both amenable, so is G.
4. Let G be a topological group with two amenable subgroups H_0 and H_1 such that H_0 is normal and $H_0 H_1 = G$. Then G is amenable.
5. The product of any family of amenable topological groups is amenable.

Some properties

Let $G = 0(2, \mathbb{R})$. Then G has a closed normal subgroup $N = SO(2, \mathbb{R}) \cong \mathbb{T}$. Since N is abelian, N is amenable. Now N has index 2 in G since for any $A \in G$ either $A \in N$ or $A = PB$ where $B \in N$ and
$P = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.
So G/N is the finite cyclic group C_2. Since C_2 is finite, C_2 is amenable. Therefore $G = O(2, \mathbb{R})$ is amenable as a discrete group.

The concept of test space

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Definition

G is amenable $\iff$ every continuous action of G on every compact space admits an invariant Borel probability measure.

Bogatyi and Fedorchuk,

Bogatyi and Ferdorchuk, 2007

For a countable discrete group G, TFAE:

- G is amenable
- Every action of G by homeomorphisms on the Cantor space $Q = [0, 1]^{\aleph_0}$ admits an invariant probability measure.

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This Answering a question by Giordano and de la Harpe.

Can we replace "countable discrete" with "Polish"?

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Remark

{countable discrete groups} $\subsetneq$ {polish groups}

Main result 1

Theorem

The Hilbert cube $Q = [0,1]^{\aleph_0}$ *is a test space for amenability of polish groups.*

Key Lemma 1

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Keller's Theorem

Keller

Each infinite-dimensional convex compact set lying in a metrizable locally convex space is homeomorphic to the Hilbert cube.

Proof of Main result 1

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□

Giordano and de la Harpe, 1997

The Cantor space $D^{\aleph_0}$ is a test space for amenability of discrete countable groups.

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It is not clear for all polish groups
But Yes for Polish SOS groups

SOS groups

Definition

A topological group G is call SOS if it has a basis at identity consisting of open subgroups.

Theorem

For any Polish group G, the following are equivalent:

1. G *is isomorphic to a closed subgroup of* S_∞*, the permutation group of* $\mathbb{N}$ *with the pointwise convergence topology*
2. G *is non-archimedean, i.e., admits a countable basis at identity consisting of open subgroups*
3. $G = Aut(A)$*, for a countable structure* A
4. $G = Aut(F)$*, for a Fraissé structure* F

Some SOS polish groups

- The group $Homeo(D^{\aleph_0})$ of all self homeomorphism of the Cantor space $D^{\aleph_0}$ equiped with the topology of uniform convergence or equivantly the compact-open topology is a polish SOS group

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- The infinite symetric group S_∞, i.e the group of all self-bijections of $\mathbb{N}$ equipped with the topology of simple convergence.

Note: This two examples are universals in the class of SOS polish groups.

Countable discrete vs polish SOS

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Main result 2

Theorem

For Polish SOS group, TFAE

- *G is amenable*
- *Every action of G by homeomorphisms on the Cantor space $D^{\aleph_0}$ admits an invariant probability measure.*

Key Lemma

Lemma

If G is a polish SOS group, then there is an inverse system of G-spaces $(X_\alpha, \pi_{\alpha\beta}, I)$ such that $X_\alpha \cong D^{\aleph_0}$ for all $\alpha \in I$ and $\mathcal{S}(G) = \varprojlim X_\alpha$.

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By hypothesis, there is on every G-space X_α an invariant probability measure μ_α.
Therefore, there is an invariant probabilty measure on $\mathcal{S}(G)$.

$\square$

extreme amenability

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A topological group G is extremely amenable if every continuous action of G on a compact space has a fixed point.

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Example

- The unitary group $\mathcal{U}(\ell^2)$, equipped with strong operator topology(Gromov)
- $Aut\,(\mathbb{Q}, \leq)$ the group of all order-preserving bijections of the rationals, with the topology of simple convergence(Pestov).

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- $Aut\left(\mathbb{Q}, \leq\right)$ the group of all order-preserving bijections of the rationals, with the topology of simple convergence(Pestov).
- $Iso(\mathbb{U})$ where $\mathbb{U}$ is the universal Urysohn space.

Veech

Every locally compact group acts freely on a suitable compact space. Therefore, no discrete (or locally compact) group is extreme amenable.

Open question

Does there exist a metrizable compact test space for extreme amenability of Polish groups?

Main result 3

Theorem

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For Polish SOS group, TFAE

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- *Every action of G by homeomorphisms on the Cantor space $D^{\aleph_0}$ admits an invariant probability measure.*

Note

Examples of extremely amenable Polish SOS groups are numerous: automorphism groups of certain Fraissé structures (Kechris-Pestov-Todorcevic), e.g. $Aut(\mathbb{Q}, \leq)$ (VP)

Topological amenability

Definition

Let G be a countable discrete group and let X be a compact Hausdorff space on which G acts by homeomorphism. The action is said to be amenable if there exists a sequence of weak*-continuous maps $b^n : X \longrightarrow prob(G)$ such that for every $g \in G$, $\lim_{n \longrightarrow \infty} \sup_{x \in X} \|gb^n_x - b^n_{gx}\|_1 = 0$

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Definition

A countable group G is called topological amenable if there exist a compact space X such that

1. G acts on X by homeomorphism
2. The action of G on X is amenable

Topological amenability

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Theorem

For a countable group G the following are equivalent:

1. *G is amenable.*
2. *The trivial action on $\{*\}$ is amenable.*

Topological amenability

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Proof

This is essentially the Reiter's condition for amenability

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Reiter's condition

Let p be any real number such that $1 \leq p \leq \infty$. For a locally compact group G the following are equivalent:

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Reiter's condition

Let p be any real number such that $1 \leq p \leq \infty$. For a locally compact group G the following are equivalent:

1. G is amenable.
2. for any compact space $C \subseteq G$ and $\varepsilon > 0$, There is $f \in \{g \in L^p(G) : g \geq 0, \ \|f\|_p = 1\}$ such that: $\|_g f - f\| < \varepsilon$ for all $g \in C$.

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In fact
The free group of two generators F_2 is topologically amenable without being amenable.

Topological amenability and action on the Stone-Čech compactification

Topological amenability and action on the Stone-Čech compactification

Lemma

If G acts amenably on a compacts space Y, and if $f : Y \longrightarrow X$ is any equivariant map, then the action of G on X is amenable.

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Proof.

Let $g \in G$, denote by $L_g : G \ni h \longmapsto gh \in G$ the left-translation by g.

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Proof.

Let $g \in G$, denote by $L_g : G \ni h \longmapsto gh \in G$ the left-translation by g. By the universal property of the Stone-Cech compactification L_g extend uniquely to $\widetilde{L}_g : \beta G \longrightarrow \beta G$ such that the following diagram: □

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$$\begin{array}{ccc} G & \xrightarrow{\quad L_g \quad} & G \\ \big\downarrow & & \big\downarrow \\ \beta G & \xrightarrow{\quad \widetilde{L}_g \quad} & \beta G \end{array}$$

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$$\begin{array}{ccc} G & \xrightarrow{L_g} & G \\ \downarrow & & \downarrow \\ \beta G & \xrightarrow{\widetilde{L}_g} & \beta G \end{array}$$

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The map $G \ni g \longmapsto \widetilde{L}_g \ni \beta G$ is an action by homeomorphism of G on βG.

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Let now X be a compact G-space such that the action of G on X is amenable.

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This action $\tau : G \times X \longrightarrow X$ permit to define an equivariant map $\widetilde{\tau} : \beta G \longrightarrow X$.

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The map $G \ni g \longmapsto \widetilde{L}_g \ni \beta G$ is an action by homeomorphism of G on βG.
Let now X be a compact G-space such that the action of G on X is amenable.
This action $\tau : G \times X \longrightarrow X$ permit to define an equivariant map $\widetilde{\tau} : \beta G \longrightarrow X$.
The action of G on βG is therefore amenable by lemma 17

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Theorem

A countable group G admits an amenable action on some compact metrizable space if an only if its action on the Stone-Čech compactification βG is amenable.

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1. $\Longrightarrow$ by corollary 18

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1. $\Longrightarrow$ by corollary 18
2. $\Longleftarrow$ Suppose that G admits an amenable action on its Stone-Čech compactification βG.
   For all $g \in G$, denote $\overline{g} : \beta \ni x \longmapsto gx \in \beta G$ the action of G on βG.

Topological amenability and action on the Stone-Čech compactification

Proof.

1. $\Longrightarrow$ by corollary 18
2. $\Longleftarrow$ Suppose that G admits an amenable action on its Stone-Čech compactification βG.
   For all $g \in G$, denote $\overline{g} : \beta \ni x \longmapsto gx \in \beta G$ the action of G on βG.
   Now, define an equivalence relation $\mathcal{R}$ on βG as follows:
   $(x, y) \in \mathcal{R} \Longleftrightarrow b^n_{gx} = b^n_{gy}$ for all $n \in \mathbb{N}$ and $g \in G$.

Topological amenability and action on the Stone-Čech compactification

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   For all $g \in G$, denote $\overline{g} : \beta \ni x \longmapsto gx \in \beta G$ the action of G on βG.
   Now, define an equivalence relation $\mathcal{R}$ on βG as follows:
   $(x, y) \in \mathcal{R} \Longleftrightarrow b^n_{gx} = b^n_{gy}$ for all $n \in \mathbb{N}$ and $g \in G$.
   G act naturally on the quotient space $\beta G/\mathcal{R}$ by the quotient action.

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Proof.

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2. $\Longleftarrow$ Suppose that G admits an amenable action on its Stone-Čech compactification βG.
   For all $g \in G$, denote $\overline{g} : \beta \ni x \longmapsto gx \in \beta G$ the action of G on βG.
   Now, define an equivalence relation $\mathcal{R}$ on βG as follows:
   $(x, y) \in \mathcal{R} \Longleftrightarrow b^n_{gx} = b^n_{gy}$ for all $n \in \mathbb{N}$ and $g \in G$.
   G act naturally on the quotient space $\beta G/\mathcal{R}$ by the quotient action.
   The quotient space $\beta G/\mathcal{R}$ is metrizable and compact.
   Besides, the canonical action of G on $\beta G/\mathcal{R}$ is amenable.

□

Main result 4

Let G be countable discret group. The following facts are equivalents:

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- G admits an amenable action on the Cantor set $D^{\aleph_0}$.

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Theorem

Let G be countable discret group. The following facts are equivalents:

- *G is topologically amenable*
- *G admits an amenable action on the Hilbert cube $I^{\aleph_0}$.*

Proof.

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therefore, there is a weak*-continuous maps $b^n : D^{\aleph_0} \longrightarrow \mathcal{P}(G)$
such that:

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$$\lim_{n\longrightarrow\infty} \sup_{x\in D^{\aleph_0}} \|gc_x^n - c_{gx}^n\|_1 = 0$$

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$$\lim_{n\longrightarrow\infty} \sup_{x\in D^{\aleph_0}} \|gc_x^n - c_{gx}^n\|_1 = 0$$
For all n, There is an affine extention $c^n : \mathcal{P}(D^{\aleph_0}) \longmapsto \mathcal{P}(G)$ of b^n.

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The maps c^n are weak*-continuous and verified

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$$\lim_{n \longrightarrow \infty} \sup_{\mu \in \mathcal{P}(D^{\aleph_0})} \|gc_\mu^n - c_{g\mu}^n\|_1 = 0.$$

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$$\lim_{n \longrightarrow \infty} \sup_{\mu \in \mathcal{P}(D^{\aleph_0})} \|gc_\mu^n - c_{g\mu}^n\|_1 = 0.$$

We conclude that the action of G on $\mathcal{P}(D^{\aleph_0})$ is amenable.

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We conclude that the action of G on $\mathcal{P}(D^{\aleph_0})$ is amenable.
By Keller's theorem, $\mathcal{P}(D^{\aleph_0})$ is homeomorhic to $I^{\aleph_0}$.

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We conclude that the action of G on $\mathcal{P}(D^{\aleph_0})$ is amenable.
By Keller's theorem, $\mathcal{P}(D^{\aleph_0})$ is homeomorhic to $I^{\aleph_0}$.
Therefore the action of G on $I^{\aleph_0}$ is amenable.

□