A hierarchy of separable commutative Calkin algebras

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Let $A$ be a Banach algebra. Does there exist a Banach space $X$ such that the Calkin algebra of $X$ is isomorphic, as a Banach algebra, to $A$?

The Calkin algebra of $X$ is defined to be the space $\mathcal{C}al(X) = \mathcal{L}(X) / \mathcal{K}(X)$, where $\mathcal{L}(X)$ denotes the space of all bounded linear operators defined on $X$ and $\mathcal{K}(X)$ denotes the spaces of all compact operators defined on $X$.

We denote by $[T]$ the equivalence class of $T \in \mathcal{L}(X)$ in $\mathcal{L}(X) / \mathcal{K}(X)$.

$\mathcal{C}al(X)$ endowed with the operation $[T] \circ [S] = [T \circ S]$ becomes a Banach algebra.
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The Calkin algebra of $X$ is defined to be the space $\mathcal{Cal}(X) = \frac{\mathcal{L}(X)}{\mathcal{K}(X)}$, where $\mathcal{L}(X)$ denotes the space of all bounded linear operators defined on $X$ and $\mathcal{K}(X)$ denotes the spaces of all compact operators defined on $X$.

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$Cal(X)$ endowed with the operation $[T] \circ [S] = [T \circ S]$ becomes a Banach algebra.
S. Argyros and R. Haydon in 2011 constructed a Banach space $X_{AH}$ that satisfies the “scalar plus compact” property.


Hence, $\text{Cal}(X_{AH})$ is one-dimensional.
Case $A = \ell_1(\mathbb{N}_0)$

M. Tarbard in 2013 constructed a Banach space $X_\infty$ such that $Cal(X_\infty)$ is isometric as a Banach algebra with the convolution algebra $\ell_1(\mathbb{N}_0)$.

Let $K$ be a countable compact metric space with finite Cantor-Bendixson index.

Then there exists a $\mathcal{L}_\infty$ space $X$ such that its Calkin algebra is isomorphic, as a Banach algebra, to $C(K)$.

Case $A = C(K)$ for $K$ countable compact metric space

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Case $A = C(K)$ for $K$ countable compact metric space

The basic ingredients of our method are the following:

- The Argyros Haydon space $X_{AH}$.

The space $X_{AH}$ is a separable $\mathcal{L}^\infty$ space with dual isomorphic to $\ell_1$.

The construction of $X_{AH}$ is a generalized modification of the Bourgain-Delbaen method which depends on a pair of sequences of natural numbers $(m_j, n_j)_j$ that satisfy certain growth conditions.

For $L \subset \mathbb{N}$ infinite, we denote by $X_{AH}(L)$ the space constructed using the subsequence $(m_j, n_j)_j \in L$.

For every $L \subset \mathbb{N}$ infinite, the space $X_{AH(L)}$ shares the same properties with $X_{AH}$.

Moreover, in the Argyros-Haydon paper it is shown that for $L \cap M$ is finite, then every $T : X_{AH}(L) \to X_{AH}(M)$ is compact.
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Finite sums of Argyros-Haydon spaces
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An observation.

Let \((X_i)_{i=1}^n\) a finite sequence of Banach spaces and assume that every bounded linear operator \(T : X_i \to X_j\) is compact for every \(i \neq j\).

Setting \(X = (X_1 \oplus \ldots \oplus X_n)_\infty\) it follows that \(\text{Cal}(X)\) is isometric with \((\text{Cal}(X_1) \oplus \ldots \oplus \text{Cal}(X_n))_\infty\).
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\[(Cal(X_1) \oplus \ldots \oplus Cal(X_n))_\infty.\]
Let $k \in \mathbb{N}$, $L_1, \ldots, L_k$ pairwise disjoint infinite subsets of $\mathbb{N}$ and

$$X = (X_1 \oplus \cdots \oplus X_k)_{\infty},$$

where $X_i = X_{AH(L_i)}$ for $i = 1, \ldots, k$.

Since $L_i$ are pairwise disjoint we have that every $T : X_{AH(L_i)} \to X_{AH(L_j)}$ is compact for every $i \neq j$.

Since $Cal(X_i)$ is one dimensional, by the above observation we obtain that $Cal(X)$ is $k$-dimensional.
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We will now see the Calkin algebra of the space

$$X = \left( \sum_n \oplus X_n \right)_{BD},$$

where $X_n = X_{AH}(L_n)$ for a sequence $(L_n)_n$ of pairwise disjoint infinite subsets of natural numbers.

For a sequence $(X_n)_n$ of separable Banach spaces, the space $X = (\sum_n \oplus X_n)_{BD}$ is called a Bourgain Delbaen $\mathcal{L}_\infty$ sum of $(X_n)_n$ and is defined as a subspace of $(\sum \oplus (X_n \oplus \ell_\infty(\Delta_n)))_\infty$.

The sets $\Delta_n$ are finite, pairwise disjoint and defined using the Bourgain-Delbaen method.
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For a sequence \((X_n)_n\) of separable Banach spaces, the space \( X = (\sum_n \oplus X_n)_{BD} \) is called a Bourgain Delbaen \( L_\infty \) sum of \((X_n)_n\) and is defined as a subspace of \( (\sum \oplus (X_n \oplus \ell_\infty(\Delta_n))_\infty)_\infty \).

The sets \( \Delta_n \) are finite, pairwise disjoint and defined using the Bourgain-Delbaen method.
In particular, we define linear extension operators

\[ i_n : \left( \sum_{k \leq n} (X_k \oplus \ell_\infty(\Delta_k))_\infty \right)_\infty \rightarrow \left( \sum \oplus (X_n \oplus \ell_\infty(\Delta_n))_\infty \right)_\infty \]

such that

\[ \sup_n \|i_n\| < \infty. \]

\[ (\sum_n \oplus X_n)_{BD} = \overline{\bigcup_n Y_n}, \text{ where} \]

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\[ Y_n = i_n \left[ \left( \sum_{k \leq n} \oplus (X_k \oplus \ell_\infty(\Delta_k))_\infty \right)_\infty \right]. \]
Let $x$ be a vector in \((\sum_{k \leq n} \oplus (X_k \oplus \ell_\infty(\Delta_n)))_\infty\)
The vector $i_n(x)$, i.e. $x$ is extended by assigning to it new values in $\ell_\infty(\bigcup_{k>n} \Delta_k)$.
The \( L^\infty \) structure of AH-sums

- The finite sets \( \Delta_n \) are defined recursively and for each \( \gamma \in \Delta_{n+1} \) we assign a linear functional \( c^*_\gamma : (\sum_{k=1}^n \oplus (X_k \oplus \ell_\infty(\Delta_k))_\infty)_\infty \to \mathbb{R} \) such that \( i_n(x)(\gamma) = c^*_\gamma(x) \).

This implies that for a fixed \( n \in \mathbb{N} \), taking \( x_k \in X_k \) with \( x_k \in \bigcap_{\gamma \in \bigcup_{i=1}^n \Delta_i} \ker c^*_\gamma \) then the extended vector \( i_k(x_k) \) does not have non zero values upon \( \Delta_i \) for every \( 1 \leq i \leq n \).
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This implies that for a fixed $n \in \mathbb{N}$, taking $x_k \in X_k$ with $x_k \in \bigcap_{\gamma \in \bigcup_{i=1}^{n} \Delta_i} \text{Kerc}_\gamma^*$ then the extended vector $i_k(x_k)$ does not have non zero values upon $\Delta_i$ for every $1 \leq i \leq n$. 
The $\mathcal{L}_\infty$ structure of AH-sums

Hence, for $x_k \in \cap_{\gamma \in \cup_{i=1}^n \Delta_i} \text{Ker} c_\gamma^*$ $\cap X_k$, $i = 1, \ldots, n$

$$\|i_1(x_1) + \ldots + i_n(x_n)\| \simeq \max_{1 \leq k \leq n} \|x_k\|.$$ 

Since $\Delta_i$ are finite, the above implies that

$$\langle [l_k] : k = 1, \ldots, n \rangle \simeq c_0(n).$$
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Hence, for $x_k \in \bigcap_{\gamma \in \bigcup_{i=1}^{n} \Delta_i} \text{Kerc}_\gamma^*$ \cap X_k$, $i = 1, \ldots, n$

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Since $\Delta_i$ are finite, the above implies that

$$\langle [l_k] : k = 1, \ldots, n \rangle \simeq c_0(n).$$
Without taking any further assumptions for the separable $X_n$, the space $X = (\sum_n \oplus X_n)_{BD}$ satisfies the following basic properties:

- $X = \sum_n \oplus i_n[X_n \oplus \ell_\infty(\Delta_n)]$.
- Each $X_n$ is isometric with $i_n[X_n]$ and complemented in $X$ via projection $l_n$.
- An operator $K$ defined on $X = (\sum_n \oplus X_n)_{BD}$ is called horizontally compact operator if

$$\|K|_{\sum_{n \geq k} \oplus i_n[X_n \oplus \ell_\infty(\Delta_n)]}\| \xrightarrow{k \to \infty} 0.$$
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AH-sums of separable Banach spaces

- The space \((\sum_n \oplus X_n)_{BD}\) that is constructed using Bourgain-Delbaen method of constructing \(X_{AH}\) is denoted by \((\sum \oplus X_n)_{AH}\).

- The construction of \((\sum \oplus X_n)_{AH}\) depends on the same sequence of parameters \((m_j, n_j)_j\) of \(X_{AH}\).

- Again for \(L \subset \mathbb{N}\) infinite, we denote by \((\sum_n \oplus X_n)_{AH(L)}\) the space \((\sum \oplus X_n)_{AH}\) constructed using the subsequence \((m_j, n_j)_{j \in L}\).
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For every $L \subset \mathbb{N}$ infinite the space $X = (\sum_n \oplus X_n)_{AH(L)}$ has the following additional properties:

- The dual $X^*$ is isomorphic with $\left(\sum_n \oplus (X_n^* \oplus \ell_1(\Delta_n))_1\right)_1$.

By considering some specific sequences $(X_n)_n$ of separable Banach spaces, the space $X$ has the "scalar-plus-horizontally compact" property, i.e. every operator $T \in \mathcal{L}(X)$ is of the form $T = \lambda I + K$ where $\lambda \in \mathbb{R}$ and $K$ a horizontally compact operator.

For example, if $X_n$ has the Schur property for every $n \in \mathbb{N}$, or $\ell_1$ does not embed isomorphically in $X_n^*$ for every $n \in \mathbb{N}$, then the above holds.
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Calkin algebras of AH-sums

Proposition

Let $L, L_n$ pairwise disjoints infinite subsets of $\mathbb{N}$ and $X_{\text{AHsum}} = (\sum \oplus X_{\text{AH}}(L_n))_{\text{AH}(L)}$. The space $X_{\text{AHsum}}$ has the \textit{"scalar-plus-horizontally compact"} property.

- Observe also that since $X_{\text{AH}}^* \simeq \ell_1$, the space $X_{\text{AHsum}}$ is $\mathcal{L}_\infty$ space.
- Moreover, Since every operator $T : X_{\text{AH}}(L_n) \rightarrow X_{\text{AH}}(L_m)$ is compact, we conclude that the space

$$\mathcal{L}(X_{\text{AHsum}}) = \langle I, (I_n)_n, \mathcal{K}(X_{\text{AHsum}}) \rangle,$$

where $I$ denotes the identity map upon $X_{\text{AHsum}}$ and for each $n$, $I_n$ is the projection defined on $X_{\text{AHsum}}$ with image isometric with $X_{\text{AH}}(L_n)$. 
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where $I$ denotes the identity map upon $X_{AHsum}$ and for each $n$, $I_n$ is the projection defined on $X_{AHsum}$ with image isometric with $X_{AH(L_n)}$.
Hence $\mathcal{C}al(X_{AH\text{sum}}) = \mathcal{L}(X_{AH\text{sum}})/\mathcal{K}(X_{AH\text{sum}}) = \langle [l], ([ln])_n \rangle$.

Using the $\mathcal{L}^\infty$ structure of the BD-sum $(\sum \oplus X_{AH}(L_n))_{AH(L)}$ described earlier we obtain that

$$\langle [l_k] : k = 1, \ldots, n \rangle \sim^{C_n} c_0(n).$$

Using the $\mathcal{L}^\infty$ structure of the spaces $X_{AH(L_n)}$, we have that $(C_n)_n$ is uniformly bounded and by the above we conclude that the Calkin algebra of $X_{AH\text{sum}}$ is isomorphic to $c$. 
Hence \( \text{Cal}(X_{AH\text{sum}}) = \mathcal{L}(X_{AH\text{sum}})/\mathcal{K}(X_{AH\text{sum}}) = \langle [I], ([I_n]_n) \rangle \).

Using the \( \mathcal{L}^\infty \) structure of the BD-sum \( (\sum \oplus X_{AH}(L_n))_{AH(L)} \) described earlier we obtain that

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Hence $\mathcal{C}al(X_{AHsum}) = \mathcal{L}(X_{AHsum})/\mathcal{K}(X_{AHsum}) = \langle [I], ([I_n])_n \rangle$.

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We generalize the above concept using well founded trees $T$ with a unique root such that every non maximal node of $T$ has infinitely countable immediate successors.

For such a tree $T$ and $L \subset \mathbb{N}$ infinite we construct Banach spaces $X_{(T,L)}$ using induction on the order of $T$. 
We generalize the above concept using well founded trees $\mathcal{T}$ with a unique root such that every non maximal node of $\mathcal{T}$ has infinitely countable immediate successors.

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The definition of the spaces $X(\mathcal{T}, L)$

For $\mathcal{T}$ is a singleton and $L \subset \mathbb{N}$ we define $X(\mathcal{T}, L)$ to be the space $X_{AH}(L)$.

Tree of rank zero:

$$\circlearrowleft \quad X_{AH}(L)$$
The definition of the spaces $X(T,L)$

For a tree of order one we define $X(T,L) = \left( \sum \bigoplus X(T_n,L_n) \right)_{AH(L_0)}$.

Tree of rank 1:
The definition of the spaces $X(\mathcal{T}, L)$

For a tree of order two we define $X(\mathcal{T}, L) = \left( \sum \bigoplus X(\mathcal{T}_n, L_n) \right)_{AH(L_0)}$

etc...

Tree of rank 2:

$$X(\mathcal{T}, L)$$

$$X(\mathcal{T}_{s_1}, L_{s_1})$$

$$X(\mathcal{T}_{s_2}, L_{s_2})$$

$$X(\mathcal{T}_{s_3}, L_{s_3})$$
Properties of the spaces $X(\mathcal{T}, L)$

There space $X(\mathcal{T}, L)$ is accompanied by a set of norm-one projections $I_s, s \in \mathcal{T}$.
Proposition

- For every tree $\mathcal{T}$ and $L \subseteq \mathbb{N}$ infinite, the space $X(\mathcal{T}, L)$ is $\mathcal{L}_\infty$ and if $o(\mathcal{T}) > 0$ it has the "scalar-plus-horizontally compact" property.

- Note that $o(\mathcal{T}) = 0$, the space $X(\mathcal{T}, L)$ has the "scalar plus compact" property as it coincides with the space $X_{AH}(L)$. 
Properties of the spaces $X_{(T, L)}$

**Proposition**

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Operators defined on $X_{\mathcal{T},L}$

- For an operator $S$ defined on $X_{\mathcal{T},L}$ we denote by $S_t$ the induced operator
  $$I_t \circ S \circ I_t$$
  which can considered upon $X_{\mathcal{T}_t,L_t}$.

- Every $S \in \mathcal{L}(X_{\mathcal{T},L})$ corresponds to a unique family $(\lambda_t)_{t \in \mathcal{T}}$ of scalars chosen to satisfy:

  - If $t$ is maximal (and hence $X_{\mathcal{T}_t,L_t} = X_{AH(L_t)}$), $S_t - \lambda_t I_t$ is compact, while
  - If $t$ non maximal, $S_t - \lambda_t I_t$ is horizontally compact.
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Operators defined on $X(\mathcal{T}, L)$

For an operator $S$ defined on $X_{\mathcal{T}, L}$ we denote by $S_t$ the induced operator

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The functional $f_S : \mathcal{T} \to \mathbb{R}$ that assigns to each $t \in \mathcal{T}$ the scalar $\lambda_t$, is continuous.

We define $\Phi(\mathcal{T}, L) : \mathcal{L}(X(\mathcal{T}, L)) \to C(\mathcal{T})$ by the rule

$$S \to f_S.$$ 

The induced operator

$$\Phi(\mathcal{T}, L) : \mathcal{L}(X(\mathcal{T}, L)) / \mathcal{K}(X(\mathcal{T}, L)) = \text{Cal}(X(\mathcal{T}, L)) \to C(\mathcal{T})$$

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The Calkin algebras of $X(\mathcal{T}, \mathcal{L})$

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Proposition

Let $T$ be a tree of finite rank and $L$ be an infinite subset of the natural numbers. Then the map

$\Phi_{T,L} : \text{Cal}(X_{(T,L)}) \to C(T)$ is bounded below.

Hence, $\text{Cal}(X_{(T,L)}) \simeq C(T)$ as a Banach algebra, if $o(T) < \omega$. 
The Calkin algebras of $X_{(\mathcal{T}, L)}$

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The main result

Theorem (P. Motakis - Daniele Puglisi - D.Z)

Let $K$ be a countable compact metric space with finite Cantor-Bendixson index.

Then there exists a $\mathcal{L}_\infty$ space $X$ such that its Calkin algebra is isomorphic, as a Banach algebra, to $C(K)$.

- By Sierpinski Mazurkiewich $K$ is homeomorphic to a countable ordinal number of the form $\omega^k \cdot n$, $k, n \in \mathbb{N}$.

- $X = \left( \sum_{i=1}^{n} \oplus X(T, L_i) \right)_\infty$, where $T = \omega^k$ and $(L_i)_i$ pairwise disjoint.
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Can it be extended?

**Question:** is the above theorem true for every countable compact metric space?

**Question:** is the map $\Phi_{(\mathcal{T},L)} : \text{Cal}(X_{(\mathcal{T},L)}) \rightarrow C(\mathcal{T})$ always onto?
**Question**: is the above theorem true for every countable compact metric space?

**Question**: is the map $\phi_{(\mathcal{T},L)} : \mathcal{C}(X_{(\mathcal{T},L)}) \to C(\mathcal{T})$ always onto?
Indications for affirmative answers

- The dual of $Cal(X_{(T,L)})$ is **separable** and has the **Schur property**.

- The Calkin algebra of $X_{(T,L)}$ is **commutative** as a Banach algebra and as a Banach space it is $c_0$ **saturated** and has the **Dunford-Pettis property**.
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Thank you!