Uniform classification of classical Banach spaces

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**Ribe ’76.** The local structure is preserved: There exists $K = K(\phi)$ such that every finite dimensional subspace of $X$ $K$-embeds into $Y$, and vice versa.
Johnson-Lindenstrauss-Schechtman ’96
Suppose \( X \) is uniformly homeomorphic to \( \ell_p \) for \( 1 < p < \infty \).
Then \( X \) is isomorphic to \( \ell_p \).
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Suppose $X$ is uniformly homeomorphic to $\ell_p$ for $1 < p < \infty$. Then $X$ is isomorphic to $\ell_p$.

Godefroy-Kalton-Lancien ’00
If $X$ is Lipschitz isomorphic $c_0$, then $X$ is isomorphic to $c_0$. If $X$ is uniformly homeomorphic to $c_0$, then $X$ is ‘almost’ isomorphic to $c_0$. 
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Open for $\ell_1$ (Lipschitz case too)
Idea of the proof for $1 < p < \infty$ case

- Enough to show $\ell_2 \not\hookrightarrow X$ (follows from Ribe and Johnson-Odell)
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- For $1 \leq p < 2$ Midpoint technique Enflo ‘69, Bourgain ‘87
- Alternatively, for $2 < p < \infty$ Gorelik principle Gorelik ‘94
- Asymptotic smoothness Kalton-Randrianarivony ‘08
- We will give another.
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We will give another.
Theorem. Suppose $\phi : X \to Y$ is a uniform homeomorphism and $Y$ is reflexive. Then there exists $K = K(\phi)$ such that for all $n$ and all asymptotic spaces $(x_i)_{i=1}^n$ of $X$ and all scalars $(a_i)_{i=1}^n$, we have

$$\| \sum_{i=1}^n a_i x_i \| \leq K \sup \| \sum_{i=1}^n a_i y_i \|$$

where sup is over all $(y_i)_{i=1}^n$ asymptotic spaces of $Y$. If $Y = \ell^p$, then this means

$$\| \sum_{i=1}^n a_i x_i \| \leq K \left( \sum_{i=1}^n |a_i|^p \right)^{1/p}$$

Thus, $X$ cannot contain $\ell^2$ if $p > 2$. 
**Theorem.** Suppose $\phi : X \to Y$ is a uniform homeomorphism and $Y$ is reflexive. Then there exists $K = K(\phi)$ such that for all $n$ and all asymptotic spaces $(x_i)_{i=1}^n$ of $X$ and all scalars $(a_i)_{i=1}^n$, we have

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Thus, $X$ cannot contain $\ell_2$ if $p > 2$. 
Maurey-Milman-Tomczak-Jaegermann ’94 Let $X$ be a Banach space with a normalized basis (or a minimal system) $(u_i)$. Write $n < x < y$ if $n < \min \text{supp} x < \max \text{supp} x < \min \text{supp} y$. 
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An $n$-dimensional space with basis $(e_i)_1^n$ is called an asymptotic space of $X$, write $(e_i)_1^n \in \{X\}_n$, if for all $\varepsilon > 0$

$$\forall m_1 \exists m_1 < x_1 \quad \forall m_2 \exists m_2 < x_2 \quad \ldots \quad \forall m_n \exists m_n < x_n$$

such that the resulting blocks (called permissible) satisfy $(x_i)_1^n \sim^{1+\varepsilon} (e_i)_1^n$. 

\((e_i)_1^n \in \{X\}_n\) means that for all \(\varepsilon > 0\) there exists a block tree of \(n\)-levels

\[ T_n = \{x(k_1, k_2, \ldots, k_j) : 1 \leq j \leq n\} \]

so that every branch \((x(k_1), x(k_1, k_2), \ldots, x(k_1, \ldots, k_n))\) is \((1 + \varepsilon)\)-equivalent to \((e_i)_1^n\).
Asymptotic-$\ell_p$ spaces

$X$ is asymptotic-$\ell_p$ (asymptotic-$c_0$ for $p = \infty$), if there exists $K \geq 1$ such that for all $n$ and $(e_i)_1^n \in \{X\}_n$, $(e_i)_1^n \sim_{uvb} \ell_p^n$. Indeed, every C-unconditional $(x_i)_1^n \subset \ell_p$ is $C\ell_p$-equivalent to some asymptotic space of $\ell_p$.

Tsirelson space $T$ is asymptotic-$\ell_1$.

$T^*$ is asymptotic-$c_0$. 
**Asymptotic-$\ell_p$ spaces**

$X$ is **asymptotic-$\ell_p$** (**asymptotic-$c_0$** for $p = \infty$), if there exists $K \geq 1$ such that for all $n$ and $(e_i)_1^n \in \{X\}_n$, $(e_i)_1^n \overset{K}{\sim} \text{uvb} \ \ell_p^n$.

- $\ell_p$ is asymptotic-$\ell_p$.
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- $\ell_p$ is asymptotic-$\ell_p$.
- $L_p$ is not. Indeed, every C-unconditional $(x_i)_1^n \subset L_p$ is $CK_p$-equivalent to some asymptotic space of $L_p$. 
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- Tsirelson space $T$ is asymptotic-$\ell_1$.
- $T^*$ is asymptotic-$c_0$. 
Define the **upper envelope** function $r_X$ on $c_{00}$ by

$$r_X(a_1, \ldots, a_n) = \sup_{(e_i)_{1 \in \{X\}}_n} \| \sum_{i=1}^n a_i e_i \|$$

and the **lower envelope** $g_X$ by

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- \( X \) is asymptotic-\( \ell_p \) iff \( g_X \asymp \| \cdot \|_p \asymp r_X \).
Define the **upper envelope** function $r_X$ on $c_{00}$ by

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- $X$ is asymptotic-$\ell_p$ iff $g_X \simeq \| . \|_p \simeq r_X$.
- $r_X \simeq \| . \|_\infty$ implies $X$ is asymptotic-$c_0$. 

Theorem. Suppose $\phi : X \to Y$ is uniform homeomorphism, and $X$ and $Y$ are reflexive. Then there exists $K = K(\phi)$ such that for all scalars $a = (a_i) \in c_{00}$, we have

$$\frac{1}{K} r_Y(a) \leq r_X(a) \leq K r_Y(a).$$
The upper envelope is invariant

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Corollary. Suppose $X$ is uniformly homeomorphic to a reflexive asymptotic-$c_0$ space. Then $X$ is asymptotic-$c_0$. 
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Corollary. Suppose $X$ is uniformly homeomorphic to a reflexive asymptotic-$c_0$ space. Then $X$ is asymptotic-$c_0$.

Example. $T^*$
The main technical theorem

**Theorem.** Suppose \( \phi : X \to Y \) is a uniform homeomorphism and \( Y \) is reflexive. Then for all \( (e_i)_1^k \in \{X\}_k \), integers \( (a_i)_1^k \) and \( \varepsilon > 0 \), there exist permissible \( (x_i)_1^k \) in \( X \) with \( (x_i)_1^k \sim^{1+\varepsilon} (e_i)_1^k \) and permissible tuple \( (h_i/\|h_i\|)_1^k \) in \( Y \) with \( \|h_i\| \leq K|a_i| \) (\( K \) depends only on \( \phi \)) such that

\[
\left\| \phi \left( \sum_{i=1}^{k} a_i x_i \right) - \sum_{i=1}^{k} h_i \right\| \leq \varepsilon.
\]