On the real polynomial Bohnenblust–Hille inequality

Daniel Pellegrino

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The multilinear Bohnenblust–Hille inequality

Let $K$ be the real or complex scalars. Bohnenblust and Hille (Annals, 1931):

There exists a sequence of positive scalars $(C_K, m)_{m=1}^\infty$ such that

\[
\sum_{i_1, \ldots, i_m=1}^N \left\| U(e_{i_1}, \ldots, e_{i_m}) \right\|_2^{m+1} \leq C_K, m \| U \|_{(1)}
\]

for all $m$-linear forms $U: l^\infty \times \cdots \times l^\infty \to K$ and every positive integer $N$.

The exponent $2^{m+1}$ is optimal...

The best constant $C_K, m$ in this inequality will be denoted by $B_{\text{mult}} K, m$.

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$$\left\| \sum_{i_1, \ldots, i_m=1}^{N} U(e_{i_1}, \ldots, e_{i_m}) \right\|_{2m+1} \leq C_\mathbb{K}, m \left\| U \right\|$$

for all $m$-linear forms $U : \ell^\infty_1 \times \cdots \times \ell^\infty_1 \to \mathbb{K}$ and every positive integer $N$. The exponent $2m+1$ is optimal...

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The multilinear Bohnenblust–Hille inequality

Let $\mathbb{K}$ be the real or complex scalars.

Bohnenblust and Hille (Annals, 1931):

There exists a sequence of positive scalars $(C_{\mathbb{K},m})_{m=1}^{\infty} \geq 1$ such that

$$\left( \sum_{i_1, \ldots, i_m=1}^{N} |U(e_{i_1}, \ldots, e_{i_m})|^{\frac{2m}{m+1}} \right)^{\frac{m+1}{2m}} \leq C_{\mathbb{K},m} \|U\|$$

for all $m$-linear forms $U: l_\infty \times \cdots \times l_\infty \to \mathbb{K}$ and every positive integer $N$.
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(1)

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The polynomial Bohnenblust–Hille inequality

Bohnenblust and Hille (Annals, 1931):
For any $m \geq 1$, there exists a constant $D_{K,m} \geq 1$ such that, for any $n \geq 1$, for any $m$-homogeneous polynomial $P(z) = \sum_{|\alpha| = m} a_\alpha z^\alpha$ on $l_N^\infty$, 
\[
\left( \sum_{|\alpha| = m} |a_\alpha|^2 \right)^{m+1 \over 2m} \leq D_{K,m} \|P\|_\infty, 
\]
where $\|P\|_\infty = \sup_{\|z\|_\infty \leq 1} |P(z)|$.

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The best constant $D_{K,m}$ in this inequality will be denoted by $B_{K,m}^{pol}$. 
Bohnenblust–Hille inequalities

These inequalities have been proven to be very useful and powerful in analysis, analytic number theory and physics. For instance:

1. To estimate the abscissae of convergence of Dirichlet series (this was the initial goal of Bohnenblust and Hille).
2. To estimate the Bohr radius of the $n$-dimensional polydisk.
3. In Quantum Information Theory.

It turns out that having good estimates of the constants $B_{pol}^{Km}$ and $B_{mult}^{Km}$ is crucial.
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It turns out that having good estimates of the constants $B_{K,m}^{\text{pol}}$ and $B_{K,m}^{\text{mult}}$ is crucial.
Estimates for the complex BH constants along the history

H. F. Bohnenblust and E. Hille (Annals, 1931):

$$B_{\text{mult}} C, m \leq m + \frac{1}{2} m \left(\sqrt{2}\right)^{m - 1}$$


$$q B_{\text{mult}} C, m \leq \left(\sqrt{2}\right)^{m - 1}$$

H. Queffelec (J. Analyse, 1995)

$$B_{\text{mult}} C, m \leq \left(\frac{2}{\sqrt{\pi}}\right)^{m - 1}$$

D. Nunez, D.P., Serrano and Seoane (J. Functional Analysis, 2013)

....complicated recursive formula....but in any case

$$B_{\text{mult}} C, m < (m - 1)_{0}^{0.31}$$
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## Complex multilinear BH: estimates for the constants

<table>
<thead>
<tr>
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</thead>
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<td>$B_{C,3}^{\text{mult}} \leq$</td>
<td>?</td>
<td>1.2184</td>
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<td>$\approx 1.44$</td>
<td>4</td>
<td>10.51</td>
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<td>5.66</td>
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<td>8</td>
<td>24.33</td>
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<td>1.5183</td>
<td>1.63</td>
<td>2.63</td>
<td>16</td>
<td>54.24</td>
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<td>1.68</td>
<td>2.96</td>
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<td>80.29</td>
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<tr>
<td>$B_{C,100}^{\text{mult}} \leq$</td>
<td>?</td>
<td>2.5118</td>
<td>4.55</td>
<td>$1.56 \cdot 10^5$</td>
<td>$7.9 \cdot 10^{14}$</td>
<td>$8.14 \cdot 10^{15}$</td>
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</tbody>
</table>
Best known estimates

The best known (upper) formulas for the case of real and complex scalars, up to now, are:

(Bayart, D.P., Seoane, Advances in Mathematics 2014.

\[
B_{\mathbb{C},m}^{\text{mult}} \leq \prod_{j=2}^{m} \Gamma \left( 2 - \frac{1}{j} \right)^{\frac{j}{2-2j}}.
\]

For real scalars and \( m \geq 14, \)

\[
B_{\mathbb{R},m}^{\text{mult}} \leq 2^{\frac{446381}{55440}} - \frac{m}{2} \prod_{j=14}^{m} \left( \frac{\Gamma \left( \frac{3}{2} - \frac{1}{j} \right)}{\sqrt{\pi}} \right)^{\frac{j}{2-2j}}
\]

and

\[
B_{\mathbb{R},m}^{\text{mult}} \leq \prod_{j=2}^{m} 2^{\frac{1}{2j-2}}.
\]

for \( 2 \leq m \leq 13. \)
Best known estimates

For instance, for real scalars,

\[ B_{\mathbb{R},10}^{\text{mult}} \leq 2.6656 \]

\[ B_{\mathbb{R},100}^{\text{mult}} \leq 6.1493 \]
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In Montanaro´s terminology, our result is:

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\beta(G) = \Omega\left(k - \frac{3}{2}n - \left(k - 1\right)^2\right).
\]
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**Theorem.** Let $G$ be a $k$-player XOR game with $n$ possible inputs per player. Then

$$\beta(G) = \Omega \left( k^{\frac{3}{2}} n^{\frac{-(k-1)}{2}} \right).$$
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**Theorem.** Let $G$ be a $k$-player XOR game with $n$ possible inputs per player. Then

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Please do not ask me what does it mean!
Lower estimates for BH multilinear constants: real case

...if we look for lower estimates then, by finding adequate $n$-linear forms, as we will see next, we get lower bounds for the BH constants (case of real scalars):

$$\sqrt{2} \leq B_{\text{mult}}^{R,2},$$
$$1.587 \leq B_{\text{mult}}^{R,3},$$
$$1.6818 \leq B_{\text{mult}}^{R,4},$$
$$1.741 \leq B_{\text{mult}}^{R,5},$$
$$2.1^{n-1} \leq B_{\text{mult}}^{R,n},$$

Lower estimates for BH multilinear constants: real case

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\sqrt{2} \leq B_{\text{mult}, 2}, \quad \sqrt{2} \leq 1.587, \quad 1.681 \leq B_{\text{mult}, 3}, \quad 1.8877 \geq 1.741, \\
2^{\frac{-1}{n}} \leq B_{\text{mult}, n} \leq 2^{44638155440}, \quad \prod_{j=1}^{14} \left( \Gamma \left( \frac{3}{2} - \frac{1}{j} \right) \right)^{-1/2} < 1.3^{0.36482}.
\]

for \( m \geq 14 \). This last expression is dominated by (D. Diniz, G. Munoz, D.P, J. Seoane, Proc. Amer. Math. Soc., to appear)
...if we look for lower estimates then, by finding adequate $n$-linear forms, as we will see next, we get lower bounds for the BH constants (case of real scalars):

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\begin{align*}
\sqrt{2} & \leq B_{\mathbb{R},2}^{\text{mult}} \leq \sqrt{2} \\
1.587 & \leq B_{\mathbb{R},3}^{\text{mult}} \leq 1.6818 \\
1.681 & \leq B_{\mathbb{R},4}^{\text{mult}} \leq 1.8877 \\
1.741 & \leq B_{\mathbb{R},5}^{\text{mult}} \leq 2.0586 \\
2^{1 - \frac{1}{n}} & \leq B_{\mathbb{R},n}^{\text{mult}} \leq 2^{\frac{446381}{55440}} - \frac{m}{2} \prod_{j=14}^{m} \left( \frac{\Gamma\left(\frac{3}{2} - \frac{1}{j}\right)}{\sqrt{\pi}} \right)^{\frac{j}{2 - 2j}} < 1.3m^{0.36482}. \\
\end{align*}
\]

How did we get these lower bounds?

Let $T_2 : \ell_2 \times \ell_2 \rightarrow \mathbb{R}$ be defined by $T_2(x,y) = x_1y_1 + x_1y_2 + x_2y_1 - x_2y_2$. Since the norm of $T_2$ is 2, from $(\sum |T_2(e_i,e_j)|^4)^{1/4} \leq B_{\text{mult} R,2} \|T_2\|$ we get $B_{\text{mult} R,2} \geq 2^{1/2} - 1/2 = \sqrt{2}$. 

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Case $m = 2$:

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Since the norm of $T_2$ is 2, from

$$\left( \sum_{i,j} |T_2(e_i, e_j)|^{\frac{4}{3}} \right)^{\frac{3}{4}} \leq B_{\mathbb{R},2}^{\text{mult}} \|T_2\|$$
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we get

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$$T_3(x, y, z) =$$

$$(z_1+z_2)(x_1y_1 + x_1y_2 + x_2y_1 - x_2y_2) + (z_1-z_2)(x_3y_3 + x_3y_4 + x_4y_3 - x_4y_4).$$
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Since $\|T_3\| = 4$ and

$$\left( \sum_{i,j,k} |T_3(e_i, e_j, e_k)|^{\frac{3}{2}} \right)^{\frac{2}{3}} \leq B_{\mathbb{R}, 3}^{\text{mult}} \|T_3\|$$
Case $m = 3$:

Consider $T_3 : \ell^4_\infty \times \ell^4_\infty \times \ell^4_\infty \rightarrow \mathbb{R}$ given by

$$T_3(x, y, z) = (z_1+z_2)(x_1y_1 + x_1y_2 + x_2y_1 - x_2y_2) + (z_1-z_2)(x_3y_3 + x_3y_4 + x_4y_3 - x_4y_4).$$

Since $\|T_3\| = 4$ and

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we get

$$B_{\mathbb{R}, 3}^{\text{mult}} \geq 2^{1-\frac{1}{3}}$$
Using an induction argument, we obtain

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This procedure is useless for the complex case....
The polynomial Bohnenblust–Hille inequality: estimates for the constants
Bohnenblust and Hille (Annals, 1931):

\[ B_{\mathbb{C}, m}^{\text{pol}} \leq \left( \sqrt{2} \right)^{m-1} \frac{m^m (m+1)^{m+1}}{2^m (m!)^{m+1/2m}} \]
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Defant et al (Annals, 2011): The polynomial BH inequality is hypercontractive.

\[ B_{\mathbb{C},m}^{\text{pol}} \leq \left( 1 + \frac{1}{m-1} \right)^{m-1} \sqrt{m} (\sqrt{2})^{m-1} \]
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\]

Bayart, D.P and Seoane (Advances in Math 2014): For any \( \varepsilon > 0 \), there is a \( N \) such that, for any \( m \geq N \),

\[
B_{\mathbb{C},m}^{\text{pol}} \leq (1 + \varepsilon)^m.
\]
Application: the Bohr radius problem

The Bohr radius $K_n$ of the $n$-dimensional polydisk is the largest positive number $r$ such that all polynomials $\sum \alpha a_\alpha z^\alpha$ on $\mathbb{C}^n$ satisfy

$$\sup_{z \in rD^n} \left| \sum \alpha a_\alpha z^\alpha \right| \leq \sup_{z \in D^n} \left| \left| \sum \alpha a_\alpha z^\alpha \right| \right|,$$

with $D_n = \{(z_1, \ldots, z_n) : \max |z_j| < 1 \text{ for all } j\}$.

The Bohr radius $K_1$ was studied and estimated by H. Bohr himself, and it was shown independently by M. Riesz, I. Schur and F. Wiener that $K_1 = \frac{1}{3}$. For $n \geq 2$, exact values of $K_n$ are unknown.
The Bohr radius $K_n$ of the $n$-dimensional polydisk is the largest positive number $r$ such that all polynomials $\sum_\alpha a_\alpha z^\alpha$ on $\mathbb{C}^n$ satisfy

$$\sup_{z \in r\mathbb{D}^n} \left| \sum_\alpha a_\alpha z^\alpha \right| \leq \sup_{z \in \mathbb{D}^n} \left| \sum_\alpha a_\alpha z^\alpha \right|,$$

with

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The Bohr radius $K_1$ was studied and estimated by H. Bohr himself, and it was shown independently by M. Riesz, I. Schur and F. Wiener that $K_1 = 1/3$. For $n \geq 2$, exact values of $K_n$ are unknown.
Our subexponential estimate for the constants of the complex BH inequality was the key for the solution of the Bohr radius problem:

\[ \lim_{m \to \infty} K_m \sqrt{\ln m} = 1. \]

This finishes a problem that numerous researchers have been chipping away at for more than fifteen years.
Application: the Bohr radius problem

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Application: the Bohr radius problem

Our subexponential estimate for the constants of the complex BH inequality was the key for the solution of the Bohr radius problem:

**Theorem (Bayart, D.P, Seoane)**

\[
\lim_{m \rightarrow \infty} \frac{K_m}{\sqrt{\ln m / m}} = 1.
\]

This finishes a problem that numerous researchers have been chipping away at for more than fifteen years.
Next result shows that real scalars behaves differently from real scalars:
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**Theorem (Campos, Jimenez, Munoz, D.P and Seoane)**

\[
B_{R,m}^{\text{pol}} > \left(\frac{2^{\sqrt{3}}}{\sqrt{5}}\right)^m > (1.17)^m
\]

*for all positive integers* \( m > 1 \).*
Let $m$ be an even integer. Consider the $m$-homogeneous polynomial

$$R_m(x_1, \ldots, x_m) = (x_1^2 - x_2^2 + x_1x_2) (x_3^2 - x_4^2 + x_3x_4) \cdots (x_{m-1}^2 - x_m^2 + x_{m-1}x_m).$$

Since $\|R_2\| = 5/4$, it is simple to see that

$$\|R_m\| = (5/4)^{m/2}.$$ 

From the BH inequality for $R_m$ we have

$$\left( \sum_{|\alpha|=m} a_\alpha^{\frac{2m}{m+1}} \right)^{\frac{m+1}{2m}} \leq D_{\mathbb{R},m} \|R_m\|,$$

that is,

$$D_{\mathbb{R},m} \geq \frac{\left( \frac{3}{2} \right)^{\frac{m+1}{2m}}}{\left( \frac{5}{4} \right)^{\frac{m}{2}}} \geq \frac{\left( \sqrt{3} \right)^{\frac{m+1}{2}}}{\left( \frac{5}{4} \right)^{\frac{m}{2}}} > \left( \frac{2 \sqrt[4]{3}}{\sqrt{5}} \right)^m.$$
Proof

The case $m$ is odd is similar. Keeping the previous notation, consider the $m$ homogeneous polynomial

$$R_m(x_1, \ldots, x_{2m})$$

$$= (x_{2m} + x_{2m-1}) R_{m-1}(x_1, \ldots, x_{m-1}) + (x_{2m} - x_{2m-1}) R_{m-1}(x_m, \ldots, x_{2m-2})$$

and we get the same estimate.
The case of real scalars

In fact we have

Theorem (Campos, Jimenez, Munoz, D.P, Seoane)

$$\limsup_{m} \left( B_{\mathbb{R},m}^{pol} \right)^{1/m} = 2.$$
If $P : l_\infty (\mathbb{R}) \to \mathbb{R}$ is an $m$-homogeneous polynomial, by a result of Visser if we consider the same polynomial $P_{\mathbb{C}} : l_\infty (\mathbb{C}) \to \mathbb{C}$ we have

$$\|P_{\mathbb{C}}\| \leq 2^m - 1 \|P\|.$$
If \( P : l_\infty(\mathbb{R}) \to \mathbb{R} \) is an \( m \)-homogeneous polynomial, by a result of Visser if we consider the same polynomial \( P_\mathbb{C} : l_\infty(\mathbb{C}) \to \mathbb{C} \) we have

\[
\|P_\mathbb{C}\| \leq 2^{m-1} \|P\|.
\]
Proof

If $P : l_\infty(\mathbb{R}) \to \mathbb{R}$ is an $m$-homogeneous polynomial, by a result of Visser if we consider the same polynomial $P_\mathbb{C} : l_\infty(\mathbb{C}) \to \mathbb{C}$ we have

$$\|P_\mathbb{C}\| \leq 2^{m-1} \|P\|.$$  

So, for a real polynomial $P : l_\infty(\mathbb{R}) \to \mathbb{R}$ given by $P = \sum a_\alpha z^\alpha$, we consider $P_\mathbb{C}$ and we easily get from our estimate for complex scalars (and big $m$),

$$\left( \sum_{|\alpha|=m} |a_\alpha| \frac{2^m}{m+1} \right)^{m+1} \leq (1 + \varepsilon)^m \|P_\mathbb{C}\| \leq (1 + \varepsilon)^m 2^{m-1} \|P\| \leq (2 + \delta)^m \|P\|$$

and we conclude that

$$\lim \sup_m \left( B_{m, pol}^{\text{pol}} \right)^{1/m} \leq 2.$$
Proof

The other inequality is a little bit more technical.
This talk contains results from the following papers:

- F. Bayart, D. Pellegrino, J. Seoane, The Bohr radius of the $n$-dimensional polydisk is equivalent to $\sqrt{\log n/n}$, Advances in Math 2014.
- D. Nuñez, D. Pellegrino and J.B. Seoane, D. M. Serrano-Rodriguez, There exist multilinear Bohnenblust-Hille constants $(C_n)_{n=1}^{\infty}$ with $\lim_{n \to \infty} (C_{n+1} - C_n) = 0$, J. Functional Analysis (2013).

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On the real polynomial Bohnenblust–Hille inequality
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...and preprints: