The elusive geometry of the Banach space $\ell_\infty/c_0$

Piotr Koszmider

IM PAN, Warsaw
Banach spaces and topological preliminaries

\[ \ell_\infty = \{ (a_n)_{n \in \mathbb{N}} : a_n \in \mathbb{R}, (a_n)_{n \in \mathbb{N}} \text{ bounded} \} \equiv C(\beta \mathbb{N}), \text{ sup norm} \]

\[ c_0 = \{ (a_n)_{n \in \mathbb{N}} \in \ell_\infty : \lim_{n \to \infty} a_n = 0 \} \equiv \{ f \in C(\beta \mathbb{N}) : f \upharpoonright (\beta \mathbb{N} \setminus \mathbb{N}) = 0 \}, \text{ sup norm} \]

\[ \mathbb{N}^* = \beta \mathbb{N} \setminus \mathbb{N} \]

\[ \ell_\infty / c_0 \equiv C(\mathbb{N}^*) \]

\[ \text{Clop}(\beta \mathbb{N}) \equiv \mathcal{P}(\mathbb{N}) \]

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Banach spaces and topological preliminaries

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$N^* = \beta\mathbb{N} \setminus \mathbb{N}$
$\ell_\infty/c_0 \equiv C(N^*)$
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Set-theoretic and logical preliminaries

CH = “All infinite subsets of $\mathbb{R}$ have the cardinality of $\mathbb{R}$ or the cardinality of $\mathbb{N}$.”

There is a well ordering of $\ell_\infty/c_0$ which has all proper initial segments countable.

$(2)$ is useful for transfinite inductive construction of length $|\ell_\infty/c_0| = |\mathcal{P}(\mathbb{N})| = |\mathbb{R}| = \omega_1$

Alternative axioms: MA+not CH, OCA, PFA.

Vocabulary: in ZFC, consistent, cannot be proved.
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Outline of the talk

1. The universality of $\ell_\infty/c_0$
2. Complemented subspaces of $\ell_\infty/c_0$
3. Complemented copies of $\ell_\infty/c_0$ in $\ell_\infty/c_0$
4. Infinite decompositions of $\ell_\infty/c_0$
5. The primaryness of $\ell_\infty/c_0$
6. Automorphisms of $\ell_\infty/c_0$

Piotr Koszmider (IM PAN, Warsaw)
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BWB, Maresias, 25-08-2014
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Universality of $\ell_\infty/c_0$

For any Banach space $X$ of density continuum, there is an isometric embedding of $X$ into $\ell_\infty/c_0$.

Spaces of density $\omega_1$ which can be embedded into $\ell_\infty/c_0$ without using CH:

1. All Banach spaces of density $\omega_1$.
2. $c_0$ (for $\omega_2$).
3. Many other $C(K)$s.
4. $\ell_p(\omega_2)$ for $1 \leq p < \infty$.
5. Some WCG or Hilbert generated $C(K)$s consistently do not embed in $\ell_\infty/c_0$.
Universality of $\ell_\infty/c_0$

1. (CH) [Esenin-Volpin, Doklady 1949] For any Banach space $X$ of density continuum, there is an isometric embedding of $X$ into $\ell_\infty/c_0$
Universality of $\ell_\infty / c_0$

1. (CH) [Esenin-Volpin, Doklady 1949] For any Banach space $X$ of density continuum, there is an isometric embedding of $X$ into $\ell_\infty / c_0$

2. Spaces of density $2^\omega$ which can be embedded into $\ell_\infty / c_0$ without using CH:
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   - all Banach spaces of density $\omega_1$, 
   - all Banach spaces of density $\omega_1$, 
   - many other $C(K)$
   - $\ell_p(2^\omega)$ for $1 \leq p < \infty$,
   - $C([0,2^\omega])$ does not embed isomorphically into $\ell_\infty/c_0$
   - It is consistent that there is no isomorphically universal Banach space of density $2^\omega$
   - Some WCG or Hilbert generated $C(K)$ consistently do not embed in $\ell_\infty/c_0$ (Todorcevic - JMAA 2012, Krupski, Marciszewski - Coll. M 2012, Brech, P.K. - PAMS 2013)
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(Piotr Koszmider (IM PAN, Warsaw) Geometry of $\ell_\infty / c_0$ BWB, Maresias, 25-08-2014)
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Complemented subspaces of $\ell_\infty / c_0$

Question

Does $L_\infty ([0,1] \omega_1)$ embed into $\ell_\infty / c_0$ in ZFC?
Complemented subspaces of $\ell_\infty / c_0$

1. $\ell_\infty / c_0 \equiv (\ell_\infty / c_0) \oplus (\ell_\infty / c_0)$
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Does $L_\infty([0, 1]^{\omega_1})$ embed into $\ell_\infty/c_0$ in ZFC?
Complemented copies of $\ell_\infty / c_0$

Theorem (Castillo, Plichko; JFA 2010)

(CH) There are uncomplemented subspaces of $\ell_\infty / c_0$ isomorphic to $\ell_\infty / c_0$.

Problem

Does every subspace of $\ell_\infty / c_0$ isomorphic to $\ell_\infty / c_0$ contains a further subspace isomorphic to $\ell_\infty / c_0$ which is complemented in the entire $\ell_\infty / c_0$?

Theorem (Drewnowski, Roberts - PAMS 1991)

Whenever $\ell_\infty / c_0 = A \oplus B$, then either $A$ or $B$ contains a complemented copy of $\ell_\infty / c_0$. 

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Whenever $\ell_\infty/c_0 = A \oplus B$, then either $A$ or $B$ contains a complemented copy of $\ell_\infty/c_0$. 
Theorem

Suppose that $X$, $Y$ are Banach spaces and

1. $X$ is isomorphic to a complemented subspace of $Y$,
2. $Y$ is isomorphic to a complemented subspace of $X$,
3. $X$ is isomorphic to $\ell_\infty(\ell_\infty)$,

Then $X$ and $Y$ are isomorphic.
Theorem

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Then $X$ and $Y$ are isomorphic.
$\ell_\infty$-sums
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**Theorem (Negrepontis - TAMS 1969)**

(CH)

$$\ell_\infty/c_0 \sim \ell_\infty(\ell_\infty/c_0).$$
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\[ \ell_\infty/c_0 \sim \ell_\infty(\ell_\infty/c_0). \]

**Theorem (Brech, P.K. - Fund. M. 2014)**

*It is consistent that \( \ell_\infty/c_0 \) is not isomorphic to any Banach space of the form \( \ell_\infty(X) \)
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Theorem (Negrepontis - TAMS 1969)

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$$\ell_\infty / c_0 \sim \ell_\infty (\ell_\infty / c_0).$$

Theorem (Brech, P.K. - Fund. M. 2014)

*It is consistent that $\ell_\infty / c_0$ is not isomorphic to any Banach space of the form $\ell_\infty (X)$*
The primariness of $\ell_\infty / c_0$

Theorem (Drewnowski, Roberts - PAMS 1991)

(\text{CH}) Whenever $\ell_\infty / c_0 = X \oplus Y$, then either $X$ or $Y$ is isomorphic to $\ell_\infty / c_0$.

Question

Is $\ell_\infty / c_0$ primary in ZFC?


It is consistent that there are two disjoint open subsets $U, V \subseteq \mathbb{N}^*$ with $U \cap V = \{x\}$ for some $x \in \mathbb{N}^*$ and neither $U$ nor $V$ is homeomorphic to $\mathbb{N}^*$.
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There is an automorphism of $\ell_\infty/c_0$ which is not induced by an operator on $\ell_\infty$. We develop local canonization of some automorphisms under $\text{OCA} + \text{MA}$.
Automorphisms

Joint work in preparation with Cristóbal Rodriguez Porras
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Some references:


