Construction of differentiable functions between Banach spaces.

joint work with P. Hajek, then with M. Ivanov and S. Lajara

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Relationship between the existence of non trivial real valued smooth functions on a separable Banach space $X$ and the geometry of $X$.

**Theorem.** Let $X$ be a separable Banach space. TFAE:

1. There exists on $X$ an equivalent norm diff. on $X\setminus\{0\}$.
2. There exists a $C^1$-smooth function $b : X \to \mathbb{R}$ with bounded non empty support.
3. $X^*$ is separable.

**Definition.** $X, Y$ Banach spaces. A function $f : X \to Y$ is $G$-differentiable at $x \in X$ if $\exists f'(x) \in \mathcal{L}(X,Y)$ such that for each $h \in X$, $\lim_{t \to 0} \frac{f(x+th) - f(x)}{th} = f'(x)h$.

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1. There exists on $X$ an equivalent norm $G$-diff. on $X\setminus\{0\}$.
2. There exists a $G$-diff. function $b : X \to \mathbb{R}$ with bounded non empty support.
**Theorem [Azagra-Deville].** If $X$ is an infinite dimensional Banach space with separable dual, there exists a $C^1$-smooth real valued function on $X$ with bounded support and such that $f'(X) = X^*$.  

**Theorem [Azagra,Deville and Jimenez-Sevilla].** Let $X$, $Y$ be separable Banach spaces such that $\dim(X) = \infty$. Then there exists $f : X \to Y$ Gâteaux-differentiable, such that $f'(X) = \mathcal{L}(X,Y)$. Moreover, if $\mathcal{L}(X,Y)$ is separable, $f$ can be chosen Fréchet-differentiable.  

**Theorem [Hajek].** If $f$ is a function on $c_0$ with locally uniformly continuous derivative, then $f'(c_0)$ is included in a countable union of norm compact subsets of $\ell^1$. 
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**Problem**: Let $X, Y$ be separable Banach spaces such that $\dim(X) \geq 1$, $f : X \to Y$ differentiable at every point of $X$. What is the structure of

$$f'(X) = \{ f'(x); x \in X \} \subset \mathcal{L}(X, Y)?$$

**Is $f'(X)$ connected?**

**Theorem**: (Maly 96): If $X$ is a Banach space and $f : X \to \mathbb{R}$ is Fréchet-differentiable at every point, then the set $f'(X)$ is connected in $(X^*, \|\|)$. 

Let $f : \mathbb{R}^2 \to \mathbb{R}^2$, defined by:

$$f(x, y) = \left( x^2 \sqrt{y} \cos 1/x^3, x^2 \sqrt{y} \sin 1/x^3 \right)$$

if $(x, y) \neq (0, 0)$ and $f(0, 0) = (0, 0)$.

$$\{\det(f'(x)); x \in \mathbb{R}^2\} = \{0, 3/2\} \Rightarrow f'(\mathbb{R}^2)$$ not connected.

**Theorem**: (T. Matrai): Let $X$ be a separable Banach space, and let $f$ be a real valued locally Lipschitz and Gâteaux-differentiable function on $X$. Then $f'(X)$ is connected in $(X^*, w^*)$. 

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Proposition 1: If \( f \) is a continuous and Gâteaux-differentiable bump function on \( X \), then the norm closure of \( f'(X) \) contains a ball \( B(r) \) for some \( r > 0 \).

Proposition 2:
Let \( X, Y \) be Banach spaces, \( \dim(X) \geq 1 \).
Let \( F : X \to Y \) be Lipschitz and Gâteaux-differentiable. Assume that one of the following conditions hold:

1. \( F \) is Lipschitz and \( Y = \mathbb{R} \).
2. \( F \) is Lipschitz and Fréchet-differentiable.
3. \( \mathcal{L}(X,Y) \) is separable.

Then, \( \forall x \in X, \forall \varepsilon > 0, \exists y, z \in B_X(x, \varepsilon), y \neq z \), such that
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Proposition: Let $X$ be an infinite dimensional separable Banach space. Then, $\exists f : X \to \mathbb{R}$ Gâteaux-differentiable bump, such that $f'$ is norm to weak* continuous and

$$x \neq 0 \Rightarrow \|f'(0) - f'(x)\| \geq 1$$

If $X^*$ is separable, we can assume moreover that $f$ is $C^1$ on $X \setminus \{0\}$.

Definition: Let $X, Y$ be separable Banach spaces. $(X, Y)$ has the jump property if $\exists F : X \to Y$ Lipschitz, everywhere G-differentiable, so that

$$\forall x, y \in X, x \neq y \Rightarrow \|F'(x) - F'(y)\| \geq 1$$

Question: When do $(X, Y)$ possess the jump property?
$X, Y$ separable Banach spaces.

(1) $\mathcal{L}(X, Y)$ is separable $\Rightarrow (X, Y)$ fails the jump property.

(2) $(X, \mathbb{R})$ fails the jump property.

(3) $Y \subset Z$ and $(X, Y)$ has the jump property $\Rightarrow (X, Z)$ has the jump property.

**Theorem**: $(\ell^1, \mathbb{R}^2)$ has the jump property. More precisely, there exists $F : \ell^1 \rightarrow \mathbb{R}^2$ Gateaux-differentiable, bounded, Lipschitz, such that for every $x, y \in \ell^1$, $x \neq y$, then

$$\|F'(x) - F'(y)\|_{\mathcal{L}(\ell^1, \mathbb{R}^2)} \geq 1$$

Moreover, $\forall h \in \ell^1$, $x \rightarrow F'(x).h$ is continuous from $\ell^1$ into $\mathbb{R}^2$. 

Gâteaux-differentiability criterium: Let $X$ and $Y$ be Banach spaces. Assume:

* $f_n : X \to Y$ are G-differentiable.
* $\left( \sum f_n \right)$ converges pointwise on $X$,
* For all $h$, the series $\sum_{n \geq 1} \frac{\partial f_n}{\partial h}(x)$ converges uniformly in $x$.

Then $f = \sum_{n \geq 1} f_n$ is G-differentiable on $X$, for all $x$, $f'(x) = \sum_{n \geq 1} f'_n(x)$ (where the convergence of the series is in $\mathcal{L}(X,Y)$ for the strong operator topology), and $f$ is $K$-Lipschitz.
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Moreover, if each $f'_n$ is continuous from $X$ endowed with the norm topology into $\mathcal{L}(X,Y)$ with the strong operator topology, then $f'$ shares the same continuity property.
Lemma: Given $p = (q, r) \in \mathbb{R}^2$ such that $q < r$ and $\varepsilon > 0$, there exists a $C^\infty$-function $\varphi = \varphi_{p, \varepsilon} : \mathbb{R}^2 \to \mathbb{R}^2$ such that:

(i) $|\varphi(x, y)| \leq \varepsilon$ for all $(x, y) \in \mathbb{R}^2$,
(ii) $\varphi(x, y) = 0$ if $x < q$,
(iii) $\left\| \frac{\partial \varphi}{\partial x}(x, y) \right\| \leq \varepsilon$ for all $(x, y) \in \mathbb{R}^2$,
(iv) $\left\| \frac{\partial \varphi}{\partial y}(x, y) \right\| = 1$ if $x \geq r$,
(v) $\left\| \frac{\partial \varphi}{\partial y}(x, y) \right\| \leq 1$ for all $(x, y) \in \mathbb{R}^2$,

Proof: $\varphi(x, y) = \beta(x) \left( \sin(ny), \cos(ny) \right)$,
with $\beta : \mathbb{R} \to [0, 1] \in C^\infty$, $\beta(x) = 0$ if $x \leq q$ and $\beta(x) = 1$ if $x \geq r$. 
Proof of Theorem : Let \( \mathcal{P} = \{ (q, r) \in \mathbb{Q}^2; \ q < r \} \) and 
\( k \to (n_k, (q_k, r_k)) \) be a bijection from \( \mathbb{N} \) onto \( \mathbb{N} \times \mathcal{P} \) such that 
for all \( k \), \( n_k \neq k \).

\( \varepsilon > 0 \), \( \varepsilon_k > 0 \) / \( \sum_{k=1}^{\infty} \varepsilon_k = \varepsilon \).

\( f_k : \ell^1 \to \mathbb{R}^2 \)

\[ f_k(x) = \varphi_{p_k, \varepsilon_k}(x_{n_k}, x_k) \]

\( f_k \) is a \( C^\infty \) function on \( \ell^1 \).

\( F : \ell^1 \to \mathbb{R}^2 \) is defined by : \( F(x) = \sum_{k \in \mathbb{N}} f_k(x) \)

- \( F \) is well-defined.

- \( F \) is \( G \)-differentiable on \( \ell^1 \) and \( F \) is \( (1 + \varepsilon) \)-Lipschitz on \( \ell^1 \).

Indeed \( \sum_j \sup_{x \in \ell^1} \left\| \frac{\partial f_j}{\partial x_k} \right\| \leq \sum_{j \neq k} \varepsilon_j + 1 \).
- We claim that if \( x \neq y \in \ell^1 \), then \( \|F'(x) - F'(y)\| \geq 1 - 2\varepsilon \).

\[
f_k(x) = \varphi_{p_k, \varepsilon_k}(x n_k, x_k)
\]

- If \( x \neq y \in \ell^1 \), choose \( m \) such that (for example) \( x_m \neq y_m \), then \( (q, r) \) such that \( x_m < q < r < y_m \) and finally \( k \) such that \( (n_k, q_k, r_k) = (m, q, r) \).

\[
\frac{\partial f_k}{\partial x_k}(x) = 0 \quad \left\| \frac{\partial f_k}{\partial x_k}(y) \right\| = 1
\]

and, if \( j \neq k \),

\[
\frac{\partial f_j}{\partial x_k}(x) \leq \varepsilon_j \quad \left\| \frac{\partial f_j}{\partial x_k}(y) \right\| \leq \varepsilon_j
\]

Therefore

\[
\|F'(x) - F'(y)\|_{\mathcal{L}(\ell^1, \mathbb{R}^2)} \geq \left\| \frac{\partial F}{\partial x_k}(x) - \frac{\partial F}{\partial x_k}(y) \right\|_{\mathbb{R}^2}
\]

\[
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\]
**Theorem.** Let $X, Y$ be separable Banach spaces. Assume:

$(e_n, e^*_n) \subset X \times X^*$ is a total, bounded, biorthogonal system,

$(f_n) \subset Y$ is an unconditional basic sequence such that:

$\forall h \in X, \left( \sum e^*_n(h) f_{2n-1} \right)$ and $(\sum e^*_n(h) f_{2n})$ converge in norm.

Then $(X, Y)$ has the jump property.

**Proof.** Define $z_k : X \to \mathbb{R}^2$ by $z_k(x) = (e^*_{n_k}(x), e^*_k(x))$ then

$i_k : \mathbb{R}^2 \to Y$ by $i_k(s, t) = tf_{2k-1} + sf_{2k}$,

$F_k : X \to Y$ by $F_k = i_k \circ \varphi_{p_k, \varepsilon_k} \circ z_k$ and $F = \sum F_k$.

**Corollary (Bayart).** If $X$ is a separable, infinite dimensional Banach space, then $(X, c_0)$ has the jump property.
**Corollary.** Let $X$ be a Banach space with a Schauder basis $(e_n)$, $Y$ be a Banach space and $U \in \mathcal{L}(X, Y)$ such that $(U(e_n))$ is a subsymmetric basis. Then $(X, Y)$ has the jump property.

**Example.** Let $X_p = \ell^p$ if $1 \leq p < +\infty$ and $X_\infty = c_0$. Let us fix $1 \leq p, q \leq +\infty$. TFAE:

1. $(X_p, X_q)$ has the jump property.
2. $p \leq q$.
3. $\mathcal{L}(X_p, X_q)$ is not separable.

**Example.** Let $J$ be the James’ space. Then $(J, \ell^2)$ and $(J, J)$ have the jump property.
Corollary. Let $X$ be a Banach space with a Schauder basis $(e_n)$, $Y$ be a Banach space such that $Y \cong Y \oplus Y$ and $U \in \mathcal{L}(X,Y)$ such that $(U(e_n))$ is an unconditional basis. Then $(X,Y)$ has the jump property.

Example. Assume $1 \leq q \leq p \leq 2$ and $p \neq 1$. Then $\left( L^p([0,1]), L^q([0,1]) \right)$ has the jump property.

What about the other values of $p$ and $q$?

Corollary. Let $X$ be a Banach space with an unconditional basis and such that $X \cong X \oplus X$. Then $(X,X)$ has the jump property.

Example. If $T$ is the Tsirelson space, then $(T,T)$ and $(T^*,T^*)$ have the jump property. If $X$ is the space of Argyros and Haydon, then $(X,X)$ fails the jump property.
Open questions

1) Does \((L^1([0,1]), L^1([0,1]))\) have the jump property?

2) If \(\mathcal{L}(X,Y)\) is nonseparable and \(\dim(Y) \geq 2\), does \((X,Y)\) have the jump property?

If \(\mathcal{L}(X,Y)\) icontains \(\ell^\infty\) and \(\dim(Y) \geq 2\), does \((X,Y)\) have the jump property?

3) Does \((JT, \mathbb{R}^2)\) have the jump property?

4) Describe the couples \((X,Y)\) of separable Banach spaces for which

\[\exists (e_n, e_n^*) \subset X \times X^* \text{ is a total, bounded, biorthogonal system, }\]
\[\exists (f_n) \subset Y \text{ is an unconditional basic sequence such that : }\]
\[\forall h \in X, \left(\sum e_n^*(h) f_n\right) \text{ converges in norm.}\]
\[(\text{this imply } \mathcal{L}(X,Y) \supset \ell^\infty)\]
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