Tsirelson spaces with constraints

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Schreier Families

Let
\[ S_1 = \{ F \subset \mathbb{N} : |F| \leq \min F \}. \]

If \( \alpha \) is a countable ordinal and \( S_\alpha \) has been defined let
\[ S_{\alpha+1} = \left\{ \bigcup_{i=1}^{n} F_i : F_1 < \cdots < F_n, (\min F_i)_{i=1}^{n} \in S_1, F_i \in S_\alpha \right\}. \]

If \( \xi \) is a limit ordinal let \( \xi_n \uparrow \xi \) and define
\[ S_\xi = \{ F : \exists n \leq F \in S_{\xi_n} \}. \]
A sequence \((x_n)\) in a Banach space generates an \(S_\alpha-\ell_p\) spreading model if there is a constant \(C > 0\) so that for all scalars \((a_i)_i\) and \(F \in S_\alpha\) we have

\[
\frac{1}{C} \left( \sum_{i \in F} |a_i|^p \right)^{1/p} \leq \left\| \sum_{i \in F} a_i x_i \right\| \leq C \left( \sum_{i \in F} |a_i|^p \right)^{1/p}.
\]
A operator \( T \in \mathcal{L}(X) \) is \( S_\alpha \)-strictly singular (\( T \in SS_\alpha(X) \)) if for every \( \varepsilon > 0 \) and normalized basic sequence \( (x_n)_{n \in \mathbb{N}} \) in \( X \)

\[
\exists F \in S_\alpha \text{ and } x \in S_{[x_n]_{n \in F}} \text{ satisfying } \| Tx \| < \varepsilon.
\]
Strictly Singular Operators

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\]

For each \( \alpha \) the set \( SS_\alpha(X) \) is norm closed,

\[
\mathcal{K}(X) \subset SS_\alpha(X) \subset SS_\beta(X) \subset SS(X)
\]

for \( \alpha < \beta \). For \( T \in \mathcal{L}(X) \) and \( S \in SS_\alpha(X) \) we have \( TS, ST \in SS_\alpha(X) \).

For separable \( X \) we have

\[
\bigcup_{\alpha < \omega_1} SS_\alpha(X) = SS(X).
\]
Strictly Singular Operators

A operator $T \in \mathcal{L}(X)$ is $S_\alpha$-strictly singular ($T \in SS_\alpha(X)$) if for every $\varepsilon > 0$ and normalized basic sequence $(x_n)_{n \in \mathbb{N}}$ in $X$

$$\exists F \in S_\alpha \text{ and } x \in S_{[x_n]_{n \in F}} \text{ satisfying } \|Tx\| < \varepsilon.$$ 

For each $\alpha$ the set $SS_\alpha(X)$ is norm closed,

$$\mathcal{K}(X) \subset SS_\alpha(X) \subset SS_\beta(X) \subset SS(X)$$

for $\alpha < \beta$. For $T \in \mathcal{L}(X)$ and $S \in SS_\alpha(X)$ we have $TS, ST \in SS_\alpha(X)$.

For separable $X$ we have

$$\bigcup_{\alpha < \omega_1} SS_\alpha(X) = SS(X).$$

However, it is not necessarily true that $S_1, S_2 \in SS_\alpha(X)$ implies that $S_1 + S_2 \in SS_\alpha(X)$ [Odell and Teixeira].
A Banach space \((X, \| \cdot \|)\) with a basis \([e_n]\) is \(S_\alpha\)-arbitrarily distortable if for every \(\lambda > 0\) there is an equivalent norm \(| \cdot |\) on \(X\) so that for every block sequence \((x_n)\) of \((e_n)\)

\[
\exists F \in S_\alpha \text{ and } x, y \in S_{[x_n]_{n \in F}} \text{ with } \frac{|x|}{|y|} > \lambda.
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A Banach space \((X, \| \cdot \|)\) with a basis \([e_n]\) is \(S_\alpha\)-arbitrarily distortable if for every \(\lambda > 0\) there is an equivalent norm \(| \cdot |\) on \(X\) so that for every block sequence \((x_n)\) of \((e_n)\)

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\]

Define

\[
AD(X) = \min\{\alpha : X \text{ is } S_\alpha\text{-arbitrarily distortable}\}
\]

We have \(AD(X) < \omega_1\) if and only if \(X\) is arbitrarily distortable.
Let $0 < \theta < 1$. The space $T$ is the completion of $c_{00}$ under a norm that satisfies the following implicit equation: Let $x \in c_{00}$

$$
\|x\| = \max\{\|x\|_{\infty}, \sup \theta \sum_{i=1}^{n} \|E_i x\|\}
$$

where the supremum is for $n \in \mathbb{N}$ and successive intervals $(E_i)_{i=1}^{n}$ satisfying

$$(\min E_i)_{i=1}^{n} \in S_1.$$
A Tsirelson space with Constraints

Let $0 < \theta < 1$. The space $T_{0,1}$ is the completion of $c_{00}$ with respect to the norm satisfying the following implicit equation. Let $x \in c_{00}$

$$
\|x\| = \max\{\|x\|_\infty, \sup \theta \sum_{i=1}^{n} \|E_i x\|_{m_i}\}
$$

where the supremum is for successive intervals $(E_i)_{i=1}^{n}$ and $(m_i)_{i=1}^{n}$ with

$$(\min E_i)_{i=1}^{n} \in S_1 \text{ and } m_i > \max E_{i-1}$$

where

$$
\|x\|_m = \sup \frac{\theta}{m} \sum_{j=1}^{m} \|F_j x\|.
$$

where the supremum is with respect to any successive intervals $(F_j)_{j=1}^{m}$. 

A Tsirelson space with Constraints

Let $0 < \theta < 1$. The space $T_{0,1}^k$ is the completion of $c_{00}$ with respect to the norm satisfying the following implicit equation. Let $x \in c_{00}$

$$
\|x\| = \max\{\|x\|_\infty, \sup \theta \sum_{i=1}^{n} \|E_i x\|_{m_i}\}
$$

where the supremum is for successive intervals $(E_i)_{i=1}^{n}$ and $(m_i)_{i=1}^{n}$ with

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$$

where the supremum is with respect to any successive intervals $(F_j)_{j=1}^{m}$. 
A Tsirelson space with Constraints

Let $0 < \theta < 1$ and $\alpha$ be a countable ordinal. The space $T_{0,1}^{\omega \alpha}$ is the completion of $c_{00}$ with respect to the norm satisfying the following implicit equation. Let $x \in c_{00}$

$$
\|x\| = \max\{\|x\|_\infty, \sup \theta \sum_{i=1}^{n} \|E_i x\|_{m_i}\}
$$

where the supremum is for successive intervals $(E_i)_{i=1}^{n}$ and $(m_i)_{i=1}^{n}$ with

$$(\min E_i)_{i=1}^{n} \in S_{\omega \alpha} \text{ and } m_i > \max E_{i-1}$$

where

$$
\|x\|_m = \sup \frac{\theta}{m} \sum_{j=1}^{m} \|F_j x\|.
$$

where the supremum is with respect to any successive intervals $(F_j)_{j=1}^{m}$. 
A Tsirelson space with Constraints

Let $0 < \theta < 1$ and $\alpha$ be a countable ordinal. The space $T_{0,1}^{\omega\alpha}$ is the completion of $c_{00}$ with respect to the norm satisfying the following implicit equation. Let $x \in c_{00}$

$$\|x\| = \max\{\|x\|_\infty, \sup \theta \sum_{i=1}^{n} \|E_i x\|_{m_i}\}$$

where the supremum is for successive intervals $(E_i)_{i=1}^{n}$ and $(m_i)_{i=1}^{n}$ with

$$(\min E_i)_{i=1}^{n} \in S_{\omega\alpha}$$

and $m_i > \max E_{i-1}$

where

$$\|x\|_m = \sup \frac{\theta}{m} \sum_{j=1}^{m} \|F_j x\|.$$

where the supremum is with respect to any successive intervals $(F_j)_{j=1}^{m}$. 
A Tsirelson space with Constraints

Let $0 < \theta < 1$ and $1 \leq p < q \leq \infty$. The space $T_{p,q}$ is the completion of $c_{00}$ with respect to the norm satisfying the following implicit equation. Let $x \in c_{00}$

$$\|x\| = \max\{\|x\|_{\infty}, \sup \theta \left( \sum_{i=1}^{n} \|E_i x\|_{m_i, q}^{p} \right)^{\frac{1}{p}} \}$$

where the supremum is for successive intervals $(E_i)_{i=1}^{n}$ and $(m_i)_{i=1}^{n}$ with

$$(\min E_i)_{i=1}^{n} \in S_1 \text{ and } m_i > \max E_{i-1}$$

where

$$\|x\|_{m, q} = \sup \frac{\theta}{m^{1/q'}} \sum_{j=1}^{m} \|F_j x\|.$$
A Tsirelson space with Constraints

Let $0 < \theta \leq \frac{1}{2}$ and $1 \leq p < q \leq \infty$. The space $T_{p,q}$ is the completion of $c_{00}$ with respect to the norm satisfying the following implicit equation. Let $x \in c_{00}$

$$
\|x\| = \max\{\|x\|_\infty, \sup \theta \left( \sum_{i=1}^{n} \|E_i x\|_{m_i, q}^p \right)^{\frac{1}{p}}, \sup \theta \left( \sum_{i=1}^{n} \|F_i x\|_{q} \right)^{\frac{1}{q}} \}
$$

where the supremums are for successive intervals $(E_i)_{i=1}^{n}$ and $(m_i)_{i=1}^{n}$ with

$$(\min E_i)_{i=1}^{n} \in S_1 \text{ and } m_i > \max E_{i-1}$$

and any successive intervals $(F_i)_{i=1}^{n}$. Also

$$
\|x\|_{m,q} = \sup \frac{\theta}{m^{1/q'}} \sum_{j=1}^{m} \|F_j x\|
$$

where the supremum is for any successive intervals $(F_j)_{j=1}^{m}$. 

Additional Constructions

- Odell and Schlumprecht constructed a space so that every monotone basic sequence is block finitely represented in every subspace.
- Argyros and Motakis constructed an HI space so that in every subspace every subsymmetric basic sequence is admitted as a spreading model (with constant 148, of course).
- Argyros and Motakis constructed a reflexive HI space so that any operator on any infinite dimensional subspace has a non-trivial invariant subspace.
Open Problems

1. Does there exist a space $X$ so that $AD(X) = 1$? Such a space cannot contain an $\ell_1$ or $c_0$ spreading model.
2. Does there exist an arbitrarily distortable space $X$ which does not admit a $c_0$ or $\ell_1$ spreading model satisfying $AD(X) > 1$.
3. Is $AD(\ell_2) > 1$? (Gowers)
4. Which closed subsets of $[1, \infty]$ can be hereditary Krivine $p$ sets?
5. For $1 < p < q < \infty$ is the space $T_{p,q}$ superreflexive?
Thank you for listening!