On Separable Quotients

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The Problem

Open Problem: (SQP)

Given an infinite Banach space $E$, show that there exists a closed subspace $X$ so that $E/X$ is isomorphic to an infinite-dimensional separable Banach space.
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Known Contributions

Brief History:

Bessaga-Pelczynski (1958): Spaces whose dual contain $c_0$.
Hagler-Johnson (1977): Spaces whose dual have unconditional basic sequences.
Plichko (1980): Spaces with fundamental biorthogonal systems.

Remark:
For many other results including characterizations, variants and recent progresses, see Mujica’s survey and recent work of Argyros-Dodos-Kanellopoulos and Dodos, Lopez-Abad and Todorcevic.
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Results From "MathOverFlow" - BJ

- If $X^*$ has no HI subspace, then $X$ has a separable quotient.
- If $X^*$ is weak*-separable, then $X$ has a separable quotient.
Open Question (Godunov, 1974)

Let $E$ be a Banach space. Then there exists a continuous vector field $f: \mathbb{R} \times E \to E$ so that

$$u'(t) = f(t, u(t))$$

Does not have solutions at any point.

Positive Answers:

Godunov (1974) for the Hilbert space $E = \ell^2$.

Shkarin (2003) solved for Banach space having complemented subspaces with unconditional Schauder basis.

Hájek-Johanis (Best Answer (2010)): For spaces having Nontrivial Separable Quotients.
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Theorem (Marrocos, Rebouças: Studia Math. 2013)

$E$ has a SQP iff $E^*$ has a weak$^*$-transfinite Schauder frame.

Definition (WTSF):
That means to say that there exists an ordinal number $\xi$ and a transfinite sequence $(f_\alpha)_{\alpha<\xi}$ in $E^*$ so that:

$$\forall y^* \in \text{span} \{f_\alpha: \alpha<\xi\} \exists (a_{\alpha}(y^*))_{\alpha<\xi} \in \ell_\infty(\xi) \text{ s.t. } \langle y^*, x \rangle = \lim_{\alpha \to \xi} \langle \sum_{\gamma=0}^{\alpha} a_{\gamma}(y^*) f_\gamma, x \rangle$$

$(f_\alpha)_{\alpha<\xi}$ admits a biorthogonal system $(e_\alpha)_{\alpha<\xi}$ in $E$. 

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**Definition (WTSF):** That means to say that there exists an ordinal number $\xi$ and a transfinite sequence $(f_\alpha)_{\alpha<\xi}$ in $E^*$ so that:

- $\forall y^* \in \text{span}^{w^*} \left\{ f_\alpha : \alpha < \xi \right\}$
  - $\exists \ (a_\alpha(y^*))_{\alpha<\xi} \in \ell_\infty(\xi)$ s.t.

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- $(f_\alpha)_{\alpha<\xi}$ admits a biorthogonal system $(e_\alpha)_{\alpha<\xi}$ in $E$. 

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On Separable Quotients
Idea of the Proof

Let $X := \text{span}\{e_\alpha : \alpha < \xi\}$ and $Y = \text{span}\{f_\alpha : \alpha < \xi\}$.

By the Definition of WTSF, we get $X^\perp \cap Y = \{0\}$ which implies $X + Y$ is dense in $E$. In particular, we have that $Z = \{x \in E : \sum_{\alpha < \xi} |f_\alpha(x)| \|e_\alpha\| < \infty\}$ is dense in $E$. 

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Let $X := \text{span}\left\{e_\alpha : \alpha < \xi \right\}$ and $Y = \text{span}^w \left\{f_\alpha : \alpha < \xi \right\}$

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In particular, we have that

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**Tools and Approach**

**\( \ell_1 \)-Fundamental Systems**

A biorthogonal system \( \{x_\alpha, x_\alpha^*\}_{\alpha \in \Gamma} \) in \( E \times E^* \) is called \( \ell_1 \)-fundamental if the linear space

\[
\left\{ x \in E : \sum_{\alpha \in \Gamma} |x_\alpha^*(x)||x_\alpha| < \infty \right\}
\]

is dense in \( E \).

**Remark.** Every Fundundamental Biorthogonal System if \( \ell_1 \)-fundamental.
Let $X, Y$ Hausdorff LCS.
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- (I) $U \subset X$ is called a barrel in $X$ is it is closed, absolutely convex and absorbing.
- (II) $X$ is called *barreled* if every barrel in $X$ is a neighborhood of $0$.
- (III) **Closed Graph Theorem**: If $X$ is barreled, $Y$ is Fréchet and $T : X \to Y$ is a closed linear map, then $T$ is continuous.
- (IV) **Known Characterization**: A Banach space $X$ has an infinite-dimensional separable quotient *iff* $X$ has a non-barreled proper dense subspace.
Theorem

Every Banach space $E$ with a $\ell_1$-fundamental biorthogonal system has a non-trivial separable quotient.

Suppose that this is not so. Then $E$ does not contain $\ell_1$ and, moreover, as the linear space
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is dense in $E$. One readily shows that $T$ has closed linear graph. Since $Z$ is barreled, $T$ is continuous.
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is dense in $E$ it is barreled. Define now the linear operator

$$T : Z \to \ell_1(\xi); \quad T(x) = (f_\alpha(x)||e_\alpha||)_{\alpha < \xi}, \quad x \in E.$$
Continuation of the "Proof"

**Theorem**

*Every Banach space* $E$ *with a* $\ell_1$-*fundamental biorthogonal system has a non-trivial separable quotient.*

Suppose that this is not so. Then $E$ does not contain $\ell_1$ and, moreover, as the linear space

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T : Z \to \ell_1(\xi); \quad T(x) = (f_\alpha(x)||e_\alpha||)_{\alpha<\xi}, \quad x \in E.
$$

One readily shows that $T$ has closed linear graph. Since $Z$ is barreled, $T$ is continuous.
As $T$ is bounded, it can be linearly extended to the whole space $E$.

Denote this extension by $T$, too.

Since $T$ is not compact, $T(B_E)$ contains a semi normalized sequence $(x_n)$ which is equivalent to the unit basis of $\ell_1$.

By the lifting property, the formal inverse $T^{-1}$ from $\text{span}\{x_n\}$ back to $E$ is bounded.

Thus, $T^{-1}$ is really the inverse of $T$.

$\{T^{-1}(x_n)\}$ has a subsequence equivalent to the unit basis of $\ell_1$.

Contradiction.
Thanks!