SYMMETRY AND SYNCHRONY IN COUPLED CELL NETWORKS 2: GROUP NETWORKS

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Received April 28, 2006; Revised May 25, 2006

This paper continues the study of patterns of synchrony (equivalently, balanced colorings or flow-invariant subspaces) in symmetric coupled cell networks, and their relation to fixed-point spaces of subgroups of the symmetry group. Let $\Gamma$ be a permutation group acting on the set of cells. We define the group network $G_\Gamma$, whose architecture is entirely determined by the group orbits of $\Gamma$. We prove that if $\Gamma$ has the “balanced extension property” then every balanced coloring of $G_\Gamma$ is a fixed-point coloring relative to the automorphism group of the group network. This theorem applies in particular when $\Gamma$ is cyclic or dihedral, acting on cells as the symmetries of a regular polygon, and in these cases the automorphism group is $\Gamma$ itself. In general, however, the automorphism group may be larger than $\Gamma$. Several examples of this phenomenon are discussed, including the finite simple group of order 168 in its permutation representation of degree 7. More dramatically, for some choices of $\Gamma$ there exist balanced colorings of $G_\Gamma$ that are not fixed-point colorings. For example, there exists an exotic balanced 2-coloring when $\Gamma$ is the symmetry group of the two-dimensional square lattice. This coloring is doubly periodic, and its reduction modulo 8 leads to a finite group with similar properties. Although these patterns do not arise from fixed-point spaces, we provide a group-theoretic explanation of their balance property in terms of a sublattice of index two.

Keywords: Network dynamics; symmetry; synchrony.

1. Introduction

This paper is a continuation of Part 1 [Antoneli & Stewart, 2006], which studies balanced equivalence relations (colorings) of symmetric networks. It has been written as a separate paper because it is focussed on a specific technical issue. We employ the same terminology and notation as in Part 1, and for convenience we briefly recall a few key ideas.

An equivariant dynamical system is a system of ODEs $\dot{x} = f(x)$ where the vector field $f$ is smooth ($C^\infty$) and is equivariant under the action of a group $\Gamma$ on phase space $X$; that is,

$$f(\gamma x) = \gamma f(x) \quad \forall x \in X, \quad \gamma \in \Gamma$$

In general, the space $X$ can be a smooth manifold, but for simplicity we assume it is a finite-dimensional real vector space, with $\Gamma$ acting linearly.

A key concept is the fixed-point space of a subgroup $\Sigma \subseteq \Gamma$, defined by

$$\text{Fix}(\Sigma) = \{ x \in X : \sigma x = x \quad \forall \sigma \in \Sigma \}$$

This space is flow-invariant [Golubitsky et al., 1988; Golubitsky & Stewart, 2002]. That is,

$$f(\text{Fix}(\Sigma)) \subseteq \text{Fix}(\Sigma)$$

for every smooth equivariant vector field $f$. In fact, the fixed-point spaces of subgroups of $\Gamma$ are the
only flow-invariant subspaces in the above sense: see Theorem 4.1 of [Antoneli & Stewart, 2006].

Important examples of equivariant dynamical systems are symmetric coupled cell networks. By a network we mean a directed graph [Tutte, 1984; Wilson, 1985] whose nodes and edges are distinguished by “labels” or “types”. The nodes (“cells”) of a network \( G \) represent dynamical systems, and the edges (“arrows”) represent couplings. The appropriate group \( \Gamma \) comprises those permutations of the cells that preserve the couplings and the node- and edge-types. In such examples, spaces of the form \( \text{Fix}(\Sigma) \) determine natural patterns of synchrony for cells. The natural analog of the symmetry groupoid \( B_G \) of \( G \), which comprises all label-preserving bijections between “input sets” of cells.

For the link with dynamics, equip each cell \( c \) with a phase space \( P_c \). The analog of an equivariant vector field \( f \) is an “admissible” vector field; the space of admissible vector fields is determined by the action of \( B_G \) on the variables \( x_c \). Again, there exists a canonical class of flow-invariant subspaces of \( P \). Let \( \bowtie \) be an equivalence relation on the set \( C \) of cells. The associated polydiagonal

\[
\Delta_{\bowtie} = \{ x \in P : c \bowtie d \Rightarrow x_c = x_d \}
\]

is a subspace of \( P \). Theorem 6.5 of [Stewart et al., 2003] proves (using slightly different terminology) that \( \Delta_{\bowtie} \) is flow-invariant under every admissible vector field if and only if \( \bowtie \) is “balanced”. Roughly speaking, this condition states that if \( c \bowtie d \) then the input sets of \( c \) and \( d \) are related by a bijection that preserves \( \bowtie \)-equivalence classes.

It is convenient to describe such equivalence relations in terms of an associated coloring of the network. Color the cells so that \( c \) and \( d \) receive the same color if and only if \( c \bowtie d \). Then the sets of identically-colored cells are the equivalence classes for \( \bowtie \). The coloring is balanced if and only if any two cells with the same color have “color-isomorphic” input sets—that is, there is a bijection between those sets that preserves both arrow-type and cell color. If there are \( k \) colors in total then we call such a pattern a balanced \( k \)-coloring. If \( \bowtie \) is balanced, then \( \Delta_{\bowtie} \) is a balanced polydiagonal.

All networks discussed in this paper are finite (that is, have finitely many cells and finitely many arrows), except in Sec. 6 where it is convenient to consider some infinite networks. In general, the dynamic interpretation of a coupled cell network (“admissible” vector fields) leads to ODEs only when the network is finite. It is probably fairly straightforward to extend the dynamics to locally finite networks, in which each cell is the head of a finite set of arrows, but it is clear that there can be functional-analytic complications—for example, because ODE solutions can blow up in finite time, some differential equations on locally finite networks fail to have solutions over any time interval. One way to deal with this problem may be to impose a suitable Lipschitz condition. When we discuss infinite networks in this paper, only their combinatorial features are considered, so no functional-analytic technicalities are required.

1.1. Symmetric networks

Technically, an automorphism or symmetry of a network is a pair of bijections \( \omega = (\omega_C, \omega_E) \) which collectively preserve the incidence relations between arrows and cells, and the cell- and arrow-types. Here \( \omega_C \) acts on the cells \( C \) and \( \omega_E \) acts on the arrows \( E \). However, when (as in this paper) the network has no multiple arrows, it is possible to work solely with \( \omega_C \). The set of all automorphisms of a network \( G \) forms a group, the automorphism group or symmetry group \( \text{Aut}(G) \).

We say that a coloring \( \bowtie \) of \( G \) is a fixed-point coloring if \( \Delta_{\bowtie} = \text{Fix}(H) \) for some subgroup \( H \subseteq \text{Aut}(G) \). Otherwise \( \bowtie \) is exotic.

The main question addressed in this paper is: which symmetric networks have the property that every balanced coloring is a fixed-point coloring? We showed in [Antoneli & Stewart, 2006] that this property sometimes fails to hold. But there is a sense in which all of those examples are slightly artificial. Namely, some arrows that are not in the same orbit of the symmetry group are edge-equivalent.

Networks with this artificial feature can be ruled out by restricting attention to a natural class of symmetric networks, which we call group networks. Suppose that \( \Gamma \) is a permutation group acting on a set \( S \). Then we can consider the elements of \( S \) as cells, and ordered pairs \((s_1, s_2)\) of distinct elements of \( S \) as arrows. These cells and arrows can then be labeled according to their \( \Gamma \)-orbits. The resulting network \( G_\Gamma \) is symmetric under the action of \( \Gamma \); moreover, it is all-to-all coupled, that is, any two distinct cells are connected by an arrow in the sense that if \( s_1 \neq s_2 \in S \) then there exists an arrow \( e \) with \( T(e) = s_1 \) and \( \mathcal{H}(e) = s_2 \).

Although this class of networks is related very closely to the permutation action of \( \Gamma \), several
technical issues complicate the analysis of balanced colorings. One is that equivariant maps may fail to be admissible. A simple example occurs for the dihedral group $D_5$ of order 10. Another is that the automorphism group $\text{Aut}(\mathcal{G}_\Gamma)$ may be larger than $\Gamma$. An example here is the alternating group $A_4$ acting on a network with four cells.

In the positive direction, we will give conditions on $\Gamma$ that are sufficient for every balanced coloring of $\mathcal{G}_\Gamma$ to be a fixed-point coloring. These theorems require the symmetry groupoid $B_G$ to be suitably related to the symmetry group $\text{Aut}(\mathcal{G}_\Gamma)$, and are of “local-global” type. The conditions apply, in particular, when $\Gamma = \mathbb{Z}_N$ or $D_N$ acting on the vertices of a regular $N$-gon. In these cases, $\text{Aut}(\mathcal{G}_\Gamma) = \Gamma$.

An interesting case when $\text{Aut}(\mathcal{G}_\Gamma) \supseteq \Gamma$ arises when $\Gamma$ is the finite simple group of order 168 in its permutation representation of degree 7. We discuss this as an extended example.

### 1.2. Lattice example

We now come to the main result of this paper, which is currently slightly mysterious.

It is tempting to conjecture that every balanced coloring of a group network should be of the form $\text{Fix}(H)$ for a suitable subgroup $H \subseteq \text{Aut}(\mathcal{G})$. However, this turns out to be false. There is a 64-cell group network $G$ whose cells can be identified with the discrete torus $\mathbb{Z}_8 \times \mathbb{Z}_8$, acted upon by the group $\Gamma = (\mathbb{Z}_8 \oplus \mathbb{Z}_8) \vdash D_4$ where $\vdash$ indicates a semidirect product. Here $\mathbb{Z}_8 \oplus \mathbb{Z}_8$ is the translation group of the torus and $D_4$ is the “holohedry” [Hahn, 1992], which consists of the eight automorphisms that fix the origin $(0,0)$. These automorphisms map a point $(a,b)$ to the points $(\pm a, \pm b)$ and $(\pm b, \pm a)$ for any choice of signs.

The arrows are not shown in Fig. 1, because the figure would become much too complicated, but they are determined by the group action. The simplest type of arrow corresponds to “nearest neighbor coupling” in which arrows of a single type connect adjacent cells in the horizontal and vertical (but not diagonal) directions.

The 64-cell network supports the 2-coloring shown in Fig. 1. This is Fig. 6(a) in [Wang & Golubitsky, 2005], who discovered the coloring while deriving a general classification of balanced two-dimensional lattice colorings. They show that in a planar lattice with nearest-neighbor coupling, it is not unusual to find exotic balanced colorings for combinatorial reasons. However, most colorings of this kind cease to be balanced if longer-range couplings are permitted. Now comes the surprise: one of these colorings is not rendered unbalanced by increasing the coupling range. In fact, direct calculation shows that the 2-coloring in Fig. 1 is balanced for $d$th nearest neighbor coupling, however large $d$ may be. In particular, the 2-coloring is balanced for the group network $\mathcal{G}_\Gamma$.

It follows easily that the lift of this coloring to the integer lattice $\mathbb{Z}^2$ has a similar property. Because $\Gamma$ is now an infinite group, the associated group network is not finite (nor even locally finite) and hence cannot easily be associated with a system of ODEs, but we can consider all possible connections between cells up to some bounded distance. The coloring remains balanced for all such connections. However, despite its evident regularity, this coloring is not of the form $\text{Fix}(H)$ for any subgroup $H$ of the automorphism group of the network (either in $\mathbb{Z}^2$ or $\mathbb{Z}_8^2$). We emphasize that this fact is not a consequence of special assumptions like nearest-neighbor coupling: it is valid for any system of couplings such that the action of the symmetry group preserves arrow-type.

### 1.3. Structure of the paper

Section 2 introduces the notion of the group network $\mathcal{G}_\Gamma$ corresponding to any subgroup $\Gamma \subseteq S_N$. 

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**Fig. 1.** A balanced 2-coloring of the discrete torus $\mathbb{Z}_8 \times \mathbb{Z}_8$ that does not correspond to a fixed-point space of a subgroup of the automorphism group.
Section 3 discusses two examples which clarify the general theory of group networks and raise two technical issues which should be borne in mind. First: equivariant vector fields need not be admissible. Second: even for group networks, balanced colorings need not be determined by fixed-point spaces of subgroups of $\Gamma$, as we show by several examples related to regular solids.

Section 4 provides sufficient conditions on $\Gamma$ for all balanced colorings of $G_{\Gamma}$ to be fixed-point colorings (for subgroups of $\text{Aut}(G_{\Gamma})$). In particular, we define the “balanced extension property” and apply it to $\mathbb{Z}_N$, $D_N$, and some groups of symmetries of regular solids. In these cases, $\text{Aut}(G_{\Gamma}) = \Gamma$.

Section 5 considers a significant example for which $\text{Aut}(G_{\Gamma}) \supset \Gamma$, namely, the finite simple group of order 168 acting on the 7-point projective plane.

Finally, Sec. 6 discusses the main result of this paper. It introduces the 64-cell network of Fig. 1 and its lift to the integer lattice, establishes its remarkable properties, and offers a group-theoretic explanation of them, which is closely related to, but distinct from, the standard fixed-point space argument.

2. Group Networks

The examples of exotic balanced polydiagonals in [Antoneli & Stewart, 2006] all involve arrows that are equivalent but not in the same group orbit for $\Gamma$. Another feature is that the networks concerned are not all-to-all coupled, so that the “domain condition” for admissible vector fields introduces extra constraints. More precisely, the admissible maps may be a proper subspace of the equivariant maps, and this fact opens up the possibility of flow-invariant subspaces that are not fixed-point spaces, hence of exotic colorings, since Theorem 4.1 of [Antoneli & Stewart, 2006] no longer applies.

We therefore seek a more restrictive class of group-symmetric networks that might be expected to be better behaved. To do this we introduce a class of networks that is determined, in a natural manner, by the action of a finite permutation group. Specifically, cell- and arrow types in these networks correspond to group orbits on cells, arrows respectively. Moreover, the networks are all-to-all coupled.

**Definition 2.1.** Let $\Gamma$ be a permutation group acting on $\mathcal{C} = \{1, 2, \ldots, N\}$, so that $\Gamma \subseteq S_N$.

A network $\mathcal{G}$ is a $\Gamma$-network if

(a) The cells of $\mathcal{G}$ are the elements of $\mathcal{C}$.
(b) $c \sim d \iff d = \gamma c$ for some $\gamma \in \Gamma$. That is, the cell types are the $\Gamma$-orbits on $\mathcal{C}$.
(c) $E$ contains exactly one edge $(c, d)$ for each $c \neq d \in \mathcal{C}$, where $H(c, d) = c$ and $T(c, d) = d$, and no others. In particular, there are no self-connections and no multiple arrows.
(d) $(c, d) \sim_{E} (c', d') \iff c' = \gamma c$ and $d' = \gamma d$ for some $\gamma \in \Gamma$. That is, the arrow types are the $\Gamma$-orbits on the set $A$ of pairs $(c, d)$, with $c \neq d$.

These conditions determine the network uniquely up to isomorphism. We denote it by $\mathcal{G}_{\Gamma}$.

A network $\mathcal{G}$ is a group network if it is a $\Gamma$-network for some group $\Gamma$.

For example, suppose that $\Gamma = D_7$ acting on a ring of seven cells as the symmetries of a regular heptagon. Then the associated group network is shown in Fig. 2. Here the double-arrows should be interpreted as two identical single arrows, one pointing in each direction. The colors show the different $D_7$-orbits of arrows.

Two cells $c, d$ of a coupled cell network are input-equivalent, denoted $c \sim_I d$, if the subnetwork consisting of arrows with head $c$ is isomorphic to the subnetwork consisting of arrows with head $d$. There are several equivalent ways to formalize this concept.

![Fig. 2. The group network for $D_7$.](image-url)
We record two simple implications of Definition 2.1:

**Proposition 2.2.** Let \( \mathcal{G} \) be a \( \Gamma \)-network. Then \( c \sim_I d \) if and only if \( c, d \) lie in the same \( \Gamma \)-orbit.

**Corollary 2.3.** Let \( \mathcal{G} \) be a \( \Gamma \)-network. Suppose that \( \bowtie \) is a balanced equivalence relation. Then \( c \bowtie d \) implies that \( c, d \) lie in the same \( \Gamma \)-orbit.

Proof. If \( c \bowtie d \) then \( B(c, d) \neq \emptyset \) since \( \bowtie \) is balanced, so \( c \sim_I d \). Indeed, any balanced equivalence relation refines input-equivalence [Stewart et al., 2003, Sec. 6]. ■

3. Two Examples

In this section we discuss two technical issues: The relation between admissible vector fields and equivariant vector fields for group networks, and the relation between \( \Gamma \) and \( \text{Aut}(\mathcal{G}_\Gamma) \).

### 3.1. Admissible versus equivariant

Theorem 4.1 of [Antoneli & Stewart, 2006] states that if a finite group \( \Gamma \) acts linearly on a finite-dimensional vector space \( V \) over \( \mathbb{R} \), and \( W \) is a subspace that is invariant under every smooth \( \Gamma \)-equivariant map \( f : V \to V \), then \( W = \text{Fix}(H) \) for some subgroup \( H \subseteq \Gamma \). This result might lead us to hope that when \( \mathcal{G}_\Gamma \) is a group network, the admissible maps are the same as the equivariants. This, if true, would immediately imply that every balanced coloring of \( \mathcal{G}_\Gamma \) is a fixed-point coloring, for a subgroup of \( \Gamma \). However, the admissible maps can form a proper subset of the equivariants, as the following example shows.

**Example 3.1.** Figure 3 shows the \( \mathsf{D}_5 \)-network \( \mathcal{G}_{\mathsf{D}_5} \), where \( \mathsf{D}_5 \) acts naturally on the vertices \( \{1, 2, 3, 4, 5\} \) of the regular 5-gon.

Because the symmetry groups and groupoids discussed in this paper act by permutations, the structure of admissible or equivariant maps can be reduced to that of **invariant** functions for certain group actions. Differences in these invariant functions naturally imply differences between admissible and equivariant maps.

In the groupoid case, the \( \mathsf{B}_G \)-admissible maps \( f \) are determined by their components \( f_c \) where \( c \) runs through a set of representatives for input equivalence, by Stewart et al. [2003, Proposition 4.6]. Then \( f_c \) must be **invariant** under the **vertex group** \( B(c, c) \), and the components \( f_d \) for \( d \sim_I c \) are determined by pullback: \( f_d(x_{I(d)}) = f_c(\beta^* x_{I(c)}) \) for some \( \beta \) in \( B(c, d) \). For the network \( \mathcal{G}_{\mathsf{D}_5} \), we have \( B(c, c) \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \).

Similarly, the \( \mathsf{D}_5 \)-equivariant maps are determined by their components \( f_c \) where \( c \) runs through a set of representatives for \( \mathsf{D}_5 \)-orbits. Then \( f_c \) must be **invariant** under the stabilizer \( \Sigma_c = \{ \sigma \in \mathsf{D}_5 : \sigma c = c \} \), and the components \( f_d \) for \( d = \gamma c \), where \( \gamma \) is in \( \mathsf{D}_5 \), are determined by pullback. Now \( \Sigma_c \cong \mathbb{Z}_2 \), which is different from \( B(c, c) \).

Here the \( \Sigma_1 \)-invariant functions include the quadratic \( x_1 x_3 + x_2 x_4 \). This determines (via pullback) a \( \mathsf{D}_5 \)-equivariant map

\[
\begin{align*}
f(x) &= (x_1 x_3 + x_2 x_4, x_2 x_4 + x_3 x_0 + x_4 x_1, x_4 x_1 + x_0 x_2, x_0 x_2 + x_1 x_3)
\end{align*}
\]

but this is not admissible. The admissible maps must be symmetric in the pairs \( (x_1, x_4) \) and \( (x_2, x_3) \) separately (so, for instance, a suitable first component could be \( x_1 x_2 + x_1 x_3 + x_2 x_4 + x_3 x_4 \)). The difference here occurs because \( \Sigma_1 \) is a proper subgroup of \( B(1, 1) \).

Despite this difference, all balanced colorings of \( \mathcal{G}_{\mathsf{D}_5} \), indeed of \( \mathcal{G}_{\mathsf{D}_N} \), are fixed-point colorings: see Corollary 4.7.

### 3.2. Automorphism group

It is **not** true that in a group network every balanced polydiagonal is the fixed-point space of a subgroup.
of that group. A simple example occurs for the alternating group $A_4$ acting on $\{1,2,3,4\}$ in its natural permutation action, Fig. 4.

This action can be visualized as the group of rotations of a regular tetrahedron with vertices labeled 1, 2, 3, 4. The polydiagonal $\Delta_{\infty} = \{x, x, z, w\}$ for which $1 \triangleright \triangleright 2$ is easily seen to be balanced. However, it is not the fixed-point space of any subgroup of $A_4$. The only rotations of the tetrahedron that fix the set $\{1,2\}$ are the identity and the rotation $(12)(34)$ of order 2. The fixed-point spaces related to these rotations are $\{(x,y,z,w)\}$ and $\{(x,x,z,z)\}$. However, $\text{Aut}(G_{A_4}) = S_4$, and $\triangleright \triangleright$ is a fixed-point coloring derived from a subgroup of $S_4$ that is not contained in $A_4$, namely the $Z_2$ subgroup generated by reflection $(12)$.

Similar examples can be obtained when $\Gamma$ is the group of rotations of the cube (or any other regular solid) acting on the set of vertices. Now $\text{Aut}(G_{\Gamma})$ includes reflectional symmetries, with new possibilities for fixed-point spaces.

4. Balanced Extension Property

We now seek sufficient conditions for every balanced coloring of a $\Gamma$-network to be a fixed-point coloring. Some simple examples suggest the following approach.

Let $\mathcal{G}_\Gamma$ be the group network with cells $C = \{1, \ldots, N\}$ determined by a permutation group $\Gamma \subseteq S_N$. Let $K$ be an arrow-type of $\mathcal{G}_\Gamma$, that is, a $\sim_\Gamma$-equivalence class of arrows. Define $\mathcal{G}_\Gamma^K$ to be the subnetwork of $\mathcal{G}_\Gamma$ that contains all the cells in $C$, but only the arrows in $K$. Figure 5 illustrates this construction for the group network in Fig. 2.

If $\triangleright \triangleright$ is any equivalence relation on $C$, considered as the set of cells of $\mathcal{G}_\Gamma$, then $\triangleright \triangleright$ can also be considered as an equivalence relation on the cells of $\mathcal{G}_\Gamma^K$, since this network has the same cells as $\mathcal{G}_\Gamma$. For clarity, we denote the resulting equivalence relation by $\triangleright \triangleright^K$. We then have:

**Lemma 4.1.** An equivalence relation $\triangleright \triangleright$ on $C$ is balanced for $\mathcal{G}_\Gamma$ if and only if $\triangleright \triangleright^K$ is balanced for $\mathcal{G}_\Gamma^K$ for all arrow-types $K$.

**Proof.** This is immediate because by definition input isomorphisms preserve arrow-type. 

The idea here is that the network $\mathcal{G}_\Gamma^K$ usually has fewer arrows than $\mathcal{G}_\Gamma$ and this reduces the combinatorial possibilities that must be analyzed.

As usual let

$$\mathcal{B}_{\mathcal{G}_\Gamma} = \bigcup_{c,d \in C} B(c,d)$$

be the symmetry groupoid of $\mathcal{G}_\Gamma$. Similarly, define

$$\mathcal{B}_{\mathcal{G}_\Gamma^K} = \bigcup_{c,d \in C} B^K(c,d)$$

![Fig. 5. The group networks $\mathcal{G}_{D_4}^K$. Here $K$ comprises, respectively, the nearest, next-nearest, and third nearest neighbors.](image-url)
to be the symmetry groupoid of $G^K$, where $K$ is a fixed arrow-type.

**Definition 4.2.** Let $\mathcal{C}$ be a finite set, and suppose that $\bowtie$ is an equivalence relation on $\mathcal{C}$. Let $C_1, C_2$ be subsets of $\mathcal{C}$. A bijection $\beta : C_1 \to C_2$ is $\bowtie$-preserving if

$$\beta(i) \bowtie i \quad \forall i \in C_1$$

Group networks have no multiple arrows, so we may identify the input set $I(c)$, normally defined as the set of input arrows, with the set of “input cells” $T(I(c)) \cup \{c\}$. Moreover, any automorphism $\omega$ is determined by the action of the component $\omega_{\mathcal{C}}$ on $\mathcal{C}$. For the remainder of this section, we make this identification. We may then state:

**Definition 4.3.** A permutation group $\Gamma$ has the balanced extension property if, for every balanced equivalence relation $\bowtie$ on $G^K$, there exists an arrow-type $K$ (which may depend on $\bowtie$) such that for all $c, d \in C$ every $\bowtie$-preserving input bijection $\beta \in B^K(c, d)$ is the restriction to $I(c)$ of a $\bowtie$-preserving automorphism $\tilde{\beta}$ of $G^K$.

The main result of this section is:

**Theorem 4.4.** If $\Gamma$ has the balanced extension property then every balanced coloring of $G^K$ is a fixed-point coloring.

**Proof.** Let $\bowtie$ be a balanced equivalence relation on $G^K$ and choose an arrow-type $K$ satisfying the conditions of Definition 4.3. Then $\bowtie^K$ is a balanced equivalence relation on $G^K$ by Lemma 4.1. Define

$$H = \{ \tilde{\beta} : \beta \in B^0_{\mathcal{C}}, \text{ and } \beta \text{ preserves } \bowtie \}$$

where angle-brackets indicate the subgroup generated by their contents. Clearly $H$ is a subgroup of Aut($G^K$) since by definition each $\tilde{\beta} \in$ Aut($G^K$).

We claim that

$$\Delta_{\bowtie} = \text{Fix}(H)$$

which will complete the proof. Equivalently, we must prove that for all $c, d \in C$ we have $c \bowtie d$ if and only if $d = \omega(c)$ for some $\omega \in H$.

If $d = \omega(c)$ then

$$\omega|_{I(c)} : I(c) \to I(d)$$

since $\omega$ is an automorphism of $G^K$. Since by definition $\omega$ is $\bowtie$-preserving, it follows that $c \bowtie d$.

Conversely, suppose that $c \bowtie d$. Because $\bowtie$ is balanced, there exists a $\bowtie$-preserving input bijection $\beta : I(c) \to I(d)$.

Then $\beta \in H$ and $\beta(c) = \beta(d)$. Take $\omega = \beta$.

**Remark 4.5.** It is enough to verify the existence of $\tilde{\beta}$ for a set of generators $\beta$ of the symmetry groupoid $B^0_{\mathcal{C}}$.

To show that Theorem 4.4 is not vacuous, we prove:

**Lemma 4.6.** The dihedral group $D_N$, acting on $\mathcal{C} = \{1, \ldots, N\}$ as symmetries of the regular $N$-gon, has the balanced extension property.

**Proof.** Let $G = G_{D_N}$ and let $\bowtie$ be a balanced equivalence relation on $\mathcal{C}$. The cases $N = 1, 2$ are trivial so we may assume $N > 2$. Take $K$ to be the set of nearest-neighbor arrows. Then we may identify the input set $I(i)$ of cell $i$ with the triple $(i; i - 1, i + 1)$ where $i$ is the base point. Cell $i$ receives exactly two input arrows: one from cell $i - 1$ and one from cell $i + 1$.

Consider a $\bowtie$-preserving input bijection $\beta \in B^K(c, d)$ for some $c, d$. We wish to construct a suitable $\bowtie$-preserving extension $\tilde{\beta}$. We distinguish two cases: $c = d$ and $c \neq d$.

If $c = d$ then $\beta \in B^K(c, c)$, and we may (by rotating the $N$-gon) assume without loss of generality that $c = 0$. Now $I(c) = (0; -1, 1)$. There are two possibilities for $\beta$. If

$$\beta(0) = -1 \quad \beta(1) = 1$$

then we take $\tilde{\beta}$ to be the identity on $\mathcal{C}$. Alternatively, if

$$\beta(0) = 1 \quad \beta(1) = -1$$

then we take $\tilde{\beta}$ to be the reflection

$$\tilde{\beta}(i) = -i \quad \forall i \in \mathcal{C}$$

In this case we must prove that $\tilde{\beta}$ is $\bowtie$-preserving.

Because $\beta$ is $\bowtie$-preserving, $-1 \bowtie 1$. Now $I(-1) = (1; -2, 0)$ and $I(1) = (1; 0, 2)$. Since $\bowtie$ is balanced, $-2 \bowtie 2$. Repeat the argument for $I(-2)$ and $I(2)$, to deduce that $-3 \bowtie 3$, and proceed inductively. This proves that $-i \bowtie i$ for all cells $i$, so $\beta$ is $\bowtie$-preserving as required.

The remaining case occurs when $c \neq d$. By rotating the ring if necessary we may assume that $c = 0$. Suppose that $\beta(0) = d$ where $\beta$ is $\bowtie$-preserving. Then $d \bowtie 0$. We distinguish three subcases, illustrated in Fig. 6. (Here cells drawn as distinct may be the same for small $N$, and colors drawn as distinct may be identical. Neither feature
Fig. 6. Three cases for a balanced coloring on $\mathcal{G}^K_{D_N}$.

(a) If $-1 \preceq 1$ then we define
\[ \tilde{\beta}(i) = d + i \]

(b) If $-1 \not\preceq 1$ and
\[ \beta(-1) = d - 1 \quad \beta(1) = d + 1 \]
then we again define
\[ \tilde{\beta}(i) = d + i \]

(b) If $-1 \not\preceq 1$ and
\[ \beta(-1) = d + 1 \quad \beta(1) = d - 1 \]
then we define
\[ \tilde{\beta}(i) = d - i \]

In all three cases an easy inductive argument, similar to the one given for $B^K(0,0)$, shows that $\tilde{\beta}$ is $\bowtie$-preserving. ■

**Corollary 4.7.** Every balanced coloring of $\mathcal{G}_{D_N}$ is a fixed-point coloring.

Note that in this case, $\text{Aut}(\mathcal{G}_{D_N}) \cong D_N$.

The case of a cyclic group is similar but simpler. It also follows from a simple consequence of Theorem 4.1 of [Antoneli & Stewart, 2006]:

**Proposition 4.8.** Let $\mathcal{G}$ be a group network and suppose that
\[ \text{Aut}(\mathcal{G})_c = B(c,c) \tag{1} \]
for all $c \in C$. Then every flow-invariant subspace for $\mathcal{G}$ is the fixed-point space of an isotropy subgroup of $\text{Aut}(\mathcal{G})$.

**Proof.** Denote by $\mathcal{F}_X(\mathcal{G})$ the space of $\mathcal{B}_G$-admissible maps and by $\tilde{\mathcal{E}}_X(\text{Aut}(\mathcal{G}))$ the space of $\text{Aut}(\mathcal{G})$-equivariant maps. Then
\[ \mathcal{F}_X(\mathcal{G}) \subseteq \tilde{\mathcal{E}}_X(\text{Aut}(\mathcal{G})) \]

Since $\mathcal{G}$ is an $\text{Aut}(\mathcal{G})$-network, the orbits of $\text{Aut}(\mathcal{G})$ on $C$ are exactly the equivalence classes for the relation of input equivalence. By Proposition 4.6 of [Stewart et al., 2003], the $\mathcal{B}_G$-admissible maps are determined (via pullback) by their components $f_c$ where $c$ runs through a set of representatives for input equivalence, and the functions $f_c$ are $B(c,c)$-invariant. Similarly, the $\text{Aut}(\mathcal{G})$-equivariant maps are determined (via pullback) by their components $f_c$ where $c$ runs through a set of representatives for orbits and the functions $f_c$ are $\text{Aut}(\mathcal{G})_c$-invariant. Finally, (1) implies that the set of $\text{Aut}(\mathcal{G})_c$-invariant functions and the set of $B(c,c)$-invariant functions are equal. Therefore
\[ \mathcal{F}_X(\mathcal{G}) = \tilde{\mathcal{E}}_X(\text{Aut}(\mathcal{G})) \]

Theorem 4.1 of [Antoneli & Stewart, 2006] completes the proof. ■

**Corollary 4.9.** Let $\mathbb{Z}_N$ be the cyclic group of order $N$ acting on $C = \{1, \ldots, N\}$ with generator the $N$-cycle $(12, \ldots, N)$. Let $\mathcal{G}$ be the corresponding $\mathbb{Z}_N$-network. Then every balanced polydiagonal on $\mathcal{G}$ is the fixed-point space of a subgroup of $\mathbb{Z}_N$.

**Proof.** For this group, $B(c,c)$ and $\text{Aut}(\mathcal{G})_c$ both consist solely of the identity permutation. ■

**Remark 4.10**

(1) If $\Gamma$ acts transitively on $C$ then it is enough to check condition (1) for one cell.

(2) Example 3.1 shows that condition (1) is not necessary for the conclusion to hold, so Proposition 4.8 is not best possible.
(3) On the other hand, the lattice example in Sec. 6 and its mod-8 reduction show that when the difference between $\text{Aut}(\mathcal{G})_c$ and $B(c,c)$ is big enough then there can be exotic flow-invariant subspaces.

Theorem 4.4 applies to other groups. Let $O \oplus \mathbb{Z}_2^c$ be the symmetry group of a cube, acting by permutation of the eight vertices, so that $C = \{1, \ldots, 8\}$. Here $O$ is the rotation group and $\mathbb{Z}_2^c$ is generated by inversion through the origin.

Proposition 4.10. The group $O \oplus \mathbb{Z}_2^c$ has the balanced extension property.

Proof. The Proposition follows from a routine but tedious enumeration of combinatorial possibilities. ■

Note that $O$ alone does not have the balanced extension property, much as for the tetrahedral rotation group $A_4$.

5. The Simple Group of Order 168

We now consider an extreme instance of the possibility that $\text{Aut}(\mathcal{G})_\Gamma \supseteq \Gamma$. In this example, $\Gamma$ is the simple group of order 168 acting on the 7-point projective plane, and every coloring of $C$ is balanced.

First, recall [Neumann et al., 1994] that a permutation group $\Gamma$ on a set $C$ is doubly transitive if for any $(a, b), (c, d) \in C \times C$ there exists $\gamma \in \Gamma$ such that $\gamma a = c, \gamma b = d$. We then have:

Proposition 5.1. Suppose that $\Gamma$ is doubly transitive on $C$. Then every equivalence relation on $C$ is balanced.

Proof. The $\Gamma$-network $\mathcal{G}$ is all-to-all coupled by definition. All arrows are equivalent by double-transitivity. Therefore $\mathcal{G}$ is a complete graph, so $\text{Aut}(\mathcal{G}_\Gamma) = S_7$. But every partition of $C$ is the fixed-point space of a suitable subgroup of $S_7$, see [Golubitsky & Stewart, 2002, Example 1.12]. ■

The best known doubly transitive groups other than $S_N$ are the alternating groups $A_N$. The alternating group $A_N$ is normal in $S_N$ with quotient $\mathbb{Z}_2$. The examples related to regular solids in Sec. 2 also have $\Gamma$ normal in $\text{Aut}(\mathcal{G}_\Gamma)$ with quotient $\mathbb{Z}_2$. It might be hoped that something similar holds in general, but we now show that this hope is misplaced, by considering the simple group of order 168 acting on the 7-point projective plane.

It is well known that the smallest non-Abelian simple group is $A_5$ of order 60. It is also well known that the next smallest has order 168, and is the projective special linear group $\Gamma = \text{PSL}_3(2)$ over the Galois field $\text{GF}(2)$ with two elements [Neumann et al., 1994]. The latter is justly famous: for a history see [Gray, 1982]. It can be defined in many equivalent ways; in particular, it is the group of projective transformations of the projective plane $\Pi$ over $\text{GF}(2)$, also known as the Fano plane, which consists of seven points and seven lines. Each line contains three points, and each point lies on three lines. The configuration is shown in Fig. 7: the straight lines “wrap round” to form topological circles.

The group $\Gamma$ acts as permutations on the set of points, and can thus be embedded in $S_7$. This action is doubly-transitive because any pair of distinct points in a projective plane can be mapped to any other pair by a projective transformation.

By Proposition 5.1 $\text{Aut}(\mathcal{G}_\Gamma) = S_7$, which has order $7! = 5040$. Indeed, if we consider Fig. 7 as a graph, with the short line segments between points (including the omitted “wrapped round” segments) representing bidirectional arrows, the graph is all-to-all coupled (or “complete” [Tutte, 1984; Wilson, 1985] and all arrows are $\sim_E$-equivalent. Therefore the group network for $\Gamma$ is the complete graph on seven nodes and we can interpret Fig. 7 as that graph.

Since $\text{Aut}(\mathcal{G}_\Gamma) = S_7$, any equivalence relation on $\{1, 2, 3, 4, 5, 6, 7\}$ is balanced.

Fig. 7. The 7-point projective plane.
Here $\Gamma$ is not a normal subgroup of $\text{Aut}(\mathcal{G}_\Gamma)$: the only nontrivial normal subgroup of $S_7$ is $A_7$ of order 2520. The index of $\Gamma$ in $\text{Aut}(\mathcal{G}_\Gamma)$ is 30. So this example suggests that there is no group-theoretic characterization of $\text{Aut}(\mathcal{G}_\Gamma)$ in terms of normalizers or similar concepts.

It is instructive to determine the fixed-point spaces of subgroups of $\Gamma$, to see just how many extra balanced polydiagonals are afforded by fixed-point spaces of subgroups of $\text{Aut}(\mathcal{G}_\Gamma)$. Figure 8 shows the results, up to conjugacy in $\Gamma$. Here we use colors to determine an equivalence relation $\triangleright\triangleleft$ whose polydiagonal is the fixed-point space concerned. The main step in deriving this figure is to compute the partially ordered set of conjugacy classes of subgroups of $\Gamma$. (The subgroups of a given group form a lattice in the algebraic sense: a partially ordered set in which any two elements have a unique join and meet. See [Davey & Priestley, 1990, Chapter 2]. The conjugacy classes of subgroups are partially ordered by inclusion, in the sense that

$$H_1 \leq H_2 \iff \exists \gamma \in \Gamma : \gamma^{-1} H_1 \gamma \subseteq H_2$$

This relation need not define a lattice structure, but the word “lattice” is sometimes used in this context.) The subgroups of the simple group of order 168 are well known and go back to [Klein, 1879], but we sketch a proof in more modern terminology.

**Proposition 5.2.** The partially ordered set of conjugacy classes of subgroups of $\text{PSL}_3(2)$ is as shown in Fig. 9.

**Proof.** The proof of Proposition 5.2 is based on two main facts: the Sylow Theorems, see [Hall, 1959, Chapter 4] or [Neumann et al., 1994], and the classification of the maximal subgroups of $\text{PSL}_3(2)$, see [Conway et al., 1985]. We assume familiarity with the techniques and concepts used.

The order of $\Gamma$ is $168 = 2^3 \cdot 3 \cdot 7$, so its Sylow $p$-subgroups $S_p$ have orders 8, 3, 7 when $p = 2, 3, 7$ respectively.

There is an obvious conjugacy class of subgroups of order 24, namely, the symmetry groups $\Sigma_Q$ of quadrilaterals $Q$. A quadrilateral is a set of four distinct points, no three of which are collinear. $\Gamma$ acts transitively on the set of quadrilaterals, so all such subgroups are conjugate. For example, the set of points $Q = \{1, 2, 3, 4\}$ is a quadrilateral. Any

![Fig. 8. Fixed-point spaces of subgroups of $\text{PSL}_3(2)$](image-url)
permutation of this set extends uniquely to a projective transformation of $\Pi$ because each line in $\Pi$ meets the quadrilateral in exactly two points, and the third point lies on exactly two “diagonals” of $Q$. Therefore $\Sigma_Q \cong S_4$. (The action of $\Sigma_Q$ on the complement of $Q$, which is a line containing 3 points, exemplifies the unusual nature of $S_4$: there is a homomorphism $S_4 \rightarrow S_3$, with kernel the Klein 4-group $V$.) This type of $S_4$ subgroup is also called a line stabilizer because it fixes a line (setwise) [Conway et al., 1985].

There is a second, distinct conjugacy class of subgroups of order 24, namely, the symmetry groups $\Sigma_D$ of dual quadrilaterals $D$. A dual quadrilateral is a set of four distinct lines, no three of which are concurrent. The group $\Gamma$ acts transitively on the set of dual quadrilaterals, so all such subgroups are conjugate. Again, the groups $\Sigma_D$ are isomorphic to $S_4$ since they permute the four lines arbitrarily. This type of $S_4$ subgroup is also called a point stabilizer because it fixes a point [Conway et al., 1985].

We write $S_4 = \Sigma_Q$ and $S_4^* = \Sigma_D$. By Conway et al. [1985], both of these conjugacy classes consist of maximal subgroups. There is a third conjugacy class of maximal subgroups, of order 21. By Sylow theory this class, which (along with any of its members) we denote by $G_{21}$, consists of the normalizers $N_\Gamma(S_7)$ of the Sylow 7-subgroups. These groups are isomorphic to the unique non-Abelian group of order 21, which is a semidirect product of a normal $Z_7$ subgroup by a $Z_3$ subgroup. These two groups are the only nontrivial proper subgroups of $G_{21}$.

Now let $H$ be a subgroup of $\Gamma$: we wish to classify the possibilities for $H$ up to conjugacy in $\Gamma$.

Suppose first that $|H|$ is divisible by 7. Then either $H = \Gamma$ or (up to conjugacy) $H \subseteq G_{21}$. In the latter case, $H = Z_7$, a Sylow 7-subgroup, or its normalizer $G_{21}$.

Otherwise $H$ must be conjugate to a subgroup of $S_4$ or of $S_4^*$, so $|H|$ divides 24.

The subgroups of $S_4$, and their conjugacy classes, are well known, and the partially ordered set of conjugacy classes (by elements of $S_4$) of subgroups of $S_4$ is shown in Fig. 10. Since $S_4^* \cong S_4$ there is a corresponding partially ordered set of conjugacy classes (by elements of $S_4^*$) of subgroups of $S_4^*$. We denote these by appending an asterisk to the corresponding subgroups of $S_4$. Here $V$ is the usual Klein 4-group consisting of the identity and the elements $(12)(34)$, $(13)(24)$, $(14)(23)$ and $V'$ is the other subgroup of $D_4$ that is isomorphic to $V$, consisting of the identity and the elements $(12)$, $(34)$, $(12)(34)$. The order-2 subgroups $Z_2^V$ are $Z_2^T$ generated by $(12)(34)$ and (23) respectively.

The main remaining issue is the possibility that some of these conjugacy classes fuse in $\Gamma$, that is,
become conjugate by some element of $\Gamma$. See [Gorenstein, 1968, Chapter 7].

The group $S_4$ is the normalizer in $\Gamma$ of each of its subgroups $A^*_1, V$. (These groups are normal in $S_4$, but the normalizer cannot be larger because $S_4$ is maximal and $\Gamma$ is simple.) Similarly $S_4^*$ is the normalizer in $\Gamma$ of each of its subgroups $A^*_1, V^*$. It follows that $A_4$ is not conjugate in $\Gamma$ to $A_4^*$, for if it were, then the normalizer of $A_4$ would be conjugate to that of $A_4^*$, so $S_4$ and $S_4^*$ would be conjugate, which is false. Similarly, $V$ and $V^*$ are not conjugate in $\Gamma$.

On the other hand, $Z_3$ is conjugate to $Z_3^*$ since these are both Sylow 3-subgroups of $\Gamma$. Their respective normalizers $S_3$ and $S_3^*$ are also conjugate. (The $S_3$ symmetry is visible in the “equilateral triangle” shape of Fig. 7.)

Similarly, $D_4$ is conjugate to $D_4^*$ since these are both Sylow 2-subgroups of $\Gamma$. Therefore every subgroup of $D_4$ must be conjugate in $\Gamma$ to some subgroup of $D_4^*$; the question is, which? It is easy to answer this question when the subgroup is unique up to isomorphism, so we know that $Z_4$ is conjugate in $\Gamma$ to $Z_4^*$ since these are the unique cyclic subgroups of order 4 in their respective $D_4, D_4^*$.

Since $V^*$ is not conjugate to $V$, so it must be conjugate to the only other subgroup of $D_4$ that is isomorphic to $V$, namely $V'$. We therefore employ the notation $V^*$ instead of $V'$.

It is also possible that distinct conjugacy classes of subgroups of $S_4$ may fuse in $\Gamma$, that is, become conjugate in $\Gamma$. However, $V$ and $V^*$ are not conjugate in $\Gamma$, and since conjugate subgroups are isomorphic, the only case to consider is that of subgroups of order 2. We show [Klein, 1879] that the two distinct conjugacy classes of order-2 subgroups in $S_4$ fuse into a single class in $\Gamma$, which therefore also fuses with the two distinct conjugacy classes of such elements in $S_4^*$. That is, $\Gamma$ has a single conjugacy class of involutions (elements of order 2). By Sylow theory, it is sufficient to prove that the two distinct conjugacy classes (by elements of $S_4$) of $Z_2$ subgroups of $S_4$ are conjugate in $\Gamma$. These groups are denoted $Z_2^*$ and $Z_2^*$ in Fig. 10.

In $S_4$, the group $Z_2^*$ is generated by $\sigma = (12)(34)$ and $Z_2^*$ is generated by $\tau = (23)$, say. However, the action of $S_4$ induces an action on the line $\{5, 6, 7\}$ that is stabilized by $S_4$. The geometry of the Fano plane $\Pi$ shows that $\sigma$ acts trivially on this line, but $\tau$ interchanges 5 and 6 while fixing 7. Therefore in $\Gamma$ we have $\sigma = (12)(34)$ and $\tau = (23)(56)$. Since both $\{1, 2, 3, 4\}$ and $\{2, 3, 5, 6\}$ are quadrilaterals, there exists an element $\gamma \in \Gamma$ such that $\gamma(1) = 2, \gamma(2) = 3, \gamma(3) = 5, \gamma(4) = 6$. Therefore $\sigma$ and $\tau$ are conjugate in $\Gamma$, and up to conjugacy they generate a group that we denote by $Z_2$.

This completes the classification of conjugacy classes of subgroups of $\text{PSL}_3(2)$.  

It is now straightforward to compute the fixed-point spaces of these subgroups. The Sylow 7-subgroup $Z_7$ is cyclic of order 7, and hence contains a 7-cycle in $S_7$. Therefore the corresponding polydiagonal is the full diagonal $\{(x, x, x, x, x, x, x)\}$, independently of $H$. The same holds for $G_{21}$. Therefore we may assume that the order of $H$ divides 24. It therefore suffices to enumerate the fixed-point spaces subgroups of $S_4$ and $S_4^*$. Removing duplicates, we obtain Fig. 8. There are nine distinct (group orbits of) fixed-point spaces.

In contrast, $S_7$ symmetry implies the existence of $p(7) = 15$ group orbits of fixed-point spaces, where $p(n)$ is the number of partitions of $n$. This follows from the characterization of isotropy subgroups of $S_N$ in terms of partitions of $N$ in [Golubitsky & Stewart, 2002, Example 1.12]. Table 1 lists the 15 partitions, and shows which of these correspond to the nine fixed-point spaces of $\text{PSL}_3(2)$. Six partitions do not occur in this manner.

<table>
<thead>
<tr>
<th>Partition</th>
<th>$\text{PSL}_3(2)$</th>
<th>Subgroup</th>
</tr>
</thead>
<tbody>
<tr>
<td>7</td>
<td>yes</td>
<td>$S_3^*$</td>
</tr>
<tr>
<td>6 + 1</td>
<td>yes</td>
<td>$S_3^*$</td>
</tr>
<tr>
<td>5 + 2</td>
<td>no</td>
<td>$S_4$</td>
</tr>
<tr>
<td>5 + 1 + 1</td>
<td>no</td>
<td>$S_4$</td>
</tr>
<tr>
<td>4 + 3</td>
<td>yes</td>
<td>$D_4$</td>
</tr>
<tr>
<td>4 + 2 + 1</td>
<td>yes</td>
<td>$V$</td>
</tr>
<tr>
<td>4 + 1 + 1 + 1</td>
<td>yes</td>
<td>$Z_3$</td>
</tr>
<tr>
<td>3 + 3 + 1</td>
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<td>$Z_3$</td>
</tr>
<tr>
<td>3 + 2 + 2</td>
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</tr>
<tr>
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<td>no</td>
<td>$Z_3$</td>
</tr>
<tr>
<td>3 + 1 + 1 + 1 + 1</td>
<td>no</td>
<td>$Z_2$</td>
</tr>
<tr>
<td>2 + 2 + 2 + 1</td>
<td>yes</td>
<td>$V^*$</td>
</tr>
<tr>
<td>2 + 2 + 1 + 1 + 1</td>
<td>yes</td>
<td>$Z_2$</td>
</tr>
<tr>
<td>2 + 1 + 1 + 1 + 1 + 1</td>
<td>yes</td>
<td>$1$</td>
</tr>
<tr>
<td>1 + 1 + 1 + 1 + 1 + 1</td>
<td>yes</td>
<td>$1$</td>
</tr>
</tbody>
</table>
4 + 3 obtained from $\text{PSL}_3(2)$, the class with three cells forms a line in the projective plane, whereas its $S_7$-orbit includes colorings that do not obey that constraint.

6. Lattice Example

We end the paper on an enigmatic note, by showing that balanced colorings of group networks need not be fixed-point colorings. Our example, which is instructive in its own right, is taken from the classification by Wang and Golubitsky [2005], who classify all balanced 2-colorings of the square lattice $L = \mathbb{Z}^2$ with nearest-neighbor coupling. Several of their examples are not fixed-point colorings. However, in almost all of these examples, second-nearest neighbor coupling destroys the balance property.

We consider the sole exception, Fig. 6(a) of [Wang & Golubitsky, 2005], shown in Fig. 11. Here we refer to the dark cells as “red” and the light cells as “blue.” This coloring, which we shall denote by $P$, is obtained by tiling the planar square lattice $\mathbb{Z}^2$ with copies of Fig. 1. We will show that $P$ is balanced for coupling of any (finite) range, on the natural assumption that arrows that lie in the same orbit of the automorphism group $\Gamma$ of the lattice must be edge-equivalent. This group is the semidirect product $(\mathbb{Z} \oplus \mathbb{Z}) \rtimes D_4$, where $\mathbb{Z} \oplus \mathbb{Z}$ acts by translations and the holohedry $D_4$ consists of rotations and reflections that fix the origin.

The coloring $P$ is doubly periodic, with period 8 in each direction. This double periodicity implies that the lattice $L$ can be reduced modulo 8 to give a 64-cell network whose cells correspond to points on the discrete torus $L_8 = \mathbb{Z}_8^2$. The automorphism group of this network is easily seen to be $\Gamma_8 = (\mathbb{Z}_8 \oplus \mathbb{Z}_8) + D_4$ with the action induced from that of $\Gamma = (\mathbb{Z} \oplus \mathbb{Z}) + D_4$. The coloring $P$ induces a coloring $P^8$ on $L_8$, as shown in Fig. 1 (with periodic boundary conditions). Our results imply that $P^8$ is a balanced 2-coloring of the $\Gamma$-network on $L_8$, but is not a fixed-point coloring.

It is proved in [Antoneli et al., 2005] that balanced colorings of $d$-dimensional lattices must be spatially $d$-fold periodic, provided the couplings are sufficiently long-range. This example shows that despite that periodicity, colorings that are balanced for any range of coupling need not be fixed-point colorings. Aside from trivial variants (direct products of colorings in higher-dimensional lattices) this is currently the only known lattice coloring with such a property.

6.1. Towards an explanation

There is a (not very satisfactory) group-theoretic rationale for the colorings $P$ and $P^8$ being balanced, even though they are not fixed-point colorings. We mention it here in the hope that it may lead to an improved understanding of exotic balanced colorings.

We discuss this idea for $P$. A similar discussion is valid for $P^8$, but it is simpler to work in $L$ and to deduce similar statements for $L_8$ by reducing modulo 8. Note that $L_8$ is finite, whereas the group network on $L$ is not even locally finite. For this reason we work with individual group orbits of arrows in the group network on $L$. We begin by describing these group orbits.

Take coordinates $(a, b)$ on $\mathbb{Z}^2$. Then the holohedry $D_4$ acts on $(a, b)$ by mapping it to any of $(\pm a, \pm b), (\pm b, \pm a)$ with all eight choices of signs. Denote the $D_4$-orbit of $(a, b)$ by $O_{(a,b)}(0,0)$. Each such orbit has a unique canonical element $(a, b)$ for which $0 \leq a \leq b$. $O_{(a,b)}(0,0)$ contains either one, four or eight elements depending on $(a, b)$. For canonical elements, $|O_{(a,b)}(0,0)| = 1$ if and only if $(a, b) = (0, 0)$. If $a = 0$ but $b \neq 0$, then

Fig. 11. A balanced 2-coloring of the lattice $\mathbb{Z} \times \mathbb{Z}$ that does not correspond to a fixed-point space of a subgroup of the automorphism group. Dotted lines indicate axes; bold lines are (some of the) axes of reflectional symmetry.
\[ O_{(a,b)}(0,0) = 4. \] If \( 0 < a = b \) then \( |O_{(a,b)}(0,0)| = 4. \) Otherwise \( |O_{(a,b)}(0,0)| = 8. \) See Fig. 12.

In the \( \Gamma \)-network on \( L \) there is a single input arrow to cell \((c, d)\) from any distinct cell. These arrows split into orbits for the conjugate \( D_4^{(c,d)} \) of \( D_4 \) whose elements fix \((c, d)\). These orbits are translates of the sets \( O_{(a,b)}(0,0) \), and take the form

\[ O_{(a,b)}(c, d) = (c, d) + O_{(a,b)}(0,0) \]

Being translates, they again have one, four or eight elements with the same conditions on \((a, b)\). The picture is the same as Fig. 12 where now the central cell lies at \((c, d)\).

It is worth pointing out that under reduction modulo 8, it is also possible for a \( D_4 \)-orbit to contain two elements. In canonical form, this happens for \((a, b) \equiv (0, 4) \pmod{8}\).

We claim that the coloring \( P \) is balanced, on the assumption that the input set of any cell is a union of \( D_4 \)-orbits (with origin translated to that cell). Arrow-types are assumed constant on any such \( D_4 \)-orbit. By red/blue symmetry, it suffices to prove that for each red head cell, the number of red tail cells (hence also blue) is the same for each such orbit on the input set of the head cell.

It is possible to verify this statement by reducing the coloring \( P \) modulo 8 and counting the red cells for each reduced orbit, because the reduced coloring is finite. The results can then be lifted back to the coloring \( P \) on the infinite lattice \( L \). However, this calculation provides no insight into why \( P \) is balanced. We therefore proceed in a more conceptual manner. We decompose \( P \) into two disjoint subcolorings, supported on lattices that are rotations of \( L \) through \( \pi/4 \) dilated by a factor \( \sqrt{2} \). One of these lattices is also translated by \((1, 0)\). Each subcoloring is balanced, for group-theoretic reasons: they are fixed-point colorings. Moreover, the subcolorings are related in such a manner that their union is also balanced. (There is a minor technical issue: for one of the colorings the origin of coordinates does not lie on a lattice point. This is dealt with by proving a slightly stronger property than balance, also a consequence of group-theoretic features.)

Thus \( P \) can be explained in terms of group-theoretic features, even though it is not a fixed-point coloring. Possibly this explanation could be generalized, to produce more such examples.

We begin by splitting \( L \) into a sublattice \( L_1 \) and its translate \( L_2 \), defined as follows:

\[ L_1 = \{(a, b) \in L : a \equiv b \pmod{2}\} \]
\[ L_2 = \{(a, b) \in L : a \equiv b + 1 \pmod{2}\} \]

Clearly

\[ L = L_1 \cup L_2 \]

where \( \cup \) indicates disjoint union. Clearly \( L_2 \) is the translate \( L_1 + (1, 0) \).

Similarly, the 2-coloring \( P \) decomposes into \( P_1, \) the restriction of \( P \) to \( L_1, \) and \( P_2, \) the restriction of \( P \) to \( L_2. \) We read off the colorings \( P_1, P_2 \) from Fig. 11. Figure 13 shows the lattice \( L_1 \) and the coloring \( P_1. \) Similarly, Fig. 14 shows the translated lattice \( L_2 \) and the coloring \( P_2. \)

Note that (up to a translation of the origin and a rotation) the colorings \( P_1, P_2 \) are identical. (With the chosen drawings the colors must also be swapped, but this is an artifact of the chosen boundaries. On the infinite lattice a suitable choice of origin leaves the colors unchanged.)

It is obvious that \( P_1, \) hence also \( P_2, \) is a fixed-point coloring (determined by a subgroup of the automorphism groups of \( L_1 \) and \( L_2. \) In fact, it is the coloring in Fig. 2(d) of [Wang & Golubitsky, 2005]. Therefore both subcolorings are balanced.

In general, a disjoint union of balanced colorings need not be balanced, but in this case \( P_2 \) has
Fig. 13. (Left) The sublattice $L_1$. (Right) Its rotation through $\pi/4$ redrawn on the integer lattice. Dotted lines indicate positions of coordinate axes.

Fig. 14. (Left) Complement $L_2$ of the sublattice $L_1$. (Right) Its rotation through $\pi/4$ redrawn on the integer lattice. Dotted lines indicate positions of coordinate axes. Open dots indicate special points discussed in text.

a special feature that causes the union to be balanced. Note that the axes in Fig. 14 do not pass through the center of a square. Instead, they align with the edges of a square. This happens because the origin $(0,0)$ does not lie in $L_2$. We claim that every $D_4$-orbit of cells in $L_2$, centered at any corner point of any square, contains equal numbers of red and blue cells. To establish this claim, we first observe that up to the action of the isotropy group of $P_2$ on $L_2$, there are only two such points. They
correspond to the points \((0,0)\) and \((0,2)\) in \(L\), and are marked in Fig. 14(b) by open dots.

Rotation of \(P_2\) by \(\pi\) about the left dot interchanges red and blue squares, hence their numbers are equal in any \(D_4\)-orbit centered at that point. Similarly, rotation of \(P_2\) by \(\pi/2\) about the right dot interchanges red and blue squares, hence their numbers are equal in any \(D_4\)-orbit centered at that point.

The isotropy group \(\Sigma_P\) of \(P\) is generated by reflections in the solid lines marked on Fig. 11, rotation through \(\pi/2\) fixing the open \(4 \times 4\) red square adjacent to the origin setwise, and translation by \((4,4)\). The group \(\Sigma_P\) has six orbits on \(L\), marked \(A, B, C, D, E, F\) in Fig. 15. We have chosen these orbit representatives to lie in \(L_1\), for later convenience.

Three of these orbits consist red cells and three blue cells. There is an evident “color symmetry” defined by interchanging red and blue and translating by \((4,4)\). Therefore, in order to prove that \(P\) is balanced it suffices to show that every \(D_4\)-orbit \(X_A\) centered at \(A\) contains the same number of red cells (hence also of blue) as the corresponding orbits \(X_B, X_C\) centered at \(B\) or at \(C\). Call this property \((\ast)\).

There are now two cases to consider. Either \(X_A\) lie in \(L_1\), in which case so do \(X_B, X_C\), or \(X_A\) lie in \(L_2\), in which case so do \(X_B, X_C\).

If \(X_A, X_B, X_C\) lie in \(L_1\), then property \((\ast)\) is immediate because \(X_A, X_B, X_C\) are also \(D_4\)-orbits (centered respectively at \(A, B, C\)) in \(L_1\) and \(P_1\) is a fixed-point coloring, hence balanced.

On the other hand, if \(X_A, X_B, X_C\) lie in \(L_2\), property \((\ast)\) is immediate because \(X_A, X_B, X_C\) each contain the same number of red squares as they do blue ones. This completes the proof that \(P\) is balanced on any \(D_4\)-orbit of input arrows.

For the 64-cell network, reduction modulo 8 and a few simple observations immediately prove that \(P^8\) is balanced on the \(\Gamma\)-network \(Z^8\). However, it is easy to check that the automorphism group of this network is \(\Gamma\), which has six orbits on \(P^8\). Therefore \(P^8\) is not a fixed-point coloring.

Acknowledgments

The work of INS was supported in part by NSF Grant DMS-0244529 and a grant from EPSRC. It was mainly carried out at the University of Hong Kong, under the auspices of a Royal Society Kan Tong Po visiting professorship. We thank the Isaac Newton Institute, the University of Houston, and the University of Hong Kong for hospitality and additional financial support. We also thank Martin Golubitsky and Andreij Török for helpful discussions.

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