CHAPTER 2
FOURIER SERIES

In Chapter 1 we derived three problems concerning the expansion of functions in terms of sines and cosines. The most fundamental of these is the expansion of periodic functions, which is of importance not only for boundary value problems but for the analysis of any sort of periodic phenomena, and which has provided either direct or indirect inspiration for many of the developments of modern mathematical analysis. Most of this chapter is devoted to the study of periodic functions. Once they are understood, the other two expansion problems of §1.3 can be solved without difficulty, as we shall see in §2.4.

In many respects it is simpler and neater to work with the complex exponential function $e^{i\theta}$ instead of the trigonometric functions $\cos\theta$ and $\sin\theta$. We recall that these functions are related by the formulas

$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}, \qquad \sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i},$$
$$e^{i\theta} = \cos\theta + i\sin\theta.$$

The advantages of cosine and sine are that they are real-valued and are, respectively, even and odd; the advantages of the exponential are that its differentiation formula $(e^{i\theta})' = ie^{i\theta}$ and addition formula $e^{i(\theta+\phi)} = e^{i\theta}e^{i\phi}$ are simpler than the corresponding formulas for cosine and sine. Accordingly, it is worthwhile to be able to translate one formulation into the other without much effort; we urge the readers who have not yet acquired this facility to spend a little time doing so. A more complete list of the properties of exponential and trigonometric functions of complex variables will be found in Appendix 2.

2.1 The Fourier series of a periodic function

Suppose that $f(\theta)$ is a function defined on the real line such that $f(\theta+2\pi) = f(\theta)$ for all θ. Such functions are said to be **periodic with period 2π**, or **2π-periodic** for short. We shall assume that f is Riemann integrable on every bounded interval; this will be the case if f is bounded and is continuous except perhaps at finitely many points in each bounded interval. (We shall consider various other hypotheses on f in subsequent sections.) Since we shall be using the complex exponential function, we shall allow f to be complex-valued rather than merely real-valued. This bit of extra generality causes no additional difficulties and indeed simplifies some things; moreover, in more advanced work it is often crucial to use complex functions.

We wish to know if f can be expanded in a series

$$f(\theta) = \tfrac{1}{2}a_0 + \sum_{1}^{\infty}(a_n \cos n\theta + b_n \sin n\theta). \tag{2.1}$$

Here $\frac{1}{2}a_0$ is the coefficient of the constant function $1 = \cos 0\theta$, and the factor of $\frac{1}{2}$ is incorporated in it for reasons of later convenience (see the remark following equation (2.6)). There is no b_0 because $\sin 0\theta = 0$.

In view of the formulas $\cos n\theta = (e^{in\theta} + e^{-in\theta})/2$ and $\sin n\theta = (e^{in\theta} - e^{-in\theta})/2i$, (2.1) can be rewritten as

$$f(\theta) = \sum_{-\infty}^{\infty} c_n e^{in\theta} \tag{2.2}$$

where

$$c_0 = \tfrac{1}{2}a_0; \qquad c_n = \tfrac{1}{2}(a_n - ib_n) \text{ and } c_{-n} = \tfrac{1}{2}(a_n + ib_n) \text{ for } n = 1,2,3,\ldots \tag{2.3}$$

Alternatively, if we start out with (2.2), by using the formulas $e^{in\theta} = \cos n\theta + i \sin n\theta$, $\cos(-n)\theta = \cos n\theta$, and $\sin(-n)\theta = -\sin n\theta$, we can put it in the form (2.1) where

$$a_0 = 2c_0; \qquad a_n = c_n + c_{-n} \quad \text{and} \quad b_n = i(c_n - c_{-n}) \quad \text{for } n = 1,2,3,\ldots \tag{2.4}$$

In what follows we shall work primarily with (2.2), but we shall also show how to interpret the results in terms of (2.1).

As a first step towards analyzing general periodic functions in terms of trigonometric series, let us consider the following question. If we know to begin with that $f(\theta)$ has a series expansion of the form (2.2), how can the coefficients c_n be calculated in terms of f? The answer to this question is appealingly simple. Let us multiply both sides of (2.2) by $e^{-ik\theta}$ (k being an integer) and integrate from $-\pi$ to π. Taking on faith for the moment that it is permissible to integrate the series term by term, we obtain

$$\int_{-\pi}^{\pi} f(\theta)e^{-ik\theta}d\theta = \sum_{-\infty}^{\infty} c_n \int_{-\pi}^{\pi} e^{i(n-k)\theta}d\theta.$$

But

$$\int_{-\pi}^{\pi} e^{i(n-k)\theta}d\theta = \frac{1}{i(n-k)}e^{i(n-k)\theta}\Big|_{-\pi}^{\pi} = \frac{(-1)^{n-k} - (-1)^{n-k}}{i(n-k)} = 0 \qquad \text{if } n \neq k,$$

$$\int_{-\pi}^{\pi} e^{i(n-k)\theta}d\theta = \int_{-\pi}^{\pi} d\theta = 2\pi \qquad \text{if } n = k.$$

Hence the only term in the series that survives the integration is the term with $n = k$, and we obtain

$$\int_{-\pi}^{\pi} f(\theta)e^{-ik\theta}\,d\theta = 2\pi c_k.$$

In other words, relabeling the integer k as n, we have the desired formula for the coefficients c_n:

$$c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta)e^{-in\theta}\,d\theta. \tag{2.5}$$

It is now an easy matter to find the coefficients a_n and b_n for the series (2.1):

$$a_0 = 2c_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(\theta)d\theta,$$

and for $n = 1, 2, 3, \ldots,$

$$a_n = c_n + c_{-n} = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta)(e^{-in\theta} + e^{in\theta})\,d\theta = \frac{1}{\pi}\int_{-\pi}^{\pi} f(\theta)\cos n\theta\,d\theta,$$

$$b_n = i(c_n - c_{-n}) = \frac{i}{2\pi}\int_{-\pi}^{\pi} f(\theta)(e^{-in\theta} - e^{in\theta})\,d\theta = \frac{1}{\pi}\int_{-\pi}^{\pi} f(\theta)\sin n\theta\,d\theta;$$

that is,

$$\begin{aligned} a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(\theta)\cos n\theta\,d\theta \qquad (n \geq 0); \\ b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(\theta)\sin n\theta\,d\theta \qquad (n \geq 1). \end{aligned} \tag{2.6}$$

(Note that the formula for a_n here holds also for $n = 0$; this is the reason for the factor of $\frac{1}{2}$ in (2.1).)

To recapitulate: *if* f has a series expansion of the form (2.1) (or (2.2)), and *if* the series converges decently so that term-by-term integration is permissible, *then* the coefficients a_n and b_n [or c_n] are given by (2.6) [or (2.5)]. But now if f is *any* Riemann-integrable periodic function, the integrals in (2.5) and (2.6) make perfectly good sense, and we can use them to *define* the coefficients a_n, b_n, and c_n. We are now in a position to make a formal definition.

Definition. Suppose f is periodic with period 2π and integrable over $[-\pi, \pi]$. The numbers c_n defined by (2.5), or the numbers a_n and b_n defined by (2.6), are called the **Fourier coefficients** of f, and the corresponding series

$$\sum_{-\infty}^{\infty} c_n e^{in\theta} \quad \text{or} \quad \tfrac{1}{2}a_0 + \sum_{1}^{\infty}(a_n \cos n\theta + b_n \sin n\theta)$$

is called the **Fourier series** of f.

Instead of integrating from $-\pi$ to π in (2.5) and (2.6), one could equally well integrate over any interval of length 2π, for instance from 0 to 2π. The result will be the same since the integrands are all 2π-periodic. This is an instance of the following general fact, which is sufficiently useful to merit a special mention.

Lemma 2.1. *If F is periodic with period P, then $\int_a^{a+P} F(x)\,dx$ is independent of a.*

Proof: Let

$$g(a) = \int_a^{a+P} F(x)\,dx = \int_0^{a+P} F(x)\,dx - \int_0^a F(x)\,dx.$$

By the fundamental theorem of calculus, $g'(a) = F(a+P) - F(a)$, so by the periodicity of F, g' vanishes identically. Thus g is constant. ∎

Another useful observation in this context is that

$$\int_{-a}^{a} F(x)\,dx = \begin{cases} 2\int_0^a F(x)\,dx & \text{if } F \text{ is even,} \\ 0 & \text{if } F \text{ is odd.} \end{cases}$$

(Recall that F is **even** if $F(-x) = F(x)$ and **odd** if $F(-x) = -F(x)$.) Since $\cos n\theta$ is even and $\sin n\theta$ is odd, we have the following result.

Lemma 2.2. *With reference to the formulas (2.6),*

$$\text{if } f \text{ is even,} \qquad a_n = \frac{2}{\pi}\int_0^\pi f(\theta)\cos n\theta\,d\theta \quad \text{and} \quad b_n = 0;$$

$$\text{if } f \text{ is odd,} \qquad a_n = 0 \quad \text{and} \quad b_n = \frac{2}{\pi}\int_0^\pi f(\theta)\sin n\theta\,d\theta.$$

Whether the Fourier series of a 2π-periodic function f is written in the trigonometric form (2.1) or the exponential form (2.2), the constant term in the series is

$$c_0 = \tfrac{1}{2}a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta)\,d\theta,$$

which is nothing but the average or mean value of f on the interval $[-\pi, \pi]$. By Lemma 2.1, it is also the mean value of f on *any* interval of length 2π. This fact is very useful, and it may be more easily remembered than the integral formula; accordingly, we display it as a lemma.

Lemma 2.3. *The constant term in the Fourier series of a 2π-periodic function f is the mean value of f on an interval of length 2π.*

The preceding discussion shows that if we wish to find a trigonometric series that converges to a given periodic function f, the Fourier series of f is the only reasonable candidate; but we do not yet know whether it always does the job. Before tackling this general question, let us compute a couple of examples.

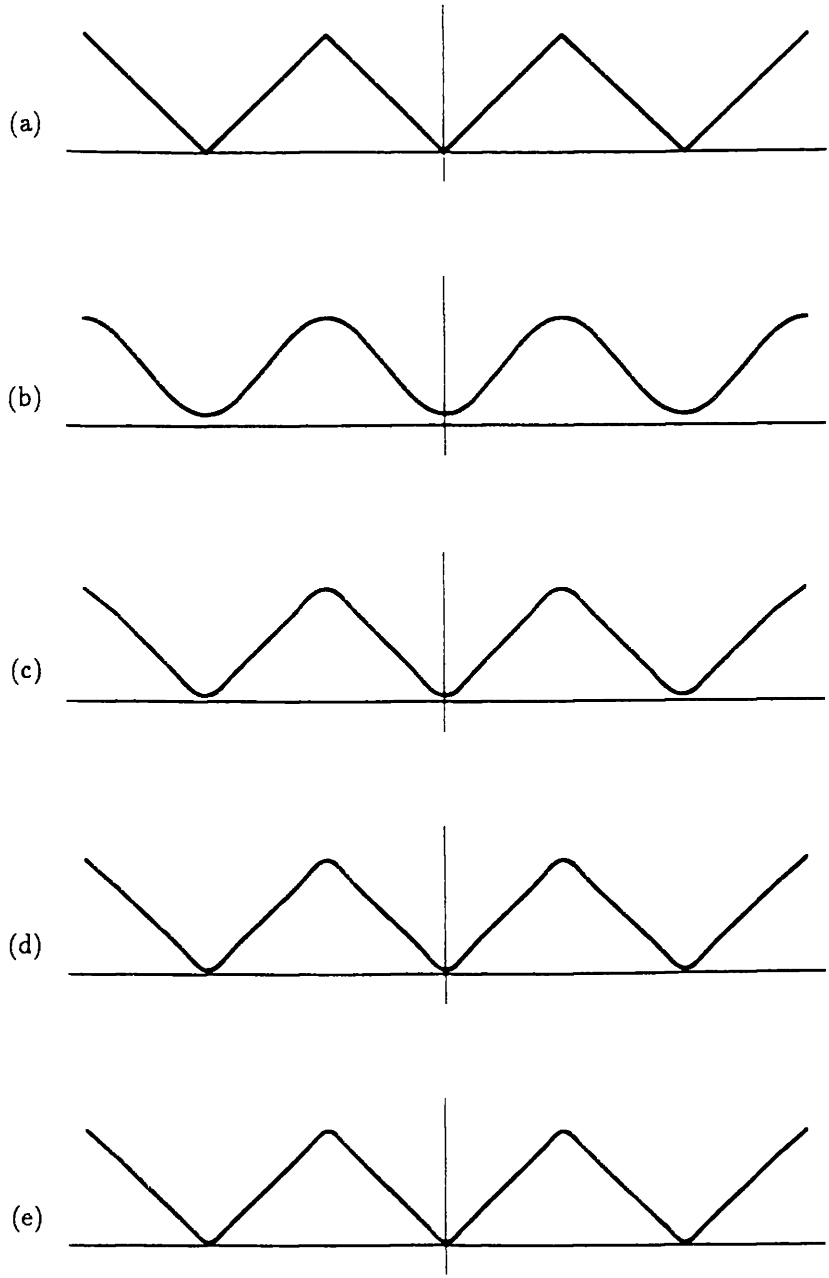


FIGURE 2.1. The triangle wave of Example 1 and some partial sums of its Fourier series: (a) the triangle wave, (b) S_1, (c) S_2, (d) S_3, and (e) S_4, where $S_K = \frac{1}{2}\pi - (4/\pi)\sum_1^K (2k-1)^{-2}\cos(2k-1)\theta$.

Example 1. Let f be the 2π-periodic function determined by the formula

$$f(\theta) = |\theta| \quad \text{for } -\pi \le \theta \le \pi;$$

that is, f is the triangle wave depicted in Figure 2.1(a). Since f is even, we can calculate the coefficients a_n and b_n by using Lemma 2.2. We have $b_n = 0$ and

$$a_n = \frac{2}{\pi}\int_0^\pi f(\theta)\cos n\theta\, d\theta = \frac{2}{\pi}\int_0^\pi \theta\cos n\theta\, d\theta.$$

Thus, for $n = 0$,

$$a_0 = \frac{2}{\pi}\int_0^\pi \theta\, d\theta = \frac{1}{\pi}\theta^2\Big|_0^\pi = \pi,$$

and for $n > 0$,

$$a_n = \frac{2}{\pi}\frac{\theta\sin n\theta}{n}\Big|_0^\pi - \frac{2}{\pi}\int_0^\pi \frac{\sin n\theta}{n}\, d\theta = \frac{2}{\pi}\frac{\cos n\theta}{n^2}\Big|_0^\pi = \frac{2}{\pi}\frac{(-1)^n - 1}{n^2},$$

since $\sin n\pi = 0$ and $\cos n\pi = (-1)^n$. Now, $(-1)^n - 1$ equals -2 when n is odd and 0 when n is even. Therefore, the Fourier series of f is

$$\frac{\pi}{2} - \frac{4}{\pi}\sum_{n=1,3,5,\ldots} \frac{1}{n^2}\cos n\theta = \frac{\pi}{2} - \frac{4}{\pi}\sum_{k=1}^{\infty}\frac{\cos(2k-1)\theta}{(2k-1)^2}. \tag{2.7}$$

The graphs of the first few partial sums of this series are shown in Figure 2.1(b–e). Evidently they provide good approximations to f: after only five terms (including the constant term), the graph of the partial sum is almost indistinguishable from the graph of f, except that the corners are a bit rounded. Moreover, we can easily see that the whole series converges absolutely, by comparison to the convergent series $\sum_1^\infty n^{-2}$.

Example 2. Let g be the 2π-periodic function determined by the formula

$$g(\theta) = \theta \quad \text{for } -\pi < \theta \le \pi.$$

In other words, g is the sawtooth wave depicted in Figure 2.2(a). We could use Lemma 2.2 to calculate a_n and b_n since g is odd, but for the sake of variety we shall use (2.5) to calculate c_n instead. For $n = 0$ we have

$$c_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} \theta\, d\theta = 0,$$

and for $n \ne 0$ we integrate by parts to obtain

$$\begin{aligned} c_n &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \theta e^{-in\theta}\, d\theta = \frac{1}{2\pi}\frac{\theta e^{-in\theta}}{-in}\Big|_{-\pi}^{\pi} - \frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{e^{-in\theta}}{-in}\, d\theta \\ &= \frac{1}{2\pi}e^{-in\theta}\left(\frac{\theta}{-in} + \frac{1}{n^2}\right)\Big|_{-\pi}^{\pi} = \frac{(-1)^{n+1}}{in}, \end{aligned}$$

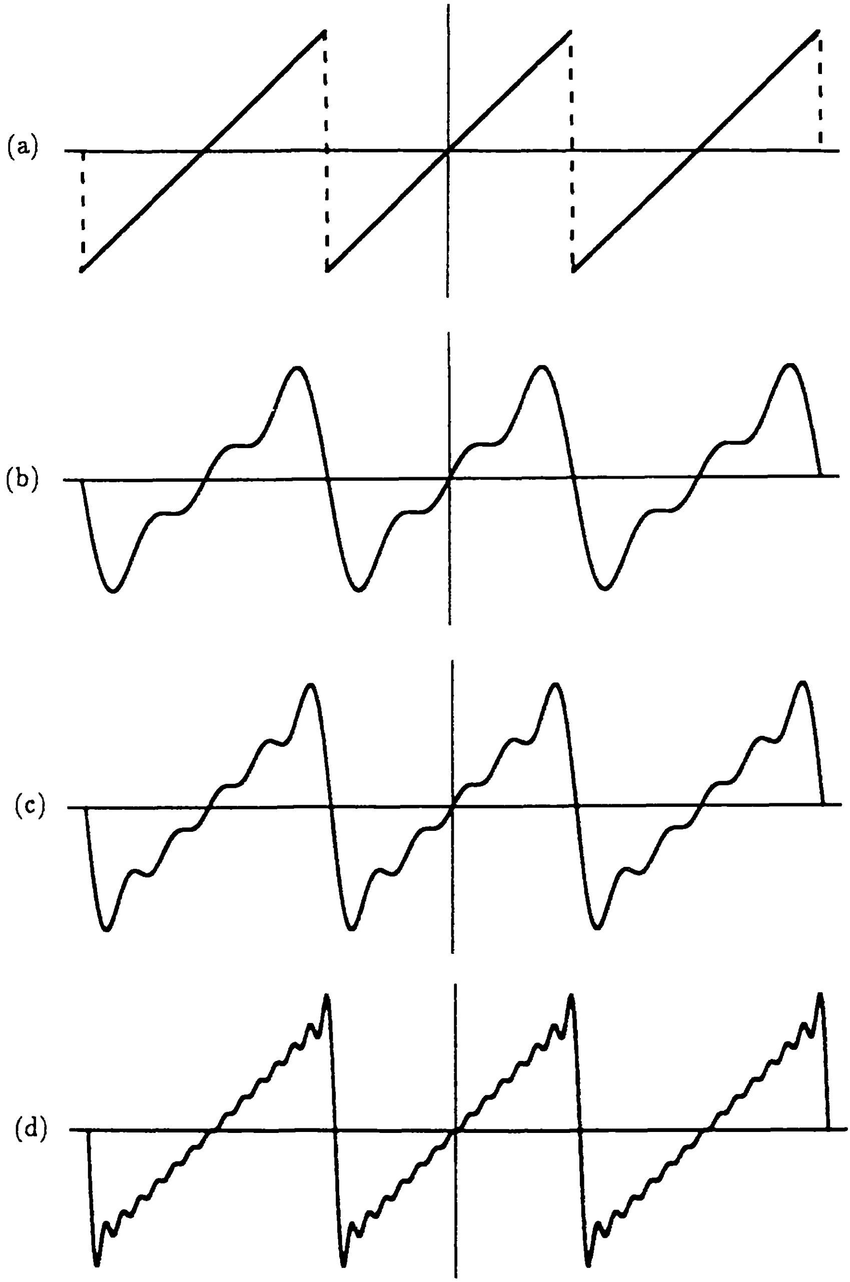


FIGURE 2.2. The sawtooth wave of Example 2 and some partial sums of its Fourier series: (a) the sawtooth wave, (b) S_3, (c) S_5, and (d) S_{14}, where $S_N = 2\sum_1^N (-1)^{n+1} n^{-1} \sin n\theta$.

since $e^{-in\pi} = (-1)^n$. Hence the Fourier series of g is

$$\sum_{n\neq 0} \frac{(-1)^{n+1}}{in} e^{in\theta}.$$

Here n runs through all positive and negative integers. Since $(-1)^n = (-1)^{-n}$, the nth and $(-n)$th terms of this series can be combined to give

$$(-1)^{n+1}\left(\frac{e^{in\theta}}{in} + \frac{e^{-in\theta}}{-in}\right) = \frac{2(-1)^{n+1}}{n} \sin n\theta,$$

and thus the Fourier series of g is

$$2\sum_{1}^{\infty} \frac{(-1)^{n+1}}{n} \sin n\theta. \tag{2.8}$$

The graphs of some partial sums of this series are shown in Figure 2.2(b–d). One can see that these partial sums do approximate the original function g, but a comparison of Figures 2.1 and 2.2 shows that the quality of the approximation here is markedly inferior to that in Example 1. One must add many more terms to the series to get a comparably close fit to the original curve, particularly near the discontinuities. (See also Figure 2.8 in §2.6, showing the 40th partial sum of the Fourier series of the reversed sawtooth wave, for an even more dramatic demonstration of this fact.)

Analytically, the reason for this is that the terms in the series (2.7) tend to zero much more rapidly than the terms in the series (2.8). Precisely, if one disregards the even-order terms in (2.7) (which are all zero), the nth term in (2.7) is of the order of magnitude of $(2n-1)^{-2}$, whereas the nth term in (2.8) is of the order of magnitude of n^{-1}. Thus, the contributions of the high-order terms is much less in (2.7) than in (2.8). As we shall see in §2.3, this phenomenon is intimately related to the fact that the triangle wave is smoother than the sawtooth wave: the former is everywhere continuous, whereas the latter has jump discontinuities. The point is that the rougher a function is, the more difficult it is to approximate it with perfectly smooth functions like linear combinations of $\cos n\theta$ and $\sin n\theta$.

In fact, there sems to be some danger that the series (2.8) will not converge: the nth term has magnitude roughly n^{-1} in general, and $\sum_1^\infty n^{-1}$ diverges. On the other hand, at a given point θ some of the functions $\sin n\theta$ will be positive and others will be negative, so there may be some cancellation effects that will prevent divergence. This is in fact the case, as we shall prove in the next section. For the moment, we simply wish to impress on the reader that the convergence of Fourier series is not a simple matter.

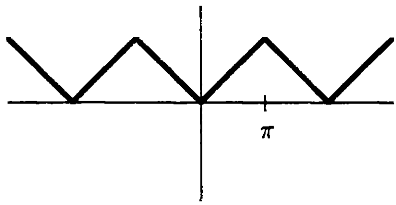
Table 1 gives a list of some elementary Fourier series. It includes all the examples we shall need later on. The fact that all the functions in this table really are the sums of their Fourier series (except perhaps at their points of discontinuity) follows from Theorem 2.1 in §2.2.

We conclude this section by deriving an estimate on the Fourier coefficients that will be needed to establish convergence results in the following sections.

TABLE 1. FOURIER SERIES

The functions f in this table are all understood to be 2π-periodic. The formula for $f(\theta)$ on either $(-\pi, \pi)$ or $(0, 2\pi)$ (except perhaps at its points of discontinuity) is given in the left column; the Fourier series of f is given in the right column; and the graph of f is sketched on the facing page.

1.	$f(\theta) = \theta \quad (-\pi < \theta < \pi)$	$2\sum_1^\infty \frac{(-1)^{n+1}}{n} \sin n\theta$
2.	$f(\theta) = \lvert\theta\rvert \quad (-\pi < \theta < \pi)$	$\frac{\pi}{2} - \frac{4}{\pi}\sum_1^\infty \frac{\cos(2n-1)\theta}{(2n-1)^2}$
3.	$f(\theta) = \pi - \theta \quad (0 < \theta < 2\pi)$	$2\sum_1^\infty \frac{\sin n\theta}{n}$
4.	$f(\theta) = \begin{cases} 0 & (-\pi < \theta < 0) \\ \theta & (0 < \theta < \pi) \end{cases}$	$\frac{\pi}{4} - \frac{2}{\pi}\sum_1^\infty \frac{\cos(2n-1)\theta}{(2n-1)^2} + \sum_1^\infty \frac{(-1)^{(n+1)}}{n} \sin n\theta$
5.	$f(\theta) = \sin^2 \theta$	$\frac{1}{2} - \frac{1}{2}\cos 2\theta$
6.	$f(\theta) = \begin{cases} -1 & (-\pi < \theta < 0) \\ 1 & (0 < \theta < \pi) \end{cases}$	$\frac{4}{\pi}\sum_1^\infty \frac{\sin(2n-1)\theta}{2n-1}$
7.	$f(\theta) = \begin{cases} 0 & (-\pi < \theta < 0) \\ 1 & (0 < \theta < \pi) \end{cases}$	$\frac{1}{2} + \frac{2}{\pi}\sum_1^\infty \frac{\sin(2n-1)\theta}{2n-1}$
8.	$f(\theta) = \lvert\sin\theta\rvert$	$\frac{2}{\pi} - \frac{4}{\pi}\sum_1^\infty \frac{\cos 2n\theta}{4n^2-1}$
9.	$f(\theta) = \lvert\cos\theta\rvert$	$\frac{2}{\pi} - \frac{4}{\pi}\sum_1^\infty \frac{(-1)^n \cos 2n\theta}{4n^2-1}$
10.	$f(\theta) = \begin{cases} 0 & (-\pi < \theta < 0) \\ \sin\theta & (0 < \theta < \pi) \end{cases}$	$\frac{1}{\pi} - \frac{2}{\pi}\sum_1^\infty \frac{\cos 2n\theta}{4n^2-1} + \frac{1}{2}\sin\theta$



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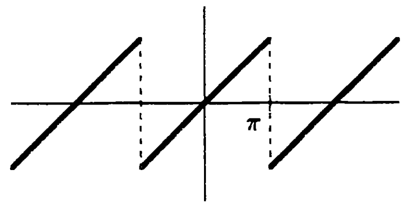

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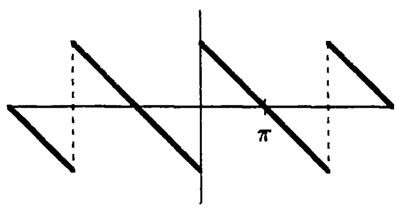

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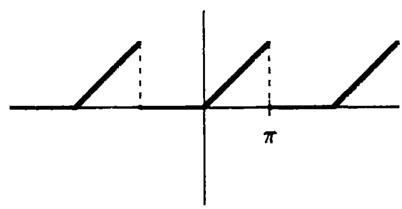

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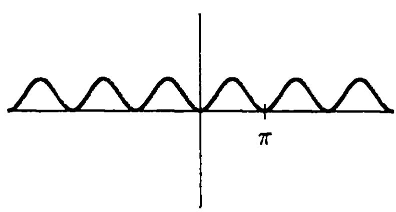

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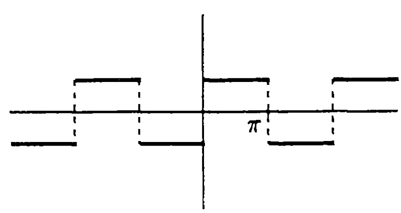

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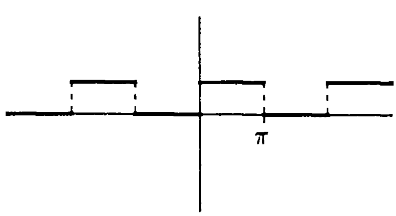

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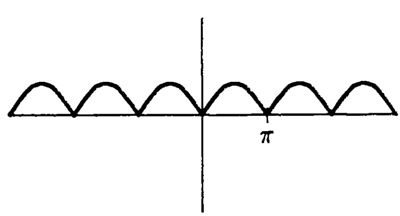

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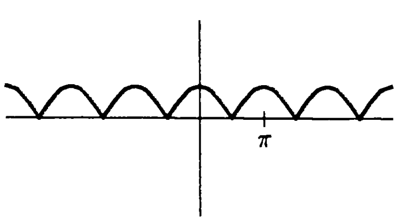

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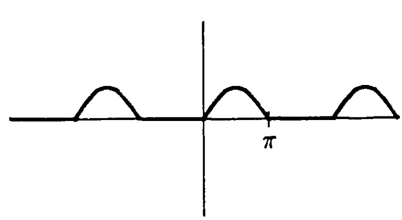

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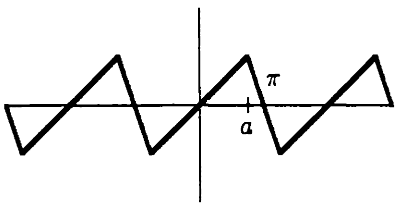
TABLE 1 (continued)

11.	$f(\theta)=\begin{cases}\theta & (-a<\theta<a)\\ a\frac{\pi-\theta}{\pi-a} & (a<\theta<\pi)\\ a\frac{\pi+\theta}{a-\pi} & (-\pi<\theta<-a)\end{cases}$	$\frac{2}{\pi-a}\sum_1^\infty \frac{\sin na}{n^2}\sin n\theta$
12.	$f(\theta)=\begin{cases}(2a)^{-1} & (\|\theta\|<a)\\ 0 & (a<\|\theta\|<\pi)\end{cases}$	$\frac{1}{2\pi}+\frac{1}{\pi}\sum_1^\infty \frac{\sin na}{na}\cos n\theta$
13.	$f(\theta)=\begin{cases}(2a)^{-1} & (\|\theta-\theta_0\|<a)\\ 0 & (a<\|\theta-\theta_0\|<\pi)\end{cases}$	$\frac{1}{2\pi}+\frac{1}{\pi}\sum_1^\infty \frac{\sin na}{na}(\cos n\theta_0\cos n\theta$ $+\sin n\theta_0\sin n\theta)$
14.	$f(\theta)=\begin{cases}1 & (-a<\theta<a)\\ -1 & (2a<\theta<4a)\\ 0 & \text{elsewhere in } (-\pi,\pi)\end{cases}$	$\sum_1^\infty \frac{\sin na}{n}\Big[(1-\cos 3na)\cos n\theta$ $-\sin 3na\sin n\theta\Big]$
15.	$f(\theta)=\begin{cases}a^{-2}(a-\|\theta\|) & (\|\theta\|<a)\\ 0 & (a<\|\theta\|<\pi)\end{cases}$	$\frac{1}{2\pi}+\frac{2}{\pi}\sum_1^\infty \frac{1-\cos na}{n^2a^2}\cos n\theta$
16.	$f(\theta)=\theta^2 \quad (-\pi<\theta<\pi)$	$\frac{\pi^2}{3}+4\sum_1^\infty \frac{(-1)^n}{n^2}\cos n\theta$
17.	$f(\theta)=\theta(\pi-\|\theta\|) \quad (-\pi<\theta<\pi)$	$\frac{8}{\pi}\sum_1^\infty \frac{\sin(2n-1)\theta}{(2n-1)^3}$
18.	$f(\theta)=e^{b\theta} \quad (-\pi<\theta<\pi)$	$\frac{\sinh b\pi}{\pi}\sum_{-\infty}^\infty \frac{(-1)^n}{b-in}e^{in\theta}$
19.	$f(\theta)=e^{b\theta} \quad (0<\theta<2\pi)$	$\frac{e^{2\pi b}-1}{2\pi}\sum_{-\infty}^\infty \frac{e^{in\theta}}{b-in}$
20.	$f(\theta)=\sinh\theta \quad (-\pi<\theta<\pi)$	$\frac{2\sinh\pi}{\pi}\sum_1^\infty \frac{(-1)^{n+1}n}{n^2+1}\sin n\theta$



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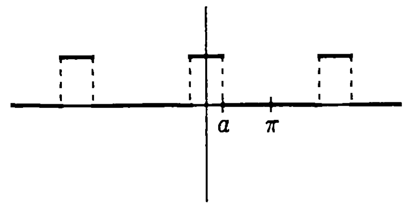

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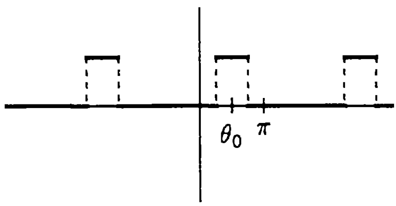

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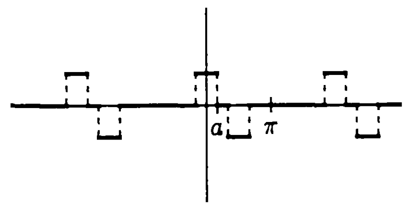

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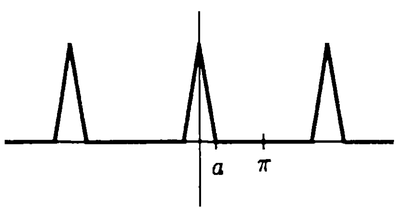

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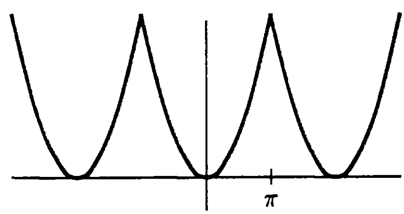

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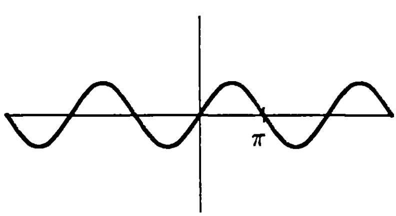

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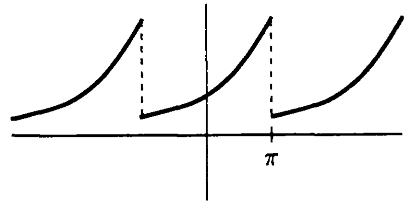

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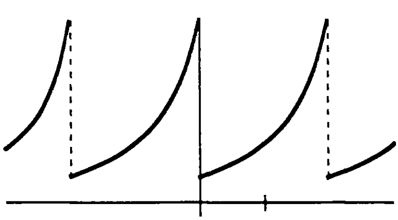

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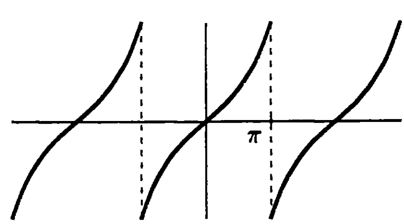


(20)

Bessel's Inequality. *If f is 2π-periodic and Riemann integrable on $[-\pi, \pi]$, and the Fourier coefficients c_n are defined by (2.5), then*

$$\sum_{-\infty}^{\infty} |c_n|^2 \le \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(\theta)|^2 d\theta.$$

Proof: Since $|z|^2 = z\bar{z}$ for any complex number z,

$$\begin{aligned}
&\left| f(\theta) - \sum_{-N}^{N} c_n e^{in\theta} \right|^2 \\
&= \left(f(\theta) - \sum_{-N}^{N} c_n e^{in\theta} \right) \left(\overline{f(\theta)} - \sum_{-N}^{N} \bar{c}_n e^{-in\theta} \right) \\
&= |f(\theta)|^2 - \sum_{-N}^{N} \left[c_n \overline{f(\theta)} e^{in\theta} + \bar{c}_n f(\theta) e^{-in\theta} \right] + \sum_{m,n=-N}^{N} c_m \bar{c}_n e^{i(m-n)\theta}.
\end{aligned}$$

Now divide both sides by 2π and integrate from $-\pi$ to π. Taking account of the formulas

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} f(\theta) e^{-in\theta} d\theta = c_n, \qquad \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{i(m-n)\theta} d\theta = \begin{cases} 0 & \text{if } m \ne n, \\ 1 & \text{if } m = n, \end{cases}$$

we obtain

$$\begin{aligned}
&\frac{1}{2\pi} \int_{-\pi}^{\pi} \left| f(\theta) - \sum_{-N}^{N} c_n e^{in\theta} \right|^2 d\theta \\
&= \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(\theta)|^2 d\theta - \sum_{-N}^{N} \left[c_n \bar{c}_n + \bar{c}_n c_n \right] + \sum_{-N}^{N} c_n \bar{c}_n \\
&= \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(\theta)|^2 d\theta - \sum_{-N}^{N} |c_n|^2.
\end{aligned}$$

But the integral on the left is certainly nonnegative, so

$$0 \le \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(\theta)|^2 d\theta - \sum_{-N}^{N} |c_n|^2.$$

Letting $N \to \infty$, we obtain the desired result. ∎

Bessel's inequality can also be stated in terms of the coefficients a_n and b_n defined by (2.6). Indeed, by equation (2.4), for $n \ge 1$ we have

$$\begin{aligned}
|a_n|^2 + |b_n|^2 &= a_n \bar{a}_n + b_n \bar{b}_n \\
&= (c_n + c_{-n})(\bar{c}_n + \bar{c}_{-n}) + i(c_n - c_{-n})(-i)(\bar{c}_n - \bar{c}_{-n}) \\
&= 2c_n \bar{c}_n + 2c_{-n} \bar{c}_{-n},
\end{aligned}$$

so that

$$|a_0|^2 = 4|c_0|^2, \qquad |a_n|^2 + |b_n|^2 = 2\Big(|c_n|^2 + |c_{-n}|^2\Big) \quad \text{for } n \geq 1.$$

Therefore,

$$\frac{1}{4}|a_0|^2 + \frac{1}{2}\sum_1^\infty \Big(|a_n|^2 + |b_n|^2\Big) = \sum_{-\infty}^{\infty} |c_n|^2 \leq \frac{1}{2\pi}\int_{-\pi}^{\pi} |f(\theta)|^2 d\theta.$$

It turns out, as we shall see later, that Bessel's inequality is actually an equality. For now, its main significance is simply the fact that the series $\sum |a_n|^2$, $\sum |b_n|^2$, and $\sum |c_n|^2$ are all convergent. As a consequence, we obtain the following result, which is a special case of a theorem known as the *Riemann-Lebesgue lemma.*

Corollary 2.1. *The Fourier coefficients a_n, b_n, and c_n all tend to zero as $n \to \infty$ (and as $n \to -\infty$ in the case of c_n).*

Proof: $|a_n|^2$, $|b_n|^2$, and $|c_n|^2$ are the nth terms of convergent series, so they tend to zero as $n \to \infty$; hence so do a_n, b_n, and c_n. ∎

EXERCISES

Verify the formulas of Table 1. That is, for $3 \leq n \leq 20$, Exercise n is to show that the series in the right column of entry n in Table 1 is the Fourier series of the function in the left column. (Entries 1 and 2 are Examples 1 and 2 in the text.) Some of these functions are related to each other, and you may be able to use this fact to avoid caclulating all the Fourier coefficients from scratch each time. Entries 3 and 4 can be derived from entries 1 and 2; entry 7 can be derived from entry 6; entries 9 and 10 can be derived from entry 8; entries 13 and 14 can be derived from entry 12; and entries 19 and 20 can be derived from entry 18.

2.2 A convergence theorem

Question: does the Fourier series of a periodic function f converge to f? The answer is certainly not obvious — for example, why should one be able to expand nonsmooth functions like the examples in §2.1 in a series whose individual terms $\cos nx$ and $\sin nx$ possess derivatives of all orders? Fourier's assertion that the answer is yes was initially greeted with a certain amount of disbelief. In fact, the answer *is* always yes provided that things are interpreted suitably, although the situation is somewhat more delicate than one might initially expect.

In this section we shall show that the Fourier series of f converges to f under certain reasonably general hypotheses on f; later, in §2.3, §2.6, §3.4, and §9.3, we shall present some variants of this result under other conditions on f. We first define the class of functions with which we shall be working.

Suppose $-\infty < a < b < \infty$. We say that a function f on the closed interval $[a,b]$ is **piecewise continuous** provided that

(i) f is continuous on $[a,b]$ except perhaps at finitely many points $x_1, \dots, x_k$;

(ii) at each of the points $x_1, \dots, x_k$, the left-hand and right-hand limits of f,

$$f(x_j-) = \lim_{h\to 0,\ h>0} f(x_j - h) \quad \text{and} \quad f(x_j+) = \lim_{h\to 0,\ h>0} f(x_j + h),$$

exist. (If the endpoint a (or b) is one of the exceptional points x_j, we require only the right-hand (or left-hand) limit to exist.)

That is, f is piecewise continuous on $[a,b]$ if f is continuous there except for finitely many finite jump discontinuities. (When we say that the limits $f(x_j\pm)$ exist, we mean in particular that they are finite: ∞ is not allowed as a value.) We denote the class of piecewise continuous functions on $[a,b]$ by $PC(a,b)$.

Next, we say that a function f on $[a,b]$ is **piecewise smooth** if f and its first derivative f' are both piecewise continuous on $[a,b]$, and we denote the class of piecewise smooth functions on $[a,b]$ by $PS(a,b)$. More precisely, $f \in PS(a,b)$ if and only if

(i) $f \in PC(a,b)$;

(ii) f' exists and is continuous on (a,b) except perhaps at finitely many points $x_1, \dots, x_K$ (which will include any points where f is discontinuous), and the one-sided limits $f'(x_j-)$ and $f'(x_j+)$ ($j = 1, \dots, K$), and also $f'(a+)$ and $f'(b-)$, exist.

In other words, f is piecewise smooth if its graph is a smooth curve except for finitely many jumps (where f is discontinuous) and corners (where f' is discontinuous). We do not allow infinite discontinuities (such as $f(x) = 1/x$ has at $x = 0$) or sharp cusps (where f' becomes infinite). See Figure 2.3.

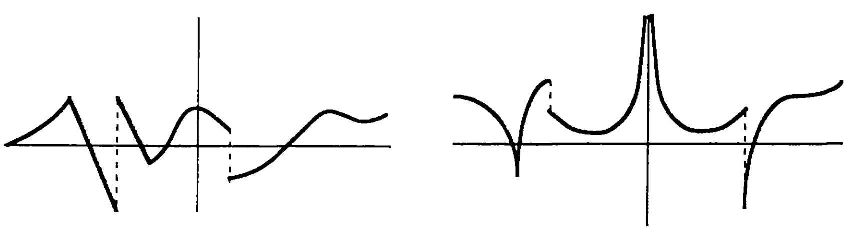


FIGURE 2.3. A piecewise smooth function (left) and a function that is not piecewise smooth (right).

One last bit of terminology: a function defined on the whole real line $\mathbf{R}$ is said to be **piecewise continuous** or **piecewise smooth** on $\mathbf{R}$ if it is so on every bounded interval $[a,b]$. (That is, f or f' may have infinitely many discontinuities on the whole line but only finitely many in any bounded interval.) We denote the spaces of piecewise continuous and piecewise smooth functions on $\mathbf{R}$ by $PC(\mathbf{R})$ and $PS(\mathbf{R})$.

We now return to consideration of the Fourier series of a 2π-periodic function $f(\theta)$. We recall that this is defined to be

$$\tfrac{1}{2}a_0 + \sum_1^\infty (a_n \cos n\theta + b_n \sin n\theta) = \sum_{-\infty}^{\infty} c_n e^{in\theta} \tag{2.9}$$

where

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(\psi)\cos n\psi\,d\psi, \qquad b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(\psi)\sin n\psi\,d\psi,$$
$$c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\psi)e^{-in\psi}\,d\psi. \tag{2.10}$$

(We have labeled the variable of integration in (2.10) as ψ simply for later convenience.)

What meaning is to be attached to this series? Of course, the sum of any infinite series is defined to be the limit of its partial sums. When we write the series (2.9) in trigonometric form, we agree always to group together the terms involving $\cos n\theta$ and $\sin n\theta$ as indicated above; correspondingly, when we write it in exponential form, we agree always to group together the terms involving $e^{in\theta}$ and $e^{-in\theta}$. (*This convention will always be in force.*) Thus we take the Nth partial sum of the series (2.9) to be the sum $S_N^f(\theta)$ defined by

$$S_N^f(\theta) = \tfrac{1}{2}a_0 + \sum_1^N (a_n \cos n\theta + b_n \sin n\theta) = \sum_{-N}^{N} c_n e^{in\theta}, \tag{2.11}$$

and our aim is to show that S_N^f converges to f as $N \to \infty$.

If we plug the definition (2.10) of c_n into (2.11), we see that

$$S_N^f(\theta) = \frac{1}{2\pi}\sum_{-N}^{N}\int_{-\pi}^{\pi} f(\psi)e^{in(\theta-\psi)}d\psi = \frac{1}{2\pi}\sum_{-N}^{N}\int_{-\pi}^{\pi} f(\psi)e^{in(\psi-\theta)}d\psi.$$

The last equality results from replacing n by $-n$; this does not affect the sum since n ranges from $-N$ to N. If we now make the change of variable $\phi = \psi - \theta$ and use Lemma 2.1, we obtain

$$S_N^f(\theta) = \frac{1}{2\pi}\sum_{-N}^{N}\int_{-\pi+\theta}^{\pi+\theta} f(\theta+\phi)e^{in\phi}d\phi = \frac{1}{2\pi}\sum_{-N}^{N}\int_{-\pi}^{\pi} f(\theta+\phi)e^{in\phi}d\phi.$$

In short,

$$S_N^f(\theta) = \int_{-\pi}^{\pi} f(\theta+\phi)D_N(\phi)\,d\phi, \quad \text{where } D_N(\phi) = \frac{1}{2\pi}\sum_{-N}^{N} e^{in\phi}. \tag{2.12}$$

The function $D_N(\phi)$ is called the Nth **Dirichlet kernel.** We can express D_N in a more computable form by recognizing that it is the sum of a finite geometric progression:

$$D_N(\phi) = \frac{1}{2\pi}e^{-iN\phi}(1 + e^{i\phi} + \cdots + e^{i2N\phi}) = \frac{1}{2\pi}e^{-iN\phi}\sum_0^{2N} e^{in\phi}.$$

Since $\sum_0^K r^n = (r^{K+1} - 1)/(r - 1)$ for any $r \neq 1$, for $\phi \neq 0$ we have

$$D_N(\phi) = \frac{1}{2\pi}e^{-iN\phi}\frac{e^{i(2N+1)\phi} - 1}{e^{i\phi} - 1} = \frac{1}{2\pi}\frac{e^{i(N+1)\phi} - e^{-iN\phi}}{e^{i\phi} - 1}. \tag{2.13}$$

Moreover, on multiplying top and bottom by $e^{-i\phi/2}$, we obtain

$$D_N(\phi) = \frac{1}{2\pi}\frac{\exp\left[i(N+\frac{1}{2})\phi\right] - \exp\left[-i(N+\frac{1}{2})\phi\right]}{\exp(i\frac{1}{2}\phi) - \exp(-i\frac{1}{2}\phi)} = \frac{1}{2\pi}\frac{\sin(N+\frac{1}{2})\phi}{\sin\frac{1}{2}\phi}. \tag{2.14}$$

From this formula it is easy to sketch the graph of D_N: it is the rapidly oscillating sine wave $y = \sin(N + \frac{1}{2})\phi$ amplitude-modulated to fit inside the envelope $y = \pm(2\pi)^{-1}\csc\frac{1}{2}\phi$. See Figure 2.4.

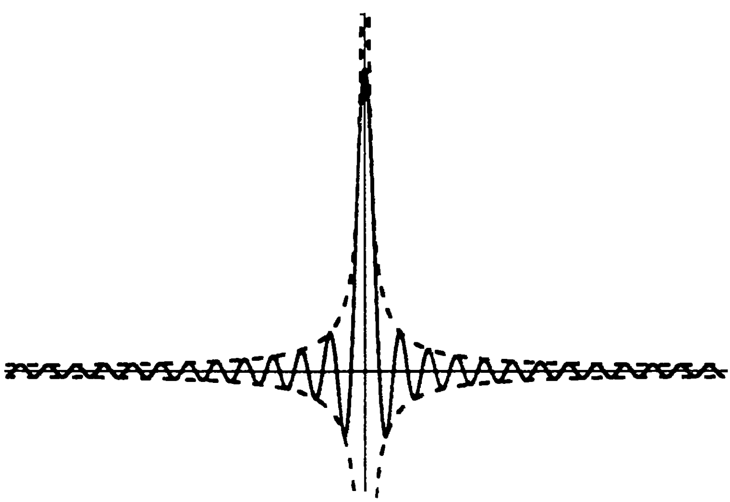


FIGURE 2.4. Graphs of the Dirichlet kernel $D_{25}(\phi)$ (solid) and its envelope $\pm(2\pi)^{-1}\csc\frac{1}{2}\phi$ (dashed) on the interval $-\pi < \phi < \pi$.

The pictorial intuition behind the fact that $S_N^f(\theta) \to f(\theta)$ is as follows: in the integral formula (2.12) for $S_N^f(\theta)$, the sharp central spike of $D_N(\phi)$ at $\phi = 0$ picks out the value $f(\theta)$, and the rapid oscillations of $D_N(\phi)$ away from $\phi = 0$ kill off most of the rest of the integral because of cancellations between positive and negative values. Before proceeding to the actual proof, however, we need one more fact about D_N.

Lemma 2.4. *For any N,*

$$\int_{-\pi}^{0} D_N(\theta)\,d\theta = \int_0^{\pi} D_N(\theta)\,d\theta = \frac{1}{2}.$$

Proof: From formula (2.12) we have

$$D_N(\theta) = \frac{1}{2\pi} + \frac{1}{\pi}\sum_1^N \cos n\theta,$$

so that

$$\int_0^{\pi} D_N(\theta)\,d\theta = \left[\frac{\theta}{2\pi} + \frac{1}{\pi}\sum_1^N \frac{\sin n\theta}{n}\right]_0^{\pi} = \frac{1}{2},$$

and likewise for the integral from $-\pi$ to 0. ∎

Here at last is our main convergence theorem. It says that the Fourier series of a function $f \in PS(\mathbf{R})$ converges pointwise to f, provided that we (re)define f at its points of discontinuity to be the average of its left- and right-hand limits.

Theorem 2.1. *If f is 2π-periodic and piecewise smooth on $\mathbf{R}$, and S_N^f is defined by (2.10) and (2.11), then*

$$\lim_{N\to\infty} S_N^f(\theta) = \tfrac{1}{2}\left[f(\theta-) + f(\theta+)\right]$$

for every θ. In particular, $\lim_{N\to\infty} S_N^f(\theta) = f(\theta)$ for every θ at which f is continuous.

Proof: By Lemma 2.4, we have

$$\tfrac{1}{2}f(\theta-) = f(\theta-)\int_{-\pi}^{0} D_N(\phi)\,d\phi, \qquad \tfrac{1}{2}f(\theta+) = f(\theta+)\int_0^{\pi} D_N(\phi)\,d\phi,$$

and hence by equation (2.12),

$$S_N^f(\theta) - \tfrac{1}{2}\left[f(\theta-) + f(\theta+)\right]$$
$$= \int_{-\pi}^{0}\left[f(\theta+\phi) - f(\theta-)\right]D_N(\phi)\,d\phi + \int_0^{\pi}\left[f(\theta+\phi) - f(\theta+)\right]D_N(\phi)\,d\phi.$$

We wish to show that for each fixed θ, this quantity approaches zero as $N \to \infty$. But by formula (2.13), we can write it as

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} g(\phi)(e^{i(N+1)\phi} - e^{-iN\phi})\,d\phi \tag{2.15}$$

where $g(\phi)$ is defined to be

$$\frac{f(\theta+\phi)-f(\theta-)}{e^{i\phi}-1} \text{ for } -\pi<\phi<0, \qquad \frac{f(\theta+\phi)-f(\theta+)}{e^{i\phi}-1} \text{ for } 0<\phi<\pi.$$

g is a well-behaved function on $[-\pi,\pi]$, as smooth as f is, except near $\phi = 0$ (where $e^{i\phi}-1$ vanishes). However, by l'Hôpital's rule,

$$\lim_{\phi\to 0+} g(\phi) = \lim_{\phi\to 0+} \frac{f(\theta+\phi)-f(\theta+)}{e^{i\phi}-1} = \lim_{\phi\to 0+} \frac{f'(\theta+\phi)}{ie^{i\varphi}} = \frac{f'(\theta+)}{i}.$$

Similarly, $g(\phi)$ approaches the finite limit $i^{-1}f'(\theta-)$ as ϕ approaches zero from the left. Hence g is actually piecewise continuous on $[-\pi,\pi]$, so by the corollary to Bessel's inequality in §2.1, its Fourier coefficients

$$C_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} g(\phi)e^{-in\phi}\,d\phi$$

tend to zero as $n \to \pm\infty$. But the expression (2.15) is nothing but $C_{-(N+1)} - C_N$, so it vanishes as $N \to \infty$; and this is what we needed to show. ∎

Let us see what this theorem says with regard to the two examples of the previous section. The function f of Example 1 is piecewise smooth and everywhere continuous, so the Fourier series of f converges to f at every point. Thus,

$$\frac{\pi}{2} - \frac{4}{\pi}\sum_{1}^{\infty}\frac{\cos(2n-1)\theta}{(2n-1)^2} = |\theta| \quad \text{for } -\pi \le \theta \le \pi. \tag{2.16}$$

On the other hand, the function g of Example 2 is piecewise smooth and continuous except at the points $\theta = k\pi$ with k odd. At these discontinuities we have $g(k\pi-) = \pi$ and $g(k\pi+) = -\pi$, so $\frac{1}{2}[g(k\pi-)+g(k\pi+)] = 0$. Thus the Fourier series of g converges to g at all points except $\theta = k\pi$ (k odd), where it converges to zero. Hence,

$$\sum_{1}^{\infty}\frac{(-1)^{n+1}}{n}\sin n\theta = \frac{\theta}{2} \quad \text{for } -\pi < \theta < \pi. \tag{2.17}$$

In particular, if we take $\theta = 0$ in (2.16), we obtain the formula

$$\sum_{1}^{\infty}\frac{1}{(2k-1)^2} = 1 + \frac{1}{9} + \frac{1}{25} + \frac{1}{49} + \cdots = \frac{\pi^2}{8}.$$

(As the reader may check, the same formula results from taking taking $\theta = \pi$.) Moreover, if we take $\theta = \frac{1}{2}\pi$ in (2.17) and use the fact that $\sin\frac{1}{2}n\pi$ is alternately 1 and -1 when n is odd and 0 when n is even, we find that

$$\sum_{1}^{\infty}\frac{(-1)^{k+1}}{2k-1} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots = \frac{\pi}{4}.$$

These are two interesting instances where numerical series can be evaluated as special values of Fourier series. Others can be found in the exercises.

Theorem 2.1 says that the Fourier series of a 2π-periodic piecewise smooth function f converges to f everywhere, provided that f is (re)defined at each of its points of discontinuity to be the average of its left- and right-hand limits there. *Henceforth, when we speak of piecewise smooth functions, we shall assume that this adjustment has been made,* unless we explicitly state otherwise. This will obviate the need to single out the points of discontinuity for special attention. In particular, with this understanding, we have the following uniqueness theorem.

Corollary 2.2. *If f and g are 2π-periodic and piecewise smooth, and f and g have the same Fourier coefficients, then $f = g$.*

Proof: f and g are both the sum of the same Fourier series. ∎

EXERCISES

1. Which of the following functions are continuous, piecewise continuous, or piecewise smooth on $[-\pi, \pi]$?
   a. $f(\theta) = \csc\theta$. b. $f(\theta) = (\sin\theta)^{1/3}$. c. $f(\theta) = (\sin\theta)^{4/3}$.
   d. $f(\theta) = \cos\theta$ if $\theta > 0$, $f(\theta) = -\cos\theta$ if $\theta \le 0$.
   e. $f(\theta) = \sin\theta$ if $\theta > 0$, $f(\theta) = \sin 2\theta$ if $\theta \le 0$.
   f. $f(\theta) = (\sin\theta)^{1/5}$ if $\theta < \frac{1}{2}\pi$, $f(\theta) = \cos\theta$ if $\theta \ge \frac{1}{2}\pi$.
2. To what values do the series in entries 6, 7, 12, and 18 of Table 1, §2.1, converge at the points where their sums are discontinuous?

The Fourier series for a number of piecewise smooth functions are listed in Table 1 of §2.1, and Theorem 2.1 tells what the sums of these series are. By using this information and choosing suitable values of θ (usually 0, $\frac{1}{2}\pi$, or π), derive the following formulas for the sums of numerical series. (The relevant entries from Table 1 are indicated in parentheses.)

3. $\displaystyle\sum_1^\infty \frac{1}{4n^2-1} = \frac{1}{2}, \quad \sum_1^\infty \frac{(-1)^{n+1}}{4n^2-1} = \frac{\pi-2}{4}$ (8).
4. $\displaystyle\sum_1^\infty \frac{1}{n^2} = \frac{\pi^2}{6}, \quad \sum_1^\infty \frac{(-1)^{n+1}}{n^2} = \frac{\pi^2}{12}$ (16).
5. $\displaystyle\sum_1^\infty \frac{(-1)^{n+1}}{(2n-1)^3} = \frac{\pi^3}{32}$ (17).
6. $\displaystyle\sum_1^\infty \frac{(-1)^n}{n^2+b^2} = \frac{\pi}{2b}\operatorname{csch} b\pi - \frac{1}{2b^2}$ (18 or 19).
7. $\displaystyle\sum_1^\infty \frac{1}{n^2+b^2} = \frac{\pi}{2b}\coth b\pi - \frac{1}{2b^2}$ (18 or 19; this is a bit tricky).

2.3 Derivatives, integrals, and uniform convergence

This section is devoted to an examination of the behavior of Fourier series in relation to the processes of calculus.

We shall be largely concerned here with periodic functions that are both continuous and piecewise smooth. Pictorially, the graph of such a function is a smooth curve except that it can have "corners" where the derivative jumps. The fundamental theorem of calculus,

$$f(b) - f(a) = \int_a^b f'(\theta)\,d\theta,$$

applies to functions f that are continuous and piecewise smooth, even though f' is undefined at the "corners." To see this it suffices to express the integral on the right as the sum of integrals over the subintervals of $[a, b]$ on which f is differentiable; the continuity of f guarantees that the endpoint evaluations at the intermediate subdivision points cancel out. For example, if f is differentiable except at the point $c \in (a, b)$, we have

$$\begin{aligned}\int_a^b f'(\theta)\,d\theta &= \int_a^c f'(\theta)\,d\theta + \int_c^b f'(\theta)d\theta \\ &= \big[f(c) - f(a)\big] + \big[f(b) - f(c)\big] = f(b) - f(a).\end{aligned}$$

This observation will be used implicitly in several of the following calculations, including the proof of Theorem 2.2.

Our first main result relates the Fourier coefficients of a function to those of its derivative. The fact that this relation is so simple is one of the main reasons underlying the utility of Fourier series.

Theorem 2.2. *Suppose f is 2π-periodic, continuous, and piecewise smooth. Let a_n, b_n, and c_n be the Fourier coefficients of f defined in (2.5) and (2.6), and let a'_n, b'_n, and c'_n be the corresponding Fourier coefficients of f'. Then*

$$a'_n = nb_n, \qquad b'_n = -na_n, \qquad c'_n = inc_n.$$

Proof: This is a simple matter of integration by parts. For example,

$$c'_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f'(\theta)e^{-in\theta}\,d\theta = \frac{1}{2\pi}f(\theta)e^{-in\theta}\Big|_{-\pi}^{\pi} - \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta)(-ine^{-in\theta})\,d\theta.$$

The first term on the right vanishes because $f(-\pi) = f(\pi)$ and $e^{in\pi} = e^{-in\pi} = (-1)^n$, and the second term is inc_n. The argument for a'_n and b'_n is the same; we leave the details to the reader. ∎

Combining this result with the theorem of the previous section, we easily obtain the basic results on differentiation and integration of Fourier series.

Theorem 2.3. *Suppose f is 2π-periodic, continuous, and piecewise smooth, and suppose also that f' is piecewise smooth. If*

$$\sum_{-\infty}^{\infty} c_n e^{in\theta} = \tfrac{1}{2}a_0 + \sum_{1}^{\infty}(a_n \cos n\theta + b_n \sin n\theta)$$

is the Fourier series of $f(\theta)$, then $f'(\theta)$ is the sum of the derived series

$$\sum_{-\infty}^{\infty} inc_n e^{in\theta} = \sum_{1}^{\infty}(nb_n \cos n\theta - na_n \sin n\theta)$$

for all θ at which $f'(\theta)$ exists. At the exceptional points where f' has jumps, the series converges to $\frac{1}{2}\left[f'(\theta-) + f'(\theta+)\right]$.

Proof: Since f' is piecewise smooth, by Theorem 2.1 it is the sum of its Fourier series at every point (with appropriate modifications at the jumps). By Theorem 2.2, the coefficients of $e^{in\theta}$, $\cos n\theta$, and $\sin n\theta$ in this series are inc_n, nb_n, and $-na_n$. The result follows. ∎

In considering integration of Fourier series, one must keep in mind that the indefinite integral of a periodic function may not be periodic. For example, the constant function $f(\theta) = 1$ is periodic, but its antiderivative $F(\theta) = \theta$ is not. However, the integral of every term in a Fourier series is periodic except for the constant term, from which we see that a periodic function has a periodic integral precisely when the constant term in its Fourier series vanishes, i.e., when its mean value on $[-\pi, \pi]$ is zero. We therefore arrive at the following result.

Theorem 2.4. *Suppose f is 2π-periodic and piecewise continuous, with Fourier coefficients a_n, b_n, c_n, and let $F(\theta) = \int_0^\theta f(\phi)\,d\phi$. If $c_0\ (= \frac{1}{2}a_0) = 0$, then for all θ we have*

$$F(\theta) = C_0 + \sum_{n\neq 0} \frac{c_n}{in} e^{in\theta} = \tfrac{1}{2}A_0 + \sum_{1}^{\infty}\left(\frac{a_n}{n}\sin n\theta - \frac{b_n}{n}\cos n\theta\right) \tag{2.18}$$

where the constant term is the mean value of F on $[-\pi, \pi]$:

$$C_0 = \tfrac{1}{2}A_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} F(\theta)\,d\theta. \tag{2.19}$$

The series on the right of (2.18) is the series obtained by formally integrating the Fourier series of f term by term, whether the latter series actually converges or not. If $c_0 \neq 0$, the sum of the series on the right of (2.18) is $F(\theta) - c_0\theta$.

Proof: F is continuous and piecewise smooth, being the integral of a piecewise continuous function. Moreover, if $c_0 = 0$, F is 2π-periodic, for

$$F(\theta + 2\pi) - F(\theta) = \int_\theta^{\theta+2\pi} f(\phi)\,d\phi = \int_{-\pi}^{\pi} f(\phi)\,d\phi = 2\pi c_0 = 0.$$

Hence, by Theorem 2.1, $F(\theta)$ is the sum of its Fourier series at every θ. But by Theorem 2.2 applied to F, the Fourier coefficients A_n, B_n, and C_n of F are related to those of f by

$$A_n = -\frac{b_n}{n}, \qquad B_n = \frac{a_n}{n}, \qquad C_n = \frac{c_n}{in} \qquad (n \neq 0).$$

The formula (2.19) for the constant C_0 or $\frac{1}{2}A_0$ is just the usual formula for the zeroth Fourier coefficient of F. If $c_0 \neq 0$, these arguments can be applied to the function $f(\theta) - c_0$ rather than $f(\theta)$, yielding the final assertion. ∎

Example. Let f be the periodic function such that $f(\theta) = 1$ for $0 < \theta < \pi$ and $f(\theta) = -1$ for $-\pi < \theta < 0$, and let $F(\theta) = \int_0^\theta f(\phi)\,d\phi$. Clearly $F(\theta) = |\theta|$ for $|\theta| \leq \pi$. By entry 4 of Table 1, §2.1, the Fourier series of f is $(4/\pi)\sum_1^\infty (2n-1)^{-1}\sin(2n-1)\theta$, so by Theorem 2.4 we have

$$F(\theta) = C_0 - \frac{4}{\pi}\sum_1^\infty \frac{\cos(2n-1)\theta}{(2n-1)^2} \quad \text{where } C_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} |\theta|\,d\theta = \frac{\pi}{2}.$$

Thus we recover the result of entry 2 of Table 1.

Theorem 2.1 gave conditions under which the Fourier series of f converges pointwise to f. However, experience in working with infinite series teaches us that simple pointwise convergence of a series can be a tricky business, and that we are much better off if the convergence is absolute and uniform. We recall the definitions: suppose the series $\sum_1^\infty g_n(x)$ converges to $g(x)$ on a set S. The convergence is *absolute* if the series $\sum_1^\infty |g_n(x)|$ also converges for $x \in S$, and *uniform* if not only does the difference $g(x) - \sum_1^N g_n(x)$ tend to zero for each $x \in S$, but so does the maximum of this difference over the whole set S:

$$\sup_{x\in S}\left| g(x) - \sum_1^N g_n(x)\right| \to 0 \quad \text{as } N \to \infty.$$

The most useful criterion for guaranteeing absolute and uniform convergence is the **Weierstrass M-test**: if there is a sequence M_n of positive constants such that

$$|g_n(x)| \leq M_n \quad \text{for } x \in S, \quad \text{and} \quad \sum_1^\infty M_n < \infty,$$

then the series $\sum_1^\infty g_n(x)$ is absolutely and uniformly convergent.

In the case of Fourier series, we have the obvious estimates

$$|a_n \cos n\theta| \leq |a_n|, \qquad |b_n \sin n\theta| \leq |b_n|, \qquad |c_n e^{in\theta}| = |c_n|.$$

Hence the Weierstrass M-test will apply to a Fourier series in trigonometric form if $\sum_0^\infty |a_n| < \infty$ and $\sum_1^\infty |b_n| < \infty$, and to a Fourier series in exponential form if

$\sum_{-\infty}^{\infty} |c_n| < \infty$. Since it follows from the equations (2.3) and (2.4) relating a_n, b_n, and c_n that

$$|c_{\pm n}| \le |a_n| + |b_n|, \qquad |a_n| \le |c_n| + |c_{-n}|, \qquad |b_n| \le |c_n| + |c_{-n}|,$$

the conditions $\sum_0^\infty |a_n| < \infty$ and $\sum_1^\infty |b_n| < \infty$ are completely equivalent to the condition $\sum_{-\infty}^{\infty} |c_n| < \infty$. We now present a sufficient (but not necessary) condition for them to hold.

Theorem 2.5. *If f is 2π-periodic, continuous, and piecewise smooth, then the Fourier series of f converges to f absolutely and uniformly on* $\mathbf{R}$.

Proof: By Theorem 2.1 and the remarks just made, it suffices to show that the series $\sum_{-\infty}^{\infty} |c_n|$ converges. Let c_n' denote the Fourier coefficients of f'. By Theorem 2.2 we know that $c_n = (in)^{-1}c_n'$ for $n \neq 0$, and by Bessel's inequality applied to f',

$$\sum_{-\infty}^{\infty} |c_n'|^2 \le \frac{1}{2\pi}\int_{-\pi}^{\pi} |f'(\theta)|^2 d\theta < \infty.$$

Hence, by the Cauchy-Schwarz inequality,

$$\sum_{-\infty}^{\infty} |c_n| = |c_0| + \sum_{n\neq 0} \left|\frac{c_n'}{n}\right| \le |c_0| + \left(\sum_{n\neq 0}\frac{1}{n^2}\right)^{1/2}\left(\sum_{n\neq 0}|c_n'|^2\right)^{1/2} < \infty,$$

since $\sum_{n\neq 0}(1/n^2) = 2\sum_1^\infty(1/n^2) < \infty$. (In case the reader needs reminding: the Cauchy-Schwarz inequality says that the dot product of two vectors is bounded by the product of their norms. It is valid in any number n of dimensions and also in the limit as $n \to \infty$. We shall discuss it in more detail in Chapter 3.) ∎

Let us return to Theorem 2.3. If f has many derivatives, Theorem 2.3 can be applied several times in succession to calculate the Fourier series of f', f'', f''', etc. Each time one takes a derivative, the magnitude of the Fourier coefficients c_n (or a_n and b_n) increases by a factor of $|n|$, which means that the derived series converges more slowly than the original series. Or, to put it another way, if the derived series converges at all, the original series must converge relatively rapidly. Thus there is a connection between the differentiability properties of a function and the rate of convergence of its Fourier series. Here is a precise result along these lines.

Theorem 2.6. *Suppose f is 2π-periodic. If f is of class $C^{(k-1)}$ and $f^{(k-1)}$ is piecewise smooth (thus $f^{(k)}$ exists except at finitely many points in each bounded interval and is piecewise continuous), then the Fourier coefficients of f satisfy*

$$\sum |n^k a_n|^2 < \infty, \qquad \sum |n^k b_n|^2 < \infty, \qquad \sum |n^k c_n|^2 < \infty.$$

In particular,

$$n^k a_n \to 0, \qquad n^k b_n \to 0, \qquad n^k c_n \to 0 \qquad \text{as } n \to \infty.$$

On the other hand, suppose the Fourier coefficients c_n ($n \neq 0$) satisfy $|c_n| \le C|n|^{-(k+\alpha)}$ (equivalently, $|a_n| \le Cn^{-(k+\alpha)}$ and $|b_n| \le Cn^{-(k+\alpha)}$) for some $C > 0$ and $\alpha > 1$. Then f is of class $C^{(k)}$.

Proof: For the first part, we apply Theorem 2.2 k times to conclude that the Fourier coefficients $c_n^{(k)}$ of $f^{(k)}$ are given by $c_n^{(k)} = (in)^k c_n$, and similarly for $a_n^{(k)}$ and $b_n^{(k)}$. The conclusions then follow from Bessel's inequality (applied to $f^{(k)}$) and its corollary. For the second part, we observe that since $\alpha > 1$,

$$\sum_{n\neq 0} |n^j c_n| \leq C \sum_{n\neq 0} |n|^{-(k-j+\alpha)} \leq 2C \sum_{n>0} n^{-\alpha} < \infty \quad \text{for } j \leq k.$$

Thus, by the Weierstrass M-test, the series $\sum_{-\infty}^{\infty}(in)^j c_n e^{in\theta}$ are absolutely and uniformly convergent for $j \leq k$. They therefore define continuous functions, which are the derivatives $f^{(j)}$ of $f(\theta) = \sum c_n e^{in\theta}$. ∎

The two halves of Theorem 2.6 are not perfect converses of each other; this is in the nature of things. (There is no simple "if and only if" theorem of this sort.) However, the moral is clear: the more derivatives a function has, the more rapidly its Fourier coefficients will tend to zero, and vice versa. In particular, f has derivatives of all orders precisely when its Fourier coefficients tend to zero more rapidly than any power of n (for example, $c_n = e^{-\epsilon|n|}$).

Another aspect of this phenomenon: the basic functions $e^{in\theta}$ or $\cos n\theta$ and $\sin n\theta$ are, of course, perfectly smooth individually, but they become more "jagged," that is, more highly oscillatory, as $n \to \infty$. In order to synthesize nonsmooth functions from these smooth ingredients, then, the proper technique is to use relatively large amounts of the high-frequency (i.e., large-n) functions.

These points are worth remembering; they are among the basic lessons of Fourier analysis. The reader can see how they work by examining the entries Table 1 in §2.1. For instance, the sawtooth wave in entry 2 is piecewise smooth but not continuous; its Fourier coefficients are on the order of n^{-1}. The triangle wave in entry 1 is one step better — continuous and piecewise smooth, with a piecewise smooth derivative; its Fourier coefficients are on the order of n^{-2}. These examples are quite typical.

EXERCISES

1. Derive the result of entry 16 of Table 1, §2.1, by using equation (2.17) and Theorem 2.4.
2. Starting from entry 16 of Table 1 and using Theorem 2.4, show that

   a. $$\theta^3 - \pi^2\theta = 12\sum_1^{\infty} \frac{(-1)^n \sin n\theta}{n^3} \qquad (-\pi \leq \theta \leq \pi);$$

   b. $$\theta^4 - 2\pi^2\theta^2 = 48\sum_1^{\infty} \frac{(-1)^{n+1} \cos n\theta}{n^4} - \frac{7\pi^4}{15} \qquad (-\pi \leq \theta \leq \pi);$$

   c. $$\sum_1^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90}.$$

3. Evaluate $\sum_1^{\infty}(2n-1)^{-4}\cos(2n-1)\theta$ by using entry 17 of Table 1.

4. By entry 8 of Table 1, we have

$$\sin\theta = \frac{2}{\pi} - \frac{4}{\pi}\sum_{1}^{\infty}\frac{\cos 2n\theta}{4n^2-1} \qquad (0 \le \theta \le \pi), \tag{$*$}$$

and we also have

$$\cos\theta = \frac{d}{d\theta}\sin\theta = -\int_{\pi/2}^{\theta}\sin\phi\, d\phi.$$

Show that the series $(*)$ can be differentiated and integrated termwise to yield two apparently different expressions for $\cos\theta$ for $0 < \theta < \pi$, and reconcile these two expressions. (Hint: Equation (2.17) is useful.)

5. Let $f(\theta)$ be the periodic function such that $f(\theta) = e^\theta$ for $-\pi < \theta \le \pi$, and let $\sum_{-\infty}^{\infty} c_n e^{in\theta}$ be its Fourier series; thus $e^\theta = \sum c_n e^{in\theta}$ for $|\theta| < \pi$. If we formally differentiate this equation, we obtain $e^\theta = \sum inc_n e^{in\theta}$. But then $c_n = inc_n$, or $(1 - in)c_n = 0$, so $c_n = 0$ for all n. This is obviously wrong; where is the mistake?

6. The Fourier series in entries 11 and 12 of Table 1 are clearly related: the second is close to being the derivative of the first. Find the exact relationship (a) by examining the series and (b) by examining the functions that the series represent.

7. How smooth are the following functions? That is, how many derivatives can you guarantee them to have?

a. $f(\theta) = \sum_{-\infty}^{\infty}\frac{e^{in\theta}}{n^{13/2} + 2n^6 - 1}$. b. $f(\theta) = \sum_{0}^{\infty}\frac{\cos n\theta}{2^n}$.

c. $f(\theta) = \sum_{0}^{\infty}\frac{\cos 2^n\theta}{2^n}$.

2.4 Fourier series on intervals

Fourier series give expansions of periodic functions on the line in terms of trigonometric functions. They can also be used to give expansions of functions defined on a finite interval in terms of trigonometric functions on that interval.

Suppose the interval in question is $[-\pi, \pi]$. (Other intervals can be transformed into this one by a linear change of variable; we shall discuss this point later.) Given a bounded, integrable function f on $[-\pi, \pi]$, we extend it to the whole real line by requiring it to be periodic of period 2π. Actually, to be completely consistent about this we should start out with f defined only on the half-open interval $(-\pi, \pi]$ or $[-\pi, \pi)$, or else (re)define f at the endpoints so that

$f(-\pi) = f(\pi)$. To be definite, we follow the first course of action; then the **periodic extension** of f to the whole line is given by

$$f(\theta + 2n\pi) = f(\theta) \quad \text{for all } \theta \in (-\pi, \pi] \text{ and all integers } n.$$

For instance, the periodic functions discussed in Examples 1 and 2 of §2.1 are the periodic extiensions of the functions $f(\theta) = |\theta|$ and $g(\theta) = \theta$ from $(-\pi, \pi]$ to the whole line.

If f is a piecewise smooth function on $(-\pi, \pi]$, we can expand its periodic extension in a Fourier series, and then by restricting the variable θ to $[-\pi, \pi]$, we obtain an expansion of the original function. All of what we have said in the previous sections applies to this situation, but there is one point that needs attention. If the original f is piecewise continuous or piecewise smooth on $[-\pi, \pi]$, then its periodic extension will be piecewise continuous or piecewise smooth on $\mathbf{R}$. However, even if f is perfectly smooth on $[-\pi, \pi]$, there will generally be discontinuities in the extended function or its derivatives at the points $(2n+1)\pi$, n an integer, where (so to speak) the copies of f are glued together. To be precise, suppose f is continuous on $[-\pi, \pi]$. Then the extension will be continuous at the points $(2n+1)\pi$ if and only if $f(-\pi) = f(\pi)$, and in this case the extension will have derivatives up to order k at $(2n+1)\pi$ if and only if $f^{(j)}(-\pi+) = f^{(j)}(\pi-)$ for $j \leq k$. (This is illustrated by the examples in §2.1: see Figures 2.1(a) and 2.2(a).) These phenomena must be kept in mind when one studies the relations between the smoothness properties of f and the size of its Fourier coefficients as in Theorem 2.6.

Two interesting variations can be made on this theme. Suppose now that we are interested in functions on the interval $[0, \pi]$ rather than $[-\pi, \pi]$. We can make such a function f into a 2π-periodic function, and hence obtain a Fourier expansions for it, by a twofold extension process: first we extend f in some simple way to the interval $[-\pi, \pi]$, then we extend the result periodically. There are two standard ways of performing the first step: we extend f to $[-\pi, \pi]$ by declaring it to be either even or odd. That is, we have the **even extension** f_{even} of f to $[-\pi, \pi]$ defined by

$$f_{\text{even}}(-\theta) = f(\theta) \text{ for } \theta \in [0, \pi]$$

and the **odd extension** f_{odd} of f to $[-\pi, \pi]$ defined by

$$f_{\text{odd}}(-\theta) = -f(\theta) \text{ for } \theta \in (0, \pi], \qquad f_{\text{odd}}(0) = 0.$$

(See Figure 2.5.) The advantage of using f_{even} or f_{odd} rather than any other extension is that the Fourier coefficients turn out very simply. Indeed, it follows from Lemma 2.2 of §2.1 that

$$\int_{-\pi}^{\pi} f_{\text{even}}(\theta) \cos n\theta \, d\theta = 2\int_{0}^{\pi} f(\theta) \cos n\theta \, d\theta, \qquad \int_{-\pi}^{\pi} f_{\text{even}}(\theta) \sin n\theta \, d\theta = 0,$$

whereas

$$\int_{-\pi}^{\pi} f_{\text{odd}}(\theta)\cos n\theta\, d\theta = 0, \qquad \int_{-\pi}^{\pi} f_{\text{odd}}(\theta)\sin n\theta\, d\theta = 2\int_0^{\pi} f(\theta)\sin n\theta\, d\theta.$$

Thus the Fourier series of f_{even} involves only cosines and the Fourier series of f_{odd} involves only sines; moreover, the Fourier coefficients for these two cases can be computed in terms of the values of the original function f on $[0, \pi]$. We are thus led to the following definitions.

Definition. Suppose f is an integrable function on $[0, \pi]$. The series

$$\tfrac{1}{2}a_0 + \sum_1^{\infty} a_n \cos n\theta, \quad \text{where } a_n = \frac{2}{\pi}\int_0^{\pi} f(\theta)\cos n\theta\, d\theta,$$

is called the **Fourier cosine series** of f. The series

$$\sum_1^{\infty} b_n \sin n\theta, \quad \text{where } b_n = \frac{2}{\pi}\int_0^{\pi} f(\theta)\sin n\theta\, d\theta,$$

is called the **Fourier sine series** of f.

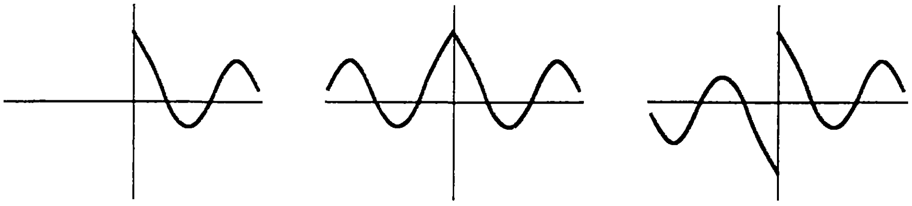


FIGURE 2.5. A function defined on $[0, \pi]$ (left), its even extension (middle), and its odd extension (right).

If f is piecewise continuous or piecewise smooth on $[0, \pi]$, its even periodic and odd periodic extensions will have the same properties on **R**, but as before, one must watch for extra discontinuities at the points $n\pi$ (n an integer) where the pieces are joined together. If f is continuous on $[0, \pi]$, the even periodic extension will be continuous everywhere, but its derivative will have jumps at the points $2n\pi$ or $(2n+1)\pi$ unless $f'(0+) = 0$ or $f'(\pi-) = 0$, respectively. The odd periodic extension is less forgiving: it will have discontinuities at the points $2n\pi$ or $(2n+1)\pi$ unless $f(0) = 0$ or $f(\pi) = 0$, respectively. (As for higher derivatives: there are potential problems with the odd-order derivatives of the even periodic extension and with the even-order derivatives of the odd periodic extension at the points $n\pi$.)

Example 1. Consider the function $f(\theta) = \theta$ on $[0, \pi]$. Its even and odd periodic extensions are given on $(-\pi, \pi)$ by $f_{\text{even}}(\theta) = |\theta|$ and $f_{\text{odd}}(\theta) = \theta$; these are the functions whose Fourier series we worked out in §2.1. Hence,

$$\theta = 2\sum_{1}^{\infty} \frac{(-1)^{n+1}\sin n\theta}{n} = \frac{\pi}{2} - \frac{4}{\pi}\sum_{1}^{\infty} \frac{\cos(2n-1)\theta}{(2n-1)^2} \qquad (0 < \theta < \pi).$$

Here f is perfectly smooth on $[0, \pi]$, but f_{odd} has discontinuities at the odd multiples of π. f_{even} is continuous everywhere, but its first derivative has discontinuities at all integer multiples of π. The reader may find other examples in Table 1.

At any rate, if we keep these remarks in mind and apply Theorem 2.1, we arrive at the following result.

Theorem 2.7. *Suppose f is piecewise smooth on $[0, \pi]$. The Fourier cosine series and the Fourier sine series of f converge to $\frac{1}{2}\left[f(\theta-) + f(\theta+)\right]$ at every $\theta \in (0, \pi)$. In particular, they converge to $f(\theta)$ at every $\theta \in (0, \pi)$ where f is continuous. The Fourier cosine series of f converges to $f(0+)$ at $\theta = 0$ and to $f(\pi-)$ at $\theta = \pi$; the Fourier sine series of f converges to 0 at both these points.*

The results of the previous section on termwise differentiation and uniform convergence can be applied to these series, provided that one takes account of the behavior at the endpoints as indicated above.

Finally, we may wish to consider periodic functions whose period is something other than 2π, or functions defined on intervals other than $[-\pi, \pi]$ or $[0, \pi]$. These situations can be reduced to the ones we have already studied by making a simple change of variable.

For instance, suppose $f(x)$ is a periodic function with period $2l$. (The factor of 2 is merely for convenience.) We make the change of variables

$$x = \frac{l\theta}{\pi}, \qquad g(\theta) = f(x) = f\left(\frac{l\theta}{\pi}\right).$$

Then g is 2π-periodic, so if it is piecewise smooth we can expand it in a Fourier series:

$$g(\theta) = \sum_{-\infty}^{\infty} c_n e^{in\theta}, \qquad c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} g(\theta)e^{-in\theta}\,d\theta.$$

If we now substitute $\theta = \pi x/l$ into these formulas, we obtain the $2l$-periodic Fourier series of the original function f:

$$f(x) = \sum_{-\infty}^{\infty} c_n e^{in\pi x/l}, \qquad c_n = \frac{1}{2l}\int_{-l}^{l} f(x)e^{-in\pi x/l}\,dx. \tag{2.20}$$

The corresponding formula in terms of cosines and sines is

$$f(x) = \tfrac{1}{2}a_0 + \sum_{1}^{\infty}\left[a_n \cos\frac{n\pi x}{l} + b_n \sin\frac{n\pi x}{l}\right], \tag{2.21}$$

where

$$a_n = \frac{1}{l}\int_{-l}^{l} f(x)\cos\frac{n\pi x}{l}\,dx, \qquad b_n = \frac{1}{l}\int_{-l}^{l} f(x)\sin\frac{n\pi x}{l}\,dx. \tag{2.22}$$

From this it follows that the Fourier cosine and sine expansions of a piecewise smooth function f on the interval $[0, l]$ are

$$f(x) = \tfrac{1}{2}a_0 + \sum_1^\infty a_n \cos\frac{n\pi x}{l}, \qquad a_n = \frac{2}{l}\int_0^l f(x)\cos\frac{n\pi x}{l}\,dx, \tag{2.23}$$

and

$$f(x) = \sum_1^\infty b_n \sin\frac{n\pi x}{l}, \qquad b_n = \frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}\,dx. \tag{2.24}$$

These formulas are probably worth memorizing; they are used very frequently. Another point worth remembering is that, just as in the case of Fourier series for periodic functions, *the constant term* $\frac{1}{2}a_0$ *in the Fourier cosine series of a function* f *on an interval is the mean value of* f *on that interval:* $\frac{1}{2}a_0 = l^{-1}\int_0^l f(x)\,dx$.

Example 2. Let us find the Fourier cosine and sine expansions of $f(x) = x$ on $[0, l]$. Having set $\theta = \pi x/l$, this amounts to finding the expansions of $g(\theta) = l\theta/\pi$ on $[0, \pi]$, which we have done above. Namely, for $0 < \theta < \pi$ we have

$$\frac{l\theta}{\pi} = \frac{2l}{\pi}\sum_1^\infty \frac{(-1)^{n+1}}{n}\sin n\theta = \frac{l}{2} - \frac{4l}{\pi^2}\sum_1^\infty \frac{1}{(2n-1)^2}\cos(2n-1)\theta,$$

so for $0 < x < l$,

$$x = \frac{2l}{\pi}\sum_1^\infty \frac{(-1)^{n+1}}{n}\sin\frac{n\pi x}{l} = \frac{l}{2} - \frac{4l}{\pi^2}\sum_1^\infty \frac{1}{(2n-1)^2}\cos\frac{(2n-1)\pi x}{l}.$$

Finally, what if we wish to use an interval of length l whose left endpoint is not 0, say $[a, a+l]$? Simply apply the preceding formulas to $g(x) = f(x+a)$; we leave it to the reader to write out the resulting formulas for $f(x)$.

EXERCISES

In Exercises 1–6, find both the Fourier cosine series and the Fourier sine series of the given function on the interval $[0, \pi]$. Try to use the results of Table 1, §2.1, rather than working from scratch. To what values do these series converge when $\theta = 0$ and $\theta = \pi$?

1. $f(\theta) = 1$.
2. $f(\theta) = \pi - \theta$.
3. $f(\theta) = \sin\theta$.
4. $f(\theta) = \cos\theta$.

5. $f(\theta) = \theta^2$. (For the sine series, use entries 1 and 17 of Table 1.)
6. $f(\theta) = \theta$ for $0 \le \theta \le \frac{1}{2}\pi$, $f(\theta) = \pi - \theta$ for $\frac{1}{2}\pi \le \theta \le \pi$. (For the sine series, use entry 11 of Table 1, and for the cosine series, entry 2.)

In Exercises 7–11, expand the function in a series of the indicated type. For example, "sine series on $[0, l]$" means a series of the form $\sum b_n \sin(n\pi x/l)$. Again, use previously derived results as much as possible.

7. $f(x) = 1$; sine series on $[0, 6\pi]$.
8. $f(x) = 1 - x$; cosine series on $[0, 1]$.
9. $f(x) = 1$ for $0 < x < 2$, $f(x) = -1$ for $2 < x < 4$; cosine series on $[0, 4]$.
10. $f(x) = lx - x^2$; sine series on $[0, l]$.
11. $f(x) = e^x$; series of the form $\sum_{-\infty}^{\infty} c_n e^{2\pi i n x}$ on $[0, 1]$.
12. Suppose f is a piecewise continuous function on $[0, \pi]$ such that $f(\theta) = f(\pi - \theta)$. (That is, the graph of f is symmetric about the line $\theta = \frac{1}{2}\pi$.) Let a_n and b_n be the Fourier cosine and sine coefficients of f. Show that $a_n = 0$ for n odd and $b_n = 0$ for n even.

2.5 Some applications

At this point we are ready to complete the solutions of the boundary value problems that were discussed in §1.3. The first of these problems was the one describing heat flow on an interval $[0, l]$, where the initial temperature is $f(x)$ and the endpoints are held at temperature zero,

$$u_t = k u_{xx}, \qquad u(x,0) = f(x) \quad \text{for } x \in [0, l], \qquad u(0,t) = u(l,t) = 0 \quad \text{for } t > 0,$$

and we derived the following series as a candidate for a solution:

$$\begin{gathered} u(x,t) = \sum_1^\infty b_n \exp\left(\frac{-n^2\pi^2 kt}{l^2}\right) \sin\frac{n\pi x}{l}, \\ \text{where} \quad f(x) = \sum_1^\infty b_n \sin\frac{n\pi x}{l}. \end{gathered} \tag{2.25}$$

The questions that we left open were: (1) Can the initial temperature f be expressed as such a sine series? (2) Does this formula for u actually define a solution of the heat equation with the given boundary conditions? We now know that the answer to the first question is *yes*, provided that f is piecewise smooth on $[0, l]$ (certainly a reasonable requirement from a physical point of view): we have merely to expand f in its Fourier sine series (2.24). Let us therefore address the second question.

The individual terms in the series for u solve the heat equation, by the way they were constructed. Moreover, when $t > 0$ the factor $\exp(-n^2\pi^2 kt/l)$ tends to zero very rapidly as $n \to \infty$, so that the series converges nicely. More precisely, since the coefficients b_n tend to zero as $n \to \infty$ and in particular are bounded by some constant C, for any positive ϵ we have

$$0 < \left| b_n \exp\left(\frac{-n^2\pi^2 kt}{l^2}\right) \sin\frac{n\pi x}{l} \right| \le Ce^{-\delta n^2} \quad \text{for } t \ge \epsilon, \text{ where } \delta = \frac{\pi^2 k\epsilon}{l^2}.$$

The same sort of estimate also holds for the first t-derivative and the first two x-derivatives of the terms of the series for u, with an extra factor of n^2 thrown in. Since $\sum_1^\infty n^k e^{-\delta n^2}$ converges for any k, we see by the Weierstrass M-test that these derived series converge absolutely and uniformly in the region $0 \le x \le l$, $t \ge \epsilon$, and we deduce that termwise differentiation of the series is permissible. Conclusion: u is a solution of the heat equation.

As for the boundary conditions, it is evident that $u(0,t) = u(l,t) = 0$, since all the terms in the series for u vanish at $x = 0, l$, and $u(x,0) = f(x)$ by the choice of the coefficients b_n. However, as we pointed out in §1.1, we really want a bit more, namely, the continuity condition that $u(x,t)$ should tend to zero as $x \to 0, l$ and to $f(x)$ as $t \to 0$. The preceding discussion shows that the first of these requirements is always satisfied: for each $t > 0$, the series for $u(x,t)$ converges uniformly on $[0,l]$,so $u(x,t)$ is a continuous function of x. (In particular, as $x \to 0$ or $x \to l$, $u(x,t)$ approaches $u(0,t)$ or $u(l,t)$, which are zero.) Moreover, if f is continuous and piecewise smooth on $[0,l]$ and $f(0) = f(l) = 0$, then the odd periodic extension of f is continuous and piecewise smooth, so $\sum |b_n| < \infty$ by Theorem 2.5. The Weierstrass M-test then shows that the series for u converges uniformly on the whole region $0 \le x \le l$, $t \ge 0$, and hence that u is continuous there; in particular, $u(x,t) \to u(x,0) = f(x)$ as $t \to 0$.

If f has discontinuities or is nonzero at the endpoints, it is still true that $u(x,t) \to f(x)$ as $t \to 0$ provided that $0 < x < l$ and f is continuous at x, but the proof is more delicate. (See Walker [53], §4.7.) We shall not concern ourselves with such technical refinements, as we have already established the main point: under reasonable assumptions on the initial temperature f, the function u satisfies all the desired conditions.

One question we have not really settled is the uniqueness of the solution. That is, we have constructed *one* solution; is it the only one? The answer is *yes*. One can argue that any solution $u(x,t)$ must be expandable in a Fourier sine series in x for each t and then use the differential equation to show that the coefficients of this series must be the ones we found above. Alternatively, one can invoke some general uniqueness theorems for solutions of the heat equation; see John [33] or Folland [24]. Similar considerations apply to the other problems we solve later, and we shall not worry about uniqueness from now on except in situations where pitfalls actually exist.

Lest the reader become too complacent, however, let us briefly consider the problem of solving the heat equation for times $t < 0$ — that is, given the temperature distribution at time $t = 0$, to reconstruct the distribution at earlier times. If we take $t < 0$ in (2.25), the factors $e^{-n^2\pi^2 kt/l^2}$ tend rapidly to *infinity* rather than zero as $n \to \infty$, with the result that the series for $u(x,t)$ will almost certainly *diverge* unless the coefficients b_n of the initial function f happen to decay extremely rapidly as $n \to \infty$. Thus (2.25), in general, does *not* give a solution to the heat equation when $t < 0$. This is not merely a failure of mathematical technique, however. The initial value problem for the time-reversed heat equation is simply not well posed, a reflection of the fundamental physical fact that the direction of time is irreversible in diffusion processes. One can mix hot water and cold water to get warm water, but one cannot then separate the warm water back into hot and cold components! More to the point, one cannot tell by examining the warm water which part was initially hot and which part was initially cold, or what their initial temperatures were.

Exactly the same considerations apply to the problem of heat flow on $[0, l]$ with insulated endpoints,

$$u_t = ku_{xx}, \qquad u(x,0) = f(x), \qquad u_x(0,t) = u_x(l,t) = 0,$$

whose solution is

$$u(x,t) = \tfrac{1}{2}a_0 + \sum_1^\infty a_n \exp\left(\frac{-n^2\pi^2kt}{l^2}\right)\cos\frac{n\pi x}{l},$$

where

$$f(x) = \tfrac{1}{2}a_0 + \sum_1^\infty a_n \cos\frac{n\pi x}{l}.$$

The only difference is that now we expand f in its Fourier cosine series (2.22).

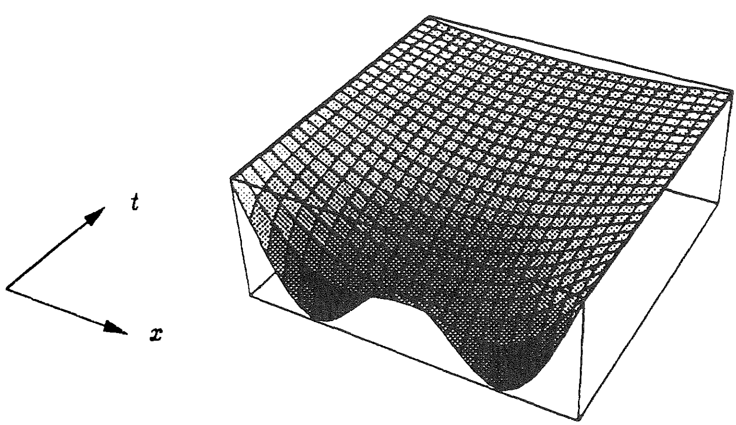


FIGURE 2.6. The solution (2.25) of the heat equation with $k = \frac{1}{4}$, $l = 1$, $b_1 = -\frac{1}{3}$, $b_2 = -\frac{1}{6}$, and $b_n = 0$ for $n > 2$, on the region $0 \le x \le 1$, $0 \le t \le 1$.

Let us pause a moment to see what these solutions tell us about the physics of the situation. In the limit as $t \to \infty$, the exponential factors all vanish, so the solution u approaches a constant — namely, 0 in the case where the endpoints are held at temperature 0 and $\frac{1}{2}a_0$ in the case of insulated endpoints. The first of these is easy to understand: the interval $[0, l]$ comes into thermal equilibrium with its surroundings. As for the second, if we recall that

$$a_0 = \frac{2}{l}\int_0^l f(x)\,dx,$$

we see that the limiting temperature $\frac{1}{2}a_0$ is simply the average value of the initial temperature. In other words, no heat enters or escapes, so the various parts of the interval simply come into thermal equilibrium with each other. Moreover, in both cases, the high-frequency terms (i.e., the terms with n large) damp out more quickly than the low-frequency terms: this expresses the fact that the diffusion of heat tends to quickly smooth out local variations in temperature. A simple illustration of these assertions can be found in Figure 2.6.

Now let us turn to the problem of the vibrating string:

$$u_{tt} = c^2 u_{xx}, \qquad u(x,0) = f(x), \quad u_t(x,0) = g(x), \quad u(0,t) = u(l,t) = 0.$$

According to the discussion in §1.3, we should expand f and g in their Fourier sine series,

$$f(x) = \sum_1^\infty b_n \sin\frac{n\pi x}{l}, \qquad g(x) = \sum_1^\infty B_n \sin\frac{n\pi x}{l}, \tag{2.26}$$

and then take

$$u(x,t) = \sum_1^\infty \sin\frac{n\pi x}{l}\left(b_n \cos\frac{n\pi ct}{l} + \frac{lB_n}{n\pi c}\sin\frac{n\pi ct}{l}\right). \tag{2.27}$$

Here the analysis is more delicate than for the heat equation, because there are no exponentially decreasing factors in this series to help the convergence. The series (2.27) for u is likely to converge about as well as the sine series for f and g, but if we differentiate it twice with respect to x or t in order to verify the wave equation, we introduce a factor of n^2; and this may well be enough to destroy the convergence.

We can avoid this difficulty by making sufficiently strong smoothness assumptions on f and g. For instance, let us suppose that f and g are of class $C^{(3)}$ and $C^{(2)}$, respectively, that f''' and g'' are piecewise smooth, and that f, g, f'', and g'' vanish at the endpoints 0 and l. These conditions guarantee that the odd periodic extensions of f and g will have the same smoothness properties (even at the points $n\pi$), and hence, by Theorem 2.6, that the coefficients b_n and B_n will satisfy

$$|b_n| \le Cn^{-4}, \qquad |B_n| \le Cn^{-3}.$$

Now the nth term in the series (2.27) will be dominated by n^{-4}, and if we differentiate it twice in either x or t it is still dominated by n^{-2}. Since $\sum_1^\infty n^{-2}$

converges, the M-test guarantees the absolute and uniform convergence of the twice-derived series, and we are in business.

This is not entirely satisfactory, however. It is physically reasonable to assume that f and g are continuous and perhaps piecewise smooth, but one may — and indeed should — have the feeling that the extra differentiability assumptions are annoyances that reflect a failure of technique rather than a real difficulty in the original problem.

We can obtain more insight into this problem by recalling the trigonometric identities

$$\sin a \cos b = \tfrac{1}{2}\Big[\sin(a+b)+\sin(a-b)\Big], \qquad \sin a \sin b = \tfrac{1}{2}\Big[\cos(a-b)-\cos(a+b)\Big],$$

by means of which the series (2.27) can be rewritten

$$u(x,t) = \frac{1}{2}\sum_1^\infty b_n \sin\frac{n\pi}{l}(x+ct) + \frac{1}{2}\sum_1^\infty b_n \sin\frac{n\pi}{l}(x-ct) + \frac{1}{2c}\sum_1^\infty \frac{lB_n}{n\pi}\cos\frac{n\pi}{l}(x-ct) - \frac{1}{2c}\sum_1^\infty \frac{lB_n}{n\pi}\cos\frac{n\pi}{l}(x+ct).$$

The first two sums on the right are just the Fourier sine series for f, evaluated at $x \pm ct$, and the last two are (up to constant factors) just the Fourier sine series for g, integrated once and then evaluated at $x \pm ct$. To restate this: let us suppose that f and g are piecewise smooth, so that the expansions (2.26) are valid on the interval $(0, l)$. We use the formulas (2.26) to extend f and g from this interval to the whole line; that is, we extend f and g to $\mathbf{R}$ by requiring them to be odd and $2l$-periodic. We then have

$$u(x,t) = \frac{1}{2}\Big[f(x+ct)+f(x-ct)\Big] + \frac{1}{2c}\Big[G(x+ct)-G(x-ct)\Big], \tag{2.28}$$

where G is any antiderivative of g.

From this closed formula it is perfectly plain that if f is twice differentiable and g is once differentiable, then u satisfies the wave equation, for

$$\frac{\partial^2}{\partial x^2} f(x \pm ct) = \frac{1}{c^2}\frac{\partial^2}{\partial t^2} f(x \pm ct) = f''(x \pm ct), \tag{2.29}$$

and likewise for G. Even here the differentiability assumptions seem a bit artificial; one would like, for example, to allow f to be a function with corners in order to model plucked strings. Indeed, in some sense the first equation in (2.29) should be correct, simply as a formal consequence of the chain rule, even if f'' is ill-defined. The idea that is crying to be set free here is the notion of a "weak solution" of a differential equation, which enables one to consider functions u defined by (2.28) as solutions of the wave equation even when the requisite derivatives of f and g do not exist. We shall say more about this in §9.5.

Another point should be raised here. One does not have to go through Fourier series to produce the formula (2.28) for the solution of the vibrating string problem; an elementary derivation is sketched in Exercise 6 of §1.1. It is then fair to ask what good the complicated-looking formula (2.27) is when the simple (2.28) is readily available. There are two good answers. First, the trick in Exercise 6, §1.1, that quickly produces the general solution of the 1-dimensional wave equation does not work for other equations (including the higher-dimensional wave equation), whereas the Fourier method and its generalizations often do. Second, although (2.28) tells you what you see if you look at a vibrating string, (2.27) *tells you what you hear when you listen to it.* The ear, unlike the eye, has a built-in Fourier analyzer that resolves sound waves into their components of different frequencies, which are perceived as musical tones.* Typically, the first term in the series (2.27) is the largest one, so one hears the note with frequency $2\pi c/l$ colored by the "overtones" at the higher frequencies $2\pi nc/l$ with $n > 1$.

The difference in the convergence properties of the series solutions (2.25) and (2.27) of the heat and wave equations reflects a difference in the physics: diffusion processes such as heat flow tend to smooth out any irregularities in the initial data, whereas wave motion propagates singularities. Thus, the solution (2.25) of the heat equation becomes smoother as t increases, and this is reflected in the exponential decay of the high-frequency terms. (See the discussion of smoothness versus rates of convergence at the end of §2.3.) However, any sharp corners in the initial configuration of a vibrating string will not disappear but merely move back and forth, as is clear from (2.28); hence there is no improvement in the rate of convergence of the solution (2.27). (Compare Figures 2.6 and 2.7, which show solutions of the heat and wave equations with the same initial values up to a constant factor and the same boundary conditions; the initial variations damp out in the first case, but not in the second.)

We shall see other applications of Fourier expansions of functions on an interval in Chapter 4. Fourier expansions are also the natural tool for analyzing periodic functions on the line. In practice, there are two principal sources of such functions. The first is the angular variable in polar or cylindrical coordinates or the longitudinal angular variable in spherical coordinates; in this context periodicity is an immediate consequence of the geometry of the situation. The other is physical phenomena that vary periodically (or approximately periodically) in time, such as certain types of electrical signals, the length of a day, daily or seasonal variations in temperature, and so forth.

As an example, let us analyze the variations in temperature beneath the ground due to the daily and seasonal fluctuations of temperature at the surface of the earth. We shall concern ourselves only with the temperature near a particular spot on the surface, over distances of (say) at most 100 meters. We therefore neglect the fact that at great depths the earth is hotter than at the surface, and we assume that (i) the earth is of uniform composition; (ii) the temperature at the surface is a function $f(t)$ of time only, not of position; (iii) $f(t)$ is periodic

* Of course, this is an oversimplification.

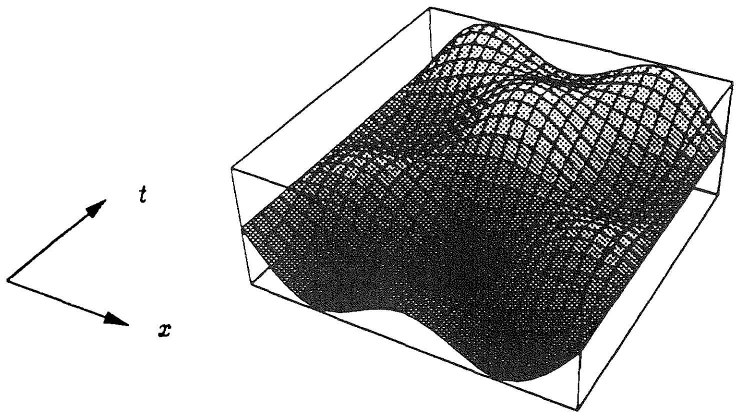


FIGURE 2.7. The solution (2.27) of the wave equation with $l = c = 1$, $b_1 = -0.2$, $b_2 = -0.1$, $b_n = 0$ for $n > 2$, and $B_n = 0$ for all n, on the region $0 \le x \le 1$, $0 \le t \le 1$.

of period 1 and so has a Fourier series

$$f(t) = \sum_{-\infty}^{\infty} c_n e^{2\pi i n t}.$$

(We may take the unit of time to be 1 year, so that the dominant terms in the series will be $n = \pm 1$, corresponding to seasonal variations, and $n = \pm 365$, corresponding to daily variations. With a bit more accuracy, we could take the unit of time to be 4 years and the dominant terms to be $n = \pm 4$ and $n = \pm 1461$ ($= \pm 4 \times 365\frac{1}{4}$). Or, we could take an even longer period to account for long-term climatic changes.) The boundary value problem to be solved is therefore

$$u_t = k u_{xx} \quad \text{for } x > 0, \qquad u(0,t) = f(t).$$

Since f is periodic in t, we expect u to have the same property, so we look for solutions of the form

$$u(x,t) = \sum_{-\infty}^{\infty} C_n(x) e^{2\pi i n t}.$$

Taking on faith that this series can be differentiated termwise, we find that

$$u_t = \sum_{-\infty}^{\infty} (2\pi i n) C_n(x) e^{2\pi i n t}, \qquad u_{xx} = \sum_{-\infty}^{\infty} C_n''(x) e^{2\pi i n t}.$$

Hence, taking into account the initial condition, we have

$$C_n''(x) - 2\pi i n k^{-1} C_n(x) = 0, \qquad C_n(0) = c_n.$$

Since the square roots of $2in$ are $\pm(1+i)n^{1/2}$ if $n > 0$ and $\pm(1-i)|n|^{1/2}$ if $n < 0$, the general solution of this differential equation is

$$\begin{aligned} a \exp\left((1+i)\sqrt{\frac{\pi n}{k}}\,x\right) + b \exp\left(-(1+i)\sqrt{\frac{\pi n}{k}}\,x\right) &\quad \text{if} \quad n > 0, \\ a \exp\left((1-i)\sqrt{\frac{\pi |n|}{k}}\,x\right) + b \exp\left(-(1-i)\sqrt{\frac{\pi |n|}{k}}\,x\right) &\quad \text{if} \quad n < 0, \\ ax + b &\quad \text{if} \quad n = 0. \end{aligned}$$

In each case we must take $a = 0$ because of the physical requirement that the temperature should remain bounded as x increases. (In effect we are imposing a boundary condition at $x = \infty$ to supplement the one at $x = 0$.) The initial condition then implies that $b = c_n$. Hence, upon grouping together the nth and $(-n)$th terms, we obtain the solution

$$\begin{aligned} u(x,t) = c_0 + \sum_1^\infty \exp\left(-\sqrt{\frac{\pi n}{k}}\,x\right) \\ \times \left[c_n \exp\left(2\pi int - i\sqrt{\frac{\pi n}{k}}\,x\right) + c_{-n} \exp\left(-2\pi int + i\sqrt{\frac{\pi n}{k}}\,x\right)\right]. \end{aligned}$$

It is now easy to check that this function u really does solve the problem.

The main features to be noted here are the following. First, all of the nonconstant terms in u (the ones with $n \neq 0$) die out exponentially fast as x increases, and the high-frequency ones die out faster than the low-frequency ones. (In actual fact, the daily variations in temperature become negligible at a depth of a few centimeters, and the seasonal ones become negligible at a depth of a few meters, where the temperature remains essentially constant at the annual mean c_0.) Second, the temperature variations at depth x are out of phase with those at the surface by an amount proportional to x and $\sqrt{|n|}$, because the heat takes time to penetrate. For example, if the $n = 1$ term, representing the main seasonal variations, is the dominant one, at depth $x = \sqrt{\pi k}$ the temperature is warmer in winter and cooler in summer.

In considering the usefulness of Fourier series or any other sort of infinite series, one should not lose sight of the fact that the partial sums of the series provide approximations to the full sum, and that such approximations may be just what one needs to obtain a computationally manageable solution to a problem. The questions about smoothness and rates of convergence that we have discussed in some detail have a computational as well as a theoretical significance: rapidly converging series such as (2.25) yield accurate answers much more readily than slowly converging ones such as (2.27). An interesting discussion of rates of convergence of infinite series, and the implications for numerical calculations, can be found in Boas [7].

On the other hand, in many situations one knows the initial data only to a finite degree of accuracy. For example, one may be studying a physical quantity

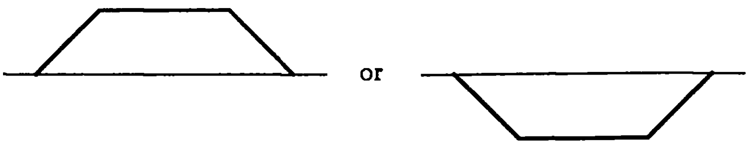
$f(t)$ that varies periodically with the time t, and one may know the values of f approximately from physical measurements. In this context the point of Fourier analysis is that it is usually appropriate to take a trigonometric polynomial of fairly low degree, whose coefficients are determined so as to fit the data well, as a mathematical model for f.

EXERCISES

1. A rod 100 cm long is insulated along its length and at both ends. Suppose that its initial temperature is $u(x,0) = x$ (x in cm, u in °C, t in sec, $0 \le x \le 100$), and that its diffusivity coefficient k is 1.1 cm^2/sec (about right if the bar is made of copper).
   a. Find the temperature $u(x,t)$ for $t > 0$. (It is something of the form $50 + \sum_1^\infty a_n(t)\cos(n\pi x/100)$, and $a_n(t) = 0$ when n is even.)
   b. Show that the first three terms of the series (i.e., $50 + a_1(t)\cos(\pi x/100) + a_3(t)\cos(3\pi x/100)$) give the temperature accurately to within 1 unit when $t = 60$. Using this fact, find $u(0,60)$, $u(10,60)$, and $u(40,60)$.

   $$\text{Hint: } \sum_1^\infty \frac{1}{(2n-1)^2} = \frac{\pi^2}{8}, \quad \text{so} \quad \sum_3^\infty \frac{1}{(2n-1)^2} = \frac{\pi^2}{8} - 1 - \frac{1}{9} \approx .123.$$

   c. Find a number $T > 0$ such that $u(x,t)$ is within 1 unit of its equilibrium value 50 for all x when $t > T$.
2. Redo Exercises 1a and 1c with $k = .01$ (a reasonable figure if the bar is made of ceramic). Now how many terms of the series are needed to get an accuracy of 1 unit when $t = 60$?
3. Consider again the copper rod of Exercise 1 ($k = 1.1$). Suppose that the rod is initially at temperature 100°C and that the ends are subsequently put into a bath of ice water (at 0°C).
   a. Assuming no heat loss along the length of the rod, find the temperature $u(x,t)$ at subsequent times.
   b. Use your answer to find $u(50,t)$ numerically when $t = 30, 60, 300, 3600$.
   c. Prove that your answers in (b) are correct to within 1 unit. (Hint: The series for $u(50,t)$ is alternating.)
4. Consider a vibrating string occupying the interval $0 \le x \le l$. Suppose the string is plucked in the middle in such a way that its initial displacement $u(x,0)$ is $2mx/l$ for $0 \le x \le \frac{1}{2}l$ and $2m(l-x)/l$ for $\frac{1}{2}l \le x \le l$ (so the maximum displacement, at $x = \frac{1}{2}l$, is m), and its initial velocity $u_t(x,0)$ is zero.
   a. Find the displacement $u(x,t)$ as a Fourier series.
   b. Describe $u(x,t)$ in the closed form (2.28) and show that at times $t > 0$, $u(x,t)$ (as a function of x) typically looks like the following figure:



5. Consider a vibrating string as in Exercise 4. Suppose the string is plucked at $x = a$ instead of $x = \frac{1}{2}l$, so the initial displacement is mx/a for $0 \le x \le a$ and $m(l-x)/(l-a)$ for $a \le x \le l$, and the initial velocity is zero.
   a. Find the displacement $u(x,t)$ as a Fourier series. (Entry 11 of Table 1, §2.1, will be helpful.)
   b. Convince yourself that the terms with large n contribute more to $u(x,t)$ when a becomes closer to l. (Musically: plucking near the end gives a tone with more higher harmonics.)
6. Suppose the string in Exercise 4 is initially struck in the middle so that its initial displacement is zero but its initial velocity $u_t(x,0)$ is 1 for $|x - \frac{1}{2}l| < \delta$ and 0 elsewhere. Find $u(x,t)$ for $t > 0$.
7. Suppose that the temperature at time t at a point on the surface of the earth is given by

$$u(0,t) = 10 - 7\cos 2\pi t - 5\cos 2\pi(365)t.$$

   (Here u is measured in °C and t is measured in years; the coefficients are roughly correct for Seattle, Washington.) Suppose that the diffusivity coefficient of the earth is $k = .003\ \text{cm}^2/\text{sec} \approx 9.46\ \text{m}^2/\text{yr}$.
   a. Find $u(x,t)$ for $x > 0$.
   b. At what depth x do the daily variations in temperature become less than 1 unit? What about the annual variations?

2.6 Further remarks on Fourier series

There is much more to be said about Fourier series than is contained in this chapter. Some good references for further information on both the theoretical aspects of the subject and its applications are the books of Dym-McKean [19], Körner [34], and Walker [53]. Also recommended is the article of Coppel [15] on the history of Fourier analysis and its influence on other branches of mathematics, and the articles by Zygmund, Hunt, and Ash in [2]. Finally, the serious student of Fourier analysis should become acquainted with the treatise of Zygmund [58], which gives an encyclopedic account of the subject.

We conclude this chapter with a brief discussion of a few other interesting aspects of Fourier series.

The transform point of view

Given a 2π-periodic function, its sequence $\{c_n\}$ of Fourier coefficients can be regarded as a function $\widehat{f}$ whose domain is the integers:

$$\widehat{f}(n) = c_n = \int_{-\pi}^{\pi} f(\theta)e^{-in\theta}d\theta.$$

The mapping $f \to \widehat{f}$ is thus a *transform* that converts periodic functions on the line to functions on the integers. The inverse transform is the operation which assigns to a function $\phi(n)$ on the integers (that decays suitably as $n \to \infty$) the function $\sum_{-\infty}^{\infty} \phi(n)e^{in\theta}$. In principle all the information in f is also contained in its transform $\widehat{f}$, and vice versa, but the information may be encoded in a more convenient form on one side or the other. For example, Theorem 2.2 shows that the transform converts differentiation into a simple algebraic operation: $\widehat{f'}(n) = in\widehat{f}(n)$. We shall return to this point of view in Chapter 7.

Comparison with Taylor series

Perhaps the most well known and widely used type of infinite series expansion for functions is the Taylor series, and it is of interest to compare the features of Taylor series and Fourier series.

In order for a function $f(x)$ to have a Taylor expansion about a point x_0,

$$f(x) = \sum_{0}^{\infty} \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n, \qquad |x - x_0| < r,$$

f must have derivatives of all orders at x_0. If it does, the coefficients of the Taylor series are determined by these derivatives, and hence by the values of f in an arbitrarily small neighborhood of x_0. The rate at which these coefficients grow or decay as $n \to \infty$ is related to the radius of convergence of the series and hence to the distance from x_0 to the nearest singularity of f (in the complex plane). In general the partial sums of the Taylor series provide excellent approximations to f near x_0 but are often of little use when $|x - x_0|$ is large.

In contrast, a function f need have only minimal smoothness properties in order to have a convergent Fourier expansion

$$f(x) = \sum_{-\infty}^{\infty} \left((2l)^{-1} \textstyle\int_a^{a+2l} f(y)e^{-in\pi y/l}dy\right) e^{in\pi x/l}, \qquad x \in (a, a+2l).$$

The coefficients of this series depend on the values of f over the entire interval $(a, a+2l)$. The rate at which they decay as $n \to \infty$ is related to the differentiability properties of f, or rather of its periodic extension. The partial sums of the Fourier series will converge to f only rather slowly if f is not very smooth, but they tend to provide good approximations over the whole interval $(a, a+2l)$.

Thus Taylor series and Fourier series are of quite different natures: the first one is intimately connected with the local properties of f near x_0, whereas the second is related to global properties of f. There is a situation, however, in which the two can be seen as aspects of the same thing. Namely, suppose f is an analytic function of the complex variable z in some disc $|z - z_0| < R$. If we write $z - z_0$ in polar coordinates as $re^{i\theta}$, the Taylor series for f about z_0 turns into a Fourier series in θ for each fixed $r < R$:

$$\sum_0^\infty a_n(z - z_0)^n = \sum_0^\infty (a_n r^n)e^{in\theta}.$$

The formula (2.5) for the Fourier coefficients, in this case, is nothing but the Cauchy integral formula for the derivatives of f at z_0. This connection between Fourier analysis and complex function theory has many interesting consequences, which are discussed in more advanced books such as Dym-McKean [19] and Zygmund [58].

Convergence of Fourier series

The study of the convergence of Fourier series has a long and complex history. The convergence theorems we have presented in §§2.2–3 are sufficient for many purposes, but they do not give the whole picture. Here we briefly indicate a few other highlights of the story. In the first place, the hypotheses of our Theorem 2.1 can be weakened. The same conclusion is obtained if we assume only that f is of "bounded variation" on the interval $[-\pi, \pi]$, which means that it can be written as the difference of two nondecreasing functions on that interval. (It is not hard to show that piecewise smooth functions have this property.) On the other hand, it has been known since 1876 that there are continuous periodic functions whose Fourier series diverge at some points, and for almost a century it was an open question whether the Fourier series of a continuous function could be guaranteed to converge at *any* point. An affirmative answer was obtained only in 1966, with a deep theorem of L. Carleson to the effect that the Fourier series of any square-integrable function f must converge to f at "almost every" point, in a sense that we shall describe in §3.3. See the article by Hunt in [2].

One fundamental fact that has emerged over the years is that, in many situations, simple pointwise convergence of a series is not the appropriate thing to look at; and there are many other notions of convergence that may be used. For example, there is uniform convergence, which is stronger than pointwise convergence; there is also "pth power mean" convergence, according to which the series $\sum_1^\infty f_n$ converges to f on the interval $[a, b]$ if

$$\lim_{N\to\infty} \int_a^b \left| \sum_1^N f_n(x) - f(x) \right|^p dx = 0.$$

We shall say much more about the case $p = 2$ in the next chapter. There are also ways of summing divergent series that can be used to advantage; we shall now briefly discuss the simplest of these.

It is easy to see that if a sequence $\{a_n\}$ converges to a, then the average $k^{-1}\sum_1^k a_n$ of its first k terms also converges to a as $k \to \infty$, but these averages may converge when the original sequence does not. For example, the sequence

$$1, 0, 1, 0, 1, 0, 1, 0, \ldots$$

is divergent; but the average of its first k terms is $(k+1)/2k$ or $1/2$ according as k is odd or even, and this tends to $1/2$ as $k \to \infty$. Now, given an infinite series $\sum_0^\infty b_n$ with partial sums $s_N = \sum_0^N b_n$, the average of its first $k+1$ partial sums,

$$\frac{1}{k+1}(s_0 + s_1 + \cdots + s_k),$$

is called its kth **Cesàro mean**, and the series is said to be **Cesàro summable** to the number s if its Cesàro means (rather than just its partial sums) converge to s. We then have the following theorem, due to L. Fejér.

Theorem 2.8. *If f is 2π-periodic and piecewise continuous on $\mathbf{R}$, then the Fourier series of f is Cesàro summable to $\frac{1}{2}\left[f(\theta-) + f(\theta+)\right]$ at every θ. Moreover, if f is everywhere continuous, the Cesàro means of the series converge to f uniformly.*

The proof of this theorem is similar in spirit to that of Theorem 2.1; it can be found, for example, in §2 of Körner [34] or §2.7 of Walker [53]. The significance of the theorem is twofold. First, it gives a way of recovering a piecewise continuous function f from its Fourier coefficients when the Fourier series fails to converge. Second, even when the Fourier series of f does converge, its Cesáro means tend to give better approximations to f than its partial sums: for example, they converge uniformly to f whenever f is continuous, whereas the partial sums converge uniformly only under stronger smoothness conditions (cf. Theorem 2.5).

The Gibbs phenomenon

Suppose f is a periodic function. If f has a discontinuity at x_0, the Fourier series of f cannot converge uniformly on any interval containing x_0, because the uniform limit of continuous functions is continuous. In fact, for the Fourier series of a piecewise smooth function f, the lack of uniformity manifests itself in a particularly dramatic way known as the **Gibbs phenomenon**: as one adds on more and more terms, the partial sums overshoot and undershoot f near the discontinuity and thus develop "spikes" that tend to zero in width but *not* in height. One can see this in Figure 2.8, which shows the fortieth partial sum of the Fourier series of the sawtooth wave function

$$f(\theta) = \pi - \theta \text{ for } 0 < \theta < 2\pi, \qquad f(\theta + 2n\pi) = f(\theta).$$

A precise statement and proof of the Gibbs phenomenon for this function is outlined in Exercise 1. It can be shown that the same behavior occurs at any discontinuity of any piecewise smooth function. See Körner [34] and Hewitt-Hewitt [28] for interesting discussions of the Gibbs phenomenon.

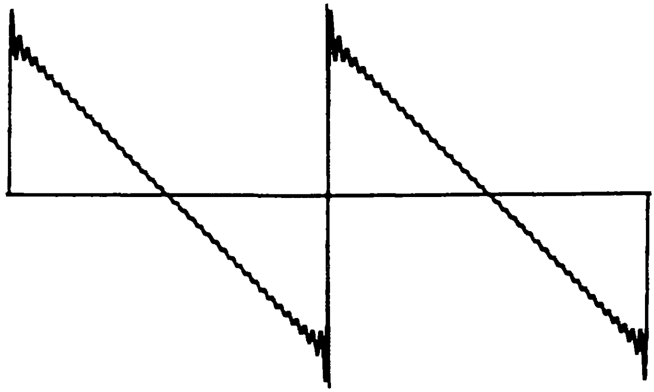


FIGURE 2.8. Graph of $2\sum_1^{40} n^{-1} \sin n\theta$, $-2\pi < \theta < 2\pi$ (an illustration of the Gibbs phenomenon).

EXERCISE

1. Recall from Table 1, §2.1, that $f(\theta) = 2\sum_1^{\infty} n^{-1} \sin n\theta$ is the 2π-periodic function that equals $\pi - \theta$ for $0 < \theta < 2\pi$. Let

$$g_N(\theta) = 2\sum_1^N \frac{\sin n\theta}{n} - (\pi - \theta),$$

so that $g(\theta)$ is the difference between $f(\theta)$ and its Nth partial sum for $0 < \theta < 2\pi$.

a. Show that $g_N'(\theta) = 2\pi D_N(\theta)$ where D_N is the Dirichlet kernel (2.10).

b. Using (2.12), show that the first critical point of $g_N(\theta)$ to the right of zero occurs at $\theta_N = \pi/(N + \frac{1}{2})$, and that

$$g_N(\theta_N) = \int_0^{\theta_N} \frac{\sin(N + \frac{1}{2})\theta}{\sin \frac{1}{2}\theta} d\theta - \pi.$$

c. Show that

$$\lim_{N\to\infty} g_N(\theta_N) = 2\int_0^{\pi} \frac{\sin\phi}{\phi} d\phi - \pi.$$

(Hint: Let $\phi = (N + \frac{1}{2})\theta$.) This limit is approximately equal to .562. Thus the difference between $f(\theta)$ and the Nth partial sum of its Fourier series develops a spike of height .562 (but of increasingly narrow width) just to the right of $\theta = 0$ as $N \to \infty$. (There is another such spike on the left.)