Global attractor and non homogeneous equilibria for a non local evolution equation in an unbounded domain

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Abstract

In this work we consider the nonlocal evolution equation \( \frac{\partial u(x,t)}{\partial t} = -u(x,t) + \tanh(\beta J * u(x,t) + h) \) which arises in models of phase separation. We prove the existence of a compact global attractor in some weighted spaces and the existence of a distinguished non homogeneous equilibrium: the ‘critical droplet’.

Key words: well posedness, global attractor, non homogeneous equilibrium.

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1 Introduction

We consider here the non local evolution equation

\[
\frac{\partial u(x,t)}{\partial t} = -u(x,t) + \tanh(\beta J * u(x,t) + h)
\]

where \( u(x,t) \) is a real function on \( \mathbb{R} \times \mathbb{R}_+; \) \( \beta > 1, J \in C^1(\mathbb{R}) \) is a non negative even function with integral equal to 1 supported in the interval \([-1,1]\), \( \beta \) and \( h \) are positive constants. The \( * \) product denotes convolution, namely:

\[
(J * u) (x) = \int_{\mathbb{R}} J(x-y)u(y) \, dy.
\]
This equation arises as a continuum limit of one-dimensional Ising spin systems with Glauber dynamics and Kac potentials \([0]\); \(u\) represents then a magnetization density and \(\beta^{-1}\) the temperature of the system.

It is a simple matter to obtain well posedness of the problem (1) in various function spaces. In fact, since the right-hand-side of (1) defines a Lipschitz map in both \(L^2(\mathbb{R})\) and the space of bounded continuous functions, for instance, the basic properties of (local) existence and uniqueness follow from standard results of O.D.Es in Banach spaces. On the other hand, the investigation of qualitative properties of the flow given by (1) is a much harder topic. To begin with, the equilibria are given by the solutions of a nonlinear integral equation for which many methods used to analyze, for example, the boundary value problems that appear in the case of semilinear parabolic problems are not available. Also, the fact that the spatial domain is unbounded makes the proof of regularization properties more difficult, since we lack some of the familiar embedding properties present in bounded domains. Another difficulty is the ‘nonlocal character’ of the problem, that is, the fact that the variation in time does not depend only on the value of the solution in an arbitrary small neighborhood. As it is shown in [0] this can give rise to complicated dynamics even in the case of scalar parabolic equations.

If \(\beta \leq 1\), equation (1) has only one (stable) equilibrium. If \(\beta > 1\) and \(0 \leq h < h^*\), where \(h^*\) is implicitly defined by equation (2) below, (1) has three spatially homogeneous equilibria \(m_\beta^0, m_\beta^+\) and \(m_\beta^-\) each of them identically equal to one of the three roots of the equation
\[
m_\beta = \tanh (\beta m_\beta + h) .
\] (2)

In the last years several works dedicated to the analysis of this model appeared in the literature. In [0] and [0], the existence and uniqueness (modulo translations) of a travelling front connecting the equilibria \(m_\beta^-\) and \(m_\beta^+\) is proved. In the case \(h = 0\) the existence of a ‘standing’ wave as well as its stability properties are analysed in [0] and [0]. In this case, many equilibria periodic in \(x\) also exist, as shown in [0] and [0] However, much remains to be done, especially to understand the ‘geometric properties’ of the flow generated by (1). As a first step in this direction we prove in sections and that, in an appropriate phase spaces, the system is dissipative in the sense of [0], that is, it has a global compact attractor. Taking into account the results [0] and [0] it can be seen this cannot be true in the space of bounded continuous functions with the sup norm, since the travelling wave connecting \(m_\beta^-\) \(m_\beta^+\) has empty \(\omega\)-limit set. Our proof uses some ideas from [0], where a reaction-diffusion equation defined in \(\mathbb{R}^n\) is considered (see also [0] and [0] for related work).

In section , we prove the existence of a non-homogeneous stationary solution referred to as the ‘bump’ or ‘critical droplet’ in the literature, which seems to
play an important role in the behavior of trajectories. The existence of such a solution has been established in [0] for \( h \) ‘sufficiently close’ to 1, by a rather lengthy argument using the Newton’s method. Our proof is based on a simple topological argument and is valid for all \( 1 < h < h^\ast \). On the other hand, we do not obtain the same precise estimates. The main ingredients in our proof are the strong monotonicity properties of the flow (see Theorem ) and the existence of the (partially defined) functional

\[
\mathcal{F}(u) = \int [f(u(x)) - f(m^-_{\beta})] \, dx + \frac{1}{4} \int \int J(x - y) [u(x) - u(y)]^2 \, dx \, dy
\]

(3)

where \( f(u) \) (the free energy density) is given by

\[
f(u) = -\frac{1}{2} u^2 - \frac{h}{\beta} u - \beta^{-1} i(u)
\]

and \( i(u) \) is the entropy density

\[
i(u) = -\frac{1 + u}{2} \log \left\{ \frac{1 + u}{2} \right\} - \frac{1 - u}{2} \log \left\{ \frac{1 - u}{2} \right\}.
\]

The functional \( \mathcal{F} \) acts as a Lyapunov functional where defined, that is, it decreases along the solutions of (1). The function \( f(u) \) is a ‘double-well potential’ with minima at \( m^+_{\beta} \) and \( m^-_{\beta} \) and, therefore, the first term in (3) penalizes other values of \( u(x) \). The second term is a measure of the energy resulting from \( u(x) \) being non constant. Observe that, if we look at the action on the ‘slowly varying functions’, the approximation

\[
\frac{1}{4} \int \int J(x - y) [u(x) - u(y)]^2 \, dx \, dy \\
\approx \frac{1}{4} \int \int J(x - y) \left( \frac{\partial u(x)}{\partial x} \right)^2 (x - y)^2 \, dx \, dy
\]

becomes accurate and this term becomes the one that appears in the familiar Lyapunov functional for the one dimensional parabolic equation.
2 Well posedness and existence of a global attractor in $L^2(\mathbb{R}, \rho)$

In this section we consider the flow generated by (1) in the space $L^2(\mathbb{R}, \rho)$ defined by

$$L^2(\mathbb{R}, \rho) = \{ u \in L^1_{\text{loc}}(\mathbb{R}) | \int_{\mathbb{R}} u(x)^2 \rho(x) dx < +\infty \}$$

with norm $\| u \|_{L^2(\mathbb{R}, \rho)} = (\int_{\mathbb{R}} u^2 \rho(x) dx)^{\frac{1}{2}}$. Here $\rho$ is an integrable function (so the constant functions are included in the space). For definiteness, we may take $\rho(x) = \frac{1}{\pi} (1 + x^2)^{-1}$ (so the constant function equal to 1 has norm 1). The corresponding higher-order Sobolev space $H^k(\mathbb{R}, \rho)$ is the space of functions $u \in L^2(\mathbb{R}, \rho)$ whose distributional derivatives up to order $k$ are also in $L^2(\mathbb{R}, \rho)$, with norm $\| u \|_{H^k(\mathbb{R}, \rho)} = \left( \sum_{j=0}^{k} \| \partial^j u \|_{L^2(\mathbb{R}, \rho)}^2 \right)^{\frac{1}{2}}$.

Let $F$ be the function in $L^2(\mathbb{R}, \rho)$ defined by the right-hand side of (1), that is, $F(u) = -u + \tanh (\beta J^* u + h)$. We show $F$ is a globally Lipschitz function in $L^2(\mathbb{R}, \rho)$. More precisely, we have

**Lemma 1** Suppose $\sup\{\rho(x) | y - 1 \leq x \leq y + 1\} \leq K \rho(y)$, for some constant $K$ and all $y \in \mathbb{R}$. Then the function $F$ is a globally Lipschitz function in $L^2(\mathbb{R}, \rho)$ with

$$||F(u) - F(v)||_{L^2(\mathbb{R}, \rho)} \leq (1 + \beta \sqrt{K}) ||u - v||_{L^2(\mathbb{R}, \rho)}.$$ 

**Proof:** Since $J$ is bounded and compact supported, $(J * u)(x)$ is well defined for $u \in L^1_{\text{loc}}$. Since $|\tanh (\beta J * u + h)(x)| \leq 1$ it follows that $F(u) \in L^2(\mathbb{R}, \rho)$ if $u \in L^2(\mathbb{R}, \rho)$. Now

$$||J * u||_{L^2(\mathbb{R}, \rho)}^2 = \int_{\mathbb{R}} |(J * u)(x)|^2 \rho(x) dx$$

$$\leq \int_{\mathbb{R}} \left( \int_{\mathbb{R}} |(J(x - y)) \frac{1}{2} (J(x - y)) \frac{1}{2} |u(y)| dy \right)^2 \rho(x) dx$$

$$\leq ||J||_{L^1} \int_{\mathbb{R}} \left( \int_{\mathbb{R}} |J(x - y)| |u(y)|^2 dy \right) \rho(x) dx$$

$$\leq \int_{\mathbb{R}} \int_{\mathbb{R}} |J(x - y)| |u(y)|^2 \rho(x) dxdy$$
\[
\int_{\mathbb{R}} \left( \int_{\mathbb{R}} J(x-y) \rho(x) \, dx \right) |u(y)|^2 \, dy 
\leq \int_{\mathbb{R}} K \rho(y) |u(y)|^2 \, dy 
\leq K \|u\|_{L^2(\mathbb{R}, \rho)}^2
\]

Therefore, we have for any \( u, v \in L^2(\mathbb{R}, \rho) \)

\[
\|F(u) - F(v)\|_{L^2(\mathbb{R}, \rho)} \leq \|u - v\|_{L^2(\mathbb{R}, \rho)} 
+ \|\tanh(\beta J * u + h) - \tanh(\beta J * v + h)\|_{L^2(\mathbb{R}, \rho)} 
\leq \|u - v\|_{L^2(\mathbb{R}, \rho)} + \|((\beta J * u + h) - (\beta J * v + h))\|_{L^2(\mathbb{R}, \rho)} 
\leq \|u - v\|_{L^2(\mathbb{R}, \rho)} + \|((\beta J * (u - v))\|_{L^2(\mathbb{R}, \rho)} 
\leq (1 + \beta \sqrt{K}) \|u - v\|_{L^2(\mathbb{R}, \rho)}
\] (4)
as claimed. \qed

**Remark 2** The hypothesis \( \sup \{ \rho(x) \mid y - 1 \leq x \leq y + 1 \} \leq K \rho(y) \) of Lemma 1 is verified, for instance, if \( \rho(x) = \frac{1}{\pi}(1 + x^2)^{-1} \), with \( K = 3 \). Also, we can take \( K \) arbitrarily close to \( 1 \) by suitably choosing \( \rho \). For instance, we can take \( \rho \) equal to a constant in the interval \([-R, R]\) and equal to \( \frac{1}{\pi}(1 + x^2)^{-\frac{1}{2}} \) outside the interval \([-R - 1, R + 1]\).

From Lemma and basic theory of O.D.Es in Banach spaces it follows that, for any \( u_0 \in L^2(\mathbb{R}, \rho) \), (1) has a unique local solution in \( C([0, \tau(u_0)], L^2(\mathbb{R}, \rho)) \cap C^1((0, \tau(u_0)], L^2(\mathbb{R}, \rho)) \) for some \( \tau(u_0) > 0 \) which is continuous with respect to \( u_0 \). By standard arguments, using the variation of constants formula and Gronwall’s inequality, it follows that these solutions are actually globally defined, that is \( \tau(u_0) = \infty \) for any \( u_0 \).

From now on, we denote by \( T(t) \) the global semi flow generated by (1) in \( L^2(\mathbb{R}, \rho) \).

We now turn to the proof of the existence of a global maximal invariant compact set \( A \subset L^2(\mathbb{R}, \rho) \) for the flow \( T \), which attracts bounded sets of \( L^2(\mathbb{R}, \rho) \) (the global attractor) (see [0] or [0]). Before presenting the details, we outline the argument. By the variation of constants formula, a solution \( u(x, t) \) of (1) with initial condition \( u_0 \) is given by

\[
u(x, t) = e^{-t}u_0(x) + \int_0^t e^{s-t} \tanh \{ \beta (J * u)(x, s) + h \} \, ds
= U(t)u_0 + K(t, u_0).\]
The first part in the decomposition above is arbitrarily small if $t$ is big. The second part is bounded and regularizes the flow. This fact does not immediately imply compactness since the spatial domain is unbounded. However, it does imply compactness for the restriction to bounded subsets of the spatial domain. On the other hand, using the fact that the norm of bounded functions in the phase space is ‘concentrated’ in a bounded part of the domain, we obtain the smallness of the remainder. In this way we are able to show that the measure of non compactness of the image under the flow of a bounded set is arbitrary small which gives the asymptotic smoothness in the sense of [0].

We recall that a set $B \subset L^2(\mathbb{R}, \rho)$ is an absorbing set for the flow $T$ if, for any bounded set $C$ in $L^2(\mathbb{R}, \rho)$, there is a $t_1 > 0$ such that $T(t)C \subset B$ for any $t \geq t_1$ (see [0]).

Let $B_r$ denote the ball with center in the origin and radius $r$. We then have

**Lemma 3** $B_{1+\varepsilon}$ is an absorbing set for the flow $T(t)$ for any $\varepsilon > 0$.

*Proof:* If $u(x, t)$ is a solution of (1) with initial condition $u_0$ we have, by the variation of constants formula

$$u(x, t) = e^{-t}u_0(x) + \int_0^t e^{s-t} \tanh \{\beta(J * u)(x, s) + h\} \, ds$$

(5)

Thus

$$|u(x, t)| \leq e^{-t}|u_0(x)| + \int_0^t e^{s-t} \, ds$$

$$\leq e^{-t}|u_0(x)| + 1$$

By the triangle inequality

$$||u(\cdot, t)||_{L^2(\mathbb{R}, \rho)} \leq e^{-t}||u_0(\cdot)||_{L^2(\mathbb{R}, \rho)} + 1$$

(6)

Therefore, $u(\cdot, t) \in B_{1+\varepsilon}$ for any $t > \ln\left(\frac{||u_0||_{L^2(\mathbb{R}, \rho)}}{\varepsilon}\right)$, and the result is proved. □

**Lemma 4**. For any $\eta > 0$, there exists $t_\eta$ such that $T(t_\eta)B_{1+\varepsilon}$ has a finite covering by balls of $L^2_\rho$ with radius smaller than $\eta$. 

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Proof: From Lemma, it follows that $B_{1+\varepsilon}$ is invariant. Given $u_0 \in B_{1+\varepsilon}$, we consider the system

$$
\begin{cases}
v_t = -v, \quad v(0) = u_0 \\
w_t = -w + \tanh \{\beta(J*(v+w)) + h\}, \quad w(0) = 0
\end{cases}
$$

(7)

If $(v, w)$ is a solution of (7) then $u = v + w$ is a solution of (1) with $u(0) = u_0$. Conversely, any solution $u$ of (1) can be written as $u = v + w$, with $(v, w)$ a solution of (7).

By the variation of constants formula

$$w(x, t) = \int_0^t e^{s-t} \tanh \{\beta(J*u)(x, s) + h\} \; ds$$

(8)

and, therefore

$$|w(x, t)| \leq \int_0^t e^{s-t} \; ds \leq 1$$

(9)

for any $t \geq 0$, $x \in \mathbb{R}$.

For any $x \in \mathbb{R}$, the convolution $(J'*u)(x, s) = \int_{\mathbb{R}} J'(x-y)u(y, s) \; dy$ is well defined if $u \in L^1_{loc}$, since $J'$ has compact support. In particular, if $u \in L^2_{loc}$

$$|J'*u(x, s)| \leq \int_{x-1}^{x+1} |J'(x-y)| |u(y, s)| \; dy \leq \left( \int_{x-1}^{x+1} |J'(x-y)|^2 \; dy \right)^{1/2} \left( \int_{x-1}^{x+1} |u(y, s)|^2 \; dy \right)^{1/2}$$

$$= ||J'||_{L^2(\mathbb{R})} \left( \int_{x-1}^{x+1} |u(y, s)|^2 \; dy \right)^{1/2}$$

If $u \in B_{1+\varepsilon}$, $R > 0$ and $x \in [-R, R]$, we have

$$|J'*u(x, s)| \leq ||J'||_{L^2(\mathbb{R})} \left( \int_{-R}^{R} |u(y, s)|^2 \; dy \right)^{1/2}$$
\[
\begin{align*}
\leq ||J'||_{L^2(R)} & \left( \int_{\mathbb{R}} |u(y, s)|^2 \chi_{R+1} \rho(y) \frac{1}{\rho_R} \, dy \right)^{\frac{1}{2}} \\
\leq \frac{1}{\sqrt{\rho_R}} ||J'||_{L^2(\mathbb{R})} & \left( \int_{\mathbb{R}} |u(y, s)|^2 \rho(y) \, dy \right)^{\frac{1}{2}} \\
\leq \frac{1}{\sqrt{\rho_R}} ||J'||_{L^2(\mathbb{R})} & ||u(\cdot, s)||_{L^2_\rho} \\
\leq \frac{1 + \varepsilon}{\sqrt{\rho_R}} ||J'||_{L^2(\mathbb{R})}
\end{align*}
\]

where \( \rho_R = \inf\{|\rho(x)| \mid x \in [-R-1, R+1]\} \), and \( \chi_R \) is the characteristic function of the interval \([-R, R]\).

Therefore, differentiating \( w \) with respect to \( x \), we obtain for \( t \geq 0 \)

\[
\frac{\partial}{\partial x} w(x, t) = \beta \int_0^t e^{s-t} \sech^2(\beta J * u(x, s) + h) \cdot (J' * u)(x, s) \, ds.
\]

Thus

\[
\left| \frac{\partial}{\partial x} w(x, t) \right| \leq \beta \int_0^t e^{s-t} |(J' * u)(x, s)| \, ds
\]

\[
\leq \frac{\beta(1 + \varepsilon)}{\sqrt{\rho_R}} ||J'||_{L^2(\mathbb{R})} \int_0^t e^{s-t} \, ds
\]

\[
\leq \frac{\beta(1 + \varepsilon)}{\sqrt{\rho_R}} ||J'||_{L^2(\mathbb{R})}
\]

(11)

Since \( v(t, \cdot) = e^{-t}u_0(\cdot) \), given \( \eta > 0 \), we may find \( t(\eta) \) such that if \( t \geq t(\eta) \) then \( ||v(t, \cdot)||_{L^2(\mathbb{R}, \rho)} \leq \frac{\eta}{2} \), for any \( u_0 \in B_{1+\varepsilon} \).

Now, let \( R > 0 \) be chosen such that \( \int_{\mathbb{R}} (1 - \chi_R) \rho(x) \, dx \leq \frac{\eta}{4} \). Then, by (9)

\[
||(1 - \chi_R)w(t, \cdot)||_{L^2(\mathbb{R}, \rho)} \leq \int_{\mathbb{R}} (1 - \chi_R) |w(x, t)| \rho(x) \, dx \leq \frac{\eta}{4}
\]

Also, by (9) and (11) the restriction of \( w(\cdot, t) \) to the interval \([-R, R]\) is bounded in \( H^1[-R, R] \) (by a constant independent of \( u \in B_{1+\varepsilon} \) and, therefore the set \( \{\chi_RW(t, \cdot)\} \) with \( w(0, \cdot) \in B_1 \) is a compact subset of \( L^2_\rho \) for any \( t > 0 \) and, thus, it can be covered by a finite number of balls with radius smaller than \( \frac{\eta}{4} \).
Therefore, since \( u(t, \cdot) = v(t, \cdot) + \chi_R w(t, \cdot) + (1 - \chi_R)w(t, \cdot) \), it follows that 
\( T_{t_0}B_{1+\varepsilon} \) has a finite covering by balls of \( L^2_\rho \) with radius smaller than \( \eta \).

\[ \square \]

We denote by \( \omega(C) \) the *omega-limit* of a set \( C \).

**Theorem 5**  The set \( A = \omega(B_{1+\varepsilon}) \) is a global attractor for the flow \( T(t) \) generated by (1) in \( L^2(\mathbb{R}, \rho) \) which is contained in the ball of radius one.

**Proof:** From Lemma , it follows immediately that \( A \) is contained in the ball of radius 1. Also, since \( A \) is invariant by the flow, it follows that \( A \subset T(t)(B_{1+\varepsilon}) \), for any \( t \geq 0 \) and then, from Lemma , we obtain that the measure of non compactness of \( A \) is zero. Thus \( A \) is relatively compact and, being closed, also compact. Finally, if \( D \) is a bounded set in \( L^2(\mathbb{R}, \rho) \) then \( T(\bar{t})D \subset B_{1+\varepsilon} \) for \( \bar{t} \) big enough and, therefore, \( \omega(D) \subset \omega(B_{1+\varepsilon}) \).

Once estimates in \( L^2(\mathbb{R}, \rho) \) have been obtained for solutions in the attractor, we can use a ‘bootstrap argument to obtain \( \mathcal{C}^k \) estimates;

**Theorem 6**  The global attractor \( A \) is bounded in \( \mathcal{C}^k \), for any integer \( k \geq 0 \).

**Proof:** If \( u(x, t) \) is a solution of (1) in \( A \), we have by the variation of constants formula

\[ u(x, t) = e^{-(t-t_0)}u(x, t_0) + \int_{t_0}^t e^{s-t} \tanh \{ \beta(J * u)(x, s) + h \} \, ds \]

Since \( ||u||_{L^2(\mathbb{R}, \rho)} \leq 1 \) for any \( u \in A \), we obtain, letting \( t_0 \to -\infty \)

\[ u(x, t) = \int_{-\infty}^t e^{s-t} \tanh \{ \beta(J * u)(x, s) + h \} \, ds \quad (12) \]

The equality is in \( L^2(\mathbb{R}, \rho) \) but, since the right-hand side is as regular as \( J \) we have

\[ |u(x, t)| \leq \int_{-\infty}^t e^{s-t} \, ds \leq 1 \quad (13) \]

for any \( t \geq 0, x \in \mathbb{R} \).

From (13), we obtain
\begin{align*}
|J' \ast u(x, s)| & \leq \int_{\mathbb{R}} |J'(x - y)u(y, s)| \, dy \\
& \leq \int_{\mathbb{R}} |J'(x - y)| \, dy \\
& = ||J'||_{L^1(\mathbb{R})} \quad \quad (14)
\end{align*}

Differentiating in (12) with respect to \( x \), we obtain for \( t \geq 0 \)

\[
\frac{\partial}{\partial x} u(x, t) = \beta \int_{-\infty}^{t} e^{s-t} \text{sech}^2(\beta J \ast u(x, s) + h) \cdot (J' \ast u)(x, s) \, ds,
\]

which is well-defined by arguments entirely similar to the ones used in the proof of Lemma.

Therefore we have, using (14)

\[
\frac{\partial}{\partial x} u(\cdot, t) \leq \beta \int_{-\infty}^{t} e^{s-t} |(J' \ast u)(x, s)| \, ds \\
\leq \beta ||J'||_{L^1(\mathbb{R})} \int_{-\infty}^{t} e^{s-t} \, ds \\
\leq \beta ||J'||_{L^1(\mathbb{R})}
\]

Observe that, from this estimate, we have

\[
|J' \ast u'(x, s)| \leq \int_{\mathbb{R}} |J'(x - y)u'(y, s)| \, dy \\
\leq \beta ||J'||_{L^1(\mathbb{R})} \int_{\mathbb{R}} |J'(x - y)| \, dy \\
= \beta ||J'||_{L^1(\mathbb{R})}^2
\]

Differentiating once more, we obtain

\[
\frac{\partial^2}{\partial x^2} u(x, t) = \\
\int_{-\infty}^{t} e^{s-t} \left\{ 2\beta^2 \text{sech}(\beta J \ast u(x, s)) \cdot \text{sech} \text{tanh}(\beta J \ast u(x, s)) + \right. \\
\left. 2 \beta^2 \text{sech}^2(\beta J \ast u(x, s)) \cdot \text{sech} \right\} \int_{-\infty}^{t} e^{s-t} \, ds.
\]

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\[(J' * u(x,s))^2 + \beta \text{sech}^2(\beta J * u(x,s))(J' * u'(x,s)) \}\ d\sigma \\

and so

\[
\left\| \frac{\partial^2}{\partial x^2} u(x,t) \right\| \leq \beta \int_{-\infty}^{t} e^{s-t} 2\beta (J' * u(x,s))^2 + J' * u'(x,s) \ d\sigma \\
\leq \beta \int_{-\infty}^{t} e^{s-t} 2\beta ||J'||^2_{L^1(\mathbb{R})} + \beta ||J'||^2_{L^1(\mathbb{R})} \ d\sigma \\
\leq 3\beta^2 ||J'||^2_{L^1(\mathbb{R})}
\]

In the same way, we can obtain bounds for the derivatives of \(u\) of any order, in terms of \(J, J'\) and derivatives of lower order of \(u\), concluding the proof. (We can also obtain these estimates in terms of \(u\) and higher order derivatives of \(J\), if \(J\) is smooth enough).

\[\square\]

**Remark 7** The function in \(L^2(\mathbb{R}, \rho)\) defined by the right-hand-side of (1) is Hadamard but not Fréchet differentiable. Using (4) (see remark ) is not difficult to show that the spectrum of the (Hadamard) linearization around the equilibrium point \(m^-\) is located on a semi plane \(\text{Re}(\lambda) < -\gamma < 0\), but \(m^-\) is not locally stable since there exists a travelling wave solution connecting \(m^-\) to \(m^+\) (see \[0\]). This example shows that the ‘principle of linearized stability’ does not hold if only Hadamard differentiability is assumed. The fact that Hadamard differentiability is not enough to obtain a Hartman-Grobman type result has already been observed in \[0\].

In spite of the above remark, it will be important in section the observation that \(m^-\) is indeed locally stable for the flow generated by (1) in the space of (weighted) bounded functions restricted to the invariant set

\[
\Sigma^0 := \{ u \in C_b(\mathbb{R}) : ||u||_{\infty} \leq 1, \ u \text{ is even, increasing in } ] - \infty, 0[ \}
\text{and } u(x) - m^- \in L^2(\mathbb{R}).
\]

We can prove this is also true in \(L^2(\mathbb{R}, \rho)\). This is the content of the next two results.

**Lemma 8** Given \(\varepsilon > 0\), there exists \(\delta > 0\) and \(\tau > 0\) such that, if \(u \in \Sigma^0\) and \(\|u - m^-\|_{L^2(\mathbb{R}, \rho)} < \delta\), then \(\|T(\tau)u - m^-\|_{\infty} \leq \varepsilon\).

**Proof:** Let \(\varepsilon > 0\) be given and fix \(\tau\) such that \(e^{-\tau} \leq \frac{\varepsilon}{4}\). Now, for any \(\eta > 0\), \(M > 0\) there exists \(\delta_1 > 0\) such that, if \(u \in \Sigma^0\) and \(\|u - m^-\|_{L^2(\mathbb{R}, \rho)} < \delta_1\), then \(|u(x) - m^-| \leq \frac{\varepsilon}{4M}\) whenever \(|x| \geq \eta\). Since the flow is continuous in \(L^2(\mathbb{R}, \rho)\),
there exists $\delta_2$ such that $||T(s)u - m^-_\beta||_{L^2(\mathbb{R}, \rho)} \leq \delta$ if $||u - m^-_\beta||_{L^2(\mathbb{R}, \rho)} \leq \delta_2$, for $0 \leq s \leq \tau$. Choose $\eta$ and $M$ such that $\beta \left( 2\eta + \frac{\varepsilon}{M} \right) \leq \frac{\delta}{2}$, and let $u_0 \in \Sigma^0$ with $||u_0 - m^-_\beta||_{L^2(\mathbb{R}, \rho)} < \delta = \delta_2$. Denoting by $u(\cdot, t) = (T(t)u)(\cdot)$ the solution of (1) with initial condition $u_0(\cdot)$, we have

$$
||T(\tau)u - m^-_\beta||_\infty = |u(0, \tau) - m^-_\beta|
\leq |e^{-\tau}(u_0(0) - m^-_\beta)|
+ \int_0^\tau e^{-(\tau-s)} \left| \tanh(\beta J * u + h)(0, s) - \tanh(\beta J * m^-_\beta + h) \right| ds
\leq \frac{\varepsilon}{2} + \int_0^\tau e^{-(\tau-s)} \beta \left| \int_{\mathbb{R}} J(y) |u(y, s) - m^0_\beta| dy \right| ds
\leq \frac{\varepsilon}{2} + \int_0^\tau e^{-(\tau-s)} \beta \left( \int_{\mathbb{R}} J(y) |u(y, s) - m^0_\beta| dy \right) ds
\leq \frac{\varepsilon}{2} + \int_0^\tau e^{-(\tau-s)} \beta \left( \int_{-\infty}^\eta J(y) |u(y, s) - m^0_\beta| dy + \int_{\eta}^{+\infty} J(y) |u(y, s) - m^0_\beta| dy \right) ds
\leq \frac{\varepsilon}{2} + \int_0^\tau e^{-(\tau-s)} \beta \left( 2\eta \int_{-\infty}^\eta J(y) dy + \frac{\varepsilon}{M} \int_{-\infty}^{+\infty} J(y) dy \right) ds
\leq \frac{\varepsilon}{2} + \beta \left( 2\eta + \frac{\varepsilon}{M} \right) \int_0^\tau e^{-(\tau-s)} ds
\leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \int_0^\tau e^{-(\tau-s)} ds
\leq \varepsilon.
$$

(15)

**Corollary 9** The equilibrium $m^-_\beta$ is locally stable for the flow $T(t)$ in $L^2(\mathbb{R}, \rho) \cap \Sigma^0$.

**Proof:** From Theorem of section it follows that the equilibrium $m^-_\beta$ is locally stable for the flow defined by (1) in $C_b(\mathbb{R})$. From this and Lemma the result follows readily.

\[\square\]
3 Well posedness and existence of a global attractor in $C_\rho(\mathbb{R})$

In this section we consider the flow generated by (1) in the space $C_\rho(\mathbb{R})$ of continuous functions in $\mathbb{R}$ with norm $||u||_{C_\rho(\mathbb{R})} = \sup\{\rho(x)|u(x)| \mid x \in \mathbb{R}\} < \infty$, where the ‘weight’ $\rho$ is a continuous positive function in $\mathbb{R}$. We may also define ‘weighted’ $C^k$ spaces in the obvious way. If $m \leq \rho(x) \leq M$, where $m, M$ are positive constants, $C_\rho(\mathbb{R})$ is the space $C_b(\mathbb{R})$ of bounded continuous functions. This case has been considered for example in [0], [0], [0] and [0]. Some properties established in these works can be extended to an arbitrary weight $\rho$ (see for example theorem at the end of this section).

We will be mainly interested, however, in the case where the weight $\rho$ satisfies $\lim_{|x| \to \pm\infty} \rho(x) = 0$, for which strong dissipativity properties can be proved. To distinguish from the $L^2(\mathbb{R}, \rho)$ case, we denote the flow generated by (1) in $C_\rho(\mathbb{R})$ by $S(t)$. The proof of well-definiteness and existence of the global attractor for $S(t)$ follows by arguments similar to the previous section. For this reason, we only indicate some points were changes are needed, leaving details to the reader.

As before, let $F$ be the function in defined by the right-hand side of (1), (in $C_\rho(\mathbb{R})$ now, though). The following result is analogous to Lemma .

**Lemma 10** The function $F$ is globally Lipschitz in $C_\rho(\mathbb{R})$ with

$$||F(u) - F(v)||_{C_\rho(\mathbb{R})} \leq (1 + \beta) ||u - v||_{C_\rho(\mathbb{R})}.$$ 

**Proof:** It is easy to see that $F$ is well-defined. Furthermore, if $u, v \in C_\rho(\mathbb{R})$, we have

$$|J \ast u(x)| \rho(x) \leq \rho(x) \int_{x-1}^{x+1} J(x-y)|u(y)| \: dy$$

$$\leq \rho(x) \sup_{y \in [x-1, x+1]} |u(y)|$$

$$\leq \sup_{y \in [x-1, x+1]} |(u(y)|\rho(y))$$

$$\leq ||u||_{C_\rho(\mathbb{R})}$$

Thus

$$|F(u(x)) - F(v(x))| \rho(x)$$

$$\leq |u(x) - v(x)| \rho(x)$$

$$+ |\tanh(\beta J \ast u(x) + h) - \tanh(\beta J \ast v(x) + h)| \rho(x)$$

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\[
\leq ||u - v||_{C^r(\mathbb{R})} + \beta |J * (u - v)(x)|\rho(x)
\]
\[
\leq ||u - v||_{C^r(\mathbb{R})} + \beta ||u - v||_{C^r(\mathbb{R})}
\]

and we obtain
\[
||F(u) - F(v)||_{C^r(\mathbb{R})} \leq (1 + \beta K)||u - v||_{C^r(\mathbb{R})}
\]
as claimed. \(\square\)

We denote by \(B_r\) the ball with center in the origin and radius \(r\) in \(C^r(\mathbb{R})\).

**Lemma 11** If \(\rho(x) \leq M\) for any \(x \in \mathbb{R}\) then \(B_{M(1+\varepsilon)}\) is an absorbing set for the flow \(S(t)\) for any \(\varepsilon > 0\).

**Proof:** The proof is very similar to the one of Lemma and will be omitted. \(\square\)

**Lemma 12** Suppose \(\rho(x) \leq M\) for any \(x \in \mathbb{R}\) and \(\lim_{|x| \to \infty} \rho(x) = 0\). Then, for any \(\eta > 0\), there exists \(t_\eta\) such that \(T_{t_\eta}B_{M(1+\varepsilon)}\) has a finite covering by balls of \(C^r\) with radius smaller than \(\eta\).

**Proof:** The proof is again similar to the corresponding result of previous section (Lemma ). We need to replace estimate (10) by (16) below.

If \(x \in [-R, R]\), and \(\rho_R = \inf\{\rho(x) : x \in [-R - 1, R + 1]\}\)

\[
|J' \ast u(x, s)| \leq ||J'||_{L^1(\mathbb{R})} \sup_{y \in [x-1, x+1]} |u(y)|
\]
\[
\leq ||J'||_{L^1(\mathbb{R})} \frac{1}{\rho_R} \sup_{y \in [x-1, x+1]} |u(y)| \rho(y)
\]
\[
\leq ||J'||_{L^1(\mathbb{R})} \frac{1}{\rho_R} ||u||_{C^r(\mathbb{R})}
\]
\[
\leq \frac{||J'||_{L^1(\mathbb{R})}}{\rho_R} (1 + \varepsilon)
\]

The estimate (11) of the derivative \(\frac{\partial}{\partial x} w(x, t)\) is then replaced by

\[
\left|\frac{\partial}{\partial x} w(x, t)\right| \leq \beta \int_0^t e^{\varepsilon-t} |(J' \ast u)(x, s)| \, ds \leq \frac{\beta (1 + \varepsilon)}{\rho_R} ||J'||_{L^1(\mathbb{R})}
\]

(17)

The remaining part of the argument follows the proof of Lemma closely, replacing the compact embedding \(H^1[-R, R] \hookrightarrow L^2[-R, R]\) by the compact embedding \(C^1[-R, R] \hookrightarrow C[-R, R]\). \(\square\)
The following comparison result has been proven in [0] (Theorem 2.7) for the case $h = 0$ and $\rho = 1$. Its extension for $h \geq 0$ and arbitrary weight $\rho$ is straightforward.

**Theorem 13** If $u, v$ are solutions of (1) in $C_\rho(\mathbb{R})$ with $u(0, x) \leq v(0, x)$, for any $x \in \mathbb{R}$ then $u(t, x) \leq v(t, x)$, for any $x \in \mathbb{R}$, and $t \geq 0$.

We observe that any two continuous functions $u, v$, are included in $C_\rho(\mathbb{R})$ for a convenient $\rho$. Therefore the above extension, though straightforward, allow us to use comparison arguments for arbitrary continuous functions.

Using (), we can prove the following estimate on the size of the attractor.

**Proposition 14** For any $\beta \geq 0$, the attractor $A$, of the flow defined by (1) in $C_\rho(\mathbb{R})$ is contained in the rectangle $\{m_\beta^- \leq u(x) \leq m_\beta^+\}$ and, if $0 \leq \beta < 1$, $A = \{m_\beta^0\}$.

**Proof:** By Theorem , there is a constant $B$ such that $\|u\|_{\infty} \leq B$ for any $u \in A$ (actually we can take $B = 1$ by estimate (13)). Now, by uniqueness, the subspace of constant functions is invariant by the flow and, if $u(x, t) = u(t) = S(t)u_0$, $u(t)$ is a solution of the ordinary differential equation $\dot{u} = -u + \tanh(\beta u + h)$. The result then follows from Theorem and the properties of the solutions of this O.D.E. \hfill \Box

**Remark 15** We have obtained existence of global attractors in two different settings: $C_\rho(\mathbb{R})$ (in this section) and in $L^2(\mathbb{R}, \rho)$ (in the previous section). However, it follows from Theorem and Proposition that they are both contained in the space of bounded continuous functions. From this and the invariance property of the attractors we conclude that they are actually the same. In particular, property also holds for the flow in $L^2(\mathbb{R}, \rho)$. (I am indebted to the referee for this observation).

To conclude this section, we observe that remark also applies to the flow in $C_\rho(\mathbb{R})$.

4 Existence of the bump

In this section, we prove the existence of a special symmetric non homogeneous equilibrium, known as the ‘bump’ or ‘critical droplet’ in the literature. It should be observed that, due to the translation invariance property of the right-hand-side of (1), the existence of such a solution implies the existence of a whole one-parameter family of equilibria given by translation in the $x$-variable. For clarity, from now on, we fix our weight $\rho$ as $\rho(x) = \frac{1}{\pi}(1 + x^2)^{-1}$. 

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As before, we denote the flow in $C_ρ(R)$ by $S(t)$ and the space of bounded continuous functions in $R$ by $C_b(R)$.

Consider the functional

$$
F(u) = \int [f(u(x)) - f(m^-_β)] \, dx + \frac{1}{4} \int \int J(x - y)[u(x) - u(y)]^2 \, dx \, dy
$$

defined in (3). This functional (or rather, a similar one) was used in [0] (with $h = 0$) to prove the existence of special solutions connecting the two stable phases (the instanton). In fact, some results we prove for $F$ are similar to those obtained in [0] and [0]. However, there are some additional difficulties here due to the fact that the integrand now is not positive. To overcome these difficulties, we consider only the restriction of $F$ to the subset $Σ$ of $C_b(R) \subset C_ρ(R)$, defined by

$$
Σ := \{u \in C_b(R) : ||u||_\infty \leq 1, \ u \text{ is even, increasing in } -\infty, 0\}.
$$

(18)

Now, there is $m^* > m^0_β$, such that $f(ξ) ≥ f(m^-_β)$ for any $ξ ≤ m^*$. Therefore, if $u \in Σ$ the integrand in $F$ is positive for $x$ outside a finite interval and thus $F$ is well defined (possibly with value $+\infty$). Moreover, $F(u) < +\infty$ if, and only if $u(x)$ is close -in a certain sense- to $m^-_β$ in a neighborhood of the infinity. More precisely, we have the following result

**Theorem 16** The functional $F(u)$ is lower semi-continuous in $Σ$ with respect to the $L^2_{loc}$ topology and $F(u) < +\infty$ if and only if

$$
u(x) - m^-_β \in L^2(R) \quad (\text{with the Lebesgue measure}).
$$

(19)

**Proof:** Observe initially that there are constants $ε > 0$ and $K > 0$ such that

$$
|f(m) - f(m^-_β)| \leq K|\tilde{m} - m^-_β|^2 \quad \text{for } |m| \leq 1.
$$

(20)

$$
|\tilde{m} - m^-_β|^2 \leq K(f(m) - f(m^-_β)) \quad \text{for } m \in [-1, m^0_β + ε].
$$

(21)

We first prove (19). Suppose $u - m^-_β \in L^2(R)$. Then $u(x) ≤ m^0_β$, for $x$ outside some finite interval $[-L, L]$. Using (20), we obtain

$$
F(u) = \int f(u(x)) - f(m^-_β) \, dx + \frac{1}{4} \int \int J(x - y)[u(x) - u(y)]^2 \, dx \, dy
$$

16
\[
\leq \int_{|x| \leq L} f(u(x)) - f(m^-) \, dx + \int_{x \geq L} f(u(x)) - f(m^-) \, dx \\
+ \frac{1}{2} \int_{\mathbb{R}} \int_{\mathbb{R}} J(x - y)(u(x) - m^-)^2 + (m^- - u(y))^2 \, dxdy \\
\leq 2L(f(0) - f(m^-)) + K\|u - m^-\|_{L^2(\mathbb{R})}^2 \\
+ \int_{\mathbb{R}} (u(x) - m^-)^2 \, dx \int_{\mathbb{R}} J(x - y) \, dy \\
\leq 2L(f(0) - f(m^-)) + (1 + K)\|u - m^-\|_{L^2(\mathbb{R})}^2.
\]

so \(F(u) < +\infty\). Conversely, if \(F(u) < +\infty\), we also have \(u(x) \leq m^-\), for \(x\) outside some finite interval \([-L, L]\) and, using (21), we have

\[
\int_{\mathbb{R}} (u(x) - m^-)^2 \, dx \leq \int_{|x| \leq L} (u(x) - m^-)^2 \, dx + \int_{|x| \geq L} (u(x) - m^-)^2 \, dx \\
\leq \int_{|x| \leq L} (u(x) - m^-)^2 \, dx + K \int_{|x| \geq L} f(u(x)) - f(m^-) \, dx \\
\leq \int_{|x| \leq L} (u(x) - m^-)^2 \, dx + K \int_{|x| \leq L} f(u(x)) - f(m^-) \, dx \\
+ K \int_{|x| \leq L} |f(u(x)) - f(m^-)| \, dx \\
\leq \int_{|x| \leq L} (u(x) - m^-)^2 \, dx + K \mathcal{F}(u) + 2KL |f(m^+) - f(m^-)|. \\
\]

and thus \(u - m^- \in L^2(\mathbb{R})\).

We now turn to the proof of the lower semi continuity of \(\mathcal{F}\). We consider two cases.

Suppose first that \(u - m^- \notin L^2(\mathbb{R})\).

Since \(u \in \Sigma\), we can choose \(L > 0\) such that \(u(x) \leq m^0 + \frac{\varepsilon}{2}\), for any \(x\), with \(|x| \geq L\) (where \(\varepsilon\) is the constant in (21). Let \(\{u_n : n \in \mathbb{N}\}\) be a sequence in \(\Sigma\) which converges to \(u\) in \(L^2_{loc}\).

We claim that there exists \(n_0 \in \mathbb{N}\) such that \(u_n(x) \leq m^0 + \varepsilon\), for any \(x \geq 2L\) and \(n \geq n_0\). Indeed, if this is not true for some \(n \in \mathbb{N}\), then \(u_n(x) \geq m^0 + \varepsilon\)
also for $L \leq |x| \leq 2L$ and thus

$$\|u_n - u\|_{L^2(\mathbb{R})}^2 \geq 2 \int_{L}^{L+M} |u_n(x) - u(x)| \, dx$$

$$\geq \varepsilon^2 L. \quad (22)$$

which cannot occur for $n$ sufficiently big.

Therefore, using (20), we have for $n \geq n_0$, and any $R > 0$

$$\mathcal{F}(u_n) \geq \int_{\mathbb{R}} f(u_n(x)) - f(m^-_\beta) \, dx$$

$$\geq \int_{|x| \geq 2L} f(u_n(x)) - f(m^-_\beta) \, dx + \int_{|x| \leq 2L} f(u_n(x)) - f(m^-_\beta) \, dx$$

$$\geq \frac{1}{K} \int_{|x| \geq 2L} (u_n(x) - m^-_\beta)^2 \, dx - 4L(f(m^+_\beta) - f(m^-_\beta))$$

$$\geq \frac{1}{K} \int_{|x| \geq 2L} (u_n(x) - m^-_\beta)^2 \, dx - 4L(f(m^+_\beta) - f(m^-_\beta))$$

$$\geq \frac{2}{K} \left(\|u - m^-_\beta\|_{L^2[2L, 2L+R]}^2 - \|u - u_n\|_{L^2[2L, 2L+R]}^2\right)$$

$$- 4L(f(m^+_\beta) - f(m^-_\beta))$$

Given $\alpha > 0$, we choose $R$ in such a way that $\|u - m^-_\beta\|_{L^2[2L, 2L+R]} > \sqrt{\frac{\alpha}{2}} [\alpha + 4L(f(m^+_\beta) - f(m^-_\beta))]^{1/2} + 1$ and then choose $N_0 \geq n_0$ with $\|u - u_n\|_{L^2[2L, 2L+R]} \leq 1$, for $n \geq N_0$. Then $\mathcal{F}(u_n) \geq \alpha$, for $n \geq N_0$. This shows that $\liminf_{n \to \infty} \mathcal{F}(u_n) = +\infty \geq \mathcal{F}(u)$.

Suppose now that $u - m^-_\beta \in L^2(\mathbb{R})$.

Let $\{u_n : n \in \mathbb{N}\}$ with $u_n \rightharpoonup u \in L^2_{loc}$. Choose $L > 0$ such that $u(x) \leq m^0_\beta$ for $|x| \geq L$ and $\|u - m^-_\beta\|_{L^2[|L-1, +\infty]} \leq \min\{\frac{\sqrt{\alpha}}{4}, \frac{\gamma}{8\pi}\}$. Then, we have

$$\mathcal{F}(u) \leq \int_{|x| \leq L} f(u(x)) - f(m^-_\beta) \, dx + \int_{|x| \geq L} f(u(x)) - f(m^-_\beta) \, dx$$

$$+ \frac{1}{4} \int_{|y| \leq L} \int_{y - 1}^{y + 1} J(x - y)(u(x) - u(y))^2 \, dx \, dy$$

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\[ \frac{1}{4} \int_{|y| \geq L}^{y+1} \int_{y} J(x-y)(u(x) - u(y))^2 \, dx \, dy + \int_{|y| \geq L}^{y+1} \int_{y} J(x-y)(u(x) - u(y))^2 \, dx \, dy \]

\[ \leq \int_{|x| \leq L} f(u(x)) - f(m_\beta^-) \, dx + \frac{1}{4} \int_{|y| \leq L}^{y+1} \int_{y} J(x-y)(u(x) - u(y))^2 \, dx \, dy \]

\[ + K \int_{|x| \geq L} |u(x) - m_\beta^-|^2 \, dx + \frac{1}{2} \int_{|y| \geq L}^{y+1} \int_{y} J(x-y)(u(y) - m_\beta^-)^2 \, dx \, dy \]

\[ + \frac{1}{2} \int_{|x| \geq L} (u(x) - m_\beta^-)^2 \int_{y} J(x-y) \, dy \, dx \]

\[ \leq \int_{|x| \leq L} f(u(x)) - f(m_\beta^-) \, dx + \frac{1}{4} \int_{|y| \leq L}^{y+1} \int_{y} J(x-y)(u(x) - u(y))^2 \, dx \, dy \]

\[ + K \int_{|x| \geq L} |u(x) - m_\beta^-|^2 \, dx + \frac{1}{2} \int_{|y| \geq L}^{y+1} \int_{y} (u(y) - m_\beta^-)^2 \, dy \]

\[ + \frac{1}{2} \int_{|x| \geq L} (u(x) - m_\beta^-)^2 \, dx \]

\[ \leq \int_{|x| \leq L} f(u(x)) - f(m_\beta^-) \, dx + \frac{1}{4} \int_{|y| \leq L}^{y+1} \int_{y} J(x-y)(u(x) - u(y))^2 \, dx \, dy \]

\[ + \frac{\eta}{2} \]

(23)

Let now \( n_1 \) be such that \( u_n(x) \leq m_\beta^0 + \varepsilon \) for \( n \geq n_1 \) (The existence of \( n_1 \) is shown as for \( n_0 \) above, using inequality (22)). For any such \( n \), we have

\[ \mathcal{F}(u_n) \geq \int_{|x| \leq L} f(u_n(x)) - f(m_\beta^-) \, dx + \int_{|x| \geq L} f(u_n(x)) - f(m_\beta^-) \, dx \]
\[ \frac{1}{4} \int_{|y| \leq L} |\int_{y-1}^{y+1} J(x-y)(u_n(x) - u_n(y))^2 \, dx \, dy \]
\[ + \frac{1}{4} \int_{|y| \geq L} |\int_{y-1}^{y+1} J(x-y)(u_n(x) - u_n(y))^2 \, dx \, dy \]
\[ \geq \int_{|x| \leq L} f(u_n(x)) - f(m^-_\beta) \, dx \]
\[ + \frac{1}{4} \int_{|y| \leq L} |\int_{y-1}^{y+1} J(x-y)(u(x) - u(y))^2 \, dx \, dy \]
\[ - \int_{|x| \leq L} \left| f(u_n(x)) - f(u(x)) \right| \, dx \]
\[ - \frac{1}{4} \int_{|y| \leq L} |\int_{y-1}^{y+1} J(x-y)|(u(x) - u(y))^2 - (u_n(x) - u_n(y))^2 | \, dx \, dy \]

(24)

Choosing \( n_2 \) such that \( ||u_n - u||_{L^2[-L,L]}^2 \leq \frac{\eta}{4K} \) for \( n \geq n_2 \), we have

\[ \int_{|x| \leq L} \left| f(u_n(x)) - f(u(x)) \right| \, dx \leq K \int_{|x| \leq L} |u_n(x) - u(x)|^2 \, dx \leq \frac{\eta}{4} \]  

(25)

Choose now \( n_3 \) such that \( ||u_n - u||_{L^2[-L-1,L+1]}^2 \leq \frac{\eta}{4\sqrt{2(L+1)}} \). Then, since

\[ \left| (u(x) - u(y))^2 - (u_n(x) - u_n(y))^2 \right| \]
\[ \leq \left| (u_n(x) + u(x) - u_n(y) + u(y)) \right| \left| (u_n(x) - u(x)) + (u(y) - u_n(y)) \right| \]
\[ \leq 4 \left| (u_n(x) - u(x)) + (u(y) - u_n(y)) \right| \]

we obtain

\[ \frac{1}{4} \int_{|y| \leq L} |\int_{y-1}^{y+1} J(x-y)|(u(x) - u(y))^2 - (u_n(x) - u_n(y))^2 | \, dx \, dy \]
\[
\begin{align*}
&\leq \frac{1}{4} \int_{|y| \leq L^{-1}} \int_{L^{-1}}^{L+1} 4J(x-y)|u(y) - u_n(y)| \, dx \\
&+ \frac{1}{4} \int_{|x| \leq L+1} \int_{-L}^{L} 4J(x-y)|u(x) - u_n(x)| \, dy \, dx \\
&\leq \int_{|y| \leq L} |u(y) - u_n(y)| \, dy + \int_{|x| \leq L+1} |u(x) - u_n(x)| \, dx \\
&\leq \sqrt{2L} ||u - u_n||_{L^2[-L,L]} + \sqrt{2(L+1)} ||u - u_n||_{L^2[-L-1,L+1]} \\
&\leq 2\sqrt{2(L+1)} ||u - u_n||_{L^2[-L-1,L+1]} \\
&\leq \frac{\eta}{4}
\end{align*}
\]

By (23), (24), (25) and (26), we obtain, if \( n \geq \max\{n_1, n_2\} \)

\[
\mathcal{F}(u_n) \geq \int_{|x| \leq L} f(u(x)) - f(m_{\beta}) \, dx \\
+ \frac{1}{4} \int_{|y| \leq L} \int_{y^{-1}}^{y+1} J(x-y)(u(x) - u(y))^2 \, dx \, dy - \eta \\
\geq \mathcal{F}(u) - \eta
\]

Since \( \eta \) is arbitrary, it follows that \( \liminf_{n \in \mathbb{N}} \mathcal{F}(u_n) \geq \mathcal{F}(u) \) and the semi-continuity property is proved. \( \square \)

The set defined by (19) is also invariant by the flow. More precisely, we have the following result, whose proof very similar to the proof of Proposition 2.5 of [0].

**Proposition 17** Assume that \( u(\cdot, 0) \in C_b(\mathbb{R}) \) and (19) holds. Then \( u(\cdot, t) - m_{\beta} \in L^2(\mathbb{R}) \) for all \( t \geq 0 \) and \( ||u(\cdot, t) - m_{\beta}||_2 \) is bounded for \( t \) in compacts.

The following result was also proved in [0] and can easily be extended for \( h \geq 0 \).

**Lemma 18** Suppose \( u(\cdot, 0) \in C_b(\mathbb{R}) \), \( ||u(\cdot, 0)||_\infty \leq 1 \) and (19) holds. Then \( \mathcal{F}(u(\cdot, t)) \) is well defined for \( t \geq 0 \), it is differentiable with respect to \( t \) if \( t > 0 \) and

\[
\frac{d}{dt} \mathcal{F}(u(\cdot, t)) = -I(u(\cdot, t)) \leq 0
\]

where, for any \( v \in C_b(\mathbb{R}) \), \( ||v||_\infty < 1 \),

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\[ I(v(\cdot)) = \int \left\{ (J * v)(x) + h - \beta^{-1}\arctanh v(x) \right\} \] 
\[ \left[ \tanh(\beta(J * v)(x) + h) - v(x) \right] \, dx. \]

The integrand in \( I(h) \) is a non negative function which is in \( L^1 \) (with the Lebesgue measure), when \( v(\cdot) = u(\cdot, t) \). Finally, for all \( t_0 \geq 0 \) and all \( t \geq t_0 \)

\[ \mathcal{F}(u(\cdot, t)) - \mathcal{F}(u(\cdot, t_0)) = - \int_{t_0}^t I(u(\cdot, s)) \, ds \leq 0. \]

Consider the following subset of \( C_b(\mathbb{R}) \):

\[ \Sigma^0 := \{ u \in C_b(\mathbb{R}) : ||u||_\infty \leq 1, \ u \text{ is even, increasing in } ] - \infty, 0] \]

and satisfies (19) \] (27)

**Lemma 19** The set \( \Sigma^0 \) defined by (27) is positively invariant under the flow \( S(t) \) defined by (1) in \( C^\rho(\mathbb{R}) \).

**Proof:** We prove \( \Sigma^0 \) is invariant by the flow defined by (1) in \( C_b(\mathbb{R}) \). The result claimed follows then immediately by uniqueness. Suppose \( u(\cdot, t) \) is a solution of (1) with \( u_0 = u(\cdot, 0) \in \Sigma^0 \). Using Theorem , it follows by comparison with \( u \equiv 1 \) and \( u \equiv -1 \), that \( ||u(\cdot, t)||_\infty \leq 1 \). Since the space of even functions is invariant by the right-hand-side of (1), it follows by uniqueness that \( u(\cdot, t) \) is even. It also satisfies (19) by Proposition (). To prove it is increasing in \( ] - \infty, 0] \) we use an argument similar to the proof of Theorem 2.7 of ([0]). For a given \( T > 0 \) we denote by \( \mathcal{M} \) the space \( C_b(\mathbb{R} \times [0, T]) \), equipped with the sup norm. Let \( G \) be the map from \( \mathcal{M} \) to itself defined by

\[ G(u)(x, t) = e^{-t}u(x, 0) + \int_0^t e^{-(t-s)} \tanh(\beta(J * u + h))(x, s) \, ds \]

It is easy to see that, if \( u(\cdot, t) \) is increasing for any \( t \in [0, T] \), the same is true for \( G(u)(\cdot, t) \), and \( G(u)(\cdot, 0) = u(\cdot, 0) \). Defining \( u_{n+1}(x, t) = G(u_n)(x, t) \), \( u_1(x, t) = u(x, t) \), we obtain, for \( (x, t) \in \mathbb{R} \times [0, T] \)

\[ u_n(x, 0) = u_{n-1}(x, 0) = \cdots = u(x, 0) \]

and

\[ |u_{n+1}(x, t) - u_n(x, t)| \]
\[ \leq \int_0^t e^{-(t-s)}|\beta||J * u_n + h)(x, s) - (J * u_{n-1} + h)(x, s)| \, ds \]

and

\[ 22 \]
\[ \leq \beta ||u_n - u_{n-1}||_\infty \int_0^t e^{-(t-s)} |ds. \]
\[ \leq \beta T ||u_n - u_{n-1}|| \]

If \( \beta T < 1 \), \( u_n \) is a Cauchy sequence and therefore, converges in \( \mathcal{M} \). Its limit \( u_\infty \) must be increasing in \( x \) for any \( t \in [0, T] \) and satisfies the integral equation \( G(u) = u \). Therefore, it coincides with the (unique) solution of (1) with initial condition \( u(0, x) = u_0 \) in the interval \([0, T]\). Thus \( u(x, t) \) is increasing in the \( x \) variable for any \( t \in [0, T] \). By iteration we can prove the property for any \( t > 0 \), proving the Lemma.

\[ \square \]

**Corollary 20** The equilibrium \( m_\beta^- \) is locally stable for the flow \( S(t) \) restricted to \( \Sigma^0 \).

**Proof:** From Theorem it follows that the equilibrium \( m_\beta^- \) is locally stable for the flow defined by (1) in \( C_b(\mathbb{R}) \). But, if \( u \in \Sigma^0 \) then \( u(x) \geq m_\beta^- \) and
\[ ||u - m_\beta^-||_\infty = |u(0) - m_\beta^-| \]
\[ \leq \sup_{x \in \mathbb{R}} \{ \rho(x) |u(x) - m_\beta^-| \} / \rho(0) \]
\[ \leq \frac{1}{\rho(0)} ||u - m_\beta^-||_{C_\rho(\mathbb{R})} \]
and the result follows immediately. \[ \square \]

**Lemma 21** Suppose \( u \in \Sigma^0 \) and \( \omega(u) \) does not contain \( m_\beta^+ \). Then there exists an equilibrium in \( \omega(u) \cap \Sigma^0 \).

**Proof:** Suppose \( \tilde{u} \in \omega(u) \) and \( \lim_{n \to \infty} S(t_n)u = \tilde{u} \). Then \( \lim_{n \to \infty} S(t_n)u = \tilde{u} \) uniformly in compact sets and, therefore \( \tilde{u} \) must be even and increasing in \([ -\infty, 0] \). Let \( L = \lim_{x \to -\infty} \tilde{u}(x) \). It must be true that \( L \leq m_\beta^0 \) for, otherwise, \( S(t)\tilde{u} \to m_\beta^+ \) by Theorem and \( m_\beta^+ \in \omega(u) \), against the hypothesis. It follows that \( \mathcal{F}(\tilde{u}) > -\infty \). Since \( \mathcal{F} \) is decreasing and lower semi-continuous, we also have \( l = \mathcal{F}(\tilde{u}) < +\infty \). From Theorem, it follows that \( \tilde{u} - m_\beta^- \in L^2(\mathbb{R}) \). Since \( \omega(u) \) is a compact set in \( C_\rho_c(\mathbb{R}) \), it also follows that \( \mathcal{F} \) achieves its infimum in \( \omega(u) \) at a point \( \tilde{u} \). Then \( \mathcal{F}(T(\mathcal{F}(\tilde{u})) = \mathcal{F}(\tilde{u}) \) and, therefore \( \tilde{F}(\tilde{u}) = 0 \), so \( \tilde{u} \) is an equilibrium by Lemma.

\[ \square \]

We are now in a position to prove the main result of this section

**Theorem 22** There exists an equilibrium \( u \) of (1) which is continuous, even, increasing in \([ -\infty, 0] \), with \( \lim_{x \to \pm \infty} u(x) = m_\beta^- \) and \( u(0) > m_\beta^0 \).

**Proof:** We claim it is enough to prove there exists \( u^* \in \Sigma^0 \) such that \( \omega(u^*) \)
does not intercept \( \{ m^+_\beta, m^-_\beta \} \). In fact, if this is true then Lemma guarantees the existence of an equilibrium \( u \) in \( \Sigma^0 \). Furthermore, if \( u(0) \leq m^0_\beta \) then it is easy to show that \( S(\tau)u(0) < m^0_\beta \) for some positive \( \tau \) and \( S(t)u \to m^-_\beta \) by Theorem, against the hypothesis.

To prove the existence of \( u^* \), consider the following subsets of \( \Sigma^0 \):

\[
\Sigma^0_1 := \{ u \in \Sigma^0 : m^+_\beta \in \omega(u) \},
\]

\[
\Sigma^0_2 := \{ u \in \Sigma^0 : m^-_\beta \in \omega(u) \}.
\]

Since, by Corollary, \( m^-_\beta \) is locally stable in \( \Sigma^0 \) it follows that \( \Sigma^0_2 = \{ u \in \Sigma^0 : \lim_{t \to \infty} S(t)u = m^-_\beta \} \) is an open subset of \( \Sigma^0 \) (in the relative topology of \( C_\rho(\mathbb{R}) \)). Also \( \Sigma^0_1 \) is nonempty, since \( m^-_\beta \in \Sigma^0_1 \).

We consider two possibilities for \( m^+_\beta \).

Suppose first that there exists \( u \in \Sigma^0 \) arbitrarily \( C_\rho(\mathbb{R}) \)-close to \( m^+_\beta \) such that \( m^+_\beta \notin \omega(u) \) By Theorem, we may suppose \( u(0) \leq m^+_\beta \). But then, given \( \varepsilon > 0 \) (arbitrarily small) and \( L > 0 \) (arbitrarily large), we may find such a \( u \) with \( m^+_\beta - \varepsilon \leq u \leq m^+_\beta \) for \( |x| \leq L \). Choose \( \varepsilon \) in such a way that \( f(m^+_\beta - \varepsilon) < f(m^-_\beta) \) and let \( \bar{u} \) be the continuous piecewise linear function given by \( \bar{u} = m^-_\beta - \varepsilon \) if \( |x| \leq L - 1 \), \( \bar{u} = m^-_\beta \) if \( |x| \geq L \). Then

\[
\mathcal{F}(\bar{u}) = \int_\mathbb{R} [f(\bar{u}) - f(m^-_\beta)] \, dx + \frac{1}{4} \int_\mathbb{R} \int_\mathbb{R} J(x - y)[\bar{u}(x) - \bar{u}(y)]^2 \, dx \, dy
\]

\[
\leq \int_{-L+1 \leq |x| \leq L} [f(\bar{u}) - f(m^-_\beta)] \, dx + \int_{|x| \leq L-1} [f(\bar{u}) - f(m^-_\beta)] \, dx
\]

\[
+ \frac{1}{4} \int_{-L \leq |x| \leq L+1 \leq |x| \leq L+1} J(x - y)[\bar{u}(x) - \bar{u}(y)]^2 \, dx \, dy
\]

\[
\leq 2(f(0) - f(m^-_\beta) + 2(L-1)(f(m^+_\beta) - f(m^-_\beta))
\]

\[
+ \frac{1}{4} \int_{-L \leq |x| \leq L+1 \leq |x| \leq L+1} \, dx \, dy
\]

\[
\leq 2(f(0) - f(m^-_\beta) + 2(L-1)(f(m^+_\beta) - f(m^-_\beta)) + 4
\]

Therefore \( \mathcal{F}(\bar{u}) < 0 \) for \( L \) sufficiently large (and actually goes to \(-\infty \) as \( L \to \infty \)). Since \( \mathcal{F} \) is decreasing and lower semi continuous and \( \mathcal{F}(m^-_\beta) = 0 \), \( S(t)(\bar{u}) \not\to m^-_\beta \) as \( t \to \infty \). Since \( \bar{u} \leq u \) it follows by comparison (Theorem ) that \( S(t)(u) \not\to m^-_\beta \) as \( t \to \infty \). Therefore, \( \omega(u) \) does not intercept \( \{ m^+_\beta, m^-_\beta \} \) and we are done.

Suppose now that \( m^+_\beta \in \omega(u) \), for any \( u \in \Sigma^0 \) sufficiently \( C_\rho(\mathbb{R}) \)-close to it.
Then, by continuity it follows that $\Sigma^0_1$ and $\Sigma^0_2$ are both open and nonempty. Since $\Sigma^0_0$ is connected there must exist $u^* \in \Sigma^0_0 \setminus (\Sigma^0_1 \cup \Sigma^0_2)$ which, by the definition of $\Sigma^0_1$ and $\Sigma^0_2$, satisfies $\omega(u^*) \cap \{m^-_\beta, m^+_\beta\} = \emptyset$. This completes the proof. \qed

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