Real Projective Space

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This text is an exercise of manifold theory and we are going show $\mathbb{R}P^n$ is a topological manifold, that is, it is topological space with Hausdorff and second-countable topological properties and it is also locally euclidean.

Definition 1. *First definition of Projective Space.* For some natural $n \geq 1$, let $\mathbb{R}P^n := \{V \subset \mathbb{R}^{n+1} : V$ vector space, $dim(V) = 1\}$. We introduce a topology on $\mathbb{R}P^n$ as follow: define $\pi : \mathbb{R}^{n+1}\backslash\{0\} \to \mathbb{R}P^n$, $\pi(x) = span\{x\}$, we say $U \subset \mathbb{R}P^n$ is open if $\pi^{-1}(U)$ is open in $\mathbb{R}^{n+1}\backslash\{0\}$, with induced topology of $\mathbb{R}^{n+1}$, in addition, denote τ subset of parts of $\mathbb{R}P^n$ such that it is the collection of open sets of $\mathbb{R}P^n$, it is easy to check $(\mathbb{R}P^n, \tau)$ is a topological space.

Definition 2. *Second definition of Projective Space.* For some natural $n \geq 1$, consider $\mathbb{S}^n := \{x \in \mathbb{R}^{n+1} : \|x\| = 1\}$, in $\mathbb{S}^n$ we define $x \sim y$ iff $x = y$ or $A(x) = y$, that is, if y is image of x by antipodal map ($A(x) = -x, x \in \mathbb{S}^n$). Obviously $\sim$ is a equivalent relation in $\mathbb{S}^n$, the map $q : \mathbb{S}^n \to \mathbb{S}^n/\sim$, $q(x) = [x] = \{-x, x\}$ is called of projection map, we set $\mathbb{S}^n/\sim := \mathbb{P}^n$, analogous to previous definition, we have U open in $\mathbb{P}^n$ if $q^{-1}(U)$ is open in $\mathbb{S}^n$, it is equipped with induced topology of $\mathbb{R}^{n+1}$.

Proposition 1. *The $\mathbb{R}P^n$ is homeomorphic to $\mathbb{P}^n$*

Proof. Let $\varphi : \mathbb{P}^n \to \mathbb{R}P^n$, $\varphi(\{-x, x\}) = span\{-x, x\}$, we are going show φ is a homeomorphism between $\mathbb{R}P^n$ and $\mathbb{P}^n$.

Claim. φ is a bijection.

We have $\varphi(\{-x, x\}) = \varphi(\{-y, y\}) \iff \{\lambda x : \lambda \in \mathbb{R}\} = \{\mu y : \mu \in \mathbb{R}\}$, if $t \in \{\lambda x : \lambda \in \mathbb{R}\} = \{\mu y : \mu \in \mathbb{R}\}$, then $t = \lambda x = \mu y$, $\lambda, \mu \in \mathbb{R}$, hence $\|t\| = |\lambda| = |\mu| \implies |\frac{\lambda}{\mu}| = 1$, that is, $\lambda = \pm\mu$, therefore $\{-x, x\} = \{-y, y\}$, it shows φ is injective. For surjection we have just consider $x \in r \cap \mathbb{S}^n$, where $r \in \mathbb{R}P^n$, then $\varphi(\{-x, x\}) = r$.

Claim. φ is an open map.

It is true $\pi|_{\mathbb{S}^n}$ is onto and continuous. Let $U \subset \mathbb{P}^n$ open, $\pi^{-1}|_{\mathbb{S}^n}(\varphi(U)) = (\varphi^{-1} \circ \pi|_{\mathbb{S}^n})^{-1}(U)$, where $\varphi^{-1}(\{\lambda x : \lambda \in \mathbb{R}\}) = \{\lambda x : \lambda \in \mathbb{R}\} \cap \mathbb{S}^n$, it is easy to see $\varphi^{-1} \circ \pi|_{\mathbb{S}^n} = q$, hence $\pi^{-1}|_{\mathbb{S}^n}(\varphi(U)) = (\varphi^{-1} \circ \pi|_{\mathbb{S}^n})^{-1}(U) = q^{-1}(U)$ is open, therefore $\varphi(U)$ is open.

By two previous claims we have φ homeomorphism. $\square$

Proposition 2. *$\mathbb{P}^n$ is Hausdorff.*

Proof. Let $x, y \in \mathbb{S}^{n+1}$, $x \neq y$, clearly y or $-y$ is in same hemisphere of x, so $\frac{\|x-y\|}{2} \leq 1$ or $\frac{\|x-(-y)\|}{2} \leq 1$, if $\epsilon < \frac{1}{2}min\{\|x-y\|, \|x+y\|\}$, we must have $B(\epsilon, x) \cap B(\epsilon, y) = \emptyset$, where $B(\cdot, \cdot)$ is a open ball, otherwise if $z \in B(\epsilon, x) \cap B(\epsilon, y)$ by triangle inequality $\|x-y\| \leq \|x-z\| + \|z-y\| < 2\epsilon < min\{\|x-y\|, \|x+y\|\} \leq \|x-y\|$, contradiction. Analogous we show $B(\epsilon, x) \cap -B(\epsilon, y) = \emptyset$.

Denote $U := B(\epsilon, x) \cap \mathbb{S}^n$ and $V := B(\epsilon, y) \cap \mathbb{S}^n$, by preceding statement we obtain $U \cap V = U \cap -V = -U \cap V = -U \cap -V = \emptyset$. The condition $U \cap -U = V \cap -V = \emptyset$ follow of $\epsilon < 1$, otherwise, we have $z \in \{\lambda x : \lambda \in \mathbb{R}\}$ and $z \in B(\epsilon, x) \cap -B(\epsilon, x)$, on the one hand $\|z-x\| + \|z-(-x)\| = 2$, on the other hand $\|z-x\|, \|z+x\| < \epsilon < 1 \implies \|z-x\| + \|z+x\| < 2$, contradiction. Follow $U, V, -U, -V$ are pairwise disjoint.

Claim. $q(U), q(V)$ are open in $\mathbb{P}^n$.

$x \in q^{-1}(q(U)) \implies q(x) \in q(U)$, but $q(x) = \{-x, x\}$ so $x \in U$ or $x \in -U$, then $q^{-1}(q(U)) \subset -U \cup U$, conversely $x \in -U \cup U \implies q(x) \in q(U) \implies x \in q^{-1}(q(U))$, hence $-U \cup U \subset q^{-1}(q(U))$, therefore $q^{-1}(q(U)) = -U \cup U$ open in $\mathbb{S}^n$, analogous for $q(V)$.

Claim. $q(U) \cap q(V) = \emptyset$.

If there is $\{-x, x\} \in q(U) \cap q(V)$, then one of $U \cap V, -U \cap V, U \cap -V$ or $-U \cap V$ is non empty, contradiction.

By the previous affirmations, follow $\mathbb{P}^n$ Hausdorff. $\square$

Definition 3. We say a topological space M is locally euclidean (of dimension n) if for every point $p \in M$, there is a open neighborhood U and a map $\varphi : U \to \mathbb{R}^n$ such that φ is homeomorphism.

Theorem 3. *$\mathbb{R}P^n$ is locally euclidean.*

Proof. Denote $V_i = \{x \in \mathbb{R}^{n+1} \backslash \{0\} : x_i \neq 0\}$ and $U_i = \pi(V_i)$.

Claim. $V_i = \pi^{-1}(\pi(V_i))$.

Clearly $V_i \subset \pi^{-1}(\pi(V_i))$, conversely $y \in \pi^{-1}(\pi(V_i)) \implies \pi(y) \in \pi(V_i)$, so exist $x \in V_i$ such that $\pi(y) = \pi(x)$, note $y \in \pi(y) = \{\mu y : \mu \in \mathbb{R}\} = \{\lambda x : \lambda \in \mathbb{R}\} = \pi(x)$, then $y = \lambda x$, hence $y_i = \lambda x_i$, if $\lambda = 0$, then $y = 0$, contraction, therefore $y_i \neq 0$ because $x \in V_i$, $y_i \neq 0 \implies y \in V_i$, that is, $V_i \subset \pi^{-1}(\pi(V_i))$ and follow the claim.

Claim. V_i is open.

Note that V_i is complement of pre-image by projection on $i-$coordinate of $\{0\}$ in $\mathbb{R}^{n+1} \backslash \{0\}$.

Claim. U_i is open in $\mathbb{R}P^n$.

Follow by $V_i = \pi^{-1}(\pi(V_i)) = \pi^{-1}(U_i)$, V_i open.

Define $\varphi_i : U_i \to \mathbb{R}^n$, $\varphi_i([x_1, \cdots, x_{n+1}]) = \left(\frac{x_1}{x_i}, \cdots, \frac{x_{i-1}}{x_i}, \frac{x_{i+1}}{x_i}, \cdots, \frac{x_{n+1}}{x_i}\right)$.

Each φ_i is continuous.

It sufficient show continuity of $\varphi_i \circ \pi : V_i \to \mathbb{R}^n$, but it is

$$(x_1, \cdots, x_{n+1}) \mapsto \left(\frac{x_1}{x_i}, \cdots, \frac{x_{i-1}}{x_i}, \frac{x_{i+1}}{x_i}, \cdots, \frac{x_{n+1}}{x_i}\right)$$

clearly continuous.

Claim. $\psi : \mathbb{R}^n \to U_i$, $\psi(u_1, \cdots, u_n) = [u_1, \cdots, u_i, 1, u_{i+1}, \cdots, u_n]$ is φ_i^{-1}.

First

$$(\varphi_i \circ \psi)(u_1, \cdots, u_n) = \varphi_i([u_1, \cdots, u_{i-1}, 1, u_{i+1}, \cdots, u_n]) = (u_1, \cdots, u_n)$$

reciprocally,

$$\begin{aligned}(\psi \circ \varphi_i)([x_1, \cdots, x_{n+1}]) &= \psi\left(\frac{x_1}{x_i}, \cdots, \frac{x_{i-1}}{x_i}, \frac{x_{i+1}}{x_i}, \cdots, \frac{x_{n+1}}{x_i}\right) \\ &= \left[\frac{x_1}{x_i}, \cdots, \frac{x_{i-1}}{x_i}, 1, \frac{x_{i+1}}{x_i}, \cdots, \frac{x_{n+1}}{x_i}\right] \\ &= [x_1, \cdots, x_{n+1}].\end{aligned}$$

Claim. φ_i^{-1} is continuous.

We have $\varphi_i^{-1}(u_1, \cdots, u_n) = (\pi \circ f_i)(u_1, \cdots, u_n)$, where $f_i : \mathbb{R}^n \to \mathbb{R}^{n+1}$, $f(u_1, \cdots, u_n) = (u_1, \cdots, u_i, 1, u_{i+1}, \cdots, u_n)$, from continuity of π and f follow φ_i^{-1} continuous.

If $v \in \mathbb{R}P^n$, then exist $x \in \mathbb{R}^{n+1}$ such that $v = [x]$, $x \neq 0$, then exist $i \in \{1, \cdots, n+1\}$ with $x_i \neq 0$, hence $v \in U_i$, therefore $\bigcup_{i=1}^{n+1} U_i = \mathbb{R}P^n$. Finally, for each point in $\mathbb{R}P^n$ we have a local homeomorphism with $\mathbb{R}^n$, that is, $\mathbb{R}P^n$ is locally euclidean. $\square$

Corollary 4. $\mathbb{R}P^n$ *is second-countable.*

Proof. Each U_i, $i \in \{1, \cdots, n+1\}$, is second-countable (homeomorphic to R^n), because $\mathbb{R}P^n = \bigcup_{i=1}^{n+1} U_i$, we have $\mathbb{R}P^n$ is second-countable. $\square$

Corollary 5. $\mathbb{R}P^n$ *is a topological manifold of dimension* n*.*

Proof. By Theorem 1 we have $\mathbb{R}P^n$ locally euclidean and second-countable, by Proposition 2 $\mathbb{R}P^n$ is Hausdorff, because $\mathbb{P}^n$ is Hausdorff and they are homeomorphic. $\square$

References

[L] Lee, J., *Introduction to Smooth Manifolds.* Graduate Texts in Mathematics, Vol. 218, Springer, 2002.