$dim(M) = dim(T_pM)$

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This text is an exercise of manifold theory and we will show $dim(M) = dim(T_pM)$, that is, the dimension of a differential manifolds $(M, \mathcal{A})$, sometimes just M - topological manifolds with a maximal atlas of compatible charts - is the same of tangent space, but what is tangent space?

In this text M is always a differential manifold.

Definition 1. Let $p \in M$, define a set

$$\Gamma_p = \{(U, f) : U \text{ open neighborhood of p and } f : U \to \mathbb{R} \text{ smooth function}\}$$

in Γ_p set the relation: $(U, f) \sim (V, g) \iff$ there is a open neighborhood W of p such that $f|_W = g|_W$, we denote $G_p \coloneqq \Gamma_p \backslash \sim$ and an element of G_p is called *germ* in p.

In each G_p we define three operations such that G_p is an algebra, that is, a vector space and a ring, the operations are define as $[(U, f)] + [(V, g)] = [(U \cap V, f + g)]$, $\lambda[(U, f)] = [(U, \lambda f)]$ (for vector space) and $[(U, f)][(V, g)] = [(U \cap V, fg)]$ (for ring), where the $[\cdot]$ is the class of $(\cdot, \cdot)$, it is easy verify the well-definition of these operations and the axioms of vector space and ring. When the domain U of some f is clear or not important we denote just $[f]$ for the class of f in G_p.

Definition 2. A derivation at $p \in M$ is a $\mathbb{R}$-linear map $v : G_p \to \mathbb{R}$, such that the Leibniz's rule is satisfied:

$$v([f][g]) = v([f])g(p) + f(p)v([g])$$

the set of all derivation in p is called tangent space and denoted T_pM. If we define sum and product by scalar pointwise, that is, $(v + \lambda w)([f]) = v[f] + \lambda w([f])$, then T_pM is a real vector space.

Remark. Now we will show $dim(M) = dim_{\mathbb{R}}T_pM$. The sketch is first show an isomorphism between T_pM and a abstract, but useful, vector space, then show the dimension of this space is finite and equal to $dimM$.

Definition 3. Let $p \in M$, define $F_p \coloneqq \{X \in G_p : f(p) = 0, \forall f \in X\}$, of course $f(p) = 0$ for some $f \in X$ so $h(p) = 0$ for all $h \in X$. In terms of F_p we define $F_p^2 \coloneqq \{\sum_{i=1}^{n}[f_i][g_i] : [f_i], [g_i] \in F_p$ and n is any positive integer$\}$.

Remark. It easy to see F_p is an ideal and subspace of G_p, analogous for F_p^2, that is, F_p^2 is an ideal and a subspace of F_p, it makes sense in the vector space and ring context F_P/F_p^2, i.e., this set is both subspace and sub ring with induced operations.

Theorem 1. *Let $p \in M$, then $\left(F_p/F_p^2\right)^*$ is isomorphic to T_pM.*

Proof. Set the function

$$\begin{aligned} \varphi : \left(F_p/F_p^2\right)^* &\longrightarrow T_pM \\ \alpha &\longmapsto \varphi(\alpha) : G_p \longrightarrow \mathbb{R} \\ &\qquad [f] \longmapsto \varphi(\alpha)([f]) = \alpha(([f] - [\underline{f(p)}]) + F_p^2) \end{aligned}$$

where $\underline{f(p)} : M \to \mathbb{R}$, $\underline{f(p)}(q) = f(p)$, $\forall q \in M$ and $[\cdot] + F_p^2$ is the class of $[f]$ in F_p/F_p^2. Because φ, $\varphi(\alpha)$ are clearly $\mathbb{R}$-linear, $[f] - [\underline{f(p)}] = [(U, f)] - [(M, \underline{f(p)})] = [(U, f - \underline{f(p)})]$ and $(f - \underline{f(p)})(p) = 0$ we have φ well-defined if $\underline{\varphi(\alpha)}$ is a derivation.

In fact it is, but we need of a trivial lemma before the checking.

Lemma 2. *Let $f, g, h, t : X \to \mathbb{R}$ functions from set a X to $\mathbb{R}$, then $fg - ht = h(g-t) + (f-h)t + (f-h)(g-t)$*

Proof.

$$\begin{aligned} h(g-t) + (f-h)t + (f-h)(g-t) &= hg - ht + ft - ht + fg - ft - hg + ht \\ &= (hg - hg) + (-ht + ht) + (ft - ft) + (-ht + fg) \\ &= fg - ht \end{aligned}$$

□

Now, let $[f], [g] \in G_p$, then

$$\begin{aligned} \varphi(\alpha)([f][g]) &= \varphi(\alpha)([fg]) \\ &= \alpha([fg] - [\underline{f(p)g(p)}] + F_p^2) \\ &= \alpha([fg - \underline{f(p)g(p)}] + F_p^2) \end{aligned}$$

by the last lemma

$$\varphi(\alpha)([fg]) = \alpha([\underline{f(p)}(g - \underline{g(p)}) + (f - \underline{f(p)})\underline{g(p)} + (f - \underline{f(p)})(g - \underline{g(p)}) + F_p^2)$$

but $(f - \underline{f(p)})(g - \underline{g(p)}) \in F_p^2$, so

$$\varphi(\alpha)([fg]) = \alpha([\underline{f(p)}(g - \underline{g(p)}) + (f - \underline{f(p)})\underline{g(p)}] + F_p^2)$$

note that $[\underline{f(p)}][h] = f(p)[h]$, for all $[h]$ in G_p, therefore

$$\alpha([\underline{f(p)}(g - \underline{g(p)}) + (f - \underline{f(p)})\underline{g(p)}] + F_p^2) = f(p)\alpha([g - \underline{g(p)}] + F_p^2) + \alpha([f - \underline{f(p)}] + F_p^2)g(p)$$

that is, $\varphi(\alpha)([fg]) = \varphi(\alpha)([f])g(p) + f(p)\varphi(\alpha)([g])$.

Set the function

$$\begin{aligned} \psi : T_pM &\longrightarrow \left(F_p/F_p^2\right)^* \\ v &\longmapsto \psi(v) : \left(F_p/F_p^2\right) \longrightarrow \mathbb{R} \\ &\qquad [f] + F_p^2 \longmapsto \psi(v)([f] + F_p^2) = v([f]) \end{aligned}$$

we have $\psi(v)$ well-defined, let is see, if $[f]$ and $[g]$ satisfy $[f] + F_p^2 = [g] + F_p^2$, then $[f] - [g]$ is in F_p^2, hence $v([f] - [g]) = 0$ once $v(X) = 0$ for every $X \in F_p^2$ by the Leibniz's rule. Clearly $\psi(v)$ is $\mathbb{R}$-linear, so ψ is well-defined too.

We complete the proof if we show $\varphi = \psi^{-1}$. Checking,

$$\begin{aligned} (\psi \circ \varphi)(\alpha)([f] + F_p^2) &= \psi(\varphi(\alpha))([f] + F_p^2) \\ &= \varphi(\alpha)([f]) \\ &= \alpha(([f] - [\underline{f(p)}]) + F_p^2), [f] \in F_p \\ &= \alpha([f] + F_p^2) \end{aligned}$$

On the other hand,

$$\begin{aligned} (\varphi \circ \psi)(v)([f]) &= \varphi(\psi(v))([f]) \\ &= \psi(v)(([f] - [\underline{f(p)}]) + F_p^2) \\ &= v([f]) \end{aligned}$$

Therefore $\varphi = \psi^{-1}$.

□

Lemma 3. *For every smooth function $f : \mathbb{R}^n \to \mathbb{R}$, we can write*

$$f(x) = f(0) + \nabla f(0) \cdot x + \sum_{i=1}^{n} \sum_{j=1}^{n} x_i x_i \int_0^1 \int_0^1 \frac{\partial^2 f}{\partial x_i \partial x_j}(tsx) t ds dx.$$

Proof. Start with

$$f(x) - f(0) = \int_0^1 (f(tx))' dt = \int_0^1 \nabla f(tx) \cdot x dt$$

then (1) $f(x) = f(0) + \sum_{i=1}^{n} x_i \int_0^1 \frac{\partial f}{\partial x_i}(tx) dt$, denote $g_i(x) = \int_0^1 \frac{\partial f}{\partial x_i} dt$, the same argument show

$$g_i(x) = g_i(0) + \sum_{j=1}^{n} x_j \int_0^1 \int_0^1 \frac{\partial^2 f}{\partial x_i \partial x_j}(stx) t ds dt$$

but $g_i(0) = \int_0^1 \frac{\partial f}{\partial x_i}(0) dt = \frac{\partial f}{\partial x_i}(0)$ plugging in (1)

$$\begin{aligned}
f(x) &= f(0) + \sum_{i=1}^{n} x_i \int_0^1 \frac{\partial f}{\partial x_i}(tx) dt \\
&= f(0) + \sum_{i=1}^{n} x_i g_i(x) \\
&= f(0) + \sum_{i=1}^{n} x_i (\frac{\partial f}{\partial x_i}(0) + \sum_{j=1}^{n} x_j \int_0^1 \int_0^1 \frac{\partial^2 f}{\partial x_i \partial x_j}(tsx) t ds dt) \\
&= f(0) + \nabla f(0) \cdot x + \sum_{i=1}^{n} \sum_{j=1}^{n} x_i x_j \frac{\partial^2 f}{\partial x_i \partial x_i}(tsx) t ds dt
\end{aligned}$$

□

Proposition 4. *Let $p \in M$, then F_p / F_p^2 is a finite dimensional vector space and dim $\left(F_p / F_p^2\right) = dim(M)$.*

Proof. Consider a chart (U, ξ) around p, we may assume $\xi(U) = \mathbb{R}^n$ and $\xi(p) = 0$. Denote $\xi(q) = (u_i(q), \cdots, u_n(q))$, $n = dim(M)$, $u_i : U \to \mathbb{R}$ coordinates functions, let $X \in G_p$, choose $f : W = int(W) \subset U \to \mathbb{R}$ such that $[f] = X$, by the previous lemma and the fact $\xi(p) = 0$ we have

$$\begin{aligned}
(f \circ \xi^{-1})(\xi(q)) &= f(\xi^{-1}(\xi(p)) + \nabla(f \circ \xi^{-1})(\xi(p)) \cdot \xi(q) \\
&\quad + \sum_{i=1}^{n} \sum_{j=1}^{n} u_i(q) u_j(q) \int_0^1 \int_0^1 \frac{\partial^2 (f \circ \xi^{-1})}{\partial x_i \partial x_j}(ts\xi(q)) t ds dt \\
&= f(p) + \sum_{i=1}^{n} u_i(q) \frac{\partial (f \circ \xi^{-1})}{\partial x_i}(\xi(p)) + \\
&\quad + \sum_{i=1}^{n} \sum_{j=1}^{n} u_i(q) u_j(q) \int_0^1 \int_0^1 \frac{\partial^2 (f \circ \xi^{-1})}{\partial x_i \partial x_j}(ts\xi(q)) t ds dt
\end{aligned}$$

Denote $h_i = \frac{\partial (f \circ \xi^{-1})}{\partial x_i}(\xi(p))$ and $g_{ij}(q) = \int_0^1 \int_0^1 \frac{\partial^2 (f \circ \xi^{-1})}{\partial x_i \partial x_j}(ts\xi(q)) t ds dt$, then

$$f(q) = f(\xi^{-1}(\xi(q))) = f(p) + \sum_{i=i}^{n} u_i(q) h_i + \sum_{i=1}^{n} \sum_{j=1}^{n} u_i(q) u_j(q) g_{ij}(q)$$

for each $q \in W$. If $[f] \in F_p^2$, we obtain

$$[f] + F_p^2 = \sum_{i=1}^{n} h_i[u_i] + \sum_{i=1}^{n} [u_i][v_i] + F_p^2$$

where $v_i(q) = \sum_{j=1}^{n} u_j(q) g_{ij}(q)$, note that $\xi(p) = 0 \implies u_k(p) = 0, \forall k$, then each $[u_i] \in F_p$, so each $u_i g_{ij} \in F_p$, hence $v_i \in F_p$, therefore $\sum_{i=1}^{n} [u_i][v_j] \in F_p^2$, that is,

$$[f] + F_p^2 = h_i[u_i] + F_p^2$$

In other words, $[f] \in span\{[u_1] + F_p^2, \cdots, [u_n] + F_p^2\}$.

Claim. $\{[u_1] + F_p^2, \cdots, [u_n] + F_p^2\}$ is linearly independent.

Consider the derivation $\frac{\partial}{\partial u_i}([f]) = \frac{\partial(f \circ \xi^{-1})}{\partial x_i}(\xi(p))$, every element of $[f]$ agree with f in a open neighborhood of p, so the value $\frac{\partial}{\partial u_i}([f])$ is well-defined. Now, suppose $\alpha_1, \cdots, \alpha_n \in \mathbb{R}$ such that $\alpha_1[u_1] + F_p^2 + \cdots + \alpha_n[u_n] + F_p^2 \in F_p^2$, then $\alpha_1[u_1] + \cdots \alpha_n[u_n] \in F_p^2$, on one hand we have $\frac{\partial}{\partial u_i}(\alpha_1[u_1] + \cdots \alpha_n[u_n]) = 0$, because $\frac{\partial}{\partial u_i}$ is a derivation evaluated in an element of F_p^2, on the other hand

$$\frac{\partial}{\partial u_i}(\alpha_1[u_1] + \cdots \alpha_n[u_n]) = \sum_{j=1} \alpha_j \delta_{ij}$$

since $\frac{\partial}{\partial u_i}([u_j]) = \frac{\partial(u_j \circ \xi^{-1})}{\partial x_i}(\xi(q)) = \frac{\partial x_j}{\partial x_i}(\xi(q)) = \delta_{ij}$, we conclude $0 = \sum_{j=1} \alpha_j \delta_{ij} = \alpha_i$, $\forall i \in \{1, \cdots, n\}$, that is, $\{[u_1] + F_p^2, \cdots, [u_n] + F_p^2\}$ is linearly independent. Follow (F_p/F_p^2) finite dimensional and $dim(F_p/F_p^2) = n = dim(M)$. □

Corollary 5. *$dim(M) = dim(T_pM)$.*

Proof.

$$\begin{aligned} dim(M) &= dim(F_p/F_p^2), \text{ finite dimensional} \\ &= dim(F_p/F_p^2)^*, \text{ Theorem 1} \\ &= dim(T_pM). \end{aligned}$$

□

Corollary 6. *In the previous context, $p \in M$, $\left\{\frac{\partial}{\partial u_1}, \cdots, \frac{\partial}{\partial u_n}\right\}$ is base of T_pM.*

Proof. By the proof of Proposition 4 we have $\{[u_1]+F_p^2, \cdots, [u_n]+F_p^2\}$, as $\frac{\partial}{\partial u_i}([u_j]) = \delta_{ij}$ follow $\psi\left(\frac{\partial}{\partial u_i}\right)([u_j] + F_p^2) = \delta_{ij}$, that is,

$$\left\{\psi\left(\frac{\partial}{\partial u_1}\right), \cdots, \psi\left(\frac{\partial}{\partial u_n}\right)\right\} \text{ is base of } (F_p/F_p^2)^*$$

φ isomorphism, then

$$\left\{\varphi\left(\psi\left(\frac{\partial}{\partial u_1}\right)\right), \cdots, \varphi\left(\psi\left(\frac{\partial}{\partial u_n}\right)\right)\right\} = \left\{\frac{\partial}{\partial u_1}, \cdots, \frac{\partial}{\partial u_n}\right\}$$

is base of T_pM. □

Definition 4. Let $\gamma : I \subset \mathbb{R} \to M$, I connected, we say γ is smooth at $t \in int(I)$ if exist $(f \circ \gamma)'(t)$ for every smooth function f defined in a neighborhood of $\gamma(t)$. It value is denoted $\dot{\gamma}(t)$. That is,

$$\begin{aligned} \dot{\gamma} : \mathbb{R} &\longrightarrow TM \\ t &\longmapsto \dot{\gamma}(t) : G_{\gamma(t)} \longrightarrow \mathbb{R} \\ &\qquad\qquad\quad [f] \longmapsto (f \circ \gamma)'(t) \end{aligned}$$

Proposition 7. *For each $p \in M$ and $v \in T_pM$ there is a smooth curve $\gamma : \mathbb{R} \to M$ such that $\gamma(0) = p$ and $\dot{\gamma}(0) = v$*

Proof. Let (φ, U) chart around p such that $\varphi(U) = \mathbb{R}^n$, $\left\{\frac{\partial}{\partial u_1}, \cdots, \frac{\partial}{\partial u_n}\right\}$ is base of T_pM, so $v = \sum_{i=1}^{n} \alpha_i \frac{\partial}{\partial u_i}$, for $\alpha_i \in \mathbb{R}$, then define $\gamma(t) = \varphi^{-1}(\varphi(p) + t(\alpha_1, \cdots, \alpha_n))$, $\gamma(0) = \varphi^{-1}(\varphi(p)) = p$. Checking the $\dot{\gamma}(0) = v$,

$$\begin{aligned} \dot{\gamma}(0)([f]) &= (f \circ \gamma)'(0) \\ &= (f \circ \varphi^{-1} \circ \alpha)'(0), \text{ where } \alpha(t) = \varphi(p) + t(\alpha_1, \cdots, \alpha_n) \\ &= \nabla(f \circ \varphi^{-1})(\varphi(p)) \cdot (\alpha_1, \cdots, \alpha_n) \\ &= \sum_{i=1}^{n} \frac{\partial}{\partial x_i}(f \circ \varphi^{-1})(\varphi(p))\alpha_i \\ &= \sum_{i=1}^{n} \alpha_i \frac{\partial}{\partial u_i}([f]) \\ &= v([f]) \end{aligned}$$

therefore $\dot{\gamma}(0) = v$.

□

References

[L] Lee, J., *Introduction to Smooth Manifolds.* Graduate Texts in Mathematics, Vol. 218, Springer, 2002.