SMALL-SCALE STRUCTURE AND RANDOMNESS: THE SCENERY FLOW IN DYNAMICS, GEOMETRY AND PROBABILITY (DECEMBER, 2005)

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1. Introduction.

The research described in this manuscript began during my undergraduate years, proceeded through postdoctoral stays on four continents, and has continued its evolution with the help of many remarkable people: teachers, mentors and coauthors, to whom I owe the deepest thanks.

The work itself has entered the areas of functional and harmonic analysis, probability theory, ergodic theory, group boundaries, dynamical systems, complex dynamics, low-dimensional topology and fractal geometry. However with this diversity, there are common themes which unite this work, both mathematical and philosophical in nature, and as regards both content and approach or point of view.

In this text I describe the individual projects, at the same time emphasizing these central themes, as they serve not only to explain the origin and motivation of the ideas, but also to indicate directions for future research. This last point will be returned to in the final section.

1.1. **Beginnings.** According to Gian-Carlo Rota, [Rot96], mathematicians can be subdivided into two types: "problem-solvers" and "theorizers". In reading his description, I found I identified myself much more with the "theorizers". What really interests me is understanding, explaining, and discovering. If solving a problem happens to be a part of that process, then it fascinates me. But finding a new problem, asking a new question, making a new and natural definition, noting an analogy between apparently different areas and chasing down a more profound explanation, building a new theory– this is what really draws me to do mathematics.

For me mathematics is in this sense much like music, art or poetry; essential is the creative aspect, because it expresses inspiration, and this inspiration comes from the heart, the spirit, and is called out by the infinite beauty of the universe. "The universe" here emphatically includes the inner universe of thought, created over centuries of deep investigation by many minds in many cultures.

Thus, my own mathematical adventures have usually begun with a question, rather than a problem. Then, in the process of investigating the question, a series of specific problems have naturally appeared, and the writing of an article has coalesced around the explanation of the resolution of one or more of these problems.

The questions in turn have ususally been somewhat philosophical in nature, often driven by an analogy noted between apparently quite different contexts, or by trying to resolve an apparent "paradox": finding a new way of looking at something that is at first sight impossible, yet where an analogy suggests there must be some hidden common meaning.

The zen master Shunryo Suzuki [Suz70] said "in the beginner's mind there are many possibilities, in the expert's are few". And so, to do good, creative mathematics, we must try again and again

Date: text updated December 16, 2005, preprint references updated March 2009.

to return to our beginner's mind, which is not clouded by achievements, not worried about fame or credit, but which simply takes a new and fresh delight in this marvelous world around us which we have the rare opportunity to explore. Then questions arise naturally, of their own accord.

1.2. **What is randomness?** Such a question which took hold of my imagination early on (while an undergraduate at UC Berkeley) can be summarized as: "What is randomness?"

We each have an intuitive understanding of chance; but when one tries to formulate this precisely, one naturally comes upon enigmas and paradoxes. It could be said that the whole structure of probability theory, and much of measure theory, functional analysis and dynamics, has been created in the attempt to answer this simple yet profound question.

I first encountered this question in a quite concrete form. A wonderful professor in a calculus course had asked me, "If you choose a point at random from the unit interval, what is the probability it will be rational?" I thought and thought; my intuition said "zero" yet this seemed a paradox (you *could* choose a rational!) and it wasn't until I encountered measure theory that I saw what a beautiful theory had been built to "answer" this puzzle.

It was natural to wonder about the same question modified slightly, replacing the interval with the real line– and this investigation led me much deeper, to graduate school and beyond.

I gathered together my initial speculations, and presented them to one of my favorite undergraduate professors (of measure theory), John Kelley. There I had realized that random probability on the real line should be represented by a translation-invariant measure of total mass one, which was then necessarily *finitely* additive. Moreover it should have an additional fascinating property: it should be *dilation* invariant. Kelley encouraged me toward functional analysis, as providing the tools I would need to proceed further.

In graduate school at the University of Washington, Seattle I discovered that such measures had a name– an invariant mean– but that they were highly non-unique. This seemed counterintuitive, as there should be a "real" answer to the question, and this new "paradox" led me on: I proceeded to devour the literature on the subject. I had made another guess– "the" mean should be exponentially invariant as well– and soon realized that no one had considered this possibility! I came with a new, more sophisticated list of conjectures to Isaac Namioka, and he said, "Prove *this* one!" (the statement about exponential invariance). That was my first theorem, and formed part of the foundation for what would be my first paper [Fis87].

Meanwhile, my fascination for randomness became matched with a fascination for dynamics, which reminded me of the beauty so often seen in the natural world: movement of water, clouds in the sky, wind on a lake. So I was drawn to ergodic theory and dynamical systems, which combined the two, and which, as a new, exploding and beautifully rich area provided a good excuse for studying a wide swath of mathematics: probability theory, algebra, algebraic topology, geometry, information theory, number theory, and the then new subject of fractal geometry.

Dynamics also happened to be the research area of another of my favorite undergraduate professors, Rufus Bowen, and of a good friend of mine, Brian Marcus, at that time in graduate school at Berkeley and just beginning his thesis work under Bowen.

I first met my PhD thesis advisor, Doug Lind, through this contact with Bowen and Marcus. Lind had been at Berkeley, where he had a postdoc after finishing his degree under Ornstein at Stanford, and had then moved to the University of Washington. Lind's presence there was one of the reasons I chose Seattle for graduate school, as it might be possible to write a thesis with him in dynamics.

After passing my qualifying exams, I took Lind's excellent course in ergodic theory, which focussed on Ornstein's then recent proof of the isomophism theorem for Bernoulli shifts of equal entropy.

Ornstein's theory was much more than just the proof of this theorem, as it brought in revolutionary new ideas which found application in a wide variety of situations. The best references remain [Orn73] and [Shi73]; in Ornstein's book the method is extended from transformations to flows; as a striking application it is shown that the geodesic flow on a compact hyperbolic manifold of constant negative curvature is Bernoulli; that is, its time-t maps are measure theoretically isomorphic to Bernoulli shifts.

The original isomorphism theorem (for Bernoulli shifts) itself was soon improved by Keane and Smorodinsky [KS79] from a merely measurable isomorphism to a locally finite "code", termed a *finitary isomorphism.*

Their method was actually easier than Ornstein's for the case of Bernoulli shifts; but extensions to the other situations covered by Ornstein proved quite difficult. Progress was made by Rudolph [Rud81] who introduced conceptually and technically sophisticated new methods.

This led to the thesis problem suggested to me by Lind: "try to find an analogue for flows of finitary isomorphism, and prove a flow version of Keane and Smorodinsky's theorem".

Finding a rigorous formulation for this problem, and then solving it [Fis83] provided me with an excellent foundation in ergodic theory, as I learned thoroughly Ornstein's discrete time and flow proofs, as well as the completely different techniques of Keane and Smorodinsky and then Rudolph.

Resolution of this problem left me with a permanent fascination for "flows"– (actions of the additive group of real numbers, that is, continuous dynamics) as well as the more usual "discrete" dynamics given by iterations of a map.

In the course of my studies with Lind, he had suggested I read and present in seminar an article by Bowen [Bow78], a beautiful paper which made use of Markov partitions and the Gibbs theory developed by Bowen, Ruelle and Sinai, together with delicate control in a coding argument. This control was provided by a Central Limit Theorem for information. Keane and Smorodinsky's proof had, by contrast, employed only the rougher estimates given by the Shannon-McMillan Breiman Theorem, which is known as the "Ergodic Theorem of Information".

Thus began for me a fascination with the limit laws of probability, which can be thought of as providing one part of an answer to the basic question of "what is randomness?". To really understand the Central Limit Theorem, you need to understand Brownian motion. So this led to more specific questions: "What is Brownian motion? What is the significance of the normal distribution? In what sense is a random walk path approximating a Brownian path?" This is of course very well-trod upon ground; however, thanks to the unusual perspective I had aquired in ergodic theory, I was soon onto something new.

I had learned of the scaling property of Brownian motion; scaling had been in my mind anyway because of invariant means, and I was writing a thesis on flows. In dynamics we studied "pathwise" statements like the Birkhoff ergodic theorem. A question occured: is there a "pathwise" version of the CLT for Brownian motion? I thought and realized, "this is a *real* question!" And then the scaling property turned in my mind: Brownian motion was a *flow*! Since the flow acted by change of scale, i.e. by dilation, this meant that the usual Cesáro time averages of the ergodic theorem turned into their exponential conjugate; and it was just these sorts of averaging operators which I had considered in studying invariant means. To apply the ergodic theorem, it remained to show the scaling flow of Brownian motion was ergodic. Indeed I was able to show, using techniques of the Ornstein theory, that it was a Bernoulli flow of infinite entropy, which is much stronger. And

I had my proof. I also had an unexpected new theorem (that Brownian motion is a Bernoulli flow under scaling) as well as a powerful new method which could be applied to produce theorems and proofs in other, sometimes apparently quite different, situations; indeed, that initial insight led to the second foundation of what was to be my first paper, and later to [Fisb] and [FT08a].

According to the Scientific Citation Index, this first paper has been cited 15 times by other authors, see below, most relating to this part on Brownian motion.

The third foundation of this paper resulted from my first postdoc, in Israel with Hillel Furstenberg and Benjie Weiss. There I became fascinated by Furstenberg's work on amenable and nonamenable groups, in particular about group boundaries. My original question "how does one choose a point randomly from an interval, or from the line?" now extended to: "from a non-amenable group?" This seems a new paradox, worse than the first, for *non-amenable* says exactly that no such average exists. Yet there is a natural procedure to attempt, based on random walk paths. Actually there are *two* ways, averaging over the collection of paths out to depth n and taking a time average, or doing this for each path individually and then integrating the resulting values. And the question becomes: "do they give you the same answer?" and: "what is the resulting average – what meaning does it have, and how does it avoid conflict with the fact that the group is non-amenable?"

With the help of Furstenberg and Weiss, I realized that what I had defined was a *projection*, from the bounded functions on the group to the harmonic functions –which are the constant functions exactly in the case of amenability. Then a clue to the resolution of the second question was provided by Weiss: "what you need is Mokobodski's notion of measure-linear means!"

A concrete formulation of this, interesting already for the case of the real line, then led to the third foundation of my first paper [Fis87].

I returned to the case of general groups, and also Markov operators, after combining forces with Vadim Kamanovich (an expert in the field) to write our joint paper [FK98].

The theme of almost-sure invariance principles in log density, introduced in [Fisb], was taken up by Berkes and Dehling [BD93]; I returned to this theme in [FT08a], in the course of a joint project with Artur Lopes, resulting in (3), where Talet and I use the theorems of Berkes and Dehling to prove a dynamical theorem in the spirit of [Fisb]. The deeper motivation for this project is explained below, as it forms part of the project with Lopes related to (14).

At this point I will present the abstracts of this series of papers and describe the contents in more detail.

1.) **Convex-invariant Means and a Pathwise Central Limit Theorem**

(Advances in Math., Vol. 63, No. 3, 1987, pp. 213-246; [Fis87])

Abstract: We prove the existence of means λ on $L^\infty(\mathbb{R})$ which are invariant with respect to translation, convolution, and an exponential change of scale, and are also measure-linear in the sense of Mokobodski. We show $exp(x)$ can be replaced in this theorem by any chosen convex change of scale. We prove a related theorem for directed sets, in the family of all convex functions, with a partial ordering given by composition. We give two applications. (*i.*) (an "infinite space Monte-Carlo method"): For all $f \in L^\infty$, $\lambda(f) = \lambda(f(B(t))$ for almost every Brownian path $B(t)$ (*ii.*) (a pathwise Central Limit Theorem for Brownian motion): For all intervals $I \subseteq \mathbb{R}$,

$$\lambda(\chi_I(B(t)/\sqrt{t})) = \frac{1}{\sqrt{2\pi}} \int_I e^{-x^2/2}\mathrm{dx}.$$

2.) **A Pathwise Central Limit Theorem for Random Walks**

(accepted for publication, Annals of Probability; 36 pages; [Fisb])

Abstract: We prove an almost sure, time-average version of Donsker's Theorem for random walks $S_n = \Sigma_{i=1}^n X_i$ where X_i is i.i.d. (independent and identically distributed with mean zero and finite variance $(= 1)$. In its dynamical formulation this states that a.e. polygonal path $S(t)$ is a generic point for the scaling flow of Brownian motion. To prove this we show first that S_n satisfies an almost sure invariance principle of order $o(n^{\frac{1}{2}})$ on a set of times of log density one; then we show that this form of a.s.i.p. implies the theorem. As a corollary one has the pathwise CLT: for every interval $I \subseteq \mathbb{R}$, for a.e. path S_n,

$$\lim_{N\to\infty} \frac{1}{\log N} \sum_{k=1}^{N} (\chi_I(S_k/\sqrt{k}))\frac{1}{k} = \frac{1}{\sqrt{2\pi}} \int_I e^{-x^2/2}\ \mathrm{dx}.$$

3.) **Dynamical attraction to stable processes**

(with Marina Talet, Université de Marseille; [FT08a])
(preprint submitted, 2009)

Abstract: We apply dynamical ideas within probability theory, proving an almost-sure invariance principle in log density for stable processes. The familiar scaling property (self-similarity) of the stable process has a stronger expression, that the scaling flow on Skorokhod path space is a Bernoulli flow. We prove that typical paths of a random walk with iid increments in the domain of attraction of a stable law can be paired with paths of a stable process so that, after applying a non-random regularly varying time change to the walk, the two paths are forward asymptotic in the flow except for a set of times of density zero. This implies that the time-changed random walk path is a generic point for the flow, i.e. it gives all the expected time averages. As a corollary one has a pathwise functional limit theorem and a pathwise stable central limit theorem.

For the Brownian case, making use of known results in the literature, one has a stronger statement: the random walk and the Brownian paths are forward asymptotic under the scaling flow (now with no exceptional set of times), at an exponential rate given by the moment assumption.

4.) **The self-similar dynamics of renewal processes**

(with Marina Talet, Université de Marseille; [FT08b])
(preprint submitted, 2009)

Abstract: We prove an almost sure invariance principle in log density for renewal processes with gaps in the domain of attraction of an α-stable law. There are three different types of behavior: attraction to a Mittag-Leffler process for $0 < \alpha < 1$, to a centered Cauchy process for $\alpha = 1$ and to a stable process for $1 < \alpha \leq 2$. Equivalently, in dynamical terms, almost every renewal path is, up to a regularly varying coordinate change of order one, and up to times of Cesáro density zero, in the stable manifold of a self-similar path for the scaling flow. As a corollary we have pathwise functional and central limit theorems.

1.3. **Discussion.** (1.)As is well known, translation-invariant means on $L^\infty(\mathbb{R})$ are far from unique. We reduce this non-uniqueness by requiring that λ satisfy the further axioms listed above. Some evidence of why these axioms may be considered natural and useful is given by the two applications. Thus e.g. (i) is proved by invariance with respect to the convex change of scale $c(x) = x^2$, together with measure-linearity; (ii) follows from application of the Hardy-Riesz log average $1/\log t \int_1^t f(x)\ 1/x \mathrm{d}x$, which is the conjugate of the Cesáro operator $f(t) \mapsto 1/t \int_0^t f(x)\ \mathrm{d}x$ by $c(x) = \exp(x)$. The Cesáro operator is, in its turn, the exponential conjugate of convolution with a probability distribution. In (1) we studied the class of all such higher exponential conjugates of convolutions.

The connection with Hardy and Riesz' work is made in (2), and was unknown to me at the time of (1). We set up the directed set machinery in order to give the general non-uniqueness question the following more explicit form. Does there exist a directed subset of Co, the family of all convex changes of scale, for which the associated averaging operators (i.e. the $c(x)$-conjugates of a convolution) will average all f in L^∞? (We show in (1) that Co itself is large enough, yet too large since consistency fails). A natural class might be the "completely convex" functions defined in (2); both consistency and sufficiency remain unknown.

This result on changes of scale has been extended to certain Hardy classes by Boshernitzan [Bos87]; personal communication.

(2.)The usual CLT says that the ensemble of random walk paths at time t has approximately a rescaled Gaussian distribution. The question here is how much time a chosen path spends in a given part of this distribution, i.e. whether the CLT can be stated in the form "time average = space average". In the first step, from (1), we showed the usual scaling property of Wiener measure can be dynamically formulated in this way: the scaling defines a flow (an $\mathbb{R}$-action) on the space of paths which is ergodic, in fact infinite entropy Bernoulli. Next, a $o(t^{\frac{1}{2}})$ almost-sure invariance principle translates into B and S being forward asymptotic in the flow, i.e. being in the same stable set. This proves the theorem, by Strassen's a.s.i.p., for finite $(2+\delta)$ moment; interestingly, for finite second moment, $o(t^{\frac{1}{2}})$ need not hold (because of counterexamples of P. Major). So we loosen this to $o(t^{\frac{1}{2}})$ in log density, proving that condition from Skorokhod's embedding; this is still enough to prove the PCLT. We mention that $(2+\delta)$- and third-moment versions were formulated independently of (1) and proved at about the same time as (2), by Brosamler and by Schatte respectively, and that Brosamler's theorem was immediately improved to second moments (our statement) by Michael Lacey and Walter Philipp. Also, unbeknownst to any of us at the time, P. Lévy seems to state a PCLT in his 1937 book "Théorie de l'Addition des Variables Aléatoires".

1.4. **Note.** These papers are a forerunner or the idea of scenery flow; in retrospect, the scaling flow on Wiener space is the scenery flow of a self-*affine* fractal set: a sample Brownian path.

The notion of *almost–sure invariance principle in log density* introduced here has since been further explored in [BD94] and now in (3); see also [BD93].

(3.) This recent paper extends the results of (2) to stable processes, and also strengthens the result of (2) for the Gaussian case, to include the entire domain of attraction. The proof builds on [BD94], which deals with discrete time. Two main new technical difficulties arise: stable paths can be discontinuous, so one must use the Skorokhod topology on path space, which is much trickier to work with than the uniform topology; furthermore, one now is dealing with the theory of regular rather than just polynomial variation. The results from paper 3 are needed here. The motivation for this paper actually came from an apparently completely different source, beginning with a question asked by Artur Lopes; for our approach to this problem we needed these theorems as tools. More information about these results will be given below when we discuss the overall project, see (5) below.

5.) A Poisson formula for harmonic projections

(with Vadim Kaimanovich, Institut de Recherche Mathématique de Rennes)
Annales de l'Institut H. Poincaré, Probabilités et Statistiques, vol. 34, no 2, 1998, 209-216; [FK98].

1.5. **Discussion.** This paper builds on and combines ideas from previous papers of the authors, [Fis87] and [Kai92], [Kai96], [Kai98].

Over the past several decades, much work has been done by many authors toward developing the related, complementary theories of amenability of groups and of boundaries at infinity.

We say here "complementary" because, roughly speaking, amenable groups are characterized as those with trivial boundary, while within the class of nonamenable groups, knowing the boundaries helps create finer distinctions; see the discussion below.

By definition, an amenable (locally compact) group G is one which has a normalized continuous translation-invariant linear functional on L^∞ of Haar measure; what this says is that there is a consistent, group-invariant way of defining the "average value" of a bounded function.

This seems, further, to imply that for a nonamenable group, the concept of average simply won't make sense.

The perspective of this paper is that, on the contrary, even for nonamenable groups, one can give meaning in a reasonable way to the notion of the average of a function. By the above remark, however, something must be lost; this will be translation-invariance by the group. This property will be replaced by a different sort of invariance, which will then (in the amenable case) reduce to translation-invariance.

Let G be a locally compact group, and let f be a bounded Haar-measurable real-valued function on G, for which we would like to find the average value.

If G is compact, defining the average value is straightforward, for bounded measurable functions: just integrate with respect to Haar measure. For non-compact groups, however, an unavoidable problem immediately arises, even for the simplest cases, $\mathbb{Z}$ or $\mathbb{R}$. There (for the purposes of defining an average) the analogue of normalized Haar measure is a translation-invariant *finitely additive* normalized measure; Haar measure itself is infinite. But now one must make a choice, as such measures are far from unique.

Usually in the theory one stops there; the existence of such a λ is all that matters and one ignores the remaining ambiguity. However a different point of view is that this simply invites one to look for additional natural and desirable properties which a notion of averaging might be required to have. A *measure-linear mean* is one that is *universally measurable* and which allows for a change in the order of integration, in other words such that Fubini's Theorem holds; for finitely additive measures, this is by no means always the case. Mokobodski's (nontrivial) theorem is that such means exist, as limit points of the Cesáro operators λ_s defined by $(\lambda_s f)(t) = 1/s \int_0^s f(x)\, \mathrm{d}x$. This additional property will be of key importance in what follows. From now on we will assume we have chosen and fixed such a Mokobodski mean on $\mathbb{R}$, on $\mathbb{Z}$ or on the semigroups $\mathbb{R}^+$, or $\mathbb{N}$ (depending on the context).

Now for G any locally compact group, with f a bounded function, let us try to do the same thing that works on $\mathbb{R}$. The idea is this. Consider the group G to be a geometrical space, and choose a point x in G. We can then always, even for G nonamenable, define the *the average value of f as seen from x*. To do this we proceed as before: first we average f over larger and larger spheres, of radius r; then we reaverage the resulting numbers $\bar{f}_r(x)$ with our chosen invariant mean λ on $\mathbb{R}^+$ (or on $\mathbb{N}$ for a discrete group).

The problem now is that for nonamenable groups, a new type of ambiguity arises: this value may depend very much on the point x we started from. In others words, our definition has produced not a single value but a *function*, $\bar{f}(x)$. The next question is, therefore, what kind of a function is it? The answer is this: it satisfies the mean value property. In other words, $\bar{f}$ is a *harmonic* function. We conclude that (for nonamenable groups) the notion of averaging makes sense, when it is interpreted as an *operator* : a projection from the bounded to the harmonic functions.

Here is another natural way to attempt to define the average value of a function f on a space X. Beginning a random walk with path $\underline{x} \equiv x_0, x_1, x_2, \ldots$ at $x = x_0$, now apply the chosen mean λ to the sequence of values sampled by the walk, setting $\bar{f}(\underline{x}) \equiv \lambda(f(x_0), f(x_1), \ldots)$. For the simplest situations, e.g. X a compact Riemannian manifold with Brownian motion, this will (for a.e. path) yield the value one wants, equal to the integral of f; moreover also for the non-compact spaces $\mathbb{Z}, \mathbb{R}$ or indeed $\mathbb{Z}^n, \mathbb{R}^n$ this procedure will work fine, giving $\bar{f}(x)$ equal to the average value $\bar{f}$ defined before, for a.e. path.

However for the case of nonamenable groups, something goes drastically wrong, as we know it must: for no procedure can produce an invariant average. But now the situation is different from the harmonic projection: the value we get does not depend on the starting point of the walk (as it did then) but rather on its final asymptotic behavior, on the eventual "direction" taken by that particular path. Boundary theory, in particular for the Furstenberg and Poisson boundary, is exactly designed to describe that behavior.

The idea here is the following. Using our chosen Mokobodski mean λ, we are defining boundary values for a general (not necessarily harmonic) function f. The questions then become: are these boundary values well-defined, i.e. equal a.s. for two paths which converge to the same boundary point? The answer is yes. This depends crucially on a theorem on the equivalence of two different definitions of Poisson boundary, see [Kai92], [Kai96]. And secondly: does the harmonic function produced by integrating these values give the same function defined using λ before, the harmonic projection? Again the answer is yes; here the crucial ingredient is Mokobodski's property of measure linearity; with the correct machinery in place, the proof amounts simply to a change in the order of integration.

2. Fractal encounters

Given my interest in scaling, and dyamics, it was natural to become fascinated by fractal geometry. In my third postdoc, at Warwick University, in 1986-87, and continuing at TU Delft the next year, I spent long hours staring at the picture in Mandelbrot's book [Man83] of the Cantor funtion ("Devil's Staircase") and trying to "understand" it. What was happening at different points; what was its "small scale struture"? Suddenly there came an insight: the illustration in the book was the “wrong” picture! The “real” Cantor function should be extended to the positive reals, and not clustered around the diagonal as in the book- rather, it should grow nested between two envelope curves, constants times t^d where $d = \log 2/\log 3$, the Hausdorff dimension! This was a natural realization given my understanding of the scaling property for Brownian paths. I constructed a colorful version of this curve with cutting and pasting blowups from a Xerox machine onto origami paper, and put it up in the apartment which I was sharing with Tim Bedford. (This picture, in black-and-white, appears in our paper [BF92].) Bedford, a student of Caroline Series who was an expert on the thermodynamic fomalism of Sinai-Bowen-Ruelle and on fractal geometry, said one day “what was that you were telling me about that curve?” And so began an exciting and fruitful collaboration.

The motivating questions here were “How can one describe the small-scale structure of a fractal set?” Another way of putting this is: “What is the analogue of the tangent space, for a fractal set?”

Again, I realized I had happened upon a “real” question, that is, it must have a rigorous mathematical answer- but one that was not obvious, and had apparently not been considered before.

Motivated by the work on Brownian motion, I realized that the small-scale structure should be modelled by a flow, the flow of dilation by the factor e^t, acting on the set centered at a chosen point: the flow of zooming into the small scale structure. I called this the “scenery flow” as the zooming process remined me of being on a train with the landscape flying past as one went down in scale.

This flow turned out to be interesting even for the simplest example (the middle-third Cantor set) as well as for another “well-understood” example – the Brownian zero set. This latter was especially intriguing because of connections with deep ideas from probability theory: Levy’s local time and an almost-sure invariance principle for returns to zero of a random walk, due to Revesz.

The third example treated in this first paper with Bedford, that of nonlinear hyperbolic Cantor sets, used the expertise and insight of Bedford regarding these sets, building on the theory of Sinai-Bowen Ruelle and on ideas of Sullivan and Rand.

In this paper the scenery flow is not yet called such by name, but implicitly this is what we are studying. Indeed a special flow (flow under a function) appears in the proof, which is isomorphic to the scenery flow and leads to the dimension formula “scenery flow entropy= Hausdorff dimension of fractal set”; see the discussion below regarding Julia sets.

Here we focused on one property, which we only later realized could be more elegantly established as a consequence of the ergodicity of the scenery flow. This was the notion of order-two or average density.

The motivation of this is reflected in the title; it was meant as an analogue of the Lebesgue density theorem on the one hand, and as a way of making precise the rough notion of “lacunarity” introduced by Mandelbrot in his book, on the other.

This property of fractals defines a new numerical invariant, which has since been studied and calculated by a number of authors working in fractal geometry; it is discussed in the books [Fal97] and [Mat95a] and in three of the papers, by Falconer, Mattila and Mandelbrot, in a volume of conference proceedings [Fal95], [Mat95b], [Man95]. The paper [BF92] has been cited at least 23 times by others (the book citations just mentioned do not appear in the Scientific Citation Index list). See included printouts from the Scientific Citation Index.

6.) Analogues of the Lebesgue density theorem for fractal sets of reals and integers

(with Tim Bedford, Delft University of Technology)
(Proceedings London M.S. Vol. 64, Jan. 1992, 95-124; [BF92]).

Abstract: We prove the following analogues of the Lebesgue density theorem for two types of fractal subsets of $\mathbb{R}$—hyperbolic $\mathcal{C}^{1+\gamma}$ "cookie-cutter" Cantor sets and the zero set of a Brownian path. Write C for the set, and μ for the positive finite Hausdorff measure on C. Then there exists a constant c (depending on the set C) such that for μ a.e. $x \in C$,

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T \frac{\mu(B(x,e^{-t}))}{e^{-td}}\,dt = c$$

where $B(x,\varepsilon)$ is the ε–ball around x and d is the Hausdorff dimension of C. We also define analogues of Hausdorff dimension and Lebesgue density for subsets of the integers, and prove that a typical zero set of the simple random walk has dimension $\frac{1}{2}$ and density $\sqrt{\frac{2}{\pi}}$.

2.1. **Discussion.** The Lebesgue density theorem states that as one zooms down toward a.e. point in a set E of positive measure in $\mathbb{R}^n$, the fraction of a ball occupied by E converges to one. For sets of non-integral Hausdorff dimension, the analogous statement can never be true, as was shown by Marstrand. Our idea is to nevertheless compute a density, by taking a time average. For sufficiently nice fractal sets like those discussed here this converges, and hence provides another characteristic number in addition to the Hausdorff dimension. (Dimension is a biLipschitz invariant; order-two density is in $\mathbb{R}$ a $\mathcal{C}^1$ invariant, and in $\mathbb{R}^n$ a conformal invariant). Aaronson, Kamae, and Patzschke and Zähle have each now given an approximate value for c for the middle-third set. No formula is known for hyperbolic Cantor sets; that would be the analogue of Bowen's formula for dimension.

3. Infinite measures and fractal sets of integers

In the paper (6), we introduced the notion of integer Cantor sets, using the example of the returns to zero of a simple random walk. We review that motivating example, and then describe how this led naturally to some new questions about infinite measure ergodic theory.

In a picturesque description, in the second edition of his famous text, Feyller [Fel49] discusses a paradoxical aspect of coin-tossing: "We feel intuitively that is Peter and Paul toss a coin for a long time $2n$, *the number of ties* (moments when the cumulative scores are equal) *should be roughly proportional to* $2n$. *But this is not so.* Actually the number of ties increases in probability only as $(2n)^{1/2}$; that is, with increasing duration of the game the frequency of ties decreases rapidly, and the "waves" increase in length." (Italics as in original.)

One mathematical expression of this strange behavior is given by the *arc sin law*: as Feller explains, as a consequence of that law, the fraction of time when, say, Paul is winning is, for fixed ε small, much more likely to be ε-close to 0 or 1 than to the value one expects to occur on average, 1/2. (Indeed, the ratio between these probablities goes to ∞ as $\varepsilon \to 0$.)

The number of ties N_n is of course the number of returns to zero of a random walk S_n, the sum of random variables X_i taking the values ± 1 with equal and independent probabilities (a *simple random walk*).

In particular, Feller's observation indicates that the set of returns is a very sparse set.

The above behavior contrasts distinctly with, for example, the set of return times of a (finite measure) ergodic transformation to a proper subset. There, twice as long a time interval *will* have twice as many returns on average. In other words, there is a direct proportionality between observed events and elapsed time.

A further indication of the strange behavior of the zero set is given by a little-known and at first sight quite mysterious theorem of Chung and Erdös [CE51]. As a consequence of this theorem (their average is different in appearance but equivalent for this example), one has that for almost every path,

$$\lim_{m\to\infty} 1/\log m \sum_{0}^{m} (N_n/n^{\frac{1}{2}}) \frac{1}{n} \to \sqrt{2/\pi}.$$

From a present-day perspective both phenomena have a common explanation. What is really going on is that there is an underlying conservative (i.e. *recurrent*) ergodic transformation preserving an infinite measure, the shift on path space; the zero set consists exactly of the times of return to a subset of finite measure ($= 1$), when the path is at location zero at time zero. The infinite measure forces the density of these times to be zero, accounting for the sparseness in a rough sense. The polynomial rate $n^{1/2}$ gives us finer information.

Perhaps this is why the random walk return behavior seems odd to us: that when considering events in time we tend to assume a scaling with exponent one, as we are used to finite invariant measures. By contrast, in geometry power laws are very familiar, as the scaling of volume is given by an exponent equal to the dimension of the object. This relationship holds as well for fractal sets, with Euclidean volume replaced by the d-dimensional Hausdorff measure of the set.

This analogy with fractal geometry suggests that perhaps at least for certain ergodic and conservative (recurrent) infinite measure preserving transformations, the number of returns to a set of finite measure behaves like a fractal subset of the integers. If so, that we could define the *dimension* of this subset of the integers (the set of return times) as $d \equiv \lim \log N_n / \log n$, where N_n counts the number of elements in the integer subset less than or equal to n, i.e. the number returns up to time n, providing this limit exists.

By Feller's observation, the random-walk zeros will have dimension 1/2 for almost every path. The Chung-Erdös average first divides the number of returns by this polynomial rate; a log average is then applied to the normalized sequence. The limit gives an integer analogue of the Hausdorff measure of that dimension. This limiting value, $\sqrt{2/\pi}$, is equal to the order-two density of the corresponding real fractal set: the zeros of a Brownian path.

We now see more clearly the nature of the "longer and longer waves" of Feller: they are the gaps of a fractal integer set; hence the "wavelengths" should actually grow exponentially.

Chung and Erdös' beautiful proof uses the standard probabilistic technique of variance estimates, and has nothing to do with the fractal or scaling explanation. To attempt a rigorous proof from this different point of view, a first observation is that a fractal set is self-similar, so one is working on a multiplicative rather than additive scale. Asymptotically, one should get a dilation-stationary process to which one could apply the ergodic theorem, to prove convergence for that action.

And applying a time average with respect to dilation is exactly the log averaging procedure, as explained in the discussion of Brownian motion paths (1) above.

Now which process invariant for dilation should the zero sets be approximating? One knows that a typical path S_n of the simple random walk on the integers (the walk with independent, equal probabilities to take steps ± 1) approximates a Brownian path well under scaling. Therefore, intuitively, the set of return times to the origin of S_n (the zero set Z_S of the walk) should approximate when rescaled the zero set Z_B of a Brownian path $B(t)$. The collection of these sets is acted upon by an ergodic flow with respect to dilation: this is a factor of the scaling flow for Brownian paths studied in [Fis87].

One's first idea to show rigorously that Z_S approximates Z_B is to use the excellent approximation one has of random walk paths by Brownian paths given by the a.s.i.p. of Strassen, which we used in (2) to prove the pathwise CLT.

However that result does not give nearly strong enough control over the return times that we need. Instead we proceed as follows. From the literature, we know that the Hausdorff ϕ- measure of $Z_B \cap [0,t]$ for a particular regularly varying function ϕ, $\phi = (2t \log\log(1/t))^{\frac{1}{2}}$, is equal to Lévy's local time of the Brownian process. By a classical theorem of Lévy, the local time can be identified with the maximum processes of Brownian motion (this identification being given by a stocastic integral). Due to a theorem of Rèvèsz (notes), something similar is true for the number of zeroes N_n of the random walk path S up to time $n = [t]$ and its maximum process: there is an a.s.i.p. of order $(1/4 + \varepsilon)$. Hence they are forward asymptotic in the scaling flow. Strassen's result for the paths carries over to the maximum processes. In conclusion we have a very strong comparison between N_n of S up to time $n = [t]$ and the Hausdorff ϕ- measure of $Z_B \cap [0,t]$.

Precisely, typical paths Z_B and Z_S can be paired so that these zero sets are forward asymptotic under scaling, with respect to a natural metric given by measure; since as noted the collection of all sets Z_B form an ergodic flow under scaling, almost every Z_S is a generic point for this flow. Hence the Birkhoff theorem applies.

This yields a new proof (discovered independently) of Chung and Erdös' theorem for the case of the simple random walk. For details see [BF92], [Fisb], [Fis04], [FT09], [FLT09].

The ergodic theory viewpoint not only gives a new proof but a new theorem. The previous result used the scaling flow on a space with a finite measure; but the statement has consequences for a different transformation (the shift map) which preserves an infinite measure. The result is a new type of ergodic theorem for the associated infinite measure preserving transformation. This "order two" ergodic theorem shall be described next, for an apparently quite different example.

3.1. **An integer version of the middle-third Cantor set.** As we have just seen, the times of return to a set of finite measure for a certain infinite measure preserving transformation, the shift on paths of a simple random walk, can reasonably be termed a fractal subset of the integers.

Now we try to turn this idea on its head: let us begin with a fractal subset of the reals, such as the middle-third Cantor set C; construct an integer analogue, try to build an associated infinite measure transformation and see if the same theorems go through. The set C is the the simplest non-random example of a real fractal set so it is a good candidate with which to begin.

Thus, we define the following subset $[C]$ of the integers: $[C] \equiv \{0, 2, 6, 8, \dots\}$. What we have done is to remove the middle thirds from the set $\{0, 1, 2, \dots, 3^n - 1\}$ and then to take the union over n. Each member of this set has a triadic expansion much like that of the real points in C.

We check first that, as for the random walk zeroes, $[C]$ has density zero (i.e. $N_n/n \to 0$ where $N_n \equiv \#[C] \cap [0, n-1]$), which gives the analogue of the real set C having Lebesgue measure zero. And moreover, as one can show, the dimension $\lim_{n\to\infty} \log N_n / \log n$ converges to $d = \log 2/\log 3$, which is the Hausdorff dimension of C.

To construct an ergodic transformation whose return-time structure imitates this integer fractal geometry, we employ a method familiar from ergodic theory: we take the associated sequence of $0's$ and $1's$, and form the closure in the product topology of all shifts of this sequence.

It turns out that there is a unique normalized invariant measure, so we have a well-defined ergodic transformation.

This is an infinite measure, however, and indeed that was the whole point of studying this example! Recall that from the Birkhoff theorem, return times to a subset of a finite measure space have frequency equal to the relative measure of that set. In other words, "the time average equals the space average". We wanted to prove a similar theorem, at least for some examples, for infinite measure. However infinite measure forces the density to be zero.

Instead (as for random walk zeros) we consider the d-dimensional density of $E \subseteq \mathbb{N}$ which is the log average of the normalized, but still oscillating, sequence N_n/n^d. One can think of this as a (finitely additive) version of d-dimensional Hausdorff measure for integer sets.

Using scaling, we prove convergence of this limit for the particular case of $E = \{x : \ x_0 = 1\}$ in our shift space. Next we apply Hopf's ratio ergodic theorem. This implies convergence for all $f \in L^1$, and we have proved the theorem we wanted, an analogue of "time average = space average" in this setting. The constant c enters in two ways: it is the (one-sided) order-two density of the set C, and it is also (along with d) an isomorphism invariant for the transformation.

Here is the abstract of the paper:

7.) **Integer Cantor sets and an order– two ergodic theorem**

(Ergodic Theory and Dynamical Systems, Vol. 13, 1992, 45-64; [Fis92]).

Abstract: Let $M \subseteq \prod_{-\infty}^{\infty}\{0,1\}$ denote the orbit closure, under the left shift σ, of the sequence ... (all zeroes) ...101000101000000000101... corresponding to the integer Cantor set $[C] = \{\Sigma_{i=0}^{N} a_i 3^i : a_i = 0 \text{ or } 2, N \in \mathbb{N}\}$. We prove that with respect to the infinite invariant measure ρ, which is the unique normalized non–atomic invariant measure on M, for every $f \in L^1(M,\rho)$, for $\rho -$ a.e. $x \in M$,

$$\lim_{N\to\infty} \frac{1}{\log N} \sum_{k=1}^{N} \frac{S_k(f(x))}{k^d} \frac{1}{k} = c \int_M f \ d\rho,$$

where $d = \log 2/\log 3$, and c is the almost–sure value of the right–hand order–two density of the middle–third Cantor set. The proof uses renormalization to a scaling flow, plus identification of (M,σ) as a tower over the Kakutani–von Neumann dyadic odometer.

3.2. **More probabilistic examples.** We had rediscovered Chung and Erdös' limit theorem for one of the simplest examples of a countable state recurrent Markov shift, the simple random walk on $\mathbb{Z}$. We had given a completely new proof based on scaling and the fractal analogy. After proving an initial extension of this to other Markov shifts, together with Jon Aaronson and Manfred Denker, we stumbled across Chung and Erdös' theorem which proved a very similar theorem on the density of the set of zeroes of a (quite general) class of random walks- but in 1950!

We then set out to understand the Chung-Erdös proof as well as the exact relation to our statement (which we will explain below) and then to prove a stronger theorem, with application to a wide class of infinite measure preserving transformations previously studied by Aaronson.

These transformations include the Markov chains and also a family of inner functions, like Boole's transformation $x \mapsto x - 1/x$ on $\mathbb{R}$ (which preserves Lebesgue measure). Due to previous work by Aaronson and Denker, these maps are Markov-like in a certain strong sense, which enables a general method to be used in the proofs.

Then, just as in (7), the Hopf ratio ergodic theorem implies the log-average ergodic theorem for all $f \in L^1$. For this case, since the "return sequence" $a(n)$ (this is defined to be the expected number of returns to a set of measure one) is asymptotically equal to cn^d, the first formula in the Abstract gives exactly this statement.

The Chung-Erdös average has a different form from ours, see (8). But as is proved in that paper, a sufficient condition for the two types of average to be equivalent is that $a(n)$ be regularly varying. (In particular, this is the case for the simple random walk).

The present proofs unify these two approaches and cover both types of examples. An interesting question is whether one can prove a.s.i.p.'s like Révész' for these other cases. That the return sequences are fractal-like sets in the fairly weak sense guaranteed by our theorem, suggests this may be possible. Further new evidence in this direction is provided by (18), see below: we now know there are a.s.i.p.'s in log density.

Here is the abstract:

8.) **Second order ergodic theorems for ergodic transformations of infinite measure spaces**

(with Jon Aaronson and Manfred Denker)
(Tel Aviv University; University of Göttingen)
(Proceedings AMS Vol. 114, No. 1, Jan. 1992, 115-128; [ADF92]).

Abstract: For certain pointwise dual ergodic transformations T we prove almost-sure convergence of the log-averages

$$\frac{1}{\log N}\sum_{n=1}^{N}\frac{1}{na(n)}\sum_{k=1}^{n} f\circ T^k \qquad (f\in L_1)$$

and the Chung-Erdös averages

$$\frac{1}{\log a(N)}\sum_{k=1}^{N}\frac{1}{a(k)} f\circ T^k \qquad (f\in L_1^+)$$

towards $\int f$, where $a(n)$ denotes the return sequence of T.

4. A pair of flows

In proving the theorems of (7) on the integer Cantor set, the flow viewpoint was crucial: there were now *two* flows, given by dilation and by translation. They satisfied a commutation relation. A hint by Dan Rudolph in a conversation at Göttingen (1988) nudged me to the next step: this was the same commutation relation as that satisfied by the shift and the odometer– and by the geodesic and horocyclic flows! See (7).

Geodesic flows were of course in the air at Göttingen, because of the presence of Paddy Patterson. I was there at the Stochastics Institute on a postdoc at the invitation of Manfred Denker and Ulrich Krengel; to my great good fortune, another visitor at the time, Mariusz Urbanski, knew a lot about Patterson and Sullivan's work on these flows and helped me to understand parts of this beautiful theory. And Urbanski, Bedford and myself began to delve deeper into Sullivan's nonlinear "cookie-cutter" Cantor sets and the closely related theory of hyperbolic Julia sets.

Thus I had three examples in mind, all essentially at the same time: one related to nonlinear hyperbolic Cantor sets, one coming from probability theory (the Brownian zero set), and one from Fuchsian groups. In all three cases there is a pair of flows, playing the role of the geodesic and horocyclic flows, and there are two measures, one finite and one infinite. In particular, the interaction between the infinite and finite measures is crucial, and provides a rich and fascinating series of examples.

The geodesic and (unstable) horocycle flows g_t and h_t satisfy the following commutation relation:

$$h_b g_s = g_s h_{e^{-s}b}.$$

We describe this for perhaps the strangest of the three examples, a β-self-similar process with stationary increments. This is a probability measure on a subset of the functions from $\mathbb{R}$ to $\mathbb{R}$ with an appropriate topology; for the simplest example, Brownian motion, we have $\beta = 1/2$ and path space is the set of continuous functions with the uniform topology. For stable and Mittag-Leffler processes, path space is D with the Skorokhod topology. See [FT08a], [FLT09].

Self-similarity means exactly that the scaling operation on path space, $f \mapsto f(at)/a^\beta$, preserves the measure for all $a > 0$. We define the *scaling flow* (of index β) to be the $\mathbb{R}$-action τ_s on D defined by

$$\tau_s : \ f(t) \mapsto f(e^s t)/e^{s\beta}.$$

Thus for a β-self-similar process, the measure is preserved by this flow. See [BF92], [Fisb], [Fis04], [FLT09], [Ver85].

The **increment flow** η is defined (as in [Fis92]) by

$$(\eta_s f)(t) = f(s+t) - f(s).$$

Having stationary increments can also be stated in terms of a flow: it is simply the requirement that the measure be preserved by the flow η.

It is easy to verify that τ, η satisfy the same commutation relation as g, h.

For the case of Brownian motion, the increment flow is naturally isomorphic to the shift flow on white noise. Indeed, this can be taken as a definition of the white noise flow. As we show, both flows are Bernoulli flows of infinite entropy. The same ideas work for all stable and Mittag-Leffler processes. In this latter case infinite measures must be used for the increment flow, making the connection with [Fis92] and [ADF92]. See [Fis87], [FT08a], [FLT09].

The case of Fuchsian and Kleinian groups is the simplest of the three but has yet to be fully written down! See [Fisa]. Some fascinating new related problems are currently being investigated in a project with Yair Minsky.

Next we consider the example of hyperbolic Cantor sets, and the related case of Julia sets.

9.) Ratio geometry, rigidity and the scenery process for hyperbolic Cantor sets

(with T. Bedford)

(Ergodic Theory and Dynamical Systems 17, June 1997, pp. 531-564; [BF97])

Abstract: Given a $\mathcal{C}^{1+\gamma}$ hyperbolic Cantor set C, we study the sequence $C_{n,x}$ of Cantor subsets which nest down toward a point x in C. We show that $C_{n,x}$ is asymptotically equal to an ergodic Cantor set valued process. The values of this process, called *limit sets*, are indexed by a Hölder continuous set-valued function defined on D. Sullivan's dual Cantor set. We show the limit sets are themselves $\mathcal{C}^{k+\gamma}$, $\mathcal{C}^\infty$ or $\mathcal{C}^\omega$ hyperbolic Cantor sets, with the highest degree of smoothness which occurs in the $\mathcal{C}^{1+\gamma}$ conjugacy class of C. The proof of this leads to the following rigidity theorem: if two $\mathcal{C}^{k+\gamma}$, $\mathcal{C}^\infty$ or $\mathcal{C}^\omega$ hyperbolic Cantor sets are $\mathcal{C}^1$-conjugate, then the conjugacy (with a different extension) is in fact already $\mathcal{C}^{k+\gamma}$, $\mathcal{C}^\infty$ or $\mathcal{C}^\omega$. Within one $\mathcal{C}^{1+\gamma}$ conjugacy class, each smoothness class is a Banach manifold, which is acted on by the semigroup given by rescaling subintervals. Conjugacy classes nest down, and contained in the intersection of them all is a compact set which is the attractor for the semigroup: the collection of limit sets. Convergence is exponentially fast, in the $\mathcal{C}^1$ norm.

10.) **On the magnification of Cantor sets and their limit models**

(with Tim Bedford)
(Monatsh. Math. 121, 1996, pp 11–40; [BF96].)

Abstract: For a $\mathcal{C}^{1+\gamma}$ hyperbolic Cantor set C we consider the limits of sequences of closed subsets of $\mathbf{R}$ obtained by arbitrarily high magnifications around different points of C. It is shown that a well defined set of limit models exists for the infinitesimal geometry, or *scenery*, in the Cantor set. If $\widetilde{C}$ is a diffeomorphic copy of C then the set of limit models of $\widetilde{C}$ is the same as that of C. Furthermore every limit model is made of Cantor sets which are $\mathcal{C}^{1+\gamma}$ diffeomorphic with C (for some $\gamma \in (0,1)$), but not all such $\mathcal{C}^{1+\gamma}$ copies of C occur in the limit models. We show the relation between this approach to the asymptotic structure of a Cantor set and D. Sullivan's "scaling function". An alternative definition of a fractal is discussed.

11.) **The scenery flow for hyperbolic Julia sets**

(with Tim Bedford, TU Delft and Mariusz Urbanski, Univ of North Texas)
(IHES preprint 1999; Proc. London Math. Soc., vol. **3**, no. 85, 2002, pp. 467-492; [BFU02])

Abstract: We define the scenery flow at a point z in the Julia set J of a hyperbolic rational map $T : \mathbb{C} \to \mathbb{C}$ with degree ≥ 2, and more generally for T a conformal mixing repellor.

We prove that, for hyperbolic rational maps, except for a few exceptional cases listed below, the scenery flow is ergodic. We also prove ergodicity for almost all conformal mixing repellors; here the statement is that the scenery flow is ergodic for the repellors which are not linear nor contained in a finite union of real-analytic curves, and furthermore that for the collection of such maps based on a fixed open set U, the ergodic cases form a dense open subset of that collection. Scenery flow ergodicity implies that one generates the same scenery flow by zooming down toward a.e. z with respect to the Hausdorff measure H^d, where $d =$ dimension (J), and that the flow has a unique measure of maximal entropy.

For all conformal mixing repellors, the flow is loosely Bernoulli and has topological entopy $\leq d$. Moreover the flow at a.e. point is the same up to a rotation, so as a corollary, one has an analogue of the Lebesgue density theorem for the fractal set, giving a different proof of a theorem of Falconer.

12.) **On invariant line fields**

(with Mariusz Urbański, University of North Texas)
Bulletin London Math. Soc. **32**, 2000 pp. 555-570; [FU00])

Abstract: We show that a rational function of degree ≥ 2 admits an invariant line field with respect to some measure μ, which is an equilibrium state of a Hölder continuous potential whose topological pressure is greater than its supremum, only in very special cases when the Julia set is either a geometric circle or an interval or it is totally disconnected and contained in a real-analytic curve.

4.1. **Statement of main result.** **Theorem** If $f : \mathbb{C} \to \mathbb{C}$ is a rational function and there exist $k \geq 1$, a measurable function $u : J(f) \to S^1$, and an ergodic f-invariant measure μ of positive entropy and full support (that is μ is assumed to be positive on nonempty open subsets of $J(f)$) such that

$$\left(\frac{f'(z)}{|f'(z)|} \right)^k = \frac{u(f(z))}{u(z)} \quad \text{for } \mu-\text{a.e.} z \in J(f),$$

then the map f is critically finite and either

$--$(a) f has a superattracting fixed point with a preimage at which f has a different degree.

$--$(b) f is critically finite with parabolic orbifold.

$--$(c) The Julia set $J(f)$ is a connected real-analytic Jordan curve and f is biholomorphically conjugate with a finite Blaschke product.

$--$(d) The Julia set $J(f)$ is a real-analytic closed segment and f is biholomorphically conjugate with a 2-to-1 factor of a finite Blaschke product.

$--$(e) $J(f)$ is totally disconnected and $J(f)$ is contained in a real-analytic curve with selfintersections (if any) outside the Julia set.

13.) **Small-scale structure via flows**

(Progress in Probability, vol. 57, 59-78, 2004)

[Fis04] **Abstract:** We study the small scale of geometric objects embedded in a Euclidean space by

means of the flow defined by zooming toward a point of the space. For a smooth embedded manifold one sees just the tangent space asymptotically, but for fractal sets and related objects (space-filling curves, nested tilings) the flow can be quite interesting, as the "scenery" one sees keeps changing.

For a Kleinian limit set the scenery flow and geodesic flows are isomorphic. This fact suggests that for a Julia set the scenery flow could provide the analogue of the hyperbolic three manifold, with its associated geodesic and horocycle actions.

A test is to see whether Sullivan's formula for dimension (Hausdorff dimension of limit set equals geodesic flow entropy) goes through for Julia sets. This is in fact true, and the resulting formula "dimension equals scenery flow entropy" unifies the formulas of Sullivan and of Bowen-Ruelle.

For changing combinatorics, considering the model case of interval exchanges, renormalization is given on parameter space by the Teichmüller flow of a surface; the scenery flow, now acting on a space of nested tilings, extends this flow to a surface fiber bundle. Thus renormalization is realized as flowing on a unification of the dynamical and the parameter space.

For fractal sets, the translation "horocycle" scenery flow has a natural conservative ergodic infinite measure. This observation builds a bridge between fractal geometry and the probability theory of recurrent events, suggesting on the one hand new theorems for the Fuchsian case and on the other a new interpretation of some results on countable state Markov chains due to Feller and Chung-Erdös. Interesting examples are seen in the intermittent return-time behavior of maps of the interval with an indifferent fixed point.

4.2. **Discussion of (9),(10), (12,) (11).** In the process of proving the existence of order-two density for hyperbolic Cantor sets in (6), we made use of a flow construction, the Bounded Distortion Property, the way Hausdorff measure transforms under conformal maps, and the Bowen-Ruelle-Sinai theory of Gibbs states. All of these themes return in this series of papers, and a much clearer picture now emerges of what is going on. The basic idea is this. We wish to model the idea of zooming down toward a point z_0 in a fractal set, by the continuous-time dynamics of a scaling flow on sets. This flow, $A \mapsto e^s \cdot A$ for e.g. $A \subseteq \mathbf{C}$ in the case of a Julia set J, is called the scenery flow of J at z_0.

The points in this flow are called *limit sets*; they are subsets of $\mathbf{C}$ which are infinite in extent and whose structure locally is like that of J, in the precise sense of being a countable holomorphic cover.. These sets can be explicitly constructed from a linearization method reminiscent of Sternberg or Grobman-Hartmann linearization. For e.g. the Julia set case, scenery flow is then naturally coded by a special flow with return time $\log|DT|$, over a base map which is the invertible version of a skew product over T with circle fibers. Having this nice symbolic representation is what allows one to study the measure theory of the flow.

The scenery flow can be given several interpretations. The collection of scenery flows at all points can be thought of as a natural tangent object to the fractal, analogous to the tangent plane bundle of a surface, since it is scaling invariant and transforms properly. Dynamically, it is a linearization of the hyperbolic map T. Also, it is an analogue of the geodesic flow on the unit tangent bundle of a manifold; this analogy will be described more clearly in (13).

In paper (12), we prove an analogue for Julia sets of a result proved by Sullivan for Kleinian groups (only the Lebesgue measure version of which is well-known). Our proof makes use of methods developed to prove a related theorem by Anna Zdunik.

Our original motivation comes from a different direction: we had constructed the scenery flow, and the first question one asks in ergodic theory is: "is it ergodic?". We prove in (11) that the invariance of line fields is (for the hyperbolic case) equivalent to the non-existence of invariant line fields for the Hausdorff measure class. Here therfore we prove the basic lemma needed for that conclusion. The context is now much wider however, both in terms of maps and measures, preparing the way for the general case of Julia set scenery (for rational maps) treated in [FU02], see (20).

4.3. **Discussion of (20).** , (13) In both cases the analogue of the geodesic flow is the scenery flow $\hat{g}_t$ of the set. Recall that this is the flow of magnification $A \mapsto e^s A$ on a collection of subsets of $\mathbb{R}$; the horocycle flow $\hat{h}_t$ will be simply the flow of translation generated by the same collection of subsets (but now allowing motion through a gap). The best way to see the analogy is in the Fuchsian case, where these flows are in fact homomorphic images of the usual flows g_t, h_t on the unit tangent bundle of the associated Riemann surface. Here is the homomorphism: a unit tangent vector $\underline{v}$ on the universal cover $\Delta =$ the unit disk, defines a unique stereographic projection which sends the tail ξ of its geodesic to ∞, its head η to 0, and the base point of the vector to i. Consider the image $\Lambda_{\underline{v}} \subseteq \mathbb{R}$ of the limit set Λ under this projection. This defines our homomorphism; it clearly takes g_t to $\hat{g}_t$ and h_t to $\hat{h}_t$. We use the image of Patterson-Sullivan measure for $\hat{g}_t$, and of Kenny measure for $\hat{h}_t$.

For Kleinian limit sets there is an extra ingredient: pictures can be rotated. As a consequence the scenery flow is modelled not on the geodesic flow but on the geodesic frame flow. Proving ergodicity is equivalent to showing there exist no invariant line fields on (limit set)$\times$(limit set). We show (making use of Margulis' volume lemma) that the homomorphism from the frame flow to the scenery flow is at most finite-to one. Hence as a corollary of a theorem of Sullivan, the entropy of the scenery flow equals the Hausdorff dimension of the limit set. (Compare the last paragraph in (11) above).

5. Return times and the transition from finite to infinite measures for maps of the interval with an indifferent fixed point

When I arrived in Porto Alegre in October of 1995, Artur Lopes presented me with a problem which he was trying to understand: what is the dynamics of a map of the interval which is hyperbolic- has derivative > 1- at every point except one, the fixed point, where it is indifferent or *neutral*– has derivative 1?

Lopes had already done extensive research on this, delving into the physics literature and writing two papers ([Lop90],[Lop93]). In particular he had noted a fascinating link with work of Hofbauer on non-Hölder potentials for the shift map [Hof77].

We began to approach the problem from two different angles. Lopes suggested that we try to imitate the standard Ruelle-Sinai-Bowen theory in this non-Hölder setting, proving polynomial (rather than exponential) decay of correlation and then proving a Central Limit Theorem by a Gordin- type approach.

The approach I came up with was completely different. It was based on the observation that return maps for our example were like those of a certain countable state Markov chain; indeed they generate exactly the same renewal process. This theory of "recurrent events" had already been worked out by Feller in a beautiful paper of 1949 [Fel49] but no one had yet made the link to dynamics. Moreover, we could prove something stronger: the inverse functions of the paths given by the return processes were stable processes. Meanwhile and Berkes and Dehling had proved an almost-sure invariance principle for stable processes in [BD93]. In fact, they had proved an *asip in log density*, based on my preprint for the Brownian case [Fisb]. The first step we should need was to prove a dynamical version of their theorem.

The progress so far is this: our first paper was published in 2001, following Lopes' idea; on the second approach I made considerable progress (with several incomplete preprints) until Marina Talet became involved in the project in 2005, upon which gaps were filled in the proofs, theorems were strengthened, a conjecture was settled, and everything was completely rewritten. The first two papers, with Talet [FT08a], [FT08b] provide results and tools needed in the third paper [FLT09] which describes the application to interval maps.

Now we shall decribe the problem and results in more detail.

5.1. **The problem.** In [Man80], the physicist Manneville observed fractal–like data in measurements of a dynamical system, the map $f_s: [0,1] \to [0,1]$ given by $x \mapsto x + x^{1+s} (\text{mod } 1)$ for $s > 1$. The basic goal of this project is to explain this phenomenon rigorously, using a combination of Gibbs theory and recent progress in the theory of infinite measure–preserving transformations, and in so doing also to place this map in a wider context of maps of the interval with similar behavior: doubling maps with indifferent fixed points and a class of unimodal maps.

In [Lop93] the properties of this map f_s (known as the Manneville–Pomeau map of index s) is studied as s changes. Gibbs theory is applied to get a zeta function for $0 < s < 1$ and to understand the "phase transition" at $s = 1$. Also a connection is made with a class of Markov chains on countably many states which behave similarly and which are in a sense linear models for f_s.

A basic motivating idea for our project is that above the critical parameter value, the natural invariant measure becomes infinite. And, as we will explain, for certain nicely behaved transformations preserving an infinite measure, return times to a set of finite measure can have a fractal–like structure. This behavior is then (by the ratio ergodic theorem) reflected in the values along an orbit of any integrable function, and this in turn would explain Manneville's observation.

In [Fis92] [BF92] examples were studied of infinite measure preserving maps which have a fractal–like structure for return times, in a very strong sense.

This is an *almost-sure invariance principle* (a.s.i.p.) for return–times, using the language of probability theory. What this means is the following. Letting $N_n(x)$ denote the number of returns of a point x up to time n to a given set A of finite measure, one considers the orbit closure under affine time–space scaling of this function. If there is a limiting self–similar process, in that the measures converge weakly on path space, then one has proved an *invariance principle*; an a.s.i.p. is the much stronger statement that paths $N_n(x)$ are forward asymptotic to the paths of the self–similar process under scaling. If this process is itself the local time of some fractal subset of the real line, then we have what we mean by the strong form of fractal–like behavior.

This strong property is demonstrated in [BF92] for a particular Markov chain, that given by a simple random walk, which is an ergodic map under the shift transformation on path space. Its return–times (to the set: value zero at time zero) are just the zeroes of the random walk, and N_n approximates under scaling the zero sets of Brownian paths, as indicated in the discussion of fractal sets of integers above; in [Fis92] this property was proved for a very different type of example, whose return–times approximate the standard middle–thirds Cantor set.

Related nonlinear examples are studied in [BF96], [BF92], see the descriptions below.

To review [ADF92] from this point of view, in that paper what can be regarded as a weak form of fractal–like behavior is proved, but for a much wider class of infinite measure–preserving transformations. This is the existence of log and Chung–Erdös averages. The cases covered include the Markov chain examples like that studied by Lopes, and also, by an application of work of Aaronson and Thaler, the Manneville–Pomeau maps for $s \geq 1$. (I wish to thank Jon Aaronson for a discussion of this point). Moreover by other work of Aaronson and Denker, related properties have already been established: an ergodic theorem in measure for s=1, and a functional law of the iterated logarithm and distributional convergence of N_n to the correct self-similar process (a positive stable law) for $s > 1$. (See [Aar97] and [ADU93] for related theorems).

In [Fisb], a weaker notion of a.s.i.p. was introduced, *a.s.i.p. in log density.* (See Berkes and Dehling [BD93] for a further exploration of this property). The example treated in [Fisb] is the approximation to Brownian motion by random walks with finite second moments (finite variance); by a counterexample of Revesz, for the stronger statement (a.s.i.p.) one needs at least finite $(2+\delta)$ moments, so the difference between these two properties is important but can be rather subtle.

We survey our main results of [FL01] and [FLT09]. For a family of maps F_α, for $\alpha \in (0,\infty)$ (here $s = 1/\alpha$), which are hyperbolic except for an indifferent fixed point at 0 and which are piecewise-linear versions of the Manneville–Pomeau maps, one has (for $\alpha > 1$), for the (unique) invariant absolutely continuous probability measure, a polynomial decay of correlations. We use this to prove, by a Gordin-type argument, the Central Limit Theorem. This is carried out in [FL01]. In the second paper [FT08a], we show, making use of the work of Berkes and Dehling cited above, a.s. invariance principles in log density hold, for all $\alpha > 0$. There are in fact three distinct regions of behavior: $\alpha \in (0,1)$, $(1,2]$ and $(2,\infty)$; for each one gets a different type of limiting self-similar process, with a Mittag-Leffler process for $\alpha \in (0,1)$, a stable process for the middle interval, in each case of parameter α, while for all $\alpha > 2$ (and also for $\alpha = 2$) one gets the same stable process, Brownian motion. The limiting processes are Bernoulli flows of infinite entropy under scaling; the theorems have a dynamical interpretation, as convergence in the scaling flow.

We conjecture that the same theorems hold for the Manneville–Pomeau maps themselves. We also conjecture that for F_α, and f_s, for $0 < \alpha < 1$ (the infinite measure setting), a stronger statement (a.s.i.p. for return–times) holds. (This is in fact known for one related case, corresponding to $\alpha = 1/2$, the occupation times of a simple random walk, see [BF92]). If one can show this strongest form of invariance principle, it will imply all the other statements as corollaries, including the theorems of Aaronson and Denker. The proof for the random walk zeroes makes use of very special properties of the walk; the general statement may prove false or quite difficult to approach.

Next, we suggest that maps of this type can be classified up to smooth conjugacy by a scaling function, as in [BF97] except with countably many symbols, or equivalently by a scenery flow. (It should be possible to construct the scenery flow due to the bounded distortion properties which are known for this type of map by related work of Renyi, Adler and Bowen). A further question will be to study the ergodic theory of this flow, e.g. to prove log averaging there.

The Manneville–Pomeau map is a prototype of $C^{1+\beta}$ doubling maps with an indifferent fixed point. An interesting general problem is to classify these maps up to smooth conjugacy, parallel to Sullivan's work for the hyperbolic case; the conjectures above would be a step in this direction.

Finally, similar theorems can be developed in the setting of unimodal maps, where infinite measures may arise because of indifferent periodic points (as for the M-P maps) but also because of properties of the critical point. As yet little has been proved about the behavior of this class of unimodal map, which makes for an important and challenging new collection of problems to work on.

Here is more regarding the first of these papers, which focusses on the case of finite variance. (In this part to avoid confusion, we change the original notation, where instead we worked with $\gamma = \alpha + 1$).

14.) **Exact bounds for the polynomial decay of correlation, 1/f noise and the CLT for the equilibrium state of a non-Hölder potential**

(with Artur Lopes, UFRGS, Porto Alegre, Brazil)
(Nonlinearity **14**, 2001, pp. 1071-1104; [FL01])

Abstract: We analyze the correlation and limit behavior of partial sums for the stationary stochastic process $(f(T^t(x)), \mu), t = 0, 1, \ldots$, for functions f of superpolynomial variation, the class $\mathcal{SP}$ defined below (which includes the Hölder functions), where $T : \Sigma^+ \to \Sigma^+$ is the left shift map on $\Sigma^+ = \Pi_0^\infty\{0, 1\}$ and μ is the non-atomic equilibrium measure of a non-Hölder potential $g = g_\alpha$ belonging to a one parameter family, indexed by $\alpha > 1$.

First, using the renewal equation, we show a polynomial rate of convergence for the associated Ruelle operator for cylinder set observables.

We then use these estimates to prove the following theorems:
∘We extend the polynomial convergence for the Ruelle operator to functions $f \in \mathcal{SP}$.
∘ We show the measure is weak Bernoulli; the bounds are polynomial.
∘ We calculate the decay of correlation of the stationary stochastic process described above, for $f \in \mathcal{SP}$. This decay is polynomial with t: we show in Theorem 4.1 an upper bound of order $Ct^{1-\alpha}$ when $\alpha > 1$; this estimate is sharp in the sense that for each α there exist functions f (in fact $f = I_{[0]}$ gives an example) for which one has the lower bound of $ct^{1-\alpha}$ for the decay of its auto-correlation. For the lower bound we use Tauberian theorems. For this example the coefficients decay monotonically, which is important for proving the lower bound.
∘ Again using Tauberian theorems together with the upper-lower bounds we show that for each $\alpha \in (1, 2)$ one has the phenomenon of $1/f$ noise for the spectral density of the funtion $I_{[0]}$.

∘ We prove the Central Limit Theorem and functional CLT for the case f is in $\mathcal{SP}$ and for $\alpha > 2$. For this we apply Gordin's method in the setting of a polynomial rate of convergence.

From the perspective of differentiable dynamical systems μ is the unique invariant measure absolutely continuous with respect to Lebesgue measure for an associated doubling map of the circle with an indifferent fixed point. This map $T_1 = T_{1,\alpha}$ is a piecewise linear version of the Maneville-Pomeau map, and the potential g_α is equal to $-\log DT_{1,\alpha}$.

We emphasize that our class $\mathcal{SP}$ is larger than the classes studied elsewhere.

5.2. **Discussion.** We comment briefly on the relationship of this paper to previous work. For hyperbolic systems and for Hölder potentials the Central Limit Theorem is known; and there one has an exponential decay of correlations. A new approach to proving the Central Limit Theorem was introduced by Gordin, see [Via97]. The idea is to use the exponential decay of correlations to produce an L^2- cohomologous observable which gives (after composition with the map) an orthogonal sequence of random variables. This sequence is not in general independent, but it is the next best thing: its partial sums form a martingale. One can then apply the martingale Central Limit Theorem, and can show that result then pushes back to the original observable, thus finishing the proof.

Our observations for the present paper are these:
−− at least for our special case, although we certainly believe it to hold in more generality, the presence of an indifferent fixed point produces a polynomial rather than exponential decay of correlation, for $\alpha > 1$. Equivalently, the polynomial decay is true for a hyperbolic map, but with a certain type of non-Hölder potential (this correspondence is explained in [FLT09]); and
−− this is then sufficient for carrying out a Gordin-type construction, at least for a certain class of observables, if the exponent is correct (here for $\alpha > 2$); all that is actually needed is for a certain series to converge, giving the cohomologous observable. As far as we know this paper represents the first time that a Gordin-type proof has been carried out in the presence of an indifferent fixed point or of a non-exponential decay of correlation.

For the stable regime ($\alpha \in (1, 2)$) we prove what physicists call $1/f$-noise, of the form $1/\text{ freq}^{2-\alpha}$; see [FL01] for a precise statement.

The CLT itself has previously been proved in a related situation (see [ADU93]: for rational maps of the complex plane with indifferent periodic points. Their observables ϕ are more general in one sense (Hölder functions are allowed, while we consider step functions) but less in another (in [ADU93] observables ϕ are zero in a neighborhood of the fixed point and we do not assume such a property). Also, we consider noninteger exponents α and in that sense a more general class of maps.

We remark that there is a finer subdivision of the region $(2, \infty)$: for $\alpha \in (2, 3)$ one has L^2-bounds on the Ruelle operator and can directly extend Gordin's method; for $\alpha \in (3, \infty)$ one has only L^1-bounds and needs to employ the subtler technique of Liverani [Liv96]. See [FL01]. A distinction also occurs in [FLT09]: for $\alpha \in (2, 3)$ one has ever better exponential rates of convergence to Brownian motion in the a.s.i.p., as α increases; however after α passes 3, this rate fails to improve, remaining at $e^{-t/4}$.

5.3. **Three phases for maps of the interval with an indifferent fixed point.** This chart, taken from [FLT09], summarizes what is known about the family of maps F_α and f_s. We think of the three regions of behavior as three *phases;* thus as the parameter changes, we move from

Gaussian to stable, correspondingly from finite to infinite variance; we then pass through all the asymmetric stable laws until the start of the next region, marked by the Cauchy process, where we make the transition to infinite mean, infinite measure, to the Mittag-Leffler processes and to fractal-like return time behavior.

Remarkably, this trichotomy is already present in [Fel49] for renewal processes, stated in terms of distributional convergence; now the link has been made to dynamics, and the challenge remains to see whether this type of "phase change" can be found in other places in mathematics (or physics).

	$\alpha = \gamma - 1$ $s = 1/\alpha$	$1 < \gamma < 2$ $0 < \alpha < 1$ $\infty > s > 1$	$2 < \gamma \leq 3$ $1 < \alpha < 2$ $1 > s \geq 1/2$	$3 < \gamma < \infty$ $2 < \alpha < \infty$ $1/2 > s > 0$
1	For F_γ and f_s	∞ abs cts meas	finite meas	finite meas
2		∞ expectation of return times	fte exp$\equiv \mu$, ∞ var of return times	fte exp and var of return times
3		upp/lower bnds	upp/lower bnds	upp/lower bnds
4		$a(n) = n^\alpha$	$a(n) = n$	$a(n) = n$
5		dim $d = \alpha \leq 1$	dim$= 1$	dim$= 1$
	scaling flow τ_β	Limit thms are for: $\beta = \alpha; N_n$	$\beta = \frac{1}{\alpha}; N_n - \frac{n}{\mu}$	$\beta = \frac{1}{2}; N_n - \frac{n}{\mu}$
6	For F_γ:	??	??	*asip*
7		*asip* in log density in sup norm	*asip* in log density in Skorokhod metric	*asip* in log density in sup norm
8		a.s. functional thm	a.s. functional thm	a.s. functional thm
9		??	poly decay of corr	poly decay of cor
10		??	??Gordin-type proof of stable CLT??	Gordin-type proof of CLT
11	For F_γ and f_s:	functional cnvg to Mitt-Leff process	fnl converg to stable prcss	fnl cnvg to Brownian
12		a.s. CLT for Mitt-Leff law	a.s. CLT for stable law	a.s. CLT for Gaussian
13		distr converg to Mitt-Leff law	distr cnvrg to stable law	distr cnvrg to Gaussian
14		log averages;	Cesàro averages;	Cesàro averages;
15		order-2 ergodic theorem for L^1 observables	Birkhoff ergodic theorem for L^1 observables	Birkhoff ergodic theorem for L^1 observables

6. Interval exchanges, renormalization and the Teichmüller flow

While I was in Paris at IHES at the invitation of Dennis Sullivan, in 1992, I gave a seminar talk at Université Paris Nord on integer Cantor sets and the scenery flow for the middle third set (papers (7),(10) above). In the audience was someone I had never met but who was following every word with great interest. After the seminar he introduced himself; it was Pierre Arnoux, who had written an expository article explaining in a clear way the fascinating but notoriously difficult work of Rauzy, Veech and Masur on interval exchanges, Keane's conjecture and the Teichmüller flow.

I had heard of the renormalization procedure of Rauzy in lectures of Dan Rudolph; I had heard from a completely different angle of the importance of Teichmüller space in renormalizations in complex and interval dynamics from Dennis Sullivan, and of the fascinating links with Kleinian groups and Thurston's incredibly beautiful work from Curt McMullen as well as Sullivan.

It seemed to me that these areas, the works of Thuston and Sullivan on the one hand, to which I had been exposed in a beautiful and challenging set of lectures by McMullen in Boston and Berkeley, as well as in Thurston's course at Berkeley the year before, and, on the other hand, the work of Veech and Masur, were the most fascinating and incredible things being done in the general area of dynamics. Yet I knew still very little about it.

I said to Arnoux, "Oh, you know about the Teichmüller flow! I think–there must be some way to put that and the scenery flow together!" Arnoux's eyes were alight, and he said, "Yes, I was thinking just the same thing!" Thus began a wonderful collaboration that has continued for more than 10 years.

After about a year we had the basic outlines of an understanding. The Teichmüller flow had an extension to the surface fiber bundle, acting there by the Teichmüller maps; this flow was ergodic; geometrically, it was the flow of zooming toward the small scale of an interval exchange. This was a scenery flow- not for a fractal, but for a space of nested tilings.

We could also work out a nonlinear theory, proving versions of the Gibbs theory and in particular extending Cawley's "Teichmüller space for Anosov maps" to this setting. That work in turn had been inspired by Sullivan's ideas on conformal structures on Riemann surface laminations- one of the tools he developed for his famous proof of Feigenbaum's conjectures. There was a chance that our ideas could lead to similar applications (this still remains to be seen!)

As McMullen put it, "it seems that you are buliding a Teichmüller space over Teichmüller space!" (he meant here: a Teichmüller space of maps, and of surfaces, respectively).

We began writing all of this as one huge paper. That became unwieldy and has been "cut in half" at least five times. Each time we have discovered other fascinating ideas along the way, which we could not resist exploring.

Finally we finished the first parts: (15) which has been published in 2001, and which emphasises the relations with substitution dynamical systems, with the tiling models, and with algebraic models for the flow; and (16), which develops the theory from a complementary perspective, emphasizing Markov partition sequences, nonstationary shift spaces and sequences of maps.

In what follows we give an overview of the project as a whole. However the new material here is extensive so rather than repeating everything the reader is invited to read the introductions of the first two papers.

6.1. The scenery flow for circle diffeomorphisms.

One can define a "scenery flow" associated to an Anosov toral diffeomorphism, in the following way. Fixing a Markov partition for the map, this partition tiles each unstable leaf. Images of the partition by negative and positive powers of the map give a nested tiling. Now imagine zooming down toward a point of the leaf. This is a submanifold and so it straightens out into a copy of the real line; geometric limits of the tilings give a collection of tilings on $\mathbb{R}$, invariant under dilation. It turns out that such a nested tiling is asymptotically self-similar in much the same sense as the fractal sets studied in (11), (13). In fact essentially the same arguments as given there allow one to exactly describe the limiting flow; we summarize these results. Symbolically it can be represented as a special flow, with Anosov base and return time equal to log of the derivative of the map in the unstable direction. There is a unique measure of maximal entropy for the flow, with entropy one. The flow is an invariant of $\mathcal{C}^1$ conjugacy (and, together with the stable scenery flow, is a complete invariant). Scaling functions exist; moreover they have a simple description in terms of conditional measures. If the (dilation) flow is topologically mixing, then the scenery horocycle flow (given by *translation* of the tilings) is uniquely ergodic.

Next suppose we are given a circle diffeomorphism f. Choosing some point, consider the nested tilings of the circle given by the orbit of the point, with hierarchies of tilings organized by each time of nearest return to the original point. Now try to build its scenery flow. Example: given an Anosov toral diffeomorphism, the stable horocycle return maps (*holonomy* maps) are circle diffeomorphisms - and in this case, their scenery flow is exactly that described above for the Markov tiling.

The special property that rotations given by the holonomy maps have is that their rotation number (i.e. average angle of rotation) is a quadratic irrational. Equivalently, it has a periodic continued fraction expansion.

But what happens for a general circle diffeomorphism? What is the scenery flow, and how does one study its ergodic theory? That is the main question which motivated this project. There are two new issues to be dealt with: possible nonlinearity of the circle diffeomorphism, and *changing combinatorics*: that is, a rotation number for the diffeomorphism whose continued fraction is nonperiodic.

In our first two of a series of papers on this subject, we give a complete answer for the linear case for all circle rotations. In later work, in the process of being written, we extend the theory to exchanges of more than two intervals, and to certain nonlinear circle diffeomorphisms.

The continued fraction expansion is acted on by the shift map; the resulting *Gauss map* is a cross-section for the classical modular flow. This was proved by Adler, Flatto, and Series. The starting point for our first paper was work of Arnoux which gave a new proof of their theorem.

The modular flow is the geodesic flow on the unit tangent bundle of the classical modular surface, i.e. is the right multiplication by $E_t \equiv \begin{bmatrix} e^{\frac{t}{2}} & 0 \\ 0 & e^{-\frac{t}{2}} \end{bmatrix}$ for $t \in \mathbb{R}$, on $SL(2,\mathbb{Z})\backslash SL(2,\mathbb{R})$. This is naturally identified with the Teichmüller flow on the torus: the flow on the space of all conformal structures on the torus given by contracting and expanding exponentially along a pair of orthogonal foliations. Such an expansion and contraction is known as a Teichmüller map between the two tori.

Now consider the scenery flow for a circle rotation, whose continued fraction expansion has a dense orbit with respect to the Gauss map. This is the flow of zooming toward small scales in space of tilings of the line nested according to the renormalization hierarchy, i.e. according to next nearest returns to the point of origin.

In zooming toward the small scale at a point, we will see the scenery which approximates arbitrarily well that of any given rotation of periodic type (because of the denseness of the orbit). It is natural to take the orbit closure of this space; the resulting flow space is what we call the *scenery flow for geometric structures on the torus* or the *Teichmüller mapping flow*. The reason for this second name is that this flow has an alternate description: it is a flow on a torus fiber bundle over the modular flow; in the fibers we flow by applying the corresponding Teichmüller maps. In our first paper (15) [AF01] we give several further equivalent descriptions of this flow.

15.) **The scenery flow for geometric structures on the torus: The linear setting**

(with Pierre Arnoux)
(Institut de Math. de Luminy (UPR 9016) Marseille, France)
(Chinese Annals of Mathematics **22B no 4**, 2001 pp. 427-470; [AF01])

Abstract: We define the scenery flow of the torus. The flow space is the union of all flat 2-dimensional tori of area 1 with a marked direction (or equivalently, on the union of all tori with a quadratic differential of norm 1). This is a 5-dimensional space, and the flow acts by following individual points under an extremal deformation of the quadratic differential. We define associated horocycle and translation flows; the latter preserve each torus and are the horizontal and vertical flows of the corresponding quadratic differential.

The scenery flow projects to the geodesic flow on the modular surface, and admits, for each orientation preserving hyperbolic toral automorphism, an invariant 3-dimensional subset on which it is the suspension flow of that map.

We first give a simple algebraic definition in terms of the group of affine maps of the plane, and prove that the flow is Anosov. We give an explicit formula for the first-return map of the flow on convenient cross-sections. Then, in the main part of the paper, we give several different models for the flow and its cross-sections, in terms of:

- stacking and rescaling periodic tilings of the plane;
- symbolic dynamics: the natural extension of the recoding of Sturmian sequences, or the S-adic system generated by two substitutions;
- zooming and subdividing quasi-periodic tilings of the real line, or aperiodic quasicrystals of minimal complexity;
- the natural extension of two-dimensional continued fractions;
- induction on exchanges of three intervals;
- rescaling on pairs of transverse measure foliations on the torus, or the Teichmüller flow on the twice-punctured torus.

6.2. **Dynamics of sequences of maps.** In the course of this research, we realized that for a general rotation, the role played by the single Anosov map is naturally taken on by a new type of dynamical system, which should be studied in its own right: a sequence of maps along a sequence of spaces, which we call a *mapping family*.

In the paper [AF05] (16), we develop a general theory of mapping families, and then focus on the specific examples related to circle rotations: the additive and multiplicative Anosov families. We extend Adler and Mannings' theorem on special codings for Anosov maps to sequences of maps; the symbolic dynamics now becomes non-stationary, leading to connections with the Bratteli diagrams of C^*- algebra theory.

16.) **Anosov families, renormalization and nonstationary subshifts**

(with P. Arnoux, Université Marseilles)
(Erg. Th. and Dyn. Sys. 2005 (25), pp. 661-709 [AF05])

Abstract: We introduce the notion of an Anosov family, a generalization of an Anosov map of a manifold.

This is a sequence of diffeomorphisms along compact Riemannian manifolds such that the tangent bundles split into expanding and contracting subspaces.

We develop the general theory, studying sequences of maps up to a notion of isomorphism and with respect to an equivalence relation generated by two natural operations, gathering and dispersal.

Then we concentrate on linear Anosov families on the two-torus. We study in detail a basic class of examples, the multiplicative families, and a canonical dispersal of these, the additive families. These form a natural completion to the collection of all linear Anosov maps.

A renormalization procedure constructs a sequence of Markov partitions consisting of two rectangles for a given additive family. This codes the family by the nonstationary subshift of finite type determined by exactly the same sequence of matrices.

Any linear positive Anosov family on the torus has a dispersal which is an additive family. The additive coding then yields a combinatorial model for the linear family, by telescoping the additive Bratteli diagram. The resulting combinatorial space is then determined by the same sequence of nonnegative matrices, as a nonstationary edge shift. This generalizes and provides a new proof for theorems of Adler and Manning.

17.) **Anosov families, the scenery flow, and the boundary at infinity**
(with P. Arnoux, Université Marseilles)
(preprint, 34 pp.) [AF])

Abstract: A mapping family is a nonstationary dynamical system: a sequence of maps along a sequence of spaces.

An Anosov family generalizes an Anosov map of a manifold to this nonstationary setting; it is a differentiable mapping family along a sequence of compact manifolds such that the tangent bundles split into expanding and contracting subspaces.

We study linear Anosov families on the two-torus up to the equivalence relation generated by partial compositions along the sequence of maps, giving this classification: a linear Anosov family is equivalent to an additive family (one generated by two positve Dehn twists) which is unique up to a shift of time.

We show that the following are equivalent. (i) A sequence (A_i) in $SL(2,\mathbb{Z})$ is an Anosov family (ii) The sequence f_{A_i} of Möbius transformations given by these matrices is a hyperbolic mapping family on the Poincaré disk. (iii) Defining a sequence (g_n) in the modular group $SL(2,\mathbb{Z})$ by taking partial products of the matrices A_i, g_n converges to distinct nonrational points η, ξ in the boundary $\mathbb{R}\cup\infty$ of $SL(2,\mathbb{Z})$ as n tends to $\pm\infty$.

The sequence $B_i \in SL(2,\mathbb{R})$ which diagonalize the Anosov family can be viewed as tangent vectors to the Teichmüller space of the torus. The matrix B_0 is a is a unit vector tangent to the geodesic connecting the boundary points η and ξ in the upper half plane, while the rest of the B_i are the projection to the modular surface of a sequence of tangent vectors to this geodesic. The proof makes use of returns to a nice crossection for the modular flow.

6.3. **Further results: higher genus, nonlinearity, nonstationary subshifts, unique ergodicity.** In this and later papers, we see that essentially all the usual theorems of dynamics go through in this more general setting of mapping families: the stable manifold theorem, openness and structural stability, the shadowing property for pseudo-orbits, and the construction of Markov partitions (this last result is joint with Mariusz Urbanski); all of Gibbs theory (the Ruelle-Perron-Frobenius Theorem, unique ergodicity of the holonomy maps as in Bowen-Marcus), and the topological classification as proved for single maps by Franks and Manning. A goal is to try to write down all these results; the two long papers just described are the first part of that project.

A further generalization is to the exchange of more than two intervals. For the linear case, everything goes through, though there are now some interesting combinatorics, due to the well-known existence of minimal non-uniquely-ergodic foliations. We treat this symbolically via the Bratteli diagrams.

A first step is (22), where we prove a key theorem needed for both (16) and (23): a Ruelle-Perron-Frobenius Theorem for nonstationary subshifts of finite type. Related theorems has already been proved for random dynamics, by Ferrero and Schmidt and Bogenschütz and Gundlach [FB79],[FB88], [BG95], [Bog93]. Our situation is somewhat different, however. Basically, any individual fiber of what Bogenschütz and Gundlach call a random subshift will be a nonstationary subshift in our sense. But their hypotheses are too strong for the main example we are interested in. By carefully (but, in retrospect, in a natural way) defining a *word metric* on the shift space, and then introducing a natural notion of *Hölder cocycles* with respect to this metric, we are able to prove the theorem even when word growth is very slow.

As we then show, a sequence of matrices, used by Veech in his proof to record a sequence of renormalizations, can be interpreted in a different way, as defining the transitions of a nonstationary subshift. Roughly speaking, the Axiom A case corresponds to a periodic sequence of transition matrices, and a periodic nonstationary sft, which can then be recoded as a sft in the usual sense, i.e. with stationary transitions. We isolate a mixing property which for nsfts implies uniqueness for the Perron-Frobenius theorem, and also unique ergodicity for the *adic transformations* of Vershik; this is automatic in the Axiom A case but holds only almost surely in the more general setting e.g. for interval exchanges.

To make the link between adic transformations and interval exchanges, we are developing two different approaches, one together with Luis Fernando Carvallo da Rocha using Veech rectangles and one with Jerome Los and Pascal Hubert using train tracks. The unique ergodicity is being studied in a further project, with Sebastien Firenczi. In all of this we of course build on the work of M. Keane, S. Kerkhoff, H. Masur, G. Rauzy, W. Thurston and W. Veech.

7. Descriptions of papers in preparation.

18.) Self-similar returns in the transition from finite to infinite measure

(with Artur Lopes, UFRGS, Porto Alegre, Brazil, and Marina Talet, Université de Marseille; [FLT09])

Abstract: We study the asymptotic scaling limits of occupation times for a one-parameter family F_α of maps of the interval, where $\alpha \in (0, \infty)$, which have an indifferent fixed point at 0. These are piecewise linear versions of the maps $f_s : x \mapsto x + x^{1+s}(\text{mod}1)$, where $s = 1/\alpha$. There is a unique (up to normalization) Lebesgue-absolutely continuous measure; for $\alpha \in (0, 1]$ this measure is infinite. For the infinite measure case with $\alpha \in (0, 1)$ the occupation times are asymptotically fractal (in a precise sense defined below), with dimension α. There are in fact three distinct regions of behavior: $\alpha \in (0, 1)$, $(1, 2)$ and $(2, \infty)$; for each one gets a different type of limiting self-similar process, with a Mittag-Leffler process for $\alpha \in (0, 1)$, a stable process for the middle interval, in each case of parameter $1/\alpha$, while for all $\alpha > 2$ (and also for $\alpha = 2$) one gets the same stable process, Brownian motion. See §10 for precise statements.

The limiting processes are Bernoulli flows of infinite entropy under scaling; our theorems have a dynamical interpretation, as convergence in the scaling flow. Thus, the occupation time paths N_n can be coupled with paths Z of the self-similar process so that N, Z are forward asymptotic under the scaling flow except for a set of times of density zero. As a corollary, almost every path N is a generic point for the scaling flow.

In probabilistic terms, our main theorem is an a.s. invariance principle in log density, and the corollary is an a.s. functional limit theorem, with convergence to Mittag-Leffler, stable and Gaussian distributions respectively.

These results provide a rigorous explanation for some numerical observations in the physics literature.

More generally, the theorems are shown for maps F_p of *return type* with $p = (p_0, p_1, \dots)$, where $p_i > 0$, $\Sigma p_i = 1$, such that $(\Sigma_k^\infty p_i)^{-1}$ is regularly varying with exponent $\alpha > 0$, or (equivalently) for renewal processes or countable state Markov chains with regularly varying tails.

Applying known results for Markov chains, for the finite measure case we identify the isomorphism type of the maps: their natural extensions are measure theoretically but not finitarily Bernoulli.

19.) Models for the scenery flow of a hyperbolic Cantor set

(in preparation)

7.1. **Description.** We construct the scenery flow of a hyperbolic $\mathcal{C}^{1+\gamma}$ Cantor set C in two ways completely different from that given in (10): first, directly from Sullivan's scaling function; secondly, by the linearization method of (11). From comparing these two, one can see the exact relationship between Sullivan's scaling function and the scenery flow of C. The log of the scaling function, $r(x)$, is cohomologous to $\log|DT|$ (where T is the map on C), and moreover, the cohomology is given by the time of return to a natural cross-section. (Cohomology of these two functions was proved in a different way independently by Y. Jiang and Sullivan).

20.) An analogue of the geodesic flow for rational maps

(with Mariusz Urbański)
(in preparation)

Abstract: We define the scenery flow of a rational map on the Riemann sphere and prove a statement which contains both the Bowen and Sullivan formulas for Hausdorff dimension: that the dynamical dimension equals the topological entropy of the scenery flow.

As a key step, we prove a lemma of abstract ergodic theory: an extension of Abramov's formula (that the entropy of a special flow equals entropy of the base divided by expected return time) to cocycles generated by functions which can also take on zero and negative values.

We show the scenery flow of a rational map is ergodic and indeed Bernoulli except for a very limited class of maps.

We also study the ergodic theory of the translation action on this scenery, showing it is uniquely ergodic in the infinite measure sense, except for the limited class.

Both this result and the entropy formula are known to hold for the scenery flow of a geometrically finite Kleinian group.

We connect our construction with the 3-manifold lamination of Minsky and Lyubich, and give a direct proof of the convergence to scenery for positive entropy measures.

21.) Infinite measure unique ergodicity for the horocycle flows of certain Fuchsian groups

(with Marc Burger, Univ of Lausanne)
(in preparation)

7.2. **Description.** Let Γ be a Fuchsian group, i.e. a discrete subgroup of $PSL(2,\mathbb{R})$, which is *geometrically finite* (equivalently, it is finitely generated). The disk mod Γ is a Riemann surface; in this paper we are interested in groups *of second type* i.e. such that the surface has infinite area; an equivalent condition is that the limit set is not the whole circle. For *finite* area, the natural invariant measure for the geodesic and horocycle flows g_t, h_t is the same: Riemannian volume. This is the unique measure of maximal entropy for g_t and the unique invariant measure (except for periodic orbits, if there are cusps) for h_t (theorems of Furstenberg, Dani). If the surface has infinite area, the two measures are however distinct: a finite measure (Patterson-Sullivan measure) for g_t, and a (unique) infinite measure for h_t (P. Kenny's unpublished thesis of 1983, for the no-cusp case). Burger and I each rediscovered Kenny's measure independently, and had together a proof of his uniqueness theorem when we heard of his work. Adapting recent methods of M. Ratner from the finite area setting, we are able to extend this result, showing uniqueness of Kenny's measure also when cusps are allowed.

We compare this with the Cantor set scenery flow. As described in 13, the translation scenery flow is the analogue of the horocycle flow. Again one can prove unique ergodicity; now the proof of is similar to that in (7), following again from the lemma of Bowen and Marcus. We mention that the main new ingredient in both cases, when compared with (7) (the middle-third set), is this: the appropriate measure class for the odometer is now given by a Gibbs state, with respect to which it is a nonsingular map. The unique ergodicity we prove can be thought of either as coming from a nonsingular unique ergodicity, i.e. the uniqueness of measures with that Radon-Nikodym derivative, or as usual unique ergodicity on the cross-section, defined to be the Maharam skew product over the odometer.

22.) The Ruelle-Perron-Frobenius Theorem for nonstationary and random subshifts

Abstract: We prove the existence of Gibbs measures, using the projective metric approch to Ruelle's Perron-Frobenius Theorem, for a class of potentials on a *nonstationary subshift of finite type*. This is a compact symbolic space defined like a subshift of finite type but with the number of symbols and the transition matrices allowed to vary. Our potentials are *Hölder cocycles* with respect to a *word metric*, for *mixing* nsft's . These definitions extend in a natural way the usual ones for a stationary subshift. The hypotheses are sufficiently general to allow for applications to interval exchange transformations and to the random subshifts of finite type studied by Bogenschütz and Gundlach.

23.) Unique ergodicity for horocycle foliations and nonlinear interval exchanges

Abstract: We show how two quite different theorems on unique ergodicity, due to Bowen and Marcus and to Masur and Veech, can be viewed together, as cases of a single theorem.

This leads to a result which generalizes both theorems. This common generalization includes the following: an extension of the Bowen–Marcus statement from Axiom A transformations and flows to Axiom A *families*, by which we mean a sequence of maps along a sequence of compact metric spaces, and a theorem like that of Masur and Veech, but for almost every member of a class of *nonlinear* interval exchange transformations. These are maps of the interval which are topologically but not smoothly conjugate to interval exchanges, and which are constructed so as to preserve differentiable structures on the interval determined by a Hölder cocycle on a corresponding pseudoAnosov family.

Our theorem has as a corollary a unique ergodicity result for certain random dynamical systems; indeed as we show, the Masur-Veech theorem can itself be viewed in this way. This makes a connection with the theory of random subshifts of finite type developed by Bogenschütz and Gundlach.

8. Directions for the future.

During the course of this research, a frequent motivating theme has been the attempt to answer apparently paradoxical philosophical questions: is there a unique natural notion of invariant mean on $\mathbb{R}$? What is the "average" of a function on a nonamenable group? What is the "tangent space" of a fractal set?

A further motivation has been to try to understand beautiful theories developed by certain outstanding mathematicians: Ornstein's isomorphism theory, the coding variants of Keane-Smorodinsky and Rudolph; aspects of probability theory, especially limit laws, invariance principles, stable processes, recurrent events; the Gibbs theory of Ruelle-Sinai-Bowen and many others; infinite measure ergodic theory, as developed by Kakutani, Aaronson and others; complex dynamics as investigated by many authors; low-dimensional topology as studied by Thurston; substitution dynamical systems, interval exchanges, and their relation with work of Veech, Masur, Rauzy and others.

Thus, these projects have given me both the excuse and opportunity to study a wide range of beautiful mathematics. Indeed, one of the reasons I chose to work in dynamics is because of the richness of the field in this sense – one can bring in almost anything one finds interesting.

I have worked with the assumption (based both on choice and experience) that a subject is best learned by doing research in it, thus making it come to life as one develops one's own unique point of view. Also I have lived with the conviction, always borne out, that if one learns anything deeply enough, interesting questions automatically will occur. A further "axiom" is that any paradox has a resolution; and that any analogy has a real explanation, occuring at a deeper common level. If it remains a paradox, if it refuses explanation, that is a temporary state, indicating rather that we have not yet thought deeply enough.

Throughout the investigations sketched above, there have been common motivating mathematical themes, and it is these unifying themes that I have tried to elucidate in this work, and which also indicate promising directions for the future.

I give only a few examples.

Thurston suggested, in a beautiful but unpublished manuscript [Thu], that there is a three-way "dictionary" between rational maps, Kleinian groups and pseudo-Anosov maps. Sullivan and McMullen have emphasised mostly the first two of these, over the course of much beautiful and deep research. I had the immense good fortune to attend seminars and courses given by all three, and found that this idea of analogies and a "dictionary" dovetailed perfectly with the notions of scenery flow and small- scale structure described above. There has been some interaction with these theories, with, I think, much more to come in the near future.

In particular, Minsky and Lyubich used our construction of the scenery flow for Julia sets of hyperbolic rational maps as one part of their remarkable paper [ML97]. They were able to take the "Julia set– Kleinian limit set" analogy much further, proving a rigidity theorem of Thurston for quadratic maps by a method similar to Mostow's for Kleinian groups. A key part of their proof involved replacing the hyperbolic three-manifold of the group by the scenery flow space. Lyubich and Kaimanovich developed this idea further in [KL01].

The project (20) together with Urbanski builds on this work, unifying it with the ergodic theoretic ideas of (11) to study the scenery flow itself, for general rational maps.

The work with Arnoux, and extensions with da Rocha, Los and Hubert and Firenczi mentioned above, is related to the third part of Thurston's dictionary, pseudo-Anosov maps.

A first question is, "what is the analogue of a Kleinian limit set or Julia set for this case?" Our answer is: nested tilings of the line, given by where an unstable leaf meets the iterates of a Markov partition; equivalently, the tilings determined by an interval exchange transformation with periodic combinatorics.

We studied this example in such depth in part because it provides a model situation in which to consider the yet more intriguing case of non-periodic, "changing" combinatorics.

A challenge for the near future is to take this last development over to the other two realms: Kleinian groups and rational maps. A project with Yair Minsky has been initiated to examine the Kleinian case; much work remains to be done as there are many nontrivial techniques and questions involved here.

One wishes to study the scenery flow of ever wilder fractal sets. There are some especially interesting cases which are not spatially homogeneous: the scenery flow is different at every point. There are several natural examples of such "sets with many sceneries", but the prize one is the most famous of all fractal sets: the Mandelbrot set boundary. There, by work of Tan Lei, [Tan90], one knows the scenery flow at a countable dense set of points, the Misiurewicz points (and there the scenery is equal to that of the corresponding Julia set!) What happens at a general point however is far from known, and is a difficult problem indeed. Even for the renormalization periodic points the answer is not clear, and for changing combinatorics, yet deeper mysteries remain.

One's first response on encountering the difficulties may be to say "it doesn't make sense"; but in keeping with the philosophy of this monograph, perhaps it is better to wonder if it appears not to make sense only because we haven't yet looked deep enough, and not yet found the correct formulations. *Something* interesting is without a doubt going on there, the analogies are indeed profound, and the challenge is to discover what lies beyond the reach of our present vision. There are hints, there are guesses, and there is much fascinating exploration to be done.

8.1. **Acknowledgements.** I wish to thank the following, listed in rough chronological order: my parents; my favorite instructors when an undergraduate, Andrew Ogg, John Kelley, Rufus Bowen, and Michel Loeve; professors in graduate school with whom I had special contact Robert Warfield, Robert Moore, Isaac Namioka; my doctoral advisor Doug Lind; persons who helped provide support and encouragement during my doctoral and postdoctoral work, Dan Rudolph, Hillel Furstenberg, Bejie Weiss, Jon Aaronson, Eli Glasner, Meir Smorodinsky, Alexandra Bellow, Bill Parry, Mike Keane, Manfered Denker, Ulrich Krengel, Jim Campbell, Paul Trow, Benoit Mandelbrot, S. Kakutani, Curt McMullen, Dennis Sullivan, Francois Parreau, Jack Milnor, Misha Lyubich, Artur Lopes, Jacob Palis, Marina Talet; my collaborators past and present, mentioned in the above text; my colleagues at the University of São Paulo; and all the other special mathematical friends without whose presence all this would not have been as much fun, but who are too numerous to mention here.

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