

THE SCENERY FLOW FOR HYPERBOLIC JULIA SETS

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ABSTRACT.

We define the scenery flow at a point z in the Julia set J of a hyperbolic rational map $T : \mathcal{C} \rightarrow \mathcal{C}$ with degree ≥ 2 , and more generally for T a conformal mixing repeller.

We prove that, for hyperbolic rational maps, except for a few exceptional cases listed below, the scenery flow is ergodic. We also prove ergodicity for almost all conformal mixing repellers; here the statement is that the scenery flow is ergodic for the repellers which are not linear nor contained in a finite union of real-analytic curves, and furthermore that for the collection of such maps based on a fixed open set U , the ergodic cases form a dense open subset of that collection. Scenery flow ergodicity implies that one generates the same scenery flow by zooming down toward a.e. z with respect to the Hausdorff measure H^d , where $d = \dim(J)$, and that the flow has a unique measure of maximal entropy.

For all conformal mixing repellers, the flow is loosely Bernoulli and has topological entropy $\leq d$. Moreover the flow at a.e. point is the same up to a rotation, so as a corollary, one has an analogue of the Lebesgue density theorem for the fractal set, giving a different proof of a theorem of Falconer.

§1. Introduction.

Fractal sets often come equipped with a discrete dynamics, like the map $T(z) = z^2 + c$ on its Julia set J . Since this map is conformal (infinitesimally orientation- and angle-preserving) whenever the derivative DT is a non-zero complex number, one can use the nonlinear scaling given by the map itself to study the geometry of J . Thus for instance assuming for a rational map T the additional hypotheses that degree $T \geq 2$ and that T is **hyperbolic**, i.e. that for some $n \geq 1$, there exists $\alpha > 1$ such that for every $z \in J$

$$|DT^n(z)| > \alpha$$

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or more generally for a *conformal mixing repeller* (see §4.3, also [Rue1] or [PUZ,I,II] and [PU] for the definition and further properties), one can prove that the set is quasi self-similar, that the Hausdorff dimension is strictly between 0 and 2, that the Hausdorff measure at this dimension d is positive and finite, and moreover that $\mu = H^d|_J$ is **geometric** in the sense that $\exists c_1, c_2 > 0$ with $c_1\varepsilon^d < \mu(B_\varepsilon(z)) < c_2\varepsilon^d$ for all sufficiently small ε (work of Bowen, Ruelle, Sullivan: [Bo2], [Rue1], [Su2] (see also [PU]); for general background on Julia sets see also [Be], [CG], and [Mi]).

In this paper we will study a linear, continuous-time dynamics which is constructed directly from the geometry of the set J . We imagine zooming toward some chosen point $z \in J$. Now for a fractal object like J , one will see a continuously changing scenery. This suggests the question which motivated this paper. Can one, at least for certain well-behaved fractal sets, model that process with the continuous dynamics of a measure-preserving ergodic flow?

To approach this question, we begin with some definitions. We will change the scale at a constant exponential rate and call $\{e^s(J - z) : s \in \mathbb{R}\}$ the **scenery** at the point z . This collection of sets is an orbit of a flow (i.e. additive \mathbb{R} -action) on the Borel subsets of the tangent space \mathcal{C} . The **scenery flow** at z will be simply the collection of limit points as $s \rightarrow +\infty$ (i.e. the omega-limit set) of $(J - z)$, the Julia set translated so as to be centered at z , in an appropriate topology to be defined in a moment.

We want to think of the scenery flow at z as some sort of derivative or tangent object to the set; this interpretation will be made precise below. Note that for a differentiable manifold embedded in \mathbb{R}^n , this does agree with the usual notion, since the scenery flow will then consist of a single point (the tangent space at z).

By contrast, for a fractal set the scenery keeps changing. As we will see, for some points $z \in J$ the scenery is periodic or almost periodic; however for almost every z (with respect to the Hausdorff measure $\mu = \text{restriction of } H^d \text{ to } J$), one gets a random set-valued process. In fact this flow is loosely Bernoulli, has entropy $\leq d$ and is, up to a rotation, the same flow for μ -a.e. z . Furthermore, for almost all the repellers, in a strong sense explained below, the scenery flow is rotation-invariant, and is therefore exactly the same, for a.e. z .

Our first task is to construct the scenery flow. We topologize the collection of Borel sets of \mathcal{C} by two natural pseudo-metrics, which will become metrics when restricted to the subclasses of sets of interest here. Limits will be proved to exist in both metrics. The topology for the **measure metric** $\rho(E, F)$ is defined by associating to $E \subseteq \mathcal{C}$ the restriction μ_E of H^d to E , and then testing against continuous real-valued functions with compact support, i.e. for the corresponding measures this is the weak-* topology in $\mathcal{C}_c^*(\mathcal{C})$. To define the **local Hausdorff metric** $\hat{\rho}(E, F)$ on the closed subsets of \mathcal{C} we fix a conformal map (the inverse of a stereographic projection) from \mathcal{C} onto the Riemann sphere $S^2 \setminus \{\infty\}$, add the point ∞ to the images of both sets, and then use the Hausdorff metric coming from the Euclidean metric on S^2 (see §1).

Now the idea for the construction of the scenery flow is (in retrospect!) extremely easy. We form the shift space $\prod_{-\infty}^{\infty} J$, with the product topology and left shift σ , and restrict to the subset $\prod_0 = \{\underline{z} = (\dots z_{-1} z_0 z_1 \dots) : T(z_j) = z_{j+1}\}$. Note that choice of a string \underline{z} corresponds to choice of an initial point z_0 (which of course determines uniquely the

“future” z_1, z_2, \dots) together with an infinite branch of preimages z_{-1}, z_{-2}, \dots . Now for each choice of z_0 and branch of pasts, we will define a Borel set $L_{\underline{z}} \subseteq \mathcal{C}$. This will simply be the limit (in either metric) of the Julia set centered at each z_{-n} and then expanded and rotated by that derivative:

$$L_{\underline{z}} \equiv \lim_{n \rightarrow \infty} DT^n(z_{-n}) \cdot (J - z_{-n}).$$

Convergence will be proved from a strong form of the Bounded Distortion Property (Theorem 2.11). We call $L_{\underline{z}}$ a **scene** or **limit set**. It is a countable holomorphic cover of the Julia set, and as such is analogous to the imaginary axis wrapping infinitely many times around the circle via the exponential map. Indeed, for the map $T(z) = z^2$ the Julia set J is the circle, the scenery of $J - 1$ at the point 0 is its tangent line, the imaginary axis, and the above limiting procedure yields the covering map $z \mapsto e^z - 1$ (see Theorem 2.14).

The limit set will be forward asymptotic to the scenery at z_0 , in the sense that, for any choice of pasts, $\text{dist}(e^s L_{\underline{z}}, e^s(J - z_0))$ will converge to 0 (in either metric) as $s \rightarrow \infty$. But what we have done by constructing $L_{\underline{z}}$ is to define a point in the scenery flow itself. To see this we have to understand the relationships between the limit sets as the initial point changes by an application of the map T . The idea is that, roughly speaking, the linear flow $E \mapsto e^s E$ will have, as cross-section map, the derivative map DT , lifted to a bundle over J whose fiber at z is that scenery flow. This fact will enable us on the one hand to use the dynamics of T itself to study each scenery flow, and on the other hand to interpret this scenery bundle as a tangent object (as with the circle example), since it transforms properly.

To explain this more precisely, note that from the construction one sees immediately that the limit sets $L_{\underline{z}}$ and $L_{\sigma \underline{z}}$ are related by

$$L_{\sigma \underline{z}} = DT(z_0)L_{\underline{z}}.$$

Let us assume for simplicity that T is **strictly** hyperbolic, i.e. $1 < \alpha < |DT| < \beta < \infty$. (Otherwise, replace T by T^n). Now, defining $r : J \rightarrow (0, \infty)$ and $\theta : J \rightarrow S^1 \equiv [0, 2\pi)$ by $r(z) = \log |DT(z)|$ and $\theta(z) = \arg DT(z)$, and taking $z = z_0$, we can write the right-hand side as:

$$DT(z_0)L_{\underline{z}} = e^{i\theta} e^r L_{\underline{z}}.$$

That is, up to a rotation, the orbit of the scenery flow at \underline{z} returns after time $r(z_0)$ to the shifted coordinates $\sigma \underline{z}$. This return map is modelled by a skew product over the shift (\prod_0, σ) with circle fiber $S^1 = \mathbb{R}/\mathbb{Z}$ and skewing function $\varphi(\underline{z}) = \theta(z_0)/2\pi$. We write this transformation as $(\widehat{\prod}_0, \widehat{\sigma})$, where $\widehat{\prod}_0 = \prod_0 \times S^1$ and $\widehat{\sigma}(\underline{z}, s) = (\sigma(\underline{z}), s + \varphi(\underline{z}))$. We build the special flow with this base and with return time $r((\underline{z}, s)) \equiv r(z_0)$. This is the **symbolic model** for the full scenery flow of J ; the scenery flow at a point z_0 , which was defined above, will then be shown to be a closed invariant set which is exactly the image of the orbit closure of the point $(\underline{z}, 0)$ under the continuous map defined by $(\underline{z}, 0) \mapsto L_{\underline{z}}$.

What now works out beautifully is that the symbolic version of the flow with base (\prod_0, σ) and return height r (i.e. where we forget about angles), has as a natural invariant

measure ν (indeed as its unique measure of maximal entropy) a measure which on the cross-section is equivalent to the Hausdorff measure μ on J . This is, in fact, exactly the Bowen-Sinai-Ruelle Gibbs state for the function $-dr$. The uniqueness of this measure passes to the skew product, by some analysis based on a method of Furstenberg. This is described in the next paragraph. The fact that μ - a.e. scenery flow is (up to rotation) the same now follows from ergodicity of $(\widehat{\prod}_0, \hat{\sigma})$ with respect to this measure. For the same reason (now with no need to worry about rotations) one has, by the Birkhoff ergodic theorem, an analogue of the Lebesgue Density Theorem for the set J (see [B-F 1]): there is a constant $c > 0$ (the “order-two density”) such that for μ - a.e. $z \in J$,

$$\lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T \frac{\mu B(z, e^{-t})}{e^{-td}} dt$$

exists and equals c (see Proposition 4.2). This is an average density of the Hausdorff measure, sandwiched between the upper and lower bounds c_1 and c_2 mentioned before for the geometric measure μ . Finally, Bowen’s well-known formula for dimension [Bo], [Rue1] now implies the upper bound of $\dim(J)$ for the topological entropy of the scenery flow.

When we analyze measures of maximal entropy for the scenery flow, it is sufficient to consider ergodic measures on $\widehat{\prod}_0$ which project to the Gibbs state ν on \prod_0 , since the circle fibers add no entropy. Now there is a dichotomy: either $\nu \times m$ is ergodic (where $m =$ Lebesgue measure on S^1), or the ergodic measure sits on k copies of the graph of a measurable function from \prod_0 to S^1 , for some k in \mathbb{Z} , and is unique up to rotation. Equivalently, $k\varphi$ is measurably cohomologous to zero. This is proved using a Fourier series method of Furstenberg [Fu], see Theorem 4.4.

In this second case, using the fact that the map T , being a conformal mixing repeller, is hyperbolic, we can, by two theorems of Livsic (see [Li] and [PUZ,I]) reduce the analysis to a study of the periodic points. The first theorem (when reproved for the present case of circle fibers) allows one to show the measurable cohomology is actually continuous. In other words, the measure sits on the graphs of k continuous functions. By the second theorem, this is equivalent to having $\arg DT^n(z_0)$ be some multiple of $2\pi/k$, for each z_0 of period n .

Now this last condition allows us to analyze how the rotational symmetries of the scenery flow change as T varies in an appropriate space of conformal mixing repellers. By some basic complex analysis (the implicit function theorem and open mapping theorem for several complex variables) and Theorem 3.1 from [MPU] the case with discrete symmetry can at most happen for the repellers which are linear or are contained in a finite union of real-analytic curves. Moreover, these exceptional cases are negligible in the sense of Baire category, as their complement forms a dense open subset with respect to appropriate topology for the repellers; see Theorem 4.8.

For the case of hyperbolic rational functions, in view of results from [FU], we have more specific information. Some examples of rational maps with discrete symmetries which easily come to mind are: $z \mapsto z^2$ and more general Blaschke products (for which the line field tangent to the circle is invariant); the Lattés map $z \mapsto (z^2 + 1)^2/4z(z^2 - 1)$, which is covered by the conformal Anosov endomorphism of the torus $z \mapsto (1 + i)z$, hence its Julia

set is the sphere and the image of any constant line field is invariant, the map $f(z) = z^2 - 2$, whose Julia set is the interval $[-2, 2]$, and $f(z) = z^2 + c$ for c real and of sufficiently large modulus, which has for its Julia set a Cantor subset of \mathbb{R} (take e.g. $c = \sqrt{3}$); in these last two cases any constant line field is invariant. As we show in Theorem 1 from [FU], this is essentially all that can happen in the general case: the discrete symmetry forces the map into a limited number of exceptional classes corresponding to these examples (plus one more for which we know of no concrete example). In our hyperbolic case the exceptional classes are even fewer in number (see Theorem 4.9).

So in conclusion, for both hyperbolic rational maps and conformal mixing repellers, for all T not in the corresponding exceptional set, we know that for ν - (hence μ -) a.e. z_0 and w_0 , their scenery flows are identical. For all T , including the exceptional cases, we know the following: the scenery flows are the same up to a fixed rotation. Moreover μ - almost surely the scenery flow at z_0 has a unique measure of maximal entropy, bounded above by $\dim(J)$. And finally, applying theorems of Rudolph [Rud], this flow is loosely Bernoulli (has a measure-theoretically Bernoulli cross-section).

We conjecture that for the fractal case ($\dim(J) \neq 1$), one always has equality of dimension and entropy, and moreover, that the continuous map from the symbolic model to the scenery flow is at most finite-to one.

One can also construct a scenery flow at a point x in e.g. a Brownian zero set [BF1], the middle-third set and more generally a hyperbolic $\mathcal{C}^{1+\gamma}$ Cantor set [BF2, 3], and in a Fuchsian or Kleinian limit set [F2]. Scenery flows of certain families of circle diffeomorphisms are studied in [AF]. (To make the transition from [BF1], [F1] to the present perspective, note that the scaling flow on local times with local uniform topology corresponds exactly to the scenery flow on sets with the measure topology).

For the present example of a hyperbolic Julia set, the dynamics of the map $T : J \rightarrow J$ is used in studying the scenery flow, as we have described. The same is the case for hyperbolic Cantor sets. We mention that for the Fuchsian case, one can take a similar approach, using the discrete dynamics of the group action on the limit set. However in this setting there is also a natural continuous-time dynamics (the geodesic flow on the unit tangent bundle) and it is much simpler to use this directly to study the scenery flow. By contrast, for the Brownian example, the scenery flow makes sense even though there is no natural dynamics on the zero set itself.

Meanwhile, Tan Lei [Ta], also see [Mi, Appendix A] has also studied rigorously the scaling structure near a point in certain fractal sets. Her theorem states, in the language of the present paper, that for a Misiurewicz point c , the scenery flows at c in J_c (for the map $z^2 + c$) and at c in ∂M (the boundary of the Mandelbrot set) are identical, and are topologically conjugate to either a single periodic orbit or an irrational flow on a torus. Misiurewicz points form a countable dense subset of ∂M . Thus a general point $z \in \partial M$ is approximated by points whose scenery flow is known to exist and to be periodic or almost periodic, with T -periods going to infinity. In light of this observation, combined with the fact that the measure theory of the Julia sets corresponding to these points z is still far from worked out, it is an intriguing problem to try to understand the scenery flow for these points.

For a beautiful application of our limiting construction of the scenery flow see [LM].

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§2. Convergence to limit sets.

Note. We assume until §4.3 that T is a rational map; aside from Theorem 2.14, all constructions, statements and proofs are valid without essential change for conformal mixing repellers, except for those statements which refer to parameter space, which is discussed in that section.

Thus, we now fix the notation and assumptions on T made in the Introduction. That is, $T : \mathcal{C} \rightarrow \mathcal{C}$ is a hyperbolic rational map of degree ≥ 2 , the constants α, β, c_1, c_2 are as defined there, $d = \dim(J)$, and $\mu = H^d \lfloor_J$.

For simplicity of notation, we again assume strict hyperbolicity. We call μ the **conformal measure**; the reason for this name is that it satisfies the **conformal transformation property**: if for some $E \subseteq \mathcal{C}$, $T : E \rightarrow \mathcal{C}$ is 1 – 1, then $\mu(T E) = \int_E |DT(z)|^d d\mu(z)$. This follows from the fact that Hausdorff measure H^d transforms by that same formula, with respect to any 1 – 1 conformal map $f : U \rightarrow \mathcal{C}$ defined on some domain (i.e. open set) $U \subseteq \mathcal{C}$. For **conformal** there are several equivalent definitions. The first is that f is 1 – 1 and complex differentiable (synonyms are: holomorphic, complex analytic) with non-zero derivative Df . By the open mapping and inverse function theorems, this is equivalent to $f : U \rightarrow V \equiv f(U)$ being biholomorphic. From Rouché’s theorem, moreover, f is conformal if and only if it is holomorphic and 1 – 1.

In the next section we will describe the ergodic theory of μ (or rather of the Gibbs state ν , its T -invariant version) but at present all we need are the facts (originally proved, by Bowen and Ruelle, from that ergodic theory) that $0 < d < 2$ and that μ is a geometric measure.

Topologies. We first define a topology \mathcal{T} on the collection Ω of closed subsets of \mathcal{C} , the **conformal map topology**. Convergence here will imply convergence in the two weaker topologies mentioned in the Introduction, which will be given by metrics when restricted to relevant subcollections of sets. This topology is also a *uniformity* (see the statements preceding Lemma 2.2), so we can e.g. speak of the uniform convergence of functions from another uniform space to this one.

Let $B_R(z_0)$ denote the open disk about z_0 of radius R . A neighborhood base for our topology will be indexed by $R, \varepsilon > 0$, with smaller neighborhoods given by large R and small $R\varepsilon$. For $E \subseteq \mathcal{C}$, we say $F \subseteq \mathcal{C}$ is (R, ε) -close to E if $\exists f : B_R(0) \rightarrow \mathcal{C}$, (1 – 1 and) conformal, such that

$$\|f(z) - z\|_\infty \leq R\varepsilon$$

(i.e. f is uniformly close to the identity map) and

$$f(B_R(0) \cap E) = (f(B_R(0))) \cap F.$$

Sometimes we will instead need \mathcal{C}^1 -closeness to the identity. The next lemma shows this follows from uniform closeness of either f or its derivative. We note that for $f(z)$ defined on B_R , to be $R\varepsilon$ -close to the identity is scaling invariant, in the sense that if we conjugated f by $g(z) = Rz$ so as to transfer it to the unit ball, then the resulting function $g^{-1} \circ f \circ g$ would be ε -close to the identity.

Lemma 2.1. *Let $f : B_R(0) \rightarrow \mathcal{C}$ be holomorphic. Then:*

(i) *If $|f(z) - z| < R\varepsilon$ for all $z \in B_R$, then*

$$|Df(z) - 1| < 4\varepsilon \quad \text{for all } z \in B_{R/2}.$$

(ii) *If*

$$\begin{aligned} f(0) &= 0 & \text{and} \\ |Df(z) - 1| &< \varepsilon & \forall z \in B_R(0) \quad \text{then we have:} \\ |f(z) - z| &< R\varepsilon. \end{aligned}$$

Proof.

(i) We write $g(z) = f(z) - z$. By the Cauchy Integral Formula,

$$Dg(z) = \frac{1}{2\pi i} \int_{\partial B_R} \frac{g(w)}{(w-z)^2} dw.$$

For $w \in \partial B_R$ and $|z| < R/2$, $|w - z| > R/2$. Therefore

$$|Dg(z)| \leq \frac{1}{2\pi} \frac{R\varepsilon 4}{R^2} 2\pi R = 4\varepsilon.$$

(ii) Since the domain $B_R(0)$ is simply connected, the path integral is well-defined and we have:

$$|f(z) - z| = \left| \int_0^z Df(\zeta) - 1 d\zeta \right| \leq \int_0^z |Df - 1| d\zeta \leq \varepsilon |z| \leq R\varepsilon.$$

□

Although (R, ε) -closeness does not quite define a metric, one does have the following:

- (a) (approximate symmetry): If F is (R, ε) -close to E , then E is $((1 - \varepsilon)R, \varepsilon)$ -close to F .
- (b) (approximate triangle inequality): If F is (R, ε_1) -close to E , and G is (R, ε_2) -close to F , then G is $((1 - \varepsilon_1)R, \varepsilon_1 + \varepsilon_2)$ -close to E .

It is also easy to check (recalling that these sets are assumed to be closed):

- (c) If E is (R, ε) -close to F for all $R, \varepsilon > 0$, then $E = F$.
- (d) \mathcal{T} is a uniformity, using (R, ε) closeness

Furthermore we have the following natural analogue of completeness, i.e. that “Cauchy sequences” converge:

Lemma 2.2. *Let E_i be a sequence of closed subsets of \mathcal{C} satisfying: for each $R, \varepsilon > 0$, $\exists n$ such that for all $m \geq n$, E_m is (R, ε) -close to E_n . Then there exists a unique closed set $E \subseteq \mathcal{C}$ such that $E_n \rightarrow E$ in \mathcal{T} . Moreover for all $m \geq n$, E is (R, ε) -close to E_m .*

Proof. Given $R, \varepsilon > 0$, by hypothesis there exists n such that for all $m \geq n$, there is a 1-1 holomorphic function $f_m : B_R \equiv B_R(0) \rightarrow \mathcal{C}$ with $\|f_m(z) - z\|_\infty \leq \varepsilon R$ and $f_m(B_R \cap E_n) = (f(B_R)) \cap E_m$. Now since $\{f_m\}_{m=0}^\infty$ is uniformly bounded and hence is

a normal family, there exists a subsequence f_{m_k} and a holomorphic function $f : B_R \rightarrow \mathcal{C}$ such that for any given $\bar{\varepsilon} > 0$, $\|f_{m_k} - f\|_\infty \leq \bar{\varepsilon}R$ for all k large. By part (i) of Lemma 2.1, $|Df| > 0$ hence f is 1-1. Therefore so is $f \circ f_{m_k}^{-1}$, which is defined on $B_{(1-\varepsilon)R}(0)$.

Writing $w = f_{m_k}(z)$, we have for any $w \in B_{(1-\varepsilon)R}$,

$$|f \circ f_{m_k}^{-1}(w) - w| = |f(z) - f_{m_k}(z)| < \bar{\varepsilon}R.$$

Therefore defining $E(R) = f(B_R \cap E_n)$, we have shown that given $\bar{\varepsilon} > 0$, $\exists k$ such that $E(R)$ is $((1-\varepsilon)R, \bar{\varepsilon})$ -close to E_{m_k} . Let M_k also be large enough that, by the Cauchy property, for all $j > m_k$ we have that E_j is $(R, \bar{\varepsilon})$ -close to E_{m_k} . Then by (a) and (b) above, $E(R)$ is $((1-\varepsilon)(1-\bar{\varepsilon})R, 2\bar{\varepsilon})$ -close to E_j . Therefore by (a), (b) and (c), the (closed) set $E(R)$ is uniquely defined in $B_{(1-\varepsilon)R}$ independent of the initial choice of n or the subsequence m_k . For the same reason, for $R' > R$ the sets $E(R')$ and $E(R)$ agree in $B_{(1-\varepsilon)R}$. Now since $\|f_{m_k}(z) - z\|_\infty \leq \varepsilon R$ and $\|f(z) - f_{m_k}(z)\|_\infty < \bar{\varepsilon}R$ for $z \in B_R$ and all $\bar{\varepsilon} > 0$, we have that $\|f(z) - z\|_\infty \leq \varepsilon R$ and hence $E(R)$ is (R, ε) -close to E_n .

Therefore defining $E = \bigcup_{R>0} E(R)$, also E is (R, ε) -close to E_n . We conclude that $E_n \rightarrow E$ in \mathcal{T} ; note that E is a closed set. By (c), the limit is unique. Finally, repeating the entire argument for each $m \geq n$, we have that E is (R, ε) close to E_m , as claimed. \square

Next, the **measure topology** is defined on the collection of Borel sets $E \subseteq \mathcal{C}$ such that H^d is locally finite, i.e. for every $R > 0$, $H^d(E \cap B_R(0)) < \infty$. In that case $\mu_E \equiv H^d|_E$ defines a continuous linear functional on $C_c(\mathcal{C})$, the continuous real-valued functions with compact support; indeed, by the Riesz representation theorem, C_c^* is exactly the locally finite signed measures. The measure topology on sets will then be that given by the weak-* topology for C_c^* , on the corresponding measures.

To get a pseudo-metric, however, we need to further restrict the class of sets. First, for fixed $b > 0$, let \mathcal{B}_b denote the collection of all Borel subsets E of \mathcal{C} such that for all $R > 0$,

$$\mu_E(B_R(0)) < bR^d.$$

Proposition 2.3. *There is a bounded pseudo-metric ρ on \mathcal{B}_b which is equivalent to the measure topology.*

Proof. For $f \in C_c$, we first define the pseudometric ρ_f by

$$\rho_f(E, F) = \left| \int f d\mu_E - \int f d\mu_F \right|.$$

Since C_c has a countable dense subset $\{f_i\}_{i=1}^\infty$ (in the sup norm), we can (because of the bound on measure) find weights $a_i > 0$ such that for $F = \emptyset$ and for any $E \in \mathcal{B}_b$ we have

$$\rho(E, F) \equiv \sum_{i=1}^{\infty} a_i \rho_{f_i}(E, F) < K$$

for some $K < \infty$.

Now we define $\rho(E, F)$ by that sum for all $E, F \in \mathcal{B}_b$. This is clearly symmetric and satisfies the triangle inequality. By the triangle inequality we have for any $E, F \in \mathcal{B}_b$

$$\rho(E, F) < 2K < \infty,$$

so ρ is a bounded pseudometric, as claimed. \square

Next, let J be our Julia set. Define $\mathcal{D} = \{a(J - z) : z \in J, a > 0\}$ and let $\overline{\mathcal{D}}$ denote the closure in the measure topology. We will now see that $\overline{\mathcal{D}} \subseteq \mathcal{B}_b$:

Lemma 2.4. *There exists $b > 0$ such that for every $E \in \overline{\mathcal{D}}$,*

$$\mu_E(B_R(0)) < bR^d.$$

Proof. A basic fact is that J is compact. Therefore there exists $b > 0$ such that for every $z \in J$, and all $\varepsilon > 0$ (not just ε sufficiently small), for $\mu \equiv \mu_J$ we have, since μ is a geometric measure,

$$\mu(B_\varepsilon(z)) < b\varepsilon^d.$$

Now if $E \in \mathcal{D}$ and so by definition can be written $E = a(J - z)$, then

$$\mu(E \cap B_R(0)) = \mu(a((J - z) \cap B_{R/a}(0))) \leq a^d(b(R/a)^d) = bR^d.$$

This inequality passes immediately over to E in the closure $\overline{\mathcal{D}}$. \square

To prove the next lemma, we need the \mathcal{C}^1 -closeness to the identity proved in (i) of Lemma 2.1. For $g : B_R(z_0) \rightarrow \mathcal{C}$, we write

$$\|g\|_{\mathcal{C}^1} \equiv \max\{\|g\|_\infty, \|Dg\|_\infty\}.$$

Lemma 2.5. *Let $E_i \in \overline{\mathcal{D}}$ be closed sets such that $E_i \rightarrow E$ in the conformal map topology. Then $E \in \overline{\mathcal{D}}$ and $E_i \rightarrow E$ in the measure metric.*

Proof. We first claim that for each $f \in C_c$, the sequence converges in the pseudometric ρ_f . We want to show that given $\varepsilon > 0$ then, for all n large enough,

$$\left| \int_{E_n} f dH^d - \int_E f dH^d \right| < \varepsilon.$$

Now $\exists R > 0$ such that $\text{supp}(f) \subseteq B_R(0)$, and in fact such that $\text{supp}(f) \subseteq (1 - \varepsilon)B_R(0) \equiv B_{(1-\varepsilon)R}(0)$. By the hypothesis, for any $\delta > 0$ we have for all n sufficiently large that there exists $\varphi_n : B_R(0) \rightarrow \mathcal{C}$ with $\|\varphi_n(z) - z\|_{\mathcal{C}^1} < \delta$ and such that $\varphi_n(E_n \cap B_R(0)) = \varphi_n(B_R(0)) \cap E$. In particular, $\|\varphi_n(z) - z\|_\infty < \delta$ and so $\varphi_n(B_R(0)) \supseteq (1 - \varepsilon)B_R(0)$. Therefore, using the integral form of the conformal transformation property for H^d ,

$$\int_E f dH^d = \int_{\varphi_n(E_n)} f dH^d = \int_{E_n} f \circ \varphi_n |D\varphi_n|^d dH^d.$$

With δ chosen small enough that $|w - z| < \delta \Rightarrow |f(w) - f(z)| < \bar{\varepsilon}$, for the difference we have a bound of $\bar{\varepsilon}(1 + \delta)^d H^d(E_n \cap B_R(0))$. So with $\bar{\varepsilon}$ chosen appropriately, this is less than ε , as we claimed.

The same estimate shows that $E \in \overline{\mathcal{D}}$, which finishes the proof. \square

Next, we define the **local Hausdorff metric** $\bar{\rho}$ on the collection of all closed subsets of \mathcal{C} as follows. We fix an invertible holomorphic map P from \mathcal{C} onto $S^2 \setminus \{\infty\}$, the Riemann sphere minus a point. (In other words, P^{-1} is a stereographic projection.) Then we define $\bar{\rho}(E, F)$ to be the distance between $P(E) \cup \{\infty\}$ and $P(F) \cup \{\infty\}$, in the Hausdorff metric on closed subsets of S^2 determined by its usual sphere metric.

Lemma 2.6. *Let $E_i \subseteq \mathcal{C}$ be closed sets such that $E_i \rightarrow E$ in the conformal map topology. Then E is closed, and $E_i \rightarrow E$ in the local Hausdorff metric.*

Proof. This is obvious from the definitions, since to get E_i to be inside a ε -neighborhood of $P(E)$, if $\overline{P(E)}$ contains ∞ we can add to all sets an ε -ball $B_\varepsilon(\infty)$ around ∞ in S^2 , and in \mathcal{C} let R be such that the map $\varphi : B_R(0) \rightarrow \mathcal{C}$ satisfies that φ is ε -close to the identity and $P((1 - \varepsilon)B_R(0)) \supseteq S^2 \setminus B_\varepsilon(\infty)$. Note that all we need is $\|\varphi - z\|_\infty \leq \varepsilon$; control of $D\varphi$ is not necessary here. \square

Bounded Distortion Property.

Our main next goal is a geometrical version of bounded distortion, which will measure nearness exactly as needed for the conformal map topology on sets. We give two proofs, one of which is very short, based on two well-known estimates from complex analysis due to Koebe and Bieberbach, and the other (which is self-contained and despite that only slightly longer) making use of a complex version of a basic principle from real dynamics. We refer to this theorem as the analytic form of the bounded distortion property. The proof basically follows the well-known geometric series argument from the real $\mathcal{C}^{1+\gamma}$ case (see e.g. [SS], [Su1]). We mention that there are two points where one should be slightly careful when extending the proof in [SS] to \mathcal{C} . The first is that the logarithm is many-valued; a simple way of dealing with series estimates for this case is to define \log to take values in the cylinder. The second point is that Mean Value Theorem estimates for \mathbb{R} are replaced by the complex Fundamental Theorem of Calculus (i.e. the Cauchy Integral Theorem), and so must always check that domains are simply connected.

Theorem 2.7. *(Bounded Distortion Property, analytic version). For a compact set $E \subseteq \mathcal{C}$ and map $T : E \rightarrow E$ which is complex differentiable (i.e. T is defined and differentiable in some neighborhood V of E), assume $\exists \alpha, \beta$ such that $1 < \alpha < |DT| < \beta < \infty$ on E . Then given $\varepsilon > 0, \exists \delta$ (independent of n) such that if $U \subseteq V$ is open and simply connected, $T^n|_U$ is $1 - \varepsilon$, and $\text{diam}(T^n U) < \delta$ then $\forall z, w \in U$,*

$$\left| \frac{DT^n(z)}{DT^n(w)} - 1 \right| < \varepsilon.$$

To prove this we need several simple lemmas.

First, we define the logarithm to take values on the cylinder $\mathcal{C}/2\pi i\mathbb{Z} \equiv \mathbb{R} \times S^1$. Thus for $z \in \mathcal{C} \setminus \{0\}$ with $z = re^{i\theta}$ for $r > 0$ and $0 \leq \theta < 2\pi$, we define $\log(re^{i\theta}) = \log r + i\theta \equiv$

$(\log r, \theta)$. The cylinder is an additive group and the metric inherited from \mathcal{C} , written $\text{dist}(a, b)$, is translation-invariant. Our series estimates will be easily handled using this metric.

Lemma 2.8. *$\log DT$ is Lipschitz, i.e. $\exists c > 0$ such that $\forall a, b \in E$, $\text{dist}(\log DT(a), \log DT(b)) < c|a - b|$.*

Proof. Since E is compact, there exists $K > 0$ such that the magnitude of the second derivative D^2T is bounded by K on some ball containing all of E . Since the ball is simply connected the line integral is well-defined and we have for all $a, b \in E$:

$$|DT(a) - DT(b)| = \left| \int_a^b D^2T(z) dz \right| \leq \int_a^b |D^2T(z)| dz \leq K|b - a|.$$

Now cylindrical distance has the formula $\text{dist}(\log a, \log b) = \inf_{\gamma} \int_{\gamma} \frac{1}{z} dz$ where the infimum is taken over all paths γ from a to b . For a rough worst-case estimate on $|\frac{1}{z}|$, we take γ to lie outside the disk of radius α (the lower bound for $|DT|$). Therefore certainly for $|a|, |b| > \alpha$ we have

$$\text{dist}(\log a, \log b) \leq \frac{\pi}{\alpha}|a - b|.$$

Combining the estimates therefore,

$$|\log DT(z) - \log DT(w)| \leq K\pi/\alpha|z - w| \quad \text{for all } z, w \in E. \quad \square$$

Lemma 2.9. *Let U, T be as in the Theorem. For $w, z \in U$, define $w_k = T^k(w)$, $z_k = T^k(z)$ for $k = 0, \dots, n$. Then*

$$|w_k - z_k| < \alpha^{-(n-k)}\delta.$$

Proof. First, we claim that for $k = 0$ and $n = 1$, i.e. for T 1-1 on U simply connected, we have for all $w, z \in U$,

$$\alpha|w - z| < |Tw - Tz| < \beta|w - z|.$$

To prove this note that for U simply connected the path integral is well defined, so we have

$$|T(w) - T(z)| = \left| \int_z^w DT(\zeta) d\zeta \right| \leq \beta|w - z|.$$

Since T is 1-1, $T(U)$ is also simply connected, and we can apply the same reasoning to T^{-1} to get the lower bound. To prove the Lemma we actually only need the lower bound. We have:

$$\alpha|w - z| < |T(w) - T(z)| < \delta.$$

Applying the same estimate to w_k, z_k and iterating the map $(n - k)$ times completes the proof. \square

Proof of Theorem 2.7. We wish to show

$$\left| \frac{DT^n(z)}{DT^n(w)} - 1 \right| < \varepsilon.$$

Since $\exp : \mathbb{R} \times S^1 \rightarrow \mathcal{C}$ is uniformly continuous on $[\log \alpha, \log \beta] \times S^1$, it is enough to show that $\text{dist}(\log T^n(w), \log DT^n(z)) < \bar{\varepsilon}$ for the appropriate $\bar{\varepsilon}$. Now since $\text{dist}(\cdot, \cdot)$ is a translation-invariant on the additive group of the cylinder, one has the triangle inequality in this form: given $a, \tilde{a}, b, \tilde{b} \in \mathcal{C}/2\pi i\mathbb{Z}$, $\text{dist}(a + b, \tilde{a} + \tilde{b}) \leq \text{dist}(a, \tilde{a}) + \text{dist}(b, \tilde{b})$. Therefore for $w_k = T^k(w), z_k = T^k(z)$ as in the Lemma, we have from the Chain Rule together with the fact that \log is a homomorphism from the multiplicative group $\mathcal{C} \setminus \{0\}$ to the cylinder:

$$\begin{aligned} & \text{dist}(\log DT^n(w), \log DT^n(z)) \\ &= \text{dist}\left(\sum_{k=0}^{n-1} \log DT(w_k), \sum_{k=0}^{n-1} \log DT(z_k)\right) \\ &\leq \sum_{k=0}^{n-1} \text{dist}(\log DT(w_k), \log DT(z_k)). \end{aligned}$$

By Lemmas 2.8 and 2.9,

$$\text{dist}(\log DT(w_k), \log DT(z_k)) < \frac{k\pi}{\alpha} |w_k - z_k| < \frac{k\pi}{\alpha} \alpha^{-(n-k)} \delta.$$

So the bound we want is

$$\frac{k\pi\delta}{\alpha} \sum_{k=0}^{n-1} \alpha^{-(n-k)} = \frac{k\pi\delta}{\alpha} \sum_{j=1}^n \alpha^{-j} \leq \frac{k\pi\delta}{\alpha - 1}.$$

Thus $\delta = \bar{\varepsilon} \frac{(\alpha-1)}{k\pi}$ will do. \square

We need one more basic ingredient.

Lemma 2.10. *Given $T : E \rightarrow E$ satisfying the hypotheses of Theorem 2.7, there exists $\delta > 0$ such that for all $z_0 \in E$ and for all $n \geq 1$, $T^{-n}(B_\delta(z_0))$ is a disjoint union of open sets $U(z_{-n})$ containing an n^{th} preimage z_{-n} , such that $|z_{-n} - w| < \delta$ for all $w \in U(z_{-n})$ and $T^n : U(z_{-n}) \rightarrow B_\delta(z_0)$ is 1 - 1.*

Proof. If we prove this for T^{-1} , we will be done for e.g. $n = 2$ by considering the composition $T^{-2} = T^{-1} \circ T^{-1}$, since the diameters decrease, and so by induction for all n .

We can assume without loss of generality that the bounds $\alpha < |DT| < \beta$ hold on a δ_0 -neighborhood of E . By the inverse function theorem for holomorphic maps, plus compactness, $\exists \delta < \delta_0$ such that for all $w_0 \in E$, T is 1 - 1 on $B_\delta(w_0)$. Now we claim that $\alpha|w - w_0| < |Tw - Tw_0| < \beta|w - w_0|$ for all $w \in B_\delta(w_0)$. This is shown as in the proof of Lemma 2.9. We can restate this as:

$$B_{\alpha\delta}(Tw_0) \subseteq T(B_\delta(w_0)) \subseteq B_{\beta\delta}(Tw_0).$$

Finally we define $\delta = \alpha\tilde{\delta}$. Now if z_{-1} is such that $T(z_{-1}) = z_0$, we have: T is 1-1 on $B_{\tilde{\delta}}(z_{-1})$ so certainly on $U(z_{-1}) \equiv B_{\tilde{\delta}}(z_{-1}) \cap T^{-1}(B_{\delta}(z_0))$. And by the same estimate applied to T^{-1} , $|w - z_{-1}| < \frac{1}{\alpha}\tilde{\delta} < \tilde{\delta}$ for all $w \in U(z_{-1})$ which finishes the proof. \square

We need a bit more notation and then will be ready to state our Theorem. Hypotheses will be the same as in Theorem 2.7. If $T^n(z_{-n}) = z_0$ (i.e. a choice of the n^{th} inverse image is understood) then we write \widetilde{DT}^n for the affine map on \mathcal{C} defined by:

$$z \mapsto ((DT^n(z_{-n}))(z - z_{-n})) + z_0.$$

Theorem 2.11. (*Bounded Distortion Property, geometric version*). For $T : E \rightarrow E$ as in Theorem 2.7, given $\varepsilon > 0$ there exists $\delta > 0$ such that for any $z_0 \in E$ and any $n \geq 0$, for any choice of preimage $z_{-n} \in E$ with $T^n(z_{-n}) = z_0$, then

$$\widetilde{DT}^n \circ T^{-n} \quad \text{defines a function on } B_{\delta}(z_0)$$

which is $\delta\varepsilon$ -close to the identity.

First Proof. We fix $z_0, z_{-n} \in E$ such that $T^n(z_{-n}) = z_0$, and choose δ less than the deltas in Theorem 2.7 (but for $\varepsilon/2$) and in Lemma 2.10. The composition in any case defines a set map, but by that Lemma it is a well-defined function, $f(z) \equiv \widetilde{DT}^n \circ T^{-n} : B_{\delta}(z_0) \rightarrow \mathcal{C}$. We claim that $f(z)$ is $\delta\varepsilon$ -close to the identity. Now $f(z_0) = z_0$ and f is 1-1. By Theorem 2.7, for any $z \in B_{\delta}(z_0)$, writing $w = T^{-n}(z)$ for its unique preimage in $U(z_{-n})$, we have:

$$|Df(z) - 1| = \left| \frac{DT^n(z)}{DT^n(w)} - 1 \right| < \varepsilon.$$

Hence by Lemma 2.1, $\|f(z) - z\|_{\infty} \leq \varepsilon\delta$ and we are done. \square

For the second proof we need Lemmas 2.1, 2.10 above plus the following lemma. A proof follows immediately from the two facts that $|Df|$ is close to 1 (Koebe's Distortion Theorem) and that $\arg(DF)$ is close to zero (Bieberbach's Rotation Theorem). See e.g. (3) and (6) of §2.3 in [Du].

Lemma 2.12. Assume $f : B_1(0) \rightarrow \mathcal{C}$ is holomorphic and 1-1, and let it satisfy $f(0) = 0$ and $Df(0) = 1$. Then given $\varepsilon > 0$, $\exists r > 0$ such that (for all such f)

$$|Df(z) - 1| < \varepsilon \quad \text{for all } |z| < r.$$

\square

Second Proof of Theorem 2.11. We fix $\tilde{\delta}$ less than the delta in Lemma 2.7. As in the first proof, given a choice z_{-n} and z_0 , we define the map

$$f : B_{\tilde{\delta}}(z_0) \rightarrow \mathcal{C}$$

by $f = \widetilde{DT}^n \circ T^{-n}$. By conjugating f with a translation and dilation so it is defined on $B_1(0)$, we see from Lemma 2.9 that given $\varepsilon > 0 \exists r > 0$ such that for all z with $|z - z_0| < r\tilde{\delta}$,

$$|Dg(z) - 1| < \varepsilon.$$

Defining $\delta \equiv r\tilde{\delta}$, by Lemma 2.1 (ii) we conclude that $\|f(z) - z\|_{\infty} < \varepsilon\delta$ which finishes the proof. \square

Limit Sets. Now for our hyperbolic Julia set (J, T) , let the shift space (\prod_0, σ) be defined as in the Introduction. We note that \prod_0 is compact, and that the left shift σ is a homeomorphism of \prod_0 ; this is topologically the natural invertible version of (J, T) .

Given $\underline{z} = (\dots z_{-1} z_0 z_1 \dots) \in \prod_0$, we define

$$L_{\underline{z}, n} \equiv DT^n(z_{-n}) \cdot (J - z_{-n}).$$

Here is the theorem we have been leading up to.

Theorem 2.13. *For each $\underline{z} \in \prod_0$, there exists a unique closed set $L_{\underline{z}} \subseteq \mathcal{C}$ such that $L_{\underline{z}, n} \rightarrow L_{\underline{z}}$ in the conformal map topology. The convergence is uniform in \underline{z} , and the function $\underline{z} \mapsto L_{\underline{z}}$ is continuous.*

Proof. By Lemma 2.2 it will be enough, for proving convergence, to show the sequence is Cauchy in the sense given there. We are to show that for this \underline{z} , given $R, \varepsilon > 0$, there exists n such that for all $m \geq n$, $L_{\underline{z}, m}$ is (R, ε) -close to $L_{\underline{z}, n}$. Let δ be good in Theorem 2.11 for error equal to $\frac{\varepsilon}{\beta}$. Let n be the least integer such that $\bar{R} \equiv DT^n(z_{-n}) \cdot \delta > R$. This implies that

$$R < \bar{R} < \beta R.$$

Now define the affine map g from $B_{\bar{R}}(0)$ to $B_{\delta}(z_{-n})$:

$$g(z) = (\delta/\bar{R}) \cdot z + z_{-n}.$$

Writing $k = m - n$, from Theorem 2.11 the map $\varphi \equiv \widetilde{DT}^k \circ T^{-k} : B_{\delta}(z_{-n}) \rightarrow \mathcal{C}$ is $\delta\varepsilon/\beta$ -close to the identity. Therefore the conjugate

$$\Phi \equiv g^{-1} \circ \varphi \circ g : B_{\bar{R}}(0) \rightarrow \mathcal{C}$$

is $\bar{R}\varepsilon/\beta$ -close to the identity. The restriction of Φ to $B_R \subseteq B_{\bar{R}}$ is thus certainly $R\varepsilon > \bar{R}\varepsilon/\beta$ -close to the identity. Now from the definitions,

$$\Phi(L_{\underline{z}, n} \cap B_R(0)) = L_{\underline{z}, m} \cap \Phi(B_R(0)).$$

So $L_{\underline{z}, m}$ is (R, ε) -close to $L_{\underline{z}, n}$, as claimed. Hence by Lemma 2.2 there exists a unique closed set $L_{\underline{z}}$ such that $L_{\underline{z}, n} \rightarrow L_{\underline{z}}$ in \mathcal{T} .

To prove this convergence is uniform (in the sense of (R, ε) -closeness), we do the above argument for each $\underline{z} \in \prod_0$. That is, we choose δ from Theorem 2.11 for error equal to ε/β , and define $n = n(\underline{z})$ to be the least integer such that $|DT^n(z_{-n})|\delta > R$. Now since $\alpha < |DT|$, this is bounded above by some n_0 . By the last sentence in the statement of Lemma 2.2, not only is $L_{\underline{z}}(R, \varepsilon)$ -close to $L_{\underline{z}, n(\underline{z})}$, but we have that for all \underline{z} , $L_{\underline{z}}$ is (R, ε) -close to $L_{\underline{z}, n_0}$. This proves uniform convergence.

Finally, to prove $L_{\underline{z}}$ is a continuous function of \underline{z} (from the product topology on $\prod_0 \subseteq \prod_{-\infty}^{\infty} \mathcal{C}$ to \mathcal{T}), note that for each fixed n , the function $\underline{z} \mapsto L_{\underline{z}, n}$ is continuous in \underline{z} . Since a uniform limit of continuous functions is continuous, we are done. \square

We have constructed the scenery $L_{\underline{z}}$ as a limit of the sets $L_{\underline{z},n}$, showing these are a Cauchy sequence by considering maps defined on successively larger balls $B(R,0)$. These maps are individually holomorphic but were defined only locally, because of the necessity to choose branches, so they do not in themselves form a Cauchy sequence of maps on \mathcal{C} . This method has the advantage of working equally well for the conformal repellers.

Once we have constructed $L_{\underline{z}}$ in this way, we can however (for the rational map case) reverse the point of view, studying a sequence of maps which is inverse to those above but now globally defined.

For fixed \underline{z} , we define for $n \geq 0$ a map $\Phi_{\underline{z},n} : \mathcal{C} \rightarrow \mathcal{C}$ by:

$$\Phi_{\underline{z},n}(w) = T^n(z_n + (DT(z_n))^{-1} \cdot w) - z_0.$$

Theorem 2.14. *For each n , $\Phi_{\underline{z},n}(L_{\underline{z},n}) = (J - z_0)$. This is a holomorphic cover of degree $(\text{degree } T)^n$. The limit $\Phi_{\underline{z}} = \lim_{n \rightarrow \infty} \Phi_{\underline{z},n}$ converges uniformly on compact sets and $\Phi_{\underline{z}} : \mathcal{C} \rightarrow \mathcal{C}$ restricted to $L_{\underline{z}}$ is a countable holomorphic cover of $(J - z_0)$. The following diagrams commute:*

$$\begin{array}{ccc} L_{\underline{z}} & \xrightarrow{w \mapsto DT(z_0) \cdot w} & L_{\sigma(\underline{z})} \\ \phi_{\underline{z}} \downarrow & & \downarrow \phi_{\sigma \underline{z}} \\ J & \xrightarrow{w \mapsto T(w)} & J \end{array}$$

and

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{w \mapsto DT(z_0) \cdot w} & \mathcal{C} \\ \phi_{\underline{z}} \downarrow & & \downarrow \phi_{\sigma \underline{z}} \\ \mathcal{C} & \xrightarrow{w \mapsto T(w)} & \mathcal{C} \end{array}$$

Proof. A point $w \in L_{\underline{z},n}$ can be expressed as $w = DT^n(z_n)(x - z_n)$ for some point $x \in J$. Applying the above definition, it follows that $\Phi_{\underline{z}}(w) = T^k(x) - z_0 \in J - z_0$, as claimed. The maps $\Phi_{\underline{z}}$ are locally inverse to the maps used to prove convergence to $L_{\underline{z}}$. By those same estimates, the sequence $\Phi_{\underline{z}}$ is Cauchy in the C^1 -norm on arbitrarily large balls. Hence the limit exists, and restricted to $L_{\underline{z}}$ gives a countable infinity-to-one holomorphic cover of $J - z_0$. \square

Example. Let z_0 be a fixed point for T^n . Then writing $\lambda = DT^n(z_0)$ (the *multiplier* at that periodic point), the commutative diagram above simplifies, since for $\Phi \equiv \Phi_{\underline{z}} = \Phi_{\sigma^n \underline{z}}$, and we have $T^n \circ \Phi(w) = \Phi(\lambda \cdot w)$.

This is a globalization (to all of \mathcal{C}) of Koenig's classical linearization theorem [Mil]; thus the construction of scenery can be seen as giving a type of Koenig's theorem for general (non-periodic) points.

Consider the particular case $T(z) = z^2$, and choose $z_{-n} = z_0 = 1$ for all n . The scenery at the fixed point 1 is the tangent line to the circle at that point, which wraps around the circle via $\mathbb{R} \rightarrow \mathbb{R}/\mathbb{Z}$. So our countable holomorphic cover should be the exponential map

$z \mapsto e^z$, shifted over to have as image the circle centered at -1 . And this indeed is the case: the formula yields

$$\Phi_{z,n}(w) + 1 = (1 + w \cdot 2^{-n})^{2^n};$$

for real w by taking logs, we can verify the classical formula $(1+w/n)^n \rightarrow e^w$; since we know $\Phi_{z,n} \rightarrow \Phi_z$ which is holomorphic, this equality extends to \mathcal{C} . Therefore $\Phi_z(w) = e^w - 1$, as it should be.

We remark that the covering maps for other Julia sets are therefore a kind of generalization of the exponential map, and so may be of interest in their own right.

§3. The symbolic model and the scenery flow.

As in §2, we write Ω for the collection of all closed subsets of \mathcal{C} , topologized by the conformal map topology \mathcal{T} . We define the **magnification flow** τ_t on Ω to be the flow (action of the additive reals) given by dilation by the real exponential factor e^t . That is, $\tau_t(E) = e^t \cdot E$ for $E \in \Omega$. Note that the flow is (jointly) continuous. For $z \in J$ we write Ω_z for the τ_t omega-limit set of the translated Julia set $J - z$ (thought of as a *point* in the space Ω). That is,

$$\Omega_z \equiv \bigcap_{t=0}^{+\infty} cl \left(\bigcup_{s=t}^{+\infty} \tau_s(J - z) \right)$$

where cl denotes \mathcal{T} -closure. This set is closed and is flow-invariant. We write

$$\Omega_J = \bigcup_{z \in J} \Omega_z.$$

Definition. (Ω_J, τ_t) is the **scenery flow of J** , and (Ω_z, τ_t) is the **scenery flow of J at z** . The **full scenery flow of J** , denoted by $(\widehat{\Omega}_J, \tau_t)$, is τ_t acting on the closure of Ω_J under rotations, $E \mapsto e^{i\theta} \cdot E$.

The symbolic model.

We will next define a **special flow**, which as we will show maps homomorphically onto the full scenery flow of J . This symbolic representation will provide the key to studying the ergodic theory of the scenery flow, carried out in the next section.

We move to an abstract setting, recalling the standard definitions for a general map. A special flow is constructed from an invertible transformation on a set called the **base** of the flow, and a strictly positive function defined on the base. For the resulting flow, the base will be a **Poincaré cross-section**, and the function will give the time of return to the base. One says the flow is **built under** the function and **over** the base map.

Our base will be a compact topological space X , with base map a homeomorphism F and with continuous **return time function** r . The **flow space** $X_{F,r}$ is the compact topological space which is the quotient of the space $X \times \mathbb{R}$ by the equivalence relation generated by the identification

$$(x, s + r(x)) \sim (F(x), s).$$

Finally the flow τ_t on $X_{F,r}$ is defined by

$$\tau_t(x, s) = (x, s + t).$$

Remark.

- (1) We mention that to a topologist, what we are calling the base of the flow will instead be a fiber of a fiber bundle. The topological conventions are as follows. In the special case where $r \equiv 1$, one calls $(X_{F,r}, \tau_t)$ the **suspension flow**. If one interested in only the topology of the flow space, one might as well make this assumption since the topology is the same for all r positive. The suspension can be thought of as a fiber bundle over the circle \mathbb{R}/\mathbb{Z} with fiber X ; note that then the circle is the base of the fiber bundle (so $X_{F,r}$ “fibers over the circle”) while X is the base of the flow. (We will follow the dynamical rather than the topological usage of “base”).
- (2) We note that the usual picture of the special flow depicts a fundamental domain for a group action. The group is \mathbb{Z} , acting on $X \times \mathbb{R}$, with the natural action generated by the identification.

It is important to note that the identification space, and the flow, make sense for arbitrary functions r . This function is a return time to a cross-section exactly when it is positive. Positivity guarantees the existence of a nice (e.g. connected) fundamental domain.

In our particular case, the positivity of r will be a consequence of (indeed equivalent to) the strict hyperbolicity of the rational map T .

The map T is however not invertible; we make it so in a canonical way described by the following simple lemma. We are just transferring to the topological category a familiar notion of ergodic theory, the natural extension. To a topologist this will be recognized as an inverse limit construction. We state this in the abstract setting.

Lemma 3.1. *Let F be a continuous map on (and not necessarily onto) a topological space X . Then (up to homeomorphism) there is a unique space \widehat{X} , with homeomorphism \widehat{F} , which factors onto (X, F) .*

Here **smallest** means any other such space factors through $(\widehat{X}, \widehat{F})$. By definition a **factor map** or **homomorphism** is a continuous onto map which semiconjugates the transformations.

As in the measure-theoretic category, we will call this the **natural extension** of the transformation.

Proof. Define $\Pi = \Pi_{-\infty}^{\infty} X$ with the product topology, and let Π_0 be the subset of all $\underline{x} = (\dots, x_{-1}, x_0, x, \dots)$ such that $F(x_j) = x_{j+1}$. One easily checks that this is homeomorphism, and that the projection $\Pi : \underline{x} \mapsto x_0$ is a factor map from $(\widehat{X}, \widehat{F})$ to (X, F) . Finally the proof of the universal property stated above is also absolutely clear: by conjugacy, preimages of a point w to x_0 , in a third space (W, S) which maps to x_0 , must map onto specific preimages of x_0 . This defines the canonical projection from W to \widehat{X} . \square

Note: In some applications (e.g. for expanding maps which define Cantor sets, or for Douady-Hubbard polynomial-like maps) one may begin with a map F which does not fit the assumption of the above lemma, since its range strictly contains its domain. In that case one should replace F by the restriction to its **eventual domain** X_0 , (those points x with $F^n(x)$ defined for all $n > 0$), and state the universal property for (X_0, F) .

Now we return to our rational map T . The base transformation for the special flow \widehat{T} can be defined geometrically, as follows. The derivative map acts naturally on the unit tangent bundle of J (by renormalizing the length of the vector); \widehat{T} denotes the natural extension of this map, i.e. its (unique) invertible version given by the above Lemma. Note that this definition is coordinate-free, i.e. it makes sense for a map on a differentiable manifold with a Riemannian (or Finsler) metric, independent of charts. First however we define this base transformation more concretely, as a skew product over a shift map. As in the Introduction, we write Π_0 for the compact subset of $\Pi \equiv \Pi_{-\infty}^{\infty} J$ consisting of those **allowed strings** of complex numbers $\underline{z} = (\dots, z_{-1}, z_0 z_1, \dots)$ such that $T(z_j) = z_{j+1}$. We write $\widehat{\Pi}_0 = \Pi_0 \times S^1$, and setting $\varphi(\underline{z}) = \arg DT(z_0)$, we define the homeomorphism $\hat{\sigma}$ on $\widehat{\Pi}_0$ to be the skew product transformation with shift base and skewing function φ .

It is clear that the map $\hat{\sigma}$ is naturally conjugate to \widehat{T} as defined above, by Lemma 3.1. We will from now on use the notation (and explicit representation) $(\widehat{\Pi}_0, \hat{\sigma})$ for \widehat{T} .

Definition. *The symbolic model for the scenery flow of J is the special flow $(\widehat{\Omega}, \hat{\tau}_t)$ with base map $(\widehat{\Pi}_0, \hat{\sigma})$ and return time function $r(\underline{z}) = \log |DT(z_0)|$.*

We define a map Φ from $\widehat{\Pi}_0 \times \mathbb{R} = \Pi_0 \times S^1 \times \mathbb{R}$ to Ω (the closed subsets of \mathcal{C}) by $(\underline{z}, \theta, t) \mapsto e^{i\theta} e^t L_{\underline{z}}$.

Theorem 3.2. *The full scenery flow (Ω_J, τ_t) is a factor of the symbolic flow $(\widehat{\Omega}, \hat{\tau}_t)$, with factor map given by Φ .*

Proof. We will first verify that Φ is well-defined as a map from $\widehat{\Omega}$ to Ω , i.e. that it respects the identifications, and that it is continuous. Then we will show the image is (the rotation closure of) that collection of closed sets which forms the omega-limit set of some $(J - z)$.

The argument that Φ respects the identifications which define the flow space $\widehat{\Omega}$ has already been given in the Introduction. Continuity of Φ restricted to the base $\widehat{\Pi}_0$ (embedded in Ω as $(\underline{z}, \theta, S)$ with $\theta, S = 0$) has been proved in Theorem 2.13. Since rotation and dilation are continuous operations in the topology \mathcal{T} , this continuity immediately extends to all of $\widehat{\Omega}$. \square

For the rest of the proof we need the following lemma.

Lemma 3.3. *For any z_0 in J and all \underline{z} in Π_0 with that zeroth coordinate, $(J - z_0)$ is forward asymptotic to $L_{\underline{z}}$ under the magnification flow τ_t on Ω (the closed subsets of \mathcal{C}), in the topology \mathcal{T} .*

Proof. From the uniform convergence of Theorem 2.13, given $R, \varepsilon > 0$ there exists n_0 such that for all \underline{z} , for all $n > n_0$,

$$L_{\underline{z}, n} \quad \text{is} \quad (R, \varepsilon) \text{-close to} \quad L_{\underline{z}}.$$

Therefore, writing " \sim " for (R, ε) -closeness,

$$DT^n(z_0)(J - z_0) \sim L_{\sigma^n(\underline{z})} = ST^n(z_0)L_{\underline{z}}.$$

Setting $t = \log |DT^n(z_0)|$ and dividing both sides by $e^{i \arg DT^n(z_0)}$, this gives

$$\tau_t(J - z_0) \sim \tau_t L_{\underline{z}} \quad \text{and we are done.} \quad \square$$

Now we finish the proof of the Theorem.

By the Lemma, the omega-limit set of $(J - z_0)$ coincides with the omega-limit set of $L_{\underline{z}}$. Therefore $\widehat{\Omega}_J$ equals the union of the omega-limit sets of the range of Φ , $\Phi(\Omega)$. This shows that $\widehat{\Omega}_J \subseteq \Gamma(\Omega)$. For the reverse containment we are to show that given \underline{z} , φ and S in Π_0 , $[0, 2\pi]$ and \mathbb{R} , there exists $\underline{\omega}$ and θ such that $e^{i\theta} \tau_t(L_{\underline{\omega}})$ comes infinitely often arbitrarily close to $e^{i\varphi} e^s L_{\underline{z}}$ as $t \rightarrow +\infty$. It will be enough to show this for $\varphi = 0$, $s = 0$. Now since there exists a point with a dense T -orbit in the Julia set, the same is true for the shift map σ on Π_0 . Let $\underline{\omega}$ be such a point. By continuity of Φ there is a sequence of flow times t_n such that $\tau_{t_n}(L_{\underline{\omega}})$ comes arbitrarily close to $e^{i\theta_n} L_{\underline{z}}$ for some angles θ_n . By compactness of the circle $\{\theta_n\}$ has a limit point θ ; we replace t_n by this subsequence. Thus $e^{-i\theta} \tau_{t_n} L_{\underline{\omega}}$ has $L_{\underline{z}}$ as a limit point, and we are done. \square

Remark. Since the symbolic model Ω_J is a compact space and the map from there to the closed sets Ω is continuous, the image is compact. Thus, the set of all scenes is compact in the topology \mathcal{T} , hence is a compact metric space in the local Hausdorff and measure metrics.

§4 Ergodic theory and rotational behavior.

§4.1 Gibbs states and the projected flow.

We begin by considering the ergodic theory of the special flow built over the base (Π_0, σ) with return time function $r(\underline{z}) = \log |DT(z_0)|$. That is, we for now are ignoring all angular information. This **projected flow** $(\overline{\Omega}, \overline{\tau}_t)$ is a factor of the symbolic model via the projection $(\hat{z}, s) \mapsto (\underline{z}, s)$. For an expanding $\mathcal{C}^{1+\alpha}$ map on a Cantor set in $[0, 1]$ the same flow played a key role in the analysis of density properties of the Cantor set, see §3 of [BF1]. Indeed it was a close reanalysis of the convergence proof given there which led us to the "linearization" construction of the scenery used in §2 above and in [BF3]. The existence of order-two density now follows as a corollary (see Proposition 4.2).

The following theorem, describing the ergodic theory of the projected flow, follows from the fundamental work of Bowen, Ruelle and Sinai. Some of the main points in the development of the "BRS theory" relevant here are: Lemma 10 of [Bo2] (for **Bowen's formula** for Hausdorff dimension); §8 of [Si] and Proposition 3.1 of [BR] (for the relationship between measures of maximal entropy for flows and Gibbs states on a cross-section).

For completeness we include the proof. In summary, two separate results from the BRS theory (the relationship between Gibbs states and the Hausdorff measures on the one hand, and a Gibbs state on the cross-section and the measure of maximal entropy for a flow on the other) are brought together, linked by the scenery flow.

For further details and background see [BF1-3], [Fi]. For exposition on Bowen's formula see [PU]. See [Bo1], [Bo2], [PP], [PU] and [Rue2] for treatments of the theory of Gibbs states.

Theorem 4.1. *Let μ be Hausdorff d -dimensional measure H^d restricted to the hyperbolic Julia set J . Then writing ν for the (unique) T -invariant probability measure which is absolutely continuous with respect to μ and extended to the invertible map (Π_0, σ) , the product $\bar{\nu}$ of ν with Lebesgue measure on \mathbb{R} (and normalized) gives the unique measure of maximal entropy for the projected flow $(\bar{\Omega}, \bar{\nu}, \bar{\tau}_t)$. This flow is ergodic. The flow entropy equals $\dim(J)$.*

Proof. Bowen's theorem [Bo2] states that there exists a unique positive number d such that the pressure $P(\varphi) = 0$ for $\varphi(z) = -d \log |DT(z)|$, that d is the Hausdorff dimension of J and that d -dimensional Hausdorff measure μ is equal to the Ruelle eigenmeasure (times a constant). This measure is ergodic for the shift map hence the flow built over it is also ergodic. Now from the variational principle [Bo1], [Rue2] (see also [PU]) using the fact that

$$P(\varphi) = 0, \quad 0 = \sup \left(h(m) + \int_J \varphi d_m \right)$$

where the sup is taken over invariant probability measures m on J . The sup is achieved uniquely by the Gibbs state ν (which is the unique invariant measure equivalent to the eigenmeasure μ). This equation holds for φ, m, ν extended to the invertible map σ on Π_0 . Because there is a zero on the left side, we can then rewrite the equation as:

$$\frac{h(\nu)}{\int -\varphi d\nu} = 1. \quad \text{Hence, } \frac{h(\nu)}{\int_{\Pi_0} \log |DT| d\nu} = d.$$

Now Abramov's formula [Ab] states that the measure theoretic entropy for a special flow is equal to (base entropy divided by expected return time). This is exactly the left-hand side of the equation. Conversely, working through the equations in the other direction, from the variational principle this gives the maximum entropy and is the unique such measure. Hence the flow entropy (by which we mean the topological entropy and equivalently the maximal measure theoretic entropy) for $(\bar{\Omega}, \bar{\tau}_t)$ is the Hausdorff dimension of J , and the unique measure of maximal entropy is $\bar{\nu}$. \square

The next theorem now follows as a corollary. Part (iii) is due, with a different proof, to Falconer [Falc].

Proposition 4.2. *For a rational map T with hyperbolic Julia set (or, more generally, for a conformal mixing repeller),*

- (i) *for ν -a.e. $\underline{z} \in \Pi_0$, the order-two density at $0 \in \mathcal{C}$ of H^d restricted to $L_{\underline{z}}$ exists, and this value is a.s. constant on Π_0 .*
- (ii) *If for some $\underline{z} \in \Pi_0$, the order-two density at 0 of $L_{\underline{z}}$ exists, then the same holds for J at z , with equal value, where $z = z_0$ is the zeroth coordinate of \underline{z} .*
- (iii) *For μ - a.e. point $z \in J$, the order-two density exists, and the value is a.s. constant.*
- (iv) *for all $\underline{z} \in \Pi_0$, the order-two density exists and is the same at H^d - a.e. point $z \in L_{\underline{z}}$.*

Proof. The basic idea is simple, though there is a subtle analysis point we remark on later, so we will be careful with the details. Part (i) is a consequence of the Birkhoff ergodic theorem applied to the projected flow (which is ergodic); (ii) holds because the two sets are forward asymptotic under scaling; (iii) follows from (i) together with (ii), and (iv) will be proved from (iii).

For $\underline{z} \in \Pi_0$ and for $t \in \mathbb{R}$, write

$$A_t = A_{\underline{z},t} = e^t L_{\underline{z}} = \tau_t L_{\underline{z}};$$

here τ_t is the magnification flow τ_t on Ω , the collection of closed subsets of \mathcal{C} . Define

$$f(\underline{z}, t) = \int_{A_t} \chi_B dH^d = \frac{H^d(B(z, e^{-t}) \cap L_{\underline{z}})}{e^{-td}}.$$

Here we have used the conformal transformation property of H^d . Thus

$$\lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T f(\underline{z}, t) dt$$

is the order-two density at $0 \in \mathcal{C}$ of $L_{\underline{z}}$. Note that $f(\underline{z}, t)$ would be unchanged if we rotated $L_{\underline{z}}$. Hence the function f is well-defined on the projected flow space $\bar{\Omega}$. We claim $f \in L^1(\bar{\Omega}, \bar{\nu})$. Indeed, it is bounded away from 0 and ∞ : from Theorem 2.14, $L_{\underline{z}}$ is locally a conformal image of a piece of J , hence $H^d|_{L_{\underline{z}}}$ is also a geometric measure, which implies boundedness. Alternatively, the bounds for $\bar{L}_{\underline{z}}$ follow directly from those for J by the estimates to follow.

From this we know by the Birkhoff ergodic theorem that since the projected flow is ergodic, the limit exists and is $\bar{\nu}$ -a.s. constant:

$$\lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T f(\underline{z}, t + s) dt = \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T f(\tau_t(\underline{z}, s)) dt = \int_{\bar{\Omega}} f(\underline{z}, s) d\bar{\nu}.$$

Taking, in particular, $s = 0$, we see that the order-two density of $L_{\underline{z}}$ at 0 is equal to this value for ν -a.e. \underline{z} , proving (i).

Next, suppose we are given that the above limit exists for some $L_{\underline{z}}$ and some \underline{z} . Fixing $z = z_0$, the zeroth coordinate of \underline{z} , define $\tilde{A}_t = e^t(J - z) = \tau_t(J - z)$, writing this time

$$\tilde{f}(t) = \tilde{f}(\underline{z}, t) = \int_{\tilde{A}_t} \chi_B dH^d.$$

We wish to show the time average for \tilde{f} exists and takes on the same value as for $f(t) = f(\underline{z}, t)$.

For $\delta > 0$, let φ_δ be a continuous function from \mathcal{C} to $[0, 1]$ such that $\varphi_\delta = 1$ on $B = B(0, 1)$ and $= 0$ outside of $B(0, e^\delta)$. Write $\tilde{f}_\delta(t) = \int_{\tilde{A}_t} \varphi_\delta(w) dH^d$ and $f_\delta(t) = \int_{A_t} \varphi_\delta(w) dH^d$. Note that $\varphi_\delta(w \cdot e^\delta) \leq \chi_B(w) \leq \varphi_\delta(w)$ for all $w \in \mathcal{C}$. Hence, again by

the conformal transformation property of H^d , $e^{-\delta} f_\delta(t + \delta) \leq f(t) \leq f_\delta(t)$ for all t, δ ; the analogous inequalities hold for $\tilde{f}_\delta, \tilde{f}$. We have

$$\begin{aligned} e^{-\delta} \liminf_{T \rightarrow \infty} \frac{1}{T} \int_0^T f_\delta(t) dt &= e^{-\delta} \liminf_{T \rightarrow \infty} \frac{1}{T} \int_0^T f_\delta(t + \delta) dt \leq \liminf_{T \rightarrow \infty} \frac{1}{T} \int_0^T f(t) dt \\ &\leq \limsup_{T \rightarrow \infty} \frac{1}{T} \int_0^T f(t) dt \leq \limsup_{T \rightarrow \infty} \frac{1}{T} \int_0^T f_\delta(t + \delta) dt = \limsup_{T \rightarrow \infty} \frac{1}{T} \int_0^T f_\delta(t) dt. \end{aligned}$$

Now by the proof of Lemma 3.3, the closed sets $(J - z)$ and $L_{\underline{z}}$ are forward asymptotic in the magnification flow τ_t with respect to the measure metric. Since this allows sampling against continuous functions with compact support, it applies to φ_δ . By our assumption the time average of $f_\delta(\underline{z}, t)$ exists, so this implies the average for \tilde{f}_δ also exists, with the same value, equal to $\int_{\Omega} f_\delta d\bar{\nu}$.

Thus

$$\begin{aligned} e^{-\delta} \int_{\Omega} f_\delta d\bar{\nu} &= e^{-\delta} \liminf_{T \rightarrow \infty} \frac{1}{T} \int_0^T \tilde{f}_\delta(t) dt \leq \liminf_{T \rightarrow \infty} \frac{1}{T} \int_0^T \tilde{f}(t) dt \\ &\leq \limsup_{T \rightarrow \infty} \frac{1}{T} \int_0^T \tilde{f}(t) dt \leq \limsup_{T \rightarrow \infty} \frac{1}{T} \int_0^T \tilde{f}_\delta(t) dt = \int_{\Omega} f_\delta d\bar{\nu}. \end{aligned}$$

Since this is true for every $\delta > 0$, the limit for \tilde{f} exists and equals that for f . This proves (ii). Part (iii) follows immediately. Finally, for (iii), when the set $L_{\underline{z}}$ is shifted to be centered at a point $w \in L_{\underline{z}}$ by $L_{\underline{z}} - w$, then from the definitions, $L_{\underline{z}} - w$ is $e^{i\theta} L_w$ for some angle θ and choice of preimages for w . Thus since the order-two density exists at 0 for $L_{\underline{z}}$ for a.e. \underline{z} , for every \underline{z} it will exist H^d -almost surely in that scene $L_{\underline{z}}$. \square

Remark. The subtle analysis point we referred to above is this. Since the function χ_B is not continuous, we cannot apply directly convergence in the measure metric. One's first attempt is probably to argue that the boundary of the disk is however a negligible set. So one covers it by n balls of radius $1/n$, estimating their measure from the geometric measure property. And indeed, this method will work fine for $d = \text{dimension}(J) \geq 1$; but for $d \in (0, 1)$ the estimate blows up for large n . Our way out is to create bounds by dilating; since this corresponds to shifting the time, it does not matter after the time average is taken.

§4.2 Rotational behavior: Furstenberg's lemma.

The main result of this and the next section will be that the scenery flow is ergodic for the repellers which are not linear nor contained in a finite union of real-analytic curves and which form a dense open subset. The same will be proved for all hyperbolic rational maps, except for a few exceptional cases listed in Theorem 4.8. Furthermore, the full scenery flow is equal to the scenery flow at μ -almost every point in J .

Ergodicity of a special flow is equivalent to ergodicity of a cross-section map, since invariant subsets correspond. In this sub-section we will study equivalent conditions for ergodicity of the base map $(\hat{\Pi}_0, \hat{\sigma})$. To prove this we make use of two methods, one due originally to Furstenberg and one to Livsic. These form part of the general developing

theory of group-valued cocycles for group actions. For completeness, we give full proofs of what we need. Once we have shown the conditions equivalent, one of them (the periodic point condition) will then be verified by methods from complex analysis, in the next subsection.

We begin with a general skew product with circle fiber. Let F be a (not necessarily invertible) measure-preserving map of a measure space X with invariant ergodic probability measure ρ . We assume that the **skewing function** $\varphi : X \rightarrow S^1 = \mathbb{R}/\mathbb{Z}$, is measurable, and that X is a compact metric space.

We write $\widehat{X} = X \times S^1$ and define \widehat{F} on \widehat{X} by $\widehat{F}(\omega, \theta) = (F(\omega), \theta + \varphi(\omega))$. Lebesgue measure on S^1 will be denoted by m .

Proposition 4.2. *The product measure $\hat{\rho} = \rho \times m$ is invariant for \widehat{F} .*

Proof. The idea will be that since fibers are rotated by φ and then simply exchanged according to the measure-preserving map F , the skew product should preserve $\hat{\rho}$ by Fubini's theorem. To make this precise, following the proof of Lemma 2.1 in [Fu], test against a function f in $L^1(\hat{\rho})$; interchanging the order of order of integration proves invariance. \square

We will write \mathcal{M}_ρ for the collection of \widehat{F} -invariant measures with **marginal** ρ (i.e. which project to ρ). We are interested in finding equivalent conditions such that there is only one such measure (i.e. \mathcal{M}_ρ is the singleton $\{\hat{\rho}\}$), in which case we will say \widehat{F} is **ρ -uniquely ergodic**. First we discuss some other forms that an invariant measure can take. Suppose there exists a measurable function $u : X \rightarrow S^1$ such that

$$(*) \quad \varphi(x) = u \circ F(x) - u(x) \quad (\text{for } \rho\text{-a.e. } x).$$

In this case one says φ is a **coboundary**, or is **cohomologous to zero**. Now notice that the graph of u is a \widehat{F} -invariant subset of \widehat{X} . Hence there exists an invariant measure which is not $\hat{\rho}$: just lift ρ to a measure supported on that graph (or more generally, add parallel bands of mass).

We mention another interpretation of equation (*). Via the map $(w, u(w) + \theta) \mapsto (w, \theta)$, \widehat{F} is isomorphic to $F \times (\text{identity})$, i.e. to a skew product which does nothing in the fibers. This isomorphism can be viewed as a fiber-preserving change of coordinates on \widehat{X} , given by choosing a new origin for each circle; the new origin is $u(w)$. Conversely, a fiber-preserving isomorphism defines such a function u .

So far we have, therefore:

Proposition 4.3. *The following are equivalent, for $\varphi : X \rightarrow S^1$ measurable:*

- (a) *there exists u measurable with $\varphi(x) = u \circ F(x) - u(x)$*
- (b) *there exists a fiber-preserving isomorphism from \widehat{F} to $F \times (\text{identity})$*
- (c) *there exists a measurable function from X to S^1 whose graph is invariant.* \square

It is clear that the example described above of an invariant measure different from $\hat{\rho}$ will generalize to that of a map which instead of fixing the circle, permutes k equal intervals. In fact conversely, as we will now see, this is all that can happen.

This represents a generalization of Lemma 2.1 from [Fu]. (Furstenberg considers the case where the base transformation itself is uniquely ergodic).

Theorem 4.4. *The following are equivalent for \widehat{F} as above.*

- (a) $\widehat{\rho}$ is not ergodic.
- (b) \widehat{F} is not ρ -uniquely ergodic.
- (c) There exist $k \in \mathbb{Z}$ and $u : X \rightarrow S^1$ measurable such that

$$(**) \quad k\varphi = u \circ F - u.$$

Proof. (b) \implies (a). We will show that if $\widehat{\rho}$ is ergodic, then $\mathcal{M}_\rho = \{\widehat{\rho}\}$. We learned this argument from Eli Glasner; another nice argument can be given using generic points, following [Fu]. We recall that since \widehat{X} is a compact metric space, the \widehat{F} -invariant measures form a weak*-compact convex set, with the ergodic measures as the extreme points. Let $\tilde{\rho} \in \mathcal{M}_\rho$. Now if we average the measure $\tilde{\rho}$ along each circle fiber by Lebesgue measure on S^1 , we get the measure $\widehat{\rho}$. Therefore, $\widehat{\rho}$ is a convex combination of the rotated (and invariant) measures $R_\theta \tilde{\rho}$. Hence if $\widehat{\rho}$ is ergodic, then $\tilde{\rho} = \widehat{\rho}$.

(a) \implies (b). Supposing $\widehat{\rho}$ is not ergodic, it can be written as a convex combination of two invariant measures, $\widehat{\rho}_1$ and $\widehat{\rho}_2$. The only thing to check (to contradict ρ -unique ergodicity) is that they project to ρ . But whatever measure they project to must be absolutely continuous with respect to the projection of $\widehat{\rho}$, i.e. ρ ; hence by ergodicity of ρ it equals ρ .

(a) \implies (c). We cannot improve on Furstenberg's beautiful little argument, which we include for completeness. (He uses multiplicative notation i.e. S^1 is the set of complex numbers with modulus one).

Assume that $\widehat{\rho}$ is not ergodic for \widehat{F} . Then (see e.g. [Wa]) there exists a non-constant (real or complex)-valued function G in $L^2(\widehat{X}, \widehat{\rho})$. By Fubini's theorem, for ρ -a.e. circle fiber, G is in L^2 of that fiber. So there are Fourier coefficients $a_n(w)$ for the fiber over w , with $G(x, \theta) = \sum_{n=-\infty}^{\infty} a_n(x) e^{in\theta}$. Now calculating $G \circ \widehat{F}$, uniqueness of the Fourier coefficients implies that $a_n(w) = a_n(Fw) e^{in\varphi(w)}$ for each n . By ergodicity of F , the modulus of each $a_n(w)$ is ρ -a.s. constant. Since G is assumed non-constant, for some $k \neq 0$, $|a_k| \neq 0$ (ρ -almost surely). So for that k , we can normalize a_k in the equation above. Then, changing to additive notation, we define $u(w)$ by $e^{-iu(w)} = a_k(w)/|a_k|$. The equation then becomes $k\varphi = u \circ F - u$, proving (c).

Finally it is clear that (c) \implies (a), from the previous Proposition, by putting mass on the graph of u . Or, we note that the function $H_k(w, \theta) \equiv a_k(w) e^{ik\theta}$ is \widehat{F} -invariant, which contradicts ergodicity. \square

Remark 4.5. The set of k such that (c) holds is an ideal in \mathbb{Z} . If, say, ℓ generates this principal ideal, we can describe the collection of all \widehat{F} -invariant functions: they can be expressed as some combination of the H_k just defined, for all multiples k of ℓ .

§4.3 Ergodicity of the scenery flow.

First we consider the case of a **conformal mixing repeller** (X, T) . We recall the definition [Rue1], [PUZ,I,II], [PU]: let $X \subseteq V \subseteq U \subseteq M$ where M is a surface with complex structure, V and U are open, X is compact, and we have a conformal map $T : U \rightarrow M$ satisfying:

- $T(X) = X$;
- T is hyperbolic on X ;
- $\cap(T|_V)^{-n}(V) = X$;
- for any W open in X , there exists n with $T^n(W) \supseteq X$.

We recall that (X, T) is called **real-analytic** if X is contained in a finite union of real-analytic curves. Recall also from [Su3], [Pr], and [MPU] that a conformal mixing repeller (X, T) is said to be **linear** if the conformal structure on X admits a conformal linear refinement so that f is linear, that is, if there exists an atlas $\{\varphi_t\}$ that is a family of conformal injections $\phi_t : U_t \rightarrow \mathcal{C}$, where $\bigcup_t U_t \supset X$ such that all the maps $\phi_t \phi_s^{-1}$ and $\phi_t f \phi_s^{-1}$ are affine Möbius transformations.

We mention that while conformal mixing repellers clearly include the hyperbolic rational maps, they are much more general. They contain for example local (in a neighbourhood of the Julia set) analytic perturbations of hyperbolic rational functions and the limit sets of Kleinian groups of Schottky type. Other nontrivial linear examples come from generalizing the map $x \mapsto 3x \pmod{1}$ mapping $[0, 1/3] \cup [2/3, 1] \rightarrow [0, 1]$; notice that this particular example is biholomorphically conjugate with an appropriate invariant set of the map $f(z) = z^3$, which is linear (!) in the sense defined above.

The results from [Pr] and [MPU] contain the following.

Theorem 4.6. *Let (X, T) be a conformal mixing repeller. Then the following conditions are equivalent.*

- (1) *The repeller (X, T) is linear.*
- (2) *The Jacobian of T with respect to the Gibbs measure μ equivalent to the Hausdorff measure H^d on X , is locally constant.*
- (3) *There exists a cover $\{B_\lambda\}_{\lambda \in \Lambda}$ of X consisting of open disks, a family of continuous functions $\gamma_\lambda : B_\lambda \rightarrow \mathbb{R}$, $\lambda \in \Lambda$, and constants $c_{\lambda, \lambda'}^{(1)}, c_{\lambda, \lambda'}^{(2)}$ such that for all $\lambda, \lambda' \in \Lambda$*

$$\gamma_\lambda - \gamma_{\lambda'} = c_{\lambda, \lambda'}^{(1)}$$

on $B_\lambda \cap B_{\lambda'}$ and

$$\arg_\lambda(DT) - \gamma_\lambda + \gamma_{\lambda'} \circ T = c_{\lambda, \lambda'}^{(2)}$$

on $B_\lambda \cap T^{-1}(B_{\lambda'})$, where $\arg_\lambda(DT) : B_\lambda \rightarrow \mathbb{R}$ is a continuous branch of the argument of DT defined on the simply connected set B_λ .

In order to provide the reader with a more complete picture, which will be used in the proof of Theorem 4.8 we quote here Remark 2 from the proof of the implication $(ec) \Rightarrow (d2)$ of Theorem 3.1 of [MPU].

Remark 4.7. *In case X is not real-analytic, having equations in item (3) and functions γ_λ respectively holding and defined on X itself (rather than on an open cover of X) would already be sufficient to prove item (1).*

Theorem 4.8. *Let U be an open set in \mathcal{C} . Let \mathcal{R}_U denote the collection of all conformal mixing repellers $f : U \rightarrow \mathcal{C}$. Then the full scenery flow is ergodic for the repellers which are not linear nor contained in a finite union of real-analytic curves. Furthermore the ergodic maps form a dense open subset of \mathcal{R}_U , with respect to the \mathcal{C}^1 - topology on \mathcal{R}_U .*

Sketch of proof. From Livsic theory [Li1,2], [PUZ, I] the following are equivalent, given a hyperbolic map f and a real-valued Hölder continuous observable φ :

- (1) φ is cohomologous to zero in the class of Hölder continuous functions
- (2) φ is cohomologous to zero in the class of measurable functions
- (3) $S^n\varphi(x) = 0$ for each periodic point (where n is the period of x).

In [Li1,2] this is proved for real-valued φ ; one can show, using a method from [PUZ,I] that for a circle-valued function the analogous statements hold: for some $k \in \mathbb{Z}$,

- (1') $k\varphi$ is cohomologous to zero in the class of Hölder continuous functions
- (2') $k\varphi$ is cohomologous to zero in the class of measurable functions
- (3') $kS^n\varphi(x) = 0$ for each periodic point (where n is the period of x).

Passing to the multiplicative notation, it follows from Theorem 4.4 that nonergodicity of the full scenery flow is equivalent to the following: that the function

$$z \mapsto \left(\frac{DT(z)}{|DT(z)|} \right)^k$$

is a measurable coboundary (in S^1) for some integer k . This is exactly condition (2') above. Since this is equivalent with condition (1'), we conclude by Remark 4.7 that if the scenery flow is not ergodic, then the repeller (T, X) is forced to be either linear or real-analytic. Furthermore, as is easy to see (by a standard argument using the open mapping and implicit function theorems), condition (3') is closed and removable by arbitrarily small perturbations in the \mathcal{C}^1 - topology on \mathcal{R}_U . This completes the proof. \square

Lastly, combining Theorem 4.4 and Theorem 1 from [FU], in the case of hyperbolic rational functions we have the following. Examples of (2) and (3) were given in the Introduction. (We do not know at present if (1) can in fact occur, as we have no specific example.)

Theorem 4.9. *The full scenery flow of a hyperbolic rational function is ergodic unless*

- (1) *T has a superattracting fixed point with a preimage at which T has a different degree.*
- (2) *The Julia set $J(T)$ is a geometric circle and T is biholomorphically conjugate to a finite Blaschke product.*
- (3) *The Julia set $J(T)$ is totally disconnected and $J(T)$ is contained in a real-analytic curve with self-intersections (if any) lying outside of the Julia set. \square*

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