SMALL-SCALE STRUCTURE VIA FLOWS

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Abstract. We study the small scale of geometric objects embedded in a Euclidean space by means of the flow defined by zooming toward a point of the space. For a smooth embedded manifold one sees just the tangent space asymptotically, but for fractal sets and related objects (space-filling curves, nested tilings) the flow can be quite interesting, as the "scenery" one sees keeps changing.

For a Kleinian limit set the scenery flow and geodesic flows are isomorphic. This fact suggests that for a Julia set the scenery flow could provide the analogue of the hyperbolic three manifold, with its associated geodesic and horocycle actions.

A test is to see whether Sullivan's formula for dimension (Hausdorff dimension of limit set equals geodesic flow entropy) goes through for Julia sets. This is in fact true, and the resulting formula "dimension equals scenery flow entropy" unifies the formulas of Sullivan and of Bowen-Ruelle.

For changing combinatorics, considering the model case of interval exchanges, renormalization is given on parameter space by the Teichmüller flow of a surface; the scenery flow, now acting on a space of nested tilings, extends this flow to a surface fiber bundle. Thus renormalization is realized as flowing on a unification of the dynamical and the parameter space.

For fractal sets, the translation "horocycle" scenery flow has a natural conservative ergodic infinite measure. This observation builds a bridge between fractal geometry and the probability theory of recurrent events, suggesting on the one hand new theorems for the Fuchsian case and on the other a new interpretation of some results on countable state Markov chains due to Feller and Chung-Erdös. Interesting examples are seen in the intermittent return-time behavior of maps of the interval with an indifferent fixed point.

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Date: November 9, 2003.

1. Introduction.

Suppose we center a mathematical microscope at some point of a fractal set, and turn the knob continuously; as we zoom down toward smaller scales, ever-changing scenes go past us, as if we were riding on a train taking us deeper and deeper into the heart of the fractal landscape. Let us try to model such a fractal excursion mathematically. The continuously changing nature of the process sugggests that a precise description will involve a continuous–time dynamical system, in other words a *flow*. In this article, we shall sketch how such a flow (the *scenery flow* of the fractal set) can be defined, and indeed, constructed rigorously for a variety of examples, and we shall see how the scenery flow can be usefully applied in studying the fractal geometry. We shall, moreover, see that this flow of magnification, and a related translation flow, provide close analogues of two familiar flows: the geodesic and horocycle flows of a Riemann surface.

To begin our story, we shall recall some basic properties of these classical flows.

2. Geodesic and horocycle flows

Let $\mathbb{C}$ denote the complex plane and $\mathbb{H} \subseteq \mathbb{C}$ the upper half plane $\mathbb{H} = \{z = x + iy : \ y \geq 0\}$. The interior of $\mathbb{H}$, those points with $y > 0$, is given the *hyperbolic metric*, defined by $ds^2 = (dx^2 + dy^2)/y^2$, which makes it isometric to the Poincaré disk Δ. The orientation-preserving isometries for this metric are the real Möbius transformations Möb($\mathbb{R}$), with

$$f_A(z) = (az + b)/(cz + d)$$

for $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ where $a, b, c, d \in \mathbb{R}$ and $\det(A) = ad - bc = 1$; that is, $A \in SL(2, \mathbb{R})$.

The matrices A and γA for $\gamma \neq 0$ give the same Möbius transformation, so Möb($\mathbb{R}$) $\cong PSL(2, \mathbb{R})$. Let $\Gamma \subseteq$ Möb($\mathbb{R}$) be a discrete subgroup. Then the identification space $\Gamma\backslash\mathbb{H}$ is a Riemann surface; this may be compact, or be noncompact with either finite or infinite area. The unit tangent bundle of $\mathbb{H}$ can be identified with $PSL(2, \mathbb{R})$. This correspondence is easily described. Take as base point the unit vector i_i which is located at the point $i \in \mathbb{H}$ and points in the vertical direction; then, given $A \in SL(2, \mathbb{R})$, let $f_A^*(i_i)$ be the image of this vector by the derivative map of f_A, that is, it is the vector located at the point $f_A(i)$ which has been been rotated appropriately by the argument of the complex derivative. This image vector also has hyperbolic length one, as Möbius transformations are isometries for the hyperbolic metric; so this defines a map from $PSL(2, \mathbb{R})$ to the unit tangent bundle $T_1(\mathbb{H})$. The group Γ acts on $PSL(2, \mathbb{R})$ by left multiplication and one sees that $\Gamma\backslash PSL(2, \mathbb{R})$ is the unit tangent bundle of the surface $\Gamma\backslash\mathbb{H}$.

The geodesic flow on the surface is by definition the flow on this unit tangent bundle which moves a vector along its tangent geodesic at unit speed. Algebraically, this is given by right multiplication by $E_t \equiv \begin{bmatrix} e^{\frac{t}{2}} & 0 \\ 0 & e^{-\frac{t}{2}} \end{bmatrix}$ on $\Gamma\backslash PSL(2,\mathbb{R})$. To understand this, note that this matrix is equivalent as a Möbius transformation to $\begin{bmatrix} e^t & 0 \\ 0 & 1 \end{bmatrix}$ which dilates the plane by the factor e^t, and hence moves the vector i_i up the imaginary axis at unit speed in the hyperbolic metric. The action on a general unit vector is then given by the conjugation by f_A^* which is a hyperbolic isometry, so this is indeed the geodesic flow. The unstable horocycle flow h_t^u is given by the right action of $H_t^u \equiv \begin{bmatrix} 1 & 0 \\ t & 1 \end{bmatrix}$; the stable flow acts by its transpose. As the names suggest, these preserve the unstable and stable horocycles (circles tangent to the boundary $\mathbb{R}$ of $\mathbb{H}$ which are the base points of the unstable and stable sets of the geodesic flow; for the point i_i, this "circle" being the line $y = 1$).

For the simplest example of a noncompact, finite area surface, see Fig.1; here (depicted in the disk model) Γ is a free group on two generators, these being two hyperbolic Möbius transformations, one which shoves the interior of the disk to the right and one which moves everything up; the curved quadrilateral in the center is a fundamental domain for this action. The left side is glued to the right, and the bottom to the top, so the resulting surface is a torus, just like for the usual gluings of a square, to get the quotient space $\mathbb{R}^2/\mathbb{Z}^2$, except that now the corner point gives a cusp, as it goes out to ∞ in the hyperbolic metric: this is a *punctured torus* (Fig. 2).

Classical results are:

Theorem 2.1. *The geodesic and horocycle flows g_t, h_t^u, h_t^s preserve Riemannian volume of the unit tangent bundle of the surface M. This measure is finite iff the surface area is finite. For this case, if M is compact (equivalently has no cusps) then:*
(i) g_t is ergodic, indeed is (finite entropy) Bernoulli (is measure-theoretically isomorphic to a Bernoulli flow);
(ii) h_t^u, h_t^s are uniquely ergodic, with entropy zero.

In the finitely generated, finite area case with cusps, all this is true except that h_t^u, h_t^s are only nearly uniquely ergodic; normalized Riemannian volume is the only nonatomic invariant probability measure if we disallow measures supported on horocycles tangent to cusps.

More interesting for us will be the infinite area case, where the cusp opens up to flare out in a hyperbolic trumpet, Figs. 3, 5; we return to this below.

The flows g_t and h_t^u do not commute, but do satisfy the following commutation relation:

$$h_b g_a = g_a h_{e^{-s}b}.$$

In other words, the following diagram commutes:

$$\begin{array}{ccc} T^1(M) & \xrightarrow{h_{e^{-a}\cdot b}} & T^1(M) \\ g_a \uparrow & & \uparrow g_a \\ T^1(M) & \xrightarrow[h_b]{} & T^1(M) \end{array}$$

One can prove this algebraically, or see it geometrically in the upper half plane (Fig. 6).

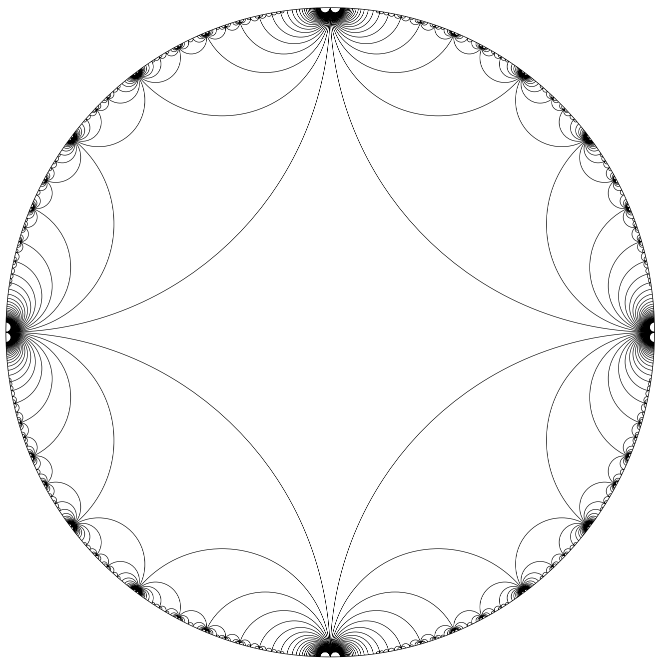

Figure 1. Covering space for punctured torus

Remark 2.1. Because of the commutation relation, the pair (geodesic flow, horocycle flow) gives an action of the $(ax + b)$-group (the real affine group) on $T^1(M)$. This already hints that there might be a relation with fractal geometry, as fractal sets genarally exhibit symmetries with respect to both dilation and translation.

Observation: The commutation relation tells us that h^u_t is isomorphic to a speeded-up version of itself. An ergodic theorist immediately will recognise that this is very special, as the entropy of a sped-up transformation or flow multiplies by that factor, so in this case:

$$\text{entropy}((h^u)_t) = e^{-a} \cdot \ \text{entropy}(h^u_t).$$

There are, thus, only two possiblities for the entropy of the flow $(h^u)_t$: 0, or ∞!

We have already seen an example of zero entropy (the finite area Riemann surface case); next we shall see a situation where infinite entropy occurs, and this example will lead us into the fractal realm.

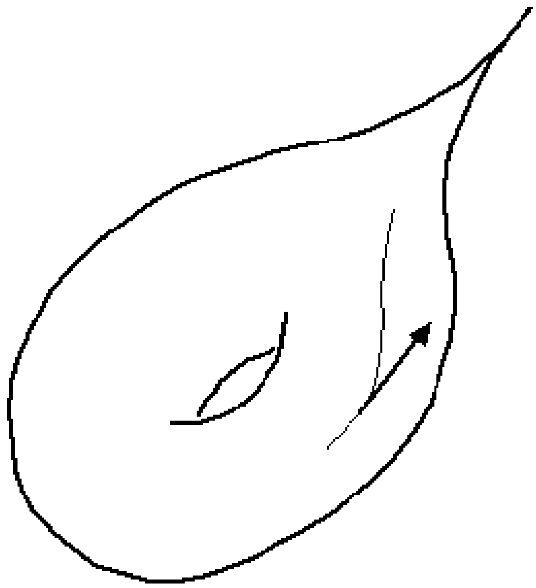

FIGURE 2. Geodesic flow on punctured torus

3. Brownian motion and stable processes

Let Ω be the space of continuous functions from $\mathbb{R}$ to $\mathbb{R}$, with the topology of uniform convergence on compact sets. This makes Ω into a Polish space, that is, a complete separable metric space, which is ideal from the point of view of measure theory.

Let μ denote Wiener measure on Ω, conditioned to be 0 at time 0, and defined both for future and past times. Then the *scaling property* of Brownian motion says: for $B \in \Omega$ and $a > 0$,

$$B(at)/\sqrt{a}$$

is "distributed like" $B(t)$. What this probability language means to an analyst is: the transformation $B \mapsto \Delta_a(B)$ defined by

$$(\Delta_a(B))(t) = B(at)/\sqrt{a}$$

preserves Wiener measure. To an ergodic theorist this suggests the following: defining

$$g_t = \Delta_{e^{-t}},$$

the action $g_t : \Omega \to \Omega$ is a measure- preserving flow! Next question: what flow is it? Answer: up to measure theoretic isomorphism, it is (the) Bernoulli flow of infinite entropy. (See [Fis87], and see [Fis04] for related work.) Call this the **scaling flow** of Brownian motion; geometrically it dilates time and rescales space appropriately to give another Brownian path. See Fig. 7.

Next, consider the **increment flow**,

$$((h^u)_a(B))(t) = B(a + t) - B(a).$$

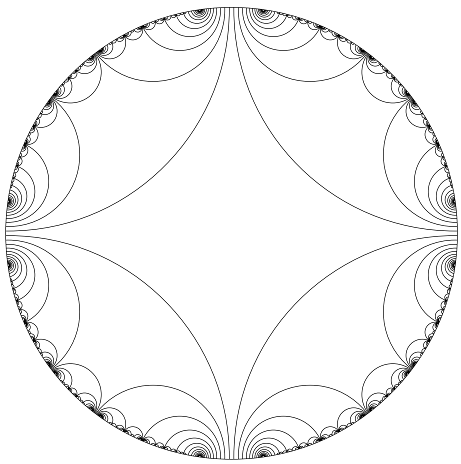

FIGURE 3. After opening the cusp.

This simply shifts the origin point $(0, 0)$ along the graph of $B(t)$, and since the process $B(t)$ has *stationary increments*, again preserves the measure μ.

Now comes the magical part: these two flows satisfy the same commutation relation as the geodesic and horocycle flows of our surface! But what is $(h^u)_t$, measure-theoretically? Answer: it is now infinite entropy. (One way to see this is to note that the increment flow is naturally isomorphic to the shift flow on white noise, which is an infinite entropy Bernoulli flow; the isomorphism is given by integration). So we have our "infinite entropy horocycle flow" example.

It is clear that the same commutation relation holds for any self-similar process with stationary increments; examples are the stable processes, see [FL02].

4. Brownian zero sets

This study of Brownian motion gives us almost for free another example, which will make the link to fractal subsets of the line. As is well known, the zero set of a Brownian path, $Z_B \subseteq \mathbb{R}$, has Hausdorff dimension $1/2$, and for the gauge function $\phi = (2t \log\log(1/t))^{\frac{1}{2}}$, has positive locally finite Hausdorff measure H_ϕ. Now define the map from Ω to itself by $B(t) \mapsto L_B(t)$ where $L_B(t) =$

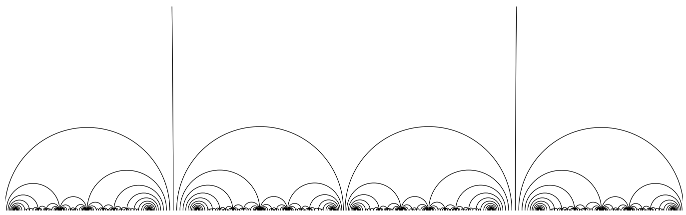

FIGURE 4. Transferred to upper half plane; "vertical lines" here are actually circles.

$H_\phi(Z_B \cap [0,t])$, the total measure up to time t (and similarly for $t < 0$, but with negative sign). Write μ_Z for the image measure on path space Ω; this is Paul Levy's *local time*. The flow g_t preserves this correspondence, hence the scaling flow on (Ω, μ_Z) is also a Bernoulli flow of infinite entropy (being a factor of the flow on (Ω, μ)) [BF92]. But what about the increment flow? Here things change: we slide into the gaps of the local time; the appropriate measure has become *infinite*. We shall return to examine the consequences of this in §12.

Note: to vizualize the local time $L_B(t)$ of a Brownian path B, it helps to know (by a theorem of Lévy) that local time paths are exactly maximum paths $M_{\widehat{B}}(t) = \sup\{B(s) : s \in [0,t]\}$ for a different Brownian path $\widehat{B}$. The correspondence $L_B \mapsto M_{\widehat{B}}$ is induced by an interesting isomorphism of Wiener space, given by a stochastic integral, see [BF92], [CW83]:

$$\widehat{B} = -\int_0^t \mathrm{sgn} B(s) dB(s)$$

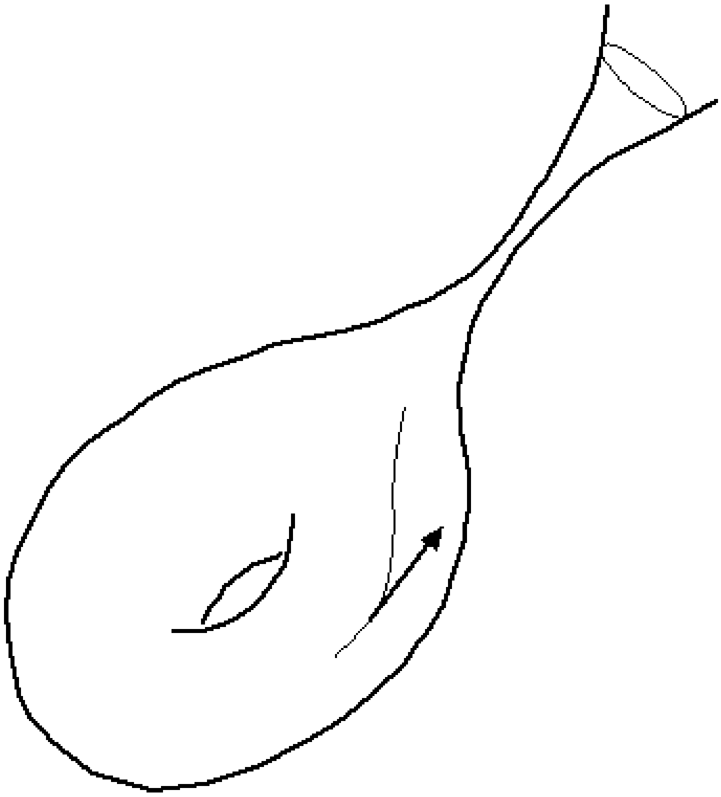

FIGURE 5. Gluing, we now have infinite area

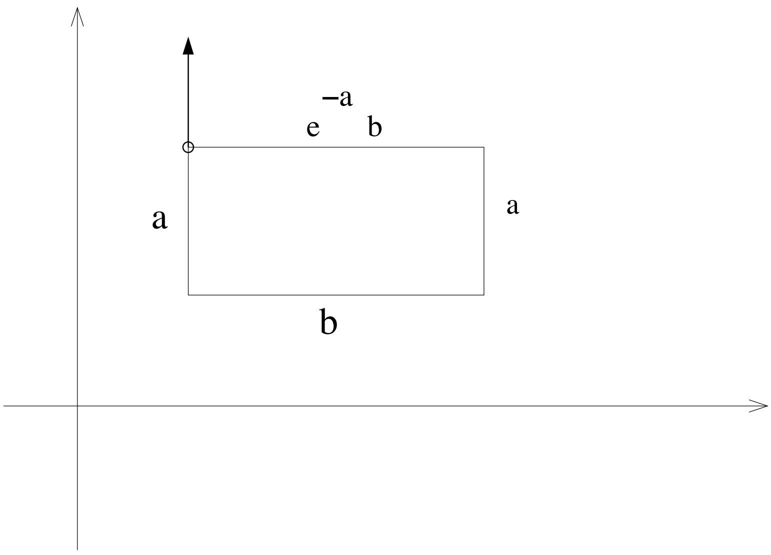

FIGURE 6. Hyperbolic distances (= flow times) in the upper half plane

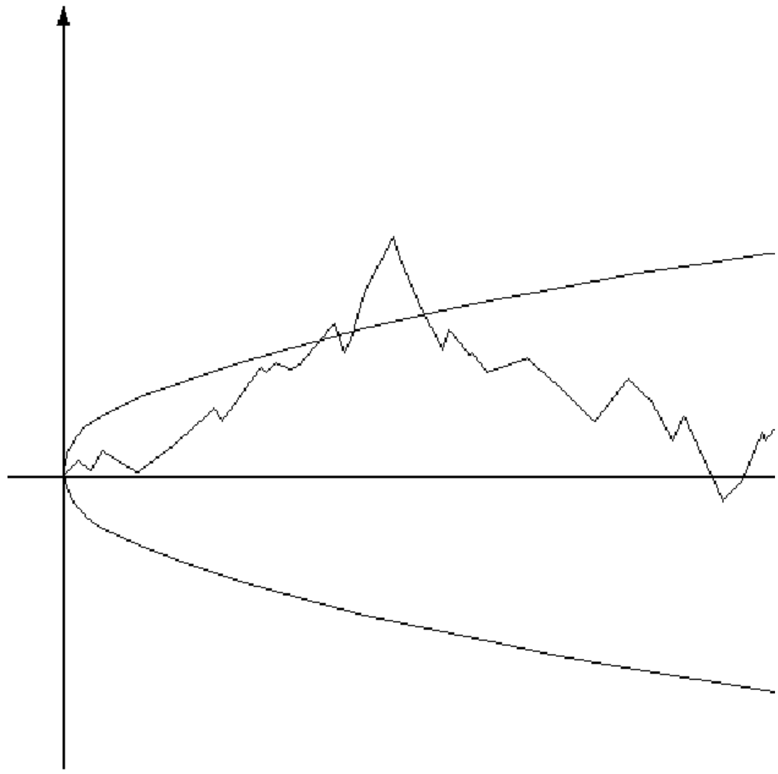

FIGURE 7. Sketch of a Brownian path; actual paths are much wilder: see [GS92]. The parabola is invariant under the scaling transformations.

5. The extended Cantor function (or Devil's Staircase)

The graph of Brownian local time is a continuous, nondecreasing function with a dense set of flat spots, reminiscent of the Cantor function– which suggests to us that we study that non-random example in a similar way. In [Fis92] we see an extended version of the usual Cantor funtion as depicted, say, in Mandelbrot's book [Man82]. Note the upper and lower envelopes of the form ct^d for $d = \log 2/\log 3$, the Hausdorff dimension of the Cantor set. Now this extended function $L(t)$, defined to be identically 0 for $t \le 0$, satisfies

$$L(3t)/2 = L(3t)/3^d = L(t);$$

this means that for the scaling flow g_t of exponent d, $g_{t_0}(L) = L$, where $t_0 = \log 3$; in other words, the scaling flow of this graph is a single periodic orbit!

Rescaling this path represents zooming down toward the point 0 in the Cantor set. But though the set C is, as we all know, "the same everywhere", on closer inspection this isn't quite so true. What do we see, for instance, if we slide the graph of f over to the point $1/4$ (which happens to be in C) by the increment flow, and then begin rescaling? In this case, the orbit is no longer periodic, but will converge asymptotically to a different periodic orbit, one with twice the previous length: $2\log 3 = \log 9$. But now differently from the point 0, there is asymptotic scenery on both sides of the point $1/4$, and so the limiting graph now is nontrivial also for $t \le 0$. (The positive part of this limiting graph is illustrated in [Fis92].)

And this is still very special: what happens when a general point of C replaces the number $1/4$? Answer: after taking the forward orbit closure (the omega-limit set), we get a mixing ergodic flow! This will be naturally isomorphic to the scenery flow of the fractal set C, with which we began the article. So now it is time to give some precise definitions.

6. The scenery flow

Let $\Omega = \Omega(\mathbb{R}^n)$ be the collection of all closed subsets of $\mathbb{R}^n$; topologize this with the Hausdorff metric on the one-point compactification (the *Attouch-Wetts* or *geometric* topology). This topology makes Ω compact; for an example consider $F_n = \{n\}$; this sequence of subsets of $\mathbb{R}$ converges in $\Omega(\mathbb{R})$ to the empty set, which is distance 0 from $\{\infty\}$ in $\mathbb{R} \cup \{\infty\}$. See e.g. §2 of [BF96].

Define the **magnification flow** g_t on Ω by $A \mapsto e^t \cdot A$. Choose a closed subset $F \subseteq \mathbb{R}^n$, choose a point $z \in F$, and define $\Omega_{(F,z)}$ to be the omega-limit set of $(F - z)$, the set translated so as to place z at the origin. Thus,

$$\Omega_{F,z} = \bigcap_{T \ge 0} \text{closure}\Big(\bigcup_{t \ge T} g_t(F - z)\Big).$$

The flow $(\Omega_{F,z}; g_t)$ is the **scenery flow** of the set F at the point z.

Example: Let C be the Cantor set $C \subseteq [0,1]$; define $\widehat{C} = \cup_{k=0}^{\infty} 3^k C$, so $3 \cdot \widehat{C} = \widehat{C}$. Then

$$g_{\log 3}(\widehat{C}) = e^{\log 3}\widehat{C} = \widehat{C},$$

so the scenery flow of C at 0 is $\{e^t \widehat{C} : t \in [0, \log 3]\}$, and this is a single periodic orbit of length $\log 3$. The total Hausdorff measure $L(t) = H_d(\widehat{C} \cap [0,t])$ gives a nondecreasing continuous function, the extended Cantor function, and this produces a flow isomorphism to the scaling flow of the Cantor function, described before.

Definition 6.1. The scenery flow of the set F is the flow of magnification by e^t on Ω_F, the union of the scenery flow spaces of each point $z \in F$ (so $\Omega_F = \cup \Omega_{F,z}$).

One can show [Fis92] that the scenery flow of the set C is naturally isomorphic to the following special flow: the height $\log 3$ suspension of the natural extension of the map $3x (\mod 1)$ on C. As a first consequence, since this flow is recurrent, a.e. orbit is dense and so the landscape encountered while zooming down toward a general point passes through all possible limiting scenes; the orbit closure is the whole space. Moreover, the flow is ergodic, so, by the Birkhoff ergodic theorem, the scenes will be encountered with the right frequency. So in particular this proves that the order-two (average) density exists and is a.s. constant. See [BF92] for a slightly different proof.

But what else can one say about this flow? The base map is isomorphic to the Bernoulli shift σ with weights $(1/2, 1/2)$ on $\Sigma = \Pi_{-\infty}^{\infty}\{0,1\}$ via ternary expansion, and this map has entropy $\log 2$. Now we see something interesting.

By Abramov's formula,

$$\text{flow entropy} = (\text{base entropy})/(\text{expected return time}),$$

so we conclude:

$$\text{entropy}(g_t) = \log 2/\log 3 = d = \dim(C)!$$

This gives us a formula for dimension:

The Hausdorff dimension of C equals the entropy of the scenery flow.

This formula is not always valid: a counterexample is a Brownian zero set; the dimension is $1/2$ but the scenery flow is isomorphic to the scaling flow on local time, which, as we have seen, has infinite entropy. Nevertheless, we wonder: is it possible that this formula is valid elsewhere?

7. The Fuchsian case

In fact we shall find just such an example in Fuchsian limit sets. Recall our modified punctured torus, where we opened up the cusp into a trumpet. The **limit set** Λ of the group Γ is now a Cantor subset of the boundary (Fig. 3; all the open intervals which are unions of the closed interval where the fundamental domain now meets the boundary are removed, leaving a topological Cantor set). So, what is its scenery flow? Well, first of all, the Cantor set is a subset of the circle, and zooming down toward this we see a tangent line: the infinitesimal scenery of a circle is a line. That's rather boring! But if we consider the scenery flow of the limit set, things become much more interesting. What we will see asymptotically is a collection of fractal subsets of the tangent space to the circle (the real line) which is invariant under dilation. But what are these sets? The answer is satisfyingly simple: they are the limit set Λ moved to the boundary $\mathbb{R}$ of $\mathbb{H}$, the upper-half space model of the hyperbolic disk. Here we have to allow all possible correspondences via Möbius transformations from Δ to $\mathbb{H}$ such that a point of the limit set occurs both at 0 and ∞. Next, what can we say about the dynamics or ergodic theory of this scenery flow? We find:

Theorem 7.1. *Let Δ be the unit disk with Poincaré metric, with boundary $\delta\Delta = S^1$. Let Γ be a finitely generated Fuchsian group of second type (that is, the limit set Λ is not all of the boundary). Then the scenery flow of Λ is a finite-to-one factor of the geodesic flow of the surface $M = \Gamma\backslash\Delta$, and the limiting scenes are images of Λ in $\mathbb{R}$ with respect to Möbius transformations from Δ to $\mathbb{H}$.*

Proof. We construct the factor map directly. Let v_p be a unit tangent vector based at the point $p \in \Delta$. Consider the geodesic tangent to v_p; this is a circle which meets $\delta\Delta$ orthogonally in two points η, ξ in the past, future directions respectively. Since a complex Möbius transformation is determined uniquely by where it sends three points, there exists a unique Möbius transformation $\alpha = \alpha_{v_p}$ such that α takes v_p to the unit vector $-i_i$ which points in the direction $-i$ at the location $i \in \mathbb{H}$, taking p to i, η, ξ to $\infty, 0$ and the geodesic to the imaginary axis.

More precisely, $\alpha(p) = i$ and $\alpha^*(v_p) = -i_i$ after normalization, where α^* is the derivative of α. Now define a map $\hat{\pi} : T^1(\Delta) \to \Omega = \{$ closed subsets of $\mathbb{R}\}$ by $v_p \mapsto \alpha_{v_p}(\Lambda)$.

Claim 1: for $\gamma \in \Gamma$, $\hat{\pi} \circ \gamma = \hat{\pi}$. This holds since $\gamma(\Lambda) = \Lambda$.

Now let $\hat{g}_t$ be the geodesic flow on $T^1(\Delta)$.

Claim 2: $\hat{\pi}(\hat{g}_t v_p) = e^t \cdot \hat{\pi}(v_p)$. The reason is that in $\mathbb{H}$, moving along the geodesic tangent to $-i_i$ (the imaginary axis) toward the point $0 \in \mathbb{H}$ is isomorphic by conjugacy to keeping the vector fixed at location i and dilating $\mathbb{H}$ by the factor e^t.

Claim 1 tells us that $\hat{\pi}$ induces a well-defined map $\pi : T^1(M) \to \Omega$ on the factor space $M = \Gamma\backslash\Delta$.

Claim 3: π is finite-to-one.

Proof: Suppose $\hat{\pi}(v_p) = \hat{\pi}(w_q)$ for some other vector w_q tangent to Δ. Then $(\alpha_{w_q}^{-1}) \circ (v_p)$ is a Möbius transformation of Δ which preserves the limit set Λ. Let $\widetilde{\Gamma}$ be the subgroup of Möb(Γ) which has been extended from Γ by adjoining all such elements. Then the limit set of $\widetilde{\Gamma}$ is also Λ. Now Margulis' Lemma implies that if $\Gamma_1 \subseteq \Gamma_2$ are groups of hyperbolic isomorphisms of $\mathbb{H}^n$ and have the same limit set, then Γ_1 is of finite index in Γ_2. Hence in the factor space $T^1(M)$there are at most finitely many such vectors w_q for a given v_p.

We thank Bernie Maskit and Peter Waterman for their help with this part of the proof.

It remains to show:

Claim 4: The asymptotic limiting sceneries of the limit set in the circle S^1 are the images by the stereographic projections.

Proof: Choose a point z in the limit set and place the circle so this point is at the origin, in the upper half space and tangent to the real axis. Zooming toward z for time t is equivalent to dilating this picture by the factor e^t; let us at the same time consider the downward-pointing unit vector at Euclidean height 1 in each picture. The Poincaré disks get larger and larger, approximating $\mathbb{H}$, and our tangent vector is moving via the geodesic flow. Now superimpose on this picture the stereographic projection determined by that vector, that is, by its image in the original Poincaré disk. The Möbius transformations converge to stereographic projections, proving the Claim. □

8. Extension to hyperbolic n-space

The above proof extends immediately to hyperbolic n- space $\mathbb{H}^n$ with the following changes: "geodesic flow" is replaced by "geodesic *frame* flow", and "finitely generated" by "geometrically finite". Consider for instance $n = 3$. Then Δ^3 is the unit ball in $\mathbb{R}^3$ with the Poincaré metric; choice of a frame defines a unique "stereographic projection" to the upper half-space model $\mathbb{H}^3$, sending the frame f_p to the standard basis frame based at the point $(0, 0, 1) \in \mathbb{H}^3 \subseteq \mathbb{R}^3$. The reason we need frames is that pictures of the scenery flow are different when rotated, and the frame flow includes this information. (For the case $n = 2$ of Fuchsian groups, there is no possibility of rotation so the frame flow is in fact isomorphic to the geodesic flow, and finitely generated is equivalent to geometrically finite.)

9. Ergodic theory and Sullivan's formula

For the case of a Fuchsian group of second type, the geodesic flow of the surface $M = \Gamma\backslash\Delta$ at first glance is not nice from the point of view of ergodic theory: not only does $T^1(M)$ have infinite volume (this is not a priori a problem, as infinite measure ergodic theory can be brought in) but, much worse than that, a.e. vector v_p is non-recurrent, i.e. eventually leaves any compact region. Sullivan's insight is that interesting dynamics can be recovered if we restrict attention precisely to those vectors which are recurrent for positive and negative times; these are exactly those vectors v_p for which the endpoints η, ξ at $-\infty, +\infty$ belong to the limit set Λ.

Next question: is there a natural measure to put on this recurrent set, replacing Riemannian volume? Sullivan's answer is a modification of Patterson's measure μ on Λ. (For nice cases, μ is the Hausdorff covering measure on Λ; for some other cases, as Sullivan showed, it is the packing measure.) Sullivan's measure on $T^1(M)$ is described as follows. It is defined first on $T^1(\Delta)$, in a Γ-equivariant way; hence it projects to the factor space $T^1(M)$. A unit tangent vector v_p is parametrized by the two endpoints $\eta \neq \xi$, plus a real number (where it is along the geodesic). Hence $T^1(\Delta)$ can be parameterized by $(S^1 \times S^1 - \text{diagonal}) \times \mathbb{R}$. The recurrent set is represented by $(\Lambda \times \Lambda - \text{diagonal}) \times \mathbb{R}$. The measure is equivalent (shares the same sets of measure zero) with $(\mu \times \mu) \times$ Lebesgue measure; the Radon-Nikodym derivative with respect to this product measure is $1/|\eta - \xi|^{2d}$, with distance measured in the Euclidean metric on the disk and d the Hausdorff dimension of Λ.

This guarantees that the resulting measure (often known as the **Patterson-Sullivan measure**) is Γ-equivariant and invariant for the geodesic flow. Sullivan then proves (we state the n-dimensional version):

Theorem 9.1. *Let Γ be a geometrically finite subgroup of Möb(Δ^n). Then the geodesic flow is ergodic for the Patterson-Sullivan measure $\hat{\mu}$, and $\hat{\mu}$ is the unique measure of maximal entropy, with entropy equal to the Hausdorff dimension dim$(\Lambda) = d$.*

Some background references are [Pat76],[Pat87], [Sul84], [Sul70].

We can hence conclude:

Corollary 9.2. *The topological entropy of the scenery flow of the limit set of a geometrically finite Kleinian group is equal to the Hausdorff dimension of the limit set.*

Proof. The scenery flow is a finite index factor of the frame flow. The entropy of the frame flow equals that of the geodesic flow, since it is an isometric extension. Since it is a finite-to-one factor, entropy is preserved. □

10. The scenery flow of a Julia set and hyperbolic Cantor set

The definition of the scenery flow (as an omega-limit set of the magnification flow acting on closed subsets of $\mathbb{R}^n$) makes sense for general fractal sets; sometimes we can identify this flow, constructing it concretely. We have already discussed the example of the middle-1/3 Cantor set; this is a linear Cantor set, and many other such linear flows (such as those generated by linear conformal IFS's with the open set condition) can be studied by a straightforward modification of this. The case of Kleinian limit sets was also easy, as this is also a "linear" case; the maps are linear fractional transformations, and, as we have seen above, the limiting sceneries are just given by stereographic projection.

We describe next the case of hyperbolic Julia sets; conformal mixing repellors and hyperbolic $\mathbb{C}^{1+\alpha}$ Cantor sets are dealt with in a similar way.

There is an analogy to the Kleinian case. There we had a homomorphism π from the hyperbolic manifold $T^1(M) = T^1(\Gamma\backslash\Delta)$ to the space Ω of scenes, conjugating the geodesic frame flow to the magnification flow on sets.

Here we construct a "model scenery flow" which will play the role of $T^1(M)$.

Let $f : \mathbb{C} \to \mathbb{C}$ be a rational map which is hyperbolic on its Julia set $\mathcal{J}$. Write Df for its derivative, and f_* for the action of the derivative on the unit tangent bundle of $\mathcal{J}$ (by which we mean the unit tangent bundle of $\mathbb{C}$, restricted to $\mathcal{J}$). Let $\tilde{f} : \widehat{\mathcal{J}} \to \widehat{\mathcal{J}}$ denote the natural extension of this map. The space $\widehat{\mathcal{J}}$ is an inverse limit and can be identified with the space of sequences $\underline{\hat{z}} = (\dots \hat{z_{-1}}\hat{z}_0\hat{z}_1 \dots)$ where $\hat{z}_i = (z_i, \theta_i) \in \mathbb{C} \times [0, 2\pi]$ represents a unit tangent vector and the sequence satisfies $\hat{z}_{i+1} = f_*(\hat{z}_i)$. Write $\widehat{\Omega}$ for this collection of sequences, with shift map $\hat{\sigma}$; let $\widetilde{\Omega}$ denote the suspension flow over $(\widehat{\Omega}, \hat{\sigma})$ with height (= return time) $\log |Df(z_0)|$. This is our model flow.

The flow homomorphism π from $\widetilde{\Omega}$ to Ω is defined as follows: on the base, $\pi(\underline{\hat{z}}) = L_{\underline{\hat{z}}} = \lim_{n\to\infty}(Df^n(z_{-n})\cdot(\mathcal{J} - z_{-n}))$. The limit is in the Hausdorff metric on closed susets of $\widehat{\mathbb{C}} \equiv \mathbb{C} \cap \{\infty\}$; thus the Julia set has been centered at the n^{th} preimage z_{-n} and expanded and rotated by the derivative map. That this always converges is a consequence of bounded distortion, see [BFU02].

We have:

Theorem 10.1. *(*[BFU02], [FU00]*). For the hyperbolic Julia set of a rational map, there is, up to rotation, a unique measure of maximal entropy for the model scenery flow. Its entropy satisfies the formula "model flow entropy equals dim($\mathcal{J}$)". When we include rotations, then for all but a few exceptional cases, there is a unique measure of maximal entropy and the model scenery flow is ergodic with respect to this measure.*

Remark on proof: We refer the reader to [BFU02], but mention two interesting points.

First, ergodicity for the model flow implies a rotational symmetry for the scenery flow; that is, when zooming down toward a.e. point we not only eventually see all possible scenes, but they occur at all angles (and with the expected frequency). This question can, by a lemma of Furstenberg [Fur61] see [BFU02] be formulated in terms of the non-existence of a certain *circle-valued cocycle* or equivalently as a problem about the non-existence of *invariant line fields* (or cross fields); so, the hard work in proving ergodicity has been transferred to that setting: see [FU00] and [May02].

Second, we sketch here the proof of the entropy formula because it is so instructive and works out so nicely. Bowen's formula for dimension [Bow79], [Rue82] reads: there exists a unique d such that $P(-\log |Df|^d) = 0$; this is $\dim(\mathcal{J})$. Here $P(\phi)$ is the pressure of a function ϕ, which can be defined as $P(\phi) = \sup(h(\mu) + \int \phi \, \mathrm{d}\mu)$ where the supremum is taken over invariant probability measures and $h(\mu)$ is the entropy of the transformation $(\mathcal{J}, f, \mu)$. We know from the theory of Sinai-Ruelle-Bowen that there exists a unique invariant measure μ such that the sup is attained. So we have for this invariant measure μ:

$$0 = h(\mu) + \int -\log |Df|^d \mathrm{d}\mu$$

hence

$$h(\mu) = \int \log |Df|^d \mathrm{d}\mu) = d \int \log |Df| \mathrm{d}\mu$$

and so

$$\frac{h(\mu)}{\int \log |Df| \mathrm{d}\mu} = d.$$

Now the formula on the left is (base entropy)/(expected return time), hence (by Abramov's formula) equals the special flow entropy.

The scenery flow is, a priori, a magnification flow defined on *sets*; however, it is natural to carry along more information, given (for the previous examples) by a labelling inherited from the Kleinian group or map f respectively. This information is provided by the model flows; we call this the *labelled* or *marked* scenery flow. We conclude with a dimension formula which unites Sullivan's formula for Kleinian limit sets, with Bowen's for Julia sets and "cookie cutter" Cantor sets:

Theorem 10.2. *For geometrically finite Kleinian groups in dimension n, hyperbolic rational maps, conformal mixing repellors, and hyperbolic $\mathbb{C}^{1+\alpha}$ Cantor sets, we have the formula:*

The entropy of the marked scenery flow of the limit set is equal to the Hausdorff dimension of the limit set.

11. Doubling maps and the Riemann surface lamination

The simplest example of a hyperbolic Julia set is for the map $f(z) = z^2 + c$ with $c = 0$; then $\mathcal{J}$ is the circle S^1 and f restricted to $\mathcal{J}$ is the usual doubling map of S^1, isomorphic to $x \mapsto 2x(\mod 1)$ on the unit interval. In this case, the natural extension of f is the hyperbolic map in the solenoid $\hat{f} : \widehat{S} \to \widehat{S}$, which in turn is an a.s. one-to-one factor of the Bernoulli shift $\sigma : \Sigma \to \Sigma$ for $\Sigma \equiv \Pi_{-\infty}^{\infty}$. Here $|Df| = 2$ so the model flow is the special flow of height $\log 2$ over the solenoid. The base entropy is $\log 2$, so the flow entropy is (by Abramov) $\log 2/\log 2 = 1$, which is, indeed, the Hausdorff dimension of the Julia set S^1. In this case the model flow space is identical to Sullivan's *Riemann surface lamination* [Sul92], [Sul91]; we thank especially D. Sullivan and J. Kahn for conversations on this point. See also Chapter VI of de Melo and van Strien [dMvS93] regarding these papers of Sullivan.

This suggests that the model flow for a general hyperbolic Julia set is, on the one hand, a generalization of the Riemann surface lamination and on the other, an analogue for rational maps of the recurrent part of the frame bundle of the hyperbolic n-manifold $\Gamma\backslash\Delta^n$. Minsky and Lyubich, building partly on our construction of sceneries, and extending that to general rational maps, showed the following remarkable result: that a rigidity theorem of Thurston for rational maps can be proved analogously to Mostow's rigidity theorem for Kleinian groups, replacing the 3- manifold by the "hyperbolic 3-manifold lamination". (They succeed moreover in extending Thurston's theorem by this approach.) See [ML97] and see [KL01], [Lyu02] for related work. All of this, also [BFU02] and [FU00], fits the philosophy of the Sullivan-Thurston "dictionary" between Kleinian groups and rational maps.

The Riemann surface lamination is a *double suspension*: the solenoid is a suspension of the adic transformation (odometer), giving a flow $(h^u)_t$, and the scenery flow space is a suspension flow g_t over that (now suspending the hyperbolic map). This space carries two flows: the flow which spins around the solenoid direction, which is just the lift of the rotation flow on the circle, and which preserves each level, and the vertical flow. This pair of flows h^u, g satisfy the same commutation relation as before–which is, indeed, exactly what should happen, since the solenoid leaves at a given height are the unstable leaves of the vertical flow.

12. The horocycle flow: infinite measures, return times and average density

Let us consider the case of a Fuchsian group of first type (the limit set is the circle); here the appropriate measure for the geodesic flow and for the horocycle flows h^u, h^s is the same: Riemannian volume on $T^1(M)$. For a group of second type, the situation is radically different: our recurrent measure is Patterson-Sullivan measure $\hat{\mu}$, but this is no longer invariant for h^u or h^s. The reason is that any given horocycle tangent to the limit set Λ meets the recurrent set in a fractal subset of the horocycle. The natural measure, therefore, is a modification of this: it is now equivalent to

(Lebesgue measure $\times \mu$) $\times$ Lebesgue measure. That is, e.g. for h^s, the boundary point ξ at $+\infty$ of a vector v_p in the support of this measure is required to be in the limit set, while the infinite past boundary point η is free to wander along the real line; these points have conditional measure given by Patterson measure μ and by Lebesgue measure on $\mathbb{R}$ respectively.

(This was noticed by myself and M. Burger independently and about the same time– both of us later than Patrick Kenny, in his very interesting but unfortunately unpublished thesis).

In nice cases, this measure (which we call **Kenny measure**), is the unique invariant Radon measure up to multiplication by a constant (since it is infinite there is always this choice of normalization). See [Ken83] for the general no-cusp case, [Bur90] for the no-cusp case with dimension $> 1/2$. With M. Burger we have a proof different from Kenny's which includes the case with cusps allowed (where we rule out atomic measures and measures supported on horocycles around the cusps in the statement of unique ergodicity); manuscript in preparation.

It is natural to wonder what happens for other fractal sets. For instance for hyperbolic $\mathbb{C}^{1+\alpha}$ Cantor sets, one can also prove infinite-measure unique ergodicity, using techniques of [BM77], combined with [BF96] and [BF97], see also [Fis03a]. (The specific case of the middle-third Cantor set is worked out in detail in [Fis92].)

The philosophy suggested by these examples is:

For some infinite measure preserving transformations,
the returns to a finite measure subset are a fractal-like subset of times.

This idea of a "fractal-like" subset of the integers can be made precise by use of the scaling flow (as $t \to -\infty$ for the way we have defined that flow here). The notion of average (or order-two) density of extends to integer fractal sets, playing the role of a finitely additive Hausdorff measure, see [BF92]. This idea then leads to a new type of ergodic theorem, given by normalization by the "dimension" followed by a log average [Fis92]. Then the average density reappears in a different guise: the limiting value of the time average is the expected value of the observable times the average density of the fractal integer set.

Further insight is given by examples coming from probability theory. Certain countable state Markov chains (called "recurrent events" by Feller [Fel49]) exhibit this type of behavior.

This led us to the rediscovery of a result of Chung and Erdös for simple random walks, and to an extension of that little-known but beautiful theorem of 1950 [CE51], see [ADF92].

Now since infinite measure-preserving transformations can have a geometrical significance, related to fractal sets, one might wonder whether there might be examples of a *transition* from to infinite measure, based on this point of view; that might be regarded as analogous to a *change of phase* in physics.

Just such a phenomenon occurs for certain maps of the interval with an indifferent (or *neutral* or *parabolic*) fixed point. Here there is a 1- parameter family of maps (see [FL01], [FL02], [FL04]), related to the Markov chain examples just discussed and also to the *Manneville-Pomeau maps* (see also [Lop93]) as well as to the interesting counterexamples of Hofbauer [Hof77], which exhibits three "phases" of behavior, as the parameter α ranges from 0 to ∞.

We consider the distribution of returns to the right half of the interval. For $\alpha \in (2, \infty)$, the mean and variance are finite; for $\alpha \in (1, 2)$, variance is infinite but mean is finite, while for $\alpha \in (0, 1)$, both are infinite. For $(1, 2]$ and $(2, \infty)$ the unique absolutely continuous invariant measure for the map is finite; for $(0, 1]$ it is infinite. The asymptotic return-time behavior for $[2, \infty)$ is Gaussian; for

$(1, 2]$ it is stable, passing through all the completely asymmetric stable laws; for $(0, 1)$ it continues on, through all the Mittag-Leffler processes.

This last region (infinite measure) is the realm of fractal-like return times. The Mittag-Leffler paths are similar to Cantor functions. The increment flow along these processes has infinite measure, and is a horocycle flow for the corresponding scaling flow (which is the "geodesic flow"). For all parameters $\alpha \neq 1$ the scaling flow on paths is infinite entropy Bernoulli (the case $\alpha = 1$ is handled in a special way as it has an extra drift parameter). This example completes a circle of ideas begun in [Fis92] and [ADF92]. For extensive background on infinite-measure ergodic theory see [Aar97] and also [Zwe95].

13. Spaces of tilings

Fractal sets are but one of the geometric forms with an interesting small-scale structure. Other examples are tilings of $\mathbb{R}^n$ which have some sort of self-similar nesting character.

The simplest example is the binary tiling of the unit interval; at level $n \geq 0$, there are 2^{n+1} tiles given by the intervals of the form $[k \cdot 2^{-(n+1)}, (k+1) \cdot 2^{-(n+1)}]$. This is a space of nested tilings with a hierarchical structure much like that of a fractal set. Indeed, the tiling structure is generated by the dynamics of the map $f : x \mapsto 2x(\mod 1)$ and can be thought of as a "cookie cutter Cantor set without the gaps". The space of nested tilings is topologized in a natural way related to the Hausdorff metric on the one-point compactification of $\mathbb{R}$, and the scenery flow for this space of tilings is modelled by the height $\log 2$ suspension flow over the natural extension of f, discussed before, i.e. by Sullivan's Riemann surface lamination.

Now the binary tilings are the joins of pullbacks of the standard Markov partition $\mathcal{P} = \{P_0, P_1\} = \{[0, 1/2], [1/2, 1]\}$ for this map. This suggests that Markov partitions give interesting candidates for a scenery flow. A next example to consider is that of an Anosov toral automorphism, such as $\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$. Here we will have two scenery flows, one for the stable and one for the unstable foliation. The resulting model scenery flow (for the unstable case) is the height $\log |Df^u|$ suspension of the Anosov diffeomorphism; the horocycle flows are the translations of the tilings, and this is modelled by the unstable flow of the suspension.

14. Markov partitions and number systems

We have seen how to model the scenery flow of some tilings of the line, associated to the doubling map of the circle and Anosov automorphisms of the two-torus; but what about tilings of the plane, such as the Penrose, Rauzy or dragon tilings? A nice class of examples are fractal tilings associated to "complex number systems"; we describe what happens for one of the best-known examples, given by the base $b = (-1 + i)$: the famous *twin dragon* tiling studied by Dekking, Gardner, Knuth, Gilbert, and others [Gil82], [Dek82] . Write $\Sigma = \Pi_{-\infty}^{\infty}\{0, 1\}$ and consider the subset Σ_0 such that for $\underline{a} = (\dots a_{-1}a_0a_1 \dots)$ there exists $M \in \mathbb{Z}$ such that $a_i = 0$ for all $i \leq M$. Then as is well-known, every $z \in \mathbb{C}$ has at least one expansion

$$z = \sum_{i=-\infty}^{\infty} a_i b^{-i}.$$

The analogue of the unit interval (for binary expansions of the real numbers) is the subset F of the plane such that $M = 0$, so $z = \sum_{i=1}^{\infty} a_i b^{-i}$. This is the **twin dragon fractal**, see e.g. [Gil82]. It is the union of two sets, $F = F_0 \cup F_1$, those points with $a_1 = 0$ and 1 respectively.

These are similar to F and are translates of each other, since $F_0 = b^{-1}F$ and $F_1 = b^{-1}F + b^{-1}$. The set F is a fundamental domain for the lattice $\mathbb{Z}+i\mathbb{Z} \simeq \mathbb{Z}+\mathbb{Z}$. The map $\mathbb{Z} \mapsto b \cdot z$ multiplies area by the factor 2, and sends this lattice into itself. Hence this map induces a transformation of the torus, $A : \mathbb{T} \to \mathbb{T}$ for $\mathbb{T} \equiv \mathbb{C}/(\mathbb{Z} \oplus i\mathbb{Z}) \simeq \mathbb{R}^2/(\mathbb{Z} \oplus \mathbb{Z})$, which is a two -to - one toral endomorphism.

To an ergodic theorist, the tiling arises in this way: $\mathcal{P} = \{F_0, F_1\}$ is a generating Markov partition for A. Indeed, a theorem of Bowen [Bow78] (see also [Caw91]) says that a Markov partition for an $n-$ toral automorphism for $n \geq 3$, or toral endomorphism for $n \geq 2$, cannot have a smooth boundary. However it *can* still be geometrically nice (and fractal!): there are examples of toral endo- and automorphisms with fractal Markov partition boundaries [Bed86]. For related work see Dekking [Dek82] (Bedford's construction made use of Dekking's approach) and also [Pra99], [KV98], [Sie00] and [Man02].

This dynamical point of view is useful for us, since we now want to build the scenery flow of the pair (map, partition), i.e. of the hierarchy of nested tilings generated by pullbacks of the Markov partition. And here we are guided by the example of the doubling map of the circle: the map A is a sort of doubling map of the torus!

Our model flow is therefore simply this: the height $\log |DA| = \log |b| = 1/2 \log 2$ suspension flow over the natural extension of the map A. This extension is coded by the full shift Σ, rather than Σ_0, and we would like to see a geometric interpretation of that larger space. It will help the explanation to define an equivalence relation on Σ: points $\underline{x}$ and $\underline{y}$ are **past equivalent** iff there exists $M \in \mathbb{Z}$ such that $x_i = y_i$ for all $i \leq M$. Fixing a point $\underline{x} \in \Sigma$, we will map its equivalence class $\langle \underline{x} \rangle$ onto the complex plane as follows. All points of the form $(\dots x_{-1} x_0 a_1 a_2 \dots)$ map to $\sum_{i=1}^{\infty} a_i b^{-i} \in F$. Now we extend this labelling to other regions: the points of the form $(\dots x_{-1} \tilde{x}_0 a_1 a_2 \dots)$ for $\tilde{x}_0 = (x_0 + 1)($ mod 2) are in a copy of F shifted by $1 = b^0$ or -1 appropriately (the first if $x_0 = 0$, the second if $x_0 = 1$). Continuing in this fashion, translating at stage n by b^n or $-b^n$, we tile the plane with tilings of each level in the hierarchy.

Dynamically, all the $\mathbb{R}^2$-translates of this tiled plane give its unstable leaf in the scenery flow. The translates are identified by choice of a point in this plane; that is, by choosing some other $\underline{y} \in \langle \underline{x} \rangle$. Now we see the significance of the past and future digits of $\underline{x}$: the digits $\leq k$ determine the large scale structure of the tiling, while those $\geq k$ specify the small-scale structure, and equivalently, determine the exact location of a point in the leaf.

The flow space is locally (disk $\times$ Cantor set $\times$ interval), the Cantor set coming from the choice of past digits in Σ. So this is locally ($\mathbb{R}^3 \times$ Cantor set), and can be given a natural hyperbolic structure; i.e. it is another example of a hyperbolic 3- manifold lamination in the sense of Minsky and Lyubich, see §11.

Note that Abramov's formula now gives flow entropy 2, the dimension of the plane, as we would expect.

15. Changing combinatorics

Another class of interesting one-dimensional tilings are those given by the renormalization hierarchy of an interval exchange transformation. If the transformation has "periodic combinatorics", in other words, if we come back to exactly the same transformation after a finite number of renormalizations (an induction on certain subintervals, called *Rauzy induction*), then the interval exchange can be realized as the return map of the stable (horocycle) flow of a pseudo-Anosov map of a surface. In this case, the scenery flow for the nested tiling is modelled much as above, by a suspension flow over this pseudo-Anosov map. But what should one do for the general case of "changing combinatorics"?

First answer: one can replace the single pseudo-Anosov map by a sequence of maps along a sequence of spaces. This defines a new type of dynamical system, a **mapping family**. See [AF03] for a development of the basic theory and an in-depth study of the simplest example (exchanges of two intervals).

Second answer: consider all such exchanges at once (with the number of intervals fixed); now the model is a suspension flow over a skew product transformation, the base of the skew product being the shift on a subshift of finite type define by a Rauzy graph. This flow can be viewed in a completely geometrical way: it is the naturally defined extension of the Teichmüller flow of the surface, to the surface fiber bundle; see [AF01]. (These are the first two of a planned series of related papers; see [Fis03b] and those papers for much more information.)

This analysis builds on and ties in with the beautiful theories developed by Rauzy, Thurston, Veech, Masur, Kerkhoff, Smillie, Vershik and many others.

16. Renormalization and scenery

This last example exhibits a link between "renormalization" and a scenery flow. Call *parameter space* the collection of exchanges of k intervals; let us term **renormalization** the transformation defined on this space, given by Rauzy induction. This transformation imbeds in a flow, as a factor of a return map to a crossection, for the Teichmüller flow of a surface. That imbedding simultaneously provides an inverse for renormalization and extends it to continuous time.

Call *dynamical space* any individual interval with its exchange transformation; its scenery is a space of nested tilings of the line. Since this scenery flow extends the Teichmüller flow, the construction places parameter space and dynamical space (the interval exchanges themselves) in a single unified picture.

Following this example, we can return to the flows we started with: the geodesic and horocycle flows of a Riemann surface, and reinterpret them in this new light. Let us recall the commutation relation. It now says: *the horocycle flow is a fixed point of renormalization*, as

The geodesic flow renormalizes the horocycle flow to itself.

This may make us think of the Coullet-Tresser- Feigenbaum map, also a fixed point for a renormalization operator; but what is its scenery flow? This map f has a Cantor set attractor, and is zero-entropy and uniquely ergodic as it is in fact topologically conjugate to the odometer T on one-side shift space Σ^+. But there is a second map g, conjugate to the shift σ, which has an analytic extension, and for which the same Cantor set is a *repellor*, making it in particular a hyperbolic $\mathbb{C}^{1+\alpha}$ Cantor set. See [Fal85], [Fei88], [Ran88], and [Sul87]. Therefore we can construct the scenery flow as in §10 above, see also §19. As for all the hyperbolic Cantor sets, the scenery horocycle flow is infinite-measure uniquely ergodic, see §12. In these constructions, as in [Fis92], a key role is played by the commutation relation $g \circ f^2 = f \circ g$ between the pair of maps, a relation which reflects their conjugacy to the shift/ odometer pair. This is a discrete time version of the commutation relation between geodesic and horocycle flows. And the analogy is now precise: the maps extend to flow cross-sections for the the dilation and translation scenery flows, and this pair of flows satisfies exactly that relation.

17. Space-filling curves

We remark that the scenery flow of a space-filling curve can be studied by combining some of the above ideas: the flow is a scaling flow on a path space, and this space is combinatorially related to the Markov partition example given above. Moreover, some nonstationary examples can be brought into play with the ideas regarding changing combinatorics. See [AF] for some more detailed studies in this direction.

18. Self-affine fractals

Here the scaling should be done by a one- parameter affine group. In fact an example has already been given above, the scaling flow on path space: e.g. the scaling flow of Brownian motion acts affinely on the graphs $(t, B(t))$ as subsets of $\mathbb{R}^2$, via the matrices $\begin{bmatrix} e^t & 0 \\ 0 & e^{\frac{t}{2}} \end{bmatrix}$.

19. Nonlinear scenery and smooth classification

When a fractal subset of the real line carries a smooth dynamics, e.g. when it is the attractor or repellor of a differentiable map, then a smooth change of coordinates gives a conjugacy between the two dynamical systems. Hence studying smooth modifications of the set is related to classifying dynamics up to smooth conjugacy. A general principle is to study first topological conjugacy, and then the finer relation of smooth conjugacy within each topological equivalence class. Knowing a smooth equivalence class is then equivalent to having an *invariant differentiable structure* for the topological map. So one is interested in classifying such differentiable structures, and at a higher level, of grouping them all together in an appropriate space. Ideally, the smooth invariants should have a nice form (perhaps as points in a subset of a function space) which make this collection easy to study. Then one would like to define some natural closure or completion, resulting in a sort of *Teichmüller space* for the space of maps.

This philosophy has been promoted and developed especially by Sullivan; originally he needed this as part of his machinery for giving a "conceptual proof" of convergence to the Feigenbaum renormalization fixed point [Sul92], [dMvS93].

Now for a fractal subset of $\mathbb{R}$, it is clear that the scenery flow is not affected by applying a smooth change of coordinates.

What we wish to sketch here is how two other, quite well-behaved invariants of smooth conjugacy are related to this flow. These are the scaling function and the invariant Gibbs measure.

In [BF96], we constructed the scenery flow of a hyperbolic $\mathbb{C}^{1+\alpha}$ Cantor set first for periodic points and then extending to general points. We now describe two other constructions for this flow. First, the model scenery flow can be constructed by the same limiting procedure as sketched for Julia sets in §10. This results in a coding by the height $\log |Df|$ suspension flow over the natural extension of the Bernoulli shift map (Σ, σ, μ) (where μ is the Gibbs measure for the potential $\varphi = -d \log |Df|$). A second way begins with a Hölder continuous *scaling function* F of Feigenbaum and Sullivan. This is a function is defined on the past and present of the shift space Σ, that is, in the digits $(\dots x_{-2}x_{-1}x_0)$, and tells the ratio of the subinterval with that label to the next larger interval labelled $(\dots x_{-2}x_{-1})$. The scenery is constructed by building up these intervals inductively in a way entirely analogous to the procedure of [Fis92]. Now the model scenery flow will have a different return time function. The natural cross-section to take is when the subinterval at a given level has length 1; this means that the flow return time will be $\log F$. Note that these functions are completely different; in particular, $\log |Df|$ is defined on the future digits while $\log F$ is defined on the past.

The extended Cantor sets (that is, extended to all of $\mathbb{R}$) built in both manners are identical, up to a change of scale, as both the scaling function and limiting scenes are produced by a similar limiting procedure, based on bounded distortion.

This change of scale is important, as it explains the exact connection between $\log |Df|$ and $\log F$. The scale change expresses the natural flow isomorphism between the two suspension flows, and amounts to a change of the cross-section; in a different language, this gives the relation of *cohomology* between these two functions. For a different proof of this see [BF96].

To show the complete equivalence of these two ideas, we need to know that an extended Cantor set built up inductively in this way from a Hölder scaling funtion in fact comes from a hyperbolic $\mathbb{C}^{1+\alpha}$ map of a Cantor set in the unit interval. Here we use a nice lemma of Sullivan [Sul87], see also [PT96], that cutting down to a window, each of the Cantor subsets we see which form part of such an extended set, is in fact a $\mathbb{C}^{1+\alpha}$ hyperbolic Cantor set with respect to the natural shift dynamics. This shows that indeed there is a one-to-one correspondence between smooth equivalences classes of maps and Hölder scaling functions.

Once we know this fact, we get something stronger: as shown in [BF97], these particular hyperbolic Cantor sets which occur in the asymptotically infinitely small scale are locally the smoothest possible representatives of their C^1 conjugacy class. This leads to a sort of rigidity theorem, see that paper for a precise statement.

All of the above material was worked out together with Tim Bedford.

A different but closely related example of this philosophy comes from the smooth classification of doubling maps.

Let $f : S^1 \to S^1$ be a degree 2 hyperbolic $\mathbb{C}^{1+\alpha}$ map of the circle. Using the same machinery just discussed for cookie-cutter Cantor sets, one constructs the scaling function and scenery flow.

The scaling function now has an interesting new interpretation: it is the conditional Gibbs measure on that leaf of the solenoid. Better explained: a solenoid leaf is a past equivalence class; the scaling function F is defined on the past, and since relative lengths of subintervals now add up to 1, knowledge of these lengths, at all future (sub)scales, defines a relative measure. Indeed, that is equivalent to such knowledge, and one can prove that this measure is the Gibbs state conditioned on the past. In other words, $\log F$ a *transition function*: what Keane calls the g-function of a g-measure [Kea72]. In this way the function gains a completely geometrical interpretation, as a scaling function for a doubling map. (We mention that this was noticed independently by Misha Lyubich, personal conversation).

Now we return to the notion of a Teichmüller theory for these maps, as sought by Sullivan. For the doubling maps, as Sullivan discovered, this connection is not merely allegorical: there is a classical (or nearly so!) Teichmüller space floating around somewhere. As Sullivan noted (see [Sul92]) there is a correspondence between conformal structures on the Riemann surface lamination and scaling functions. See Theorem VI.6.1 of [dMvS93] for a proof in the analytic case.

Here the scenery flow picture helps to make this conformal structure more easily understood: for it is just the natural conformal structure on the height $\log |Df|$ suspension flow over the solenoid.

What is remarkable is that there is a two-sided correspondence, which has been worked out by Alberto Pinto and Sullivan (preprint in progress).

Now by a theorem of Shub [Shu85], [Nit71], a nonlinear expanding map f is topologically conjugate to the linear map of the same degree, in this case to the doubling map $x \mapsto 2x(\mod 1)$. So here one has a finer classification, in the C^1 category: a smooth equivalence class of hyperbolic $C^{1+\alpha}$ maps is determined by the topological data (degree) plus any one of these:

– a Gibbs measure class

–a Hölder scaling function

– a scenery flow with Hölder return time.

Pinto and Sullivan add "a conformal structure on the RSL" plus conditions, which I don't understand well enough to quote here, extending the whole setting as far as possible. This leads them to some delicate and beautiful analysis. In this way they are able to come up with an appropriate notion of Teichmüller space for degree d maps.

Here is one way of viewing all this. Given a hyperbolic doubling map, Cantor set or Julia set, the construction of the scenery is a sort of linearization procedure – and the space of scenes is a tangent object, the analogue of a tangent space. As such, it is acted on (linearly) by the derivative of the map. This gives the return map to a cross-section of the scenery flow. The flow itself is simply dilation, and is linear as well. So to where has the all nonlinearity of the original map disappeared? Answer: it is coded into the flow space, by means of the identifications made when defining that space; and that information is, in turn, remembered by the conformal structure of the Riemann surface lamination.

We have already discussed scenery for a linear Anosov toral diffeomorphism f. For the nonlinear case, again we will have two scenery flows, one for the stable and one for the unstable foliation.

The topological classification of such maps f parallels that for expanding circle maps; by theorems of Franks and Manning [Fra69], [Fra70], [Man74] f is topologically conjugate to a linear model, the action on homology. The smooth classification now proceeds as before, and again leads to Teichmüller theory; see [Caw93] which completes a classification begun in [MM87], [dlL87]. All of this work, in turn, extends to the case of changing combinatorics. In this way, one studies certain nonlinear circle diffeomorphisms by means of a "Teichmüller space over Teichmüller space", of maps, and the torus, respectively. (Work with Arnoux, in progress; see [Fis03b]).

Smooth classification in the higher dimensional nonlinear case is a great deal more difficult, and much remains to be done.

20. Non-homogeneous fractals

In §8 of [BF96] we defined a closed subset of $\mathbb{R}^n$ to be a *spatially homogeneous fractal* iff the scenery flow is nontrivial and is the same space, for every point in a dense G_δ. This is true for all the examples discussed so far.

There are however many interesting fractal sets which are non-homogeneous in that they are "different" at every point; a precise way of saying this might be: the scenery flow exists at every point but is different everywhere. One example is this: consider the "theater curtain" pictures in [Man82], which interpolate middle- interval Cantor sets from the whole unit interval to the empty set. Now slice along a diagonal. The resulting Cantor set has asymptotically the same scenery flow as the corresponding horizontal slice at that point. There is a natural generalization of Hausdorff measure, given by a weak limit; its pointwise dimension [LY85] and order-two (average) density should accompany this change, the latter selecting a Cantor subset of the values of the graph depicted on p. 107 of [Fal97], see Patzschke and Zähle [PZ93].

Another beautiful class of examples of fractal sets which are both non-homogeneous and locally self-affine was discussed in this conference by Falconer; it is interesting to speculate on how the scenery flow varies from point to point.

But the holy grail of non-homogeneous examples is without much doubt the Mandelbrot set boundary M. As Tan Lei showed [Tan90], at certain points of M the scenery flow can be precisely analysed: for a Misiurewicz point c, the scenery flow of M at $c \in M$ is that of the Julia set $\mathcal{J}_c$ at c for the map $z \mapsto z^2 + c$. However what happens at this set of points, even though it is a countable dense subset of M, says next to nothing about the nearby points. The limiting scenery there is anybody's guess and seems to be a deep and difficult problem. (Warning: to have a meaningful result, it is probably necessary to modify the mode of convergence to the scenery flow: perhaps throwing out a set of times of density zero?) But in any case, without any doubt, there is a lot of exciting exploring yet to be done at the small scales of these and other related fractal landscapes.

21. Acknowledgements

The writing of this paper was spurred by several events: first, the invitation to this conference, and to contribute an article, for which I thank the organizers warmly. The interest of colleagues both newly and re-encountered encouraged me to put this all down on paper. Second was the recent completion and availablity in print of several of the important building blocks of this theory, [BFU02], [FL02], [AF03] which made me feel all right about presenting such an informal treatment. The third event was the occasion of writing my *livre docencia* thesis at the University of São Paulo, which caused me to look back over years of work and pull it all together. That text contains expository and background material on how the ideas developed, as well as more detailed statements and some further references; it is available in Portuguese and English on my web page or by mail upon request [Fis03b].

I wish especially to thank coworkers on these and related projects: Jon Aaronson, Pierre Arnoux, Tim Bedford, Marc Burger, Herold Dehling, Manfred Denker, Sebastien Ferenczi, Pascal Hubert, Artur Lopes, Jerome Los, Yair Minsky, Luis Fernando Carvalho da Rocha and Mariusz Urbański.

Figures 1, 3, 4 were made using McMullen's *lim* program; Tony Phillips helped with the other graphics.

There are many more people who should be acknowledged here, but see [Fis03b], where there is more space and an appropriate setting for such mention.

I only wish to say here that without the encouragement and direct and indirect participation of these friends, teachers, colleagues and above all coauthors, most of this would never have been worked out and in any case the process would not have been nearly so much fun.

It is obvious that this project- like mathematics in general- builds on and is intimately related to the work of many many people, and to a large extent this research has been my attempt to understand some small part of that body of beautiful work.

But I wish to mention specifically two ideas closely connected to that of the scenery flow, unfortunately not discussed here, and independently arrived at: Furstenberg's idea of *microsets* (unpublished lectures) and Preiss' *tangent measures*. No doubt many connections between these and other related ideas are waiting to be made!

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