A POISSON FORMULA FOR HARMONIC PROJECTIONS

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Abstract. — For an arbitrary Markov operator P on a Lebesgue measure space (X, m) we construct a projection $\mathcal{S}$ (called harmonic) from $L^\infty(X, m)$ onto the space of bounded P-harmonic functions $H^\infty(X, m, P)$. The projection $\mathcal{S} = \mathcal{S}_\lambda$ is obtained by applying a fixed measure-linear (medial) mean λ on $\mathbb{Z}_+$ to the sequence of one-dimensional distributions of the Markov measure in the path space of the Markov chain $\{x_n\}$ associated with the operator P. If there are no non-constant bounded P-harmonic functions, $\mathcal{S}$ is a projection onto the space of constants.

In the general situation when the space $H^\infty(X, m, P)$ is not necessarily trivial, the harmonic projection $\mathcal{S}f$ can still be considered as a "space average" of f. We show that for any $f \in L^\infty(X, m)$ the harmonic projection $\mathcal{S}f$ can be recovered from the averages of f (determined by the mean λ) along sample paths of the Markov chain $\{x_n\}$ by an integral Poisson formula in the same way as any bounded harmonic function is represented by the Poisson integral of its boundary values. In other words, for any $f \in L^\infty(X, m)$ its space average $\mathcal{S}f$ is the Poisson integral of the time averages along sample paths.

Key words: Measure-linear mean, finitely additive measure, Markov operator, Poisson boundary, harmonic function, Poisson formula.

Résumé. — **Une formule de Poisson pour les projections harmoniques.** Pour un opérateur de Markov P sur un espace de Lebesgue (X, m) nous construisons une projection $\mathcal{S}$ (appelée harmonique) de l'espace $L^\infty(X, m)$ sur l'espace $H^\infty(X, m, P)$ des fonctions P-harmoniques bornées. La projection $\mathcal{S} = \mathcal{S}_\lambda$ est déterminée par une moyenne médiale λ sur $\mathbb{Z}_+$ appliquée à la suite des distributions marginales de la mesure de Markov dans l'espace des chemins de la chaîne de Markov $\{x_n\}$ associée à l'opérateur P. S'il n'y a pas de fonctions P-harmoniques bornées non-constantes, $\mathcal{S}$ est une projection sur l'espace des constantes.

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Dans la situation générale, quand l'espace $H^\infty(X,m,P)$ n'est pas nécessairement trivial, la projection harmonique $\mathcal{S}f$ peut être toujours considérée comme une "moyenne spatiale" de f. Nous montrons que pour chaque $f\in L^\infty(X,m)$ la projection harmonique $\mathcal{S}f$ peut être récupérée par une formule intégrale de Poisson à partir des moyennes de f (déterminées eux aussi par la limite médiale λ) sur les chemins de la chaîne $\{x_n\}$ de la même manière qu'une fonction harmonique bornée peut être représentée par l'intégrale de Poisson de ses valeurs limites. Autrement dit, pour chaque $f\in L^\infty(X,m)$ la moyenne spatiale $\mathcal{S}f$ est l'intégrale de Poisson des moyennes temporelles.

1. Introduction

A *mean* on a measure space (X,m) is a positive normalized (hence, continuous) linear functional on the space $L^\infty(X,m)$, or, equivalently, a *finitely additive* probability measure on X absolutely continuous with respect to m. In particular, any usual absolutely continuous σ-additive probability measure on X determines a mean. However, the space $L^\infty(X,m)^*$ being much larger than $L^1(X,m)$, there are means which do not correspond to any σ-additive measure.

Obviously, there are no shift-invariant probability measures on $\mathbb{Z}_+$, but there are *invariant means* λ on $\mathbb{Z}_+$, i.e., such that

$$\lambda\{a_i\}=\lambda\{a_{i+1}\}\qquad\forall\,\boldsymbol{a}=\{a_i\}\in l^\infty(\mathbb{Z}_+)\;. \tag{1}$$

Mokobodzki (see [Me], [Fi]) proved that there exists an invariant mean λ on $\mathbb{Z}_+$ (called *measure-linear* or *medial*) with the following remarkable property: λ is *universally measurable* as a map from the product space $[-1,1]^{\mathbb{Z}_+}$ to $[-1,1]$, i.e., the integral in the R.H.S. below is well defined for any Borel probability measure μ on $[-1,1]^{\mathbb{Z}_+}$, and

$$\lambda\left\{\int\boldsymbol{a}\,d\mu(\boldsymbol{a})\right\}=\int\lambda\{\boldsymbol{a}\}\,d\mu(\boldsymbol{a})\;. \tag{2}$$

Property (2) can be considered as a "finitely additive" counterpart of the Fubini theorem. As an illustration, let us deduce from (2) a measure-linear analogue of the Birkhoff ergodic theorem [Fi]. Fix a measure-linear mean λ on $\mathbb{Z}_+$, and let T be an ergodic measure preserving transformation of a probability measure space (X,m). Assign to any function $f\in L^\infty(X,m)$ its *time averages* $\mathcal{T}f(x)=\lambda\{f(T^nx)\}$. Then the function $\mathcal{T}f$ is measurable and T-invariant, hence, a.e. constant by ergodicity of T. Thus, $\mathcal{T}$ is a *projection* of $L^\infty(X,m)$ onto the space of constants. Moreover, by (2)

$$\mathcal{T}f=\int\mathcal{T}f(x)\,dm(x)=\int\lambda\{f(T^nx)\}\,dm(x)=\lambda\left\{\int f(T^nx)\,dm(x)\right\}=\mathcal{S}f\;, \tag{3}$$

where $\mathcal{S}f=\int f(x)\,dm(x)$ is the *space average* of f with respect to the measure m, so that time averaging and space averaging lead to the same result.

The aim of this note is to find an analogue of formula (3) in the situation when the measure m is not necessarily finite, and the additional "space-time" structure on

(X, m) is provided by a Markov operator P rather than by a measure preserving transformation. This is a much weaker assumption as Markov operators naturally arise from various geometric structures on groups, Riemannian manifolds, locally finite graphs, foliations, equivalence relations, etc. (e.g., see [Ka2]). Of course, any measure type preserving transformation of the space (X, m) can be considered as a deterministic Markov operator.

For defining the "space average" $\mathcal{S}f$ we replace integrating with respect to m by averaging with respect to the one-dimensional distributions of the Markov chain on X, which gives a projection of $L^\infty(X, m)$ onto the space of bounded P-harmonic functions (we call $\mathcal{S}$ the *harmonic projection*). If there are no non-constant bounded harmonic functions, then $\mathcal{S}$ is a projection onto the space of constants, i.e., a *harmonic mean* on $L^\infty(X, m)$. The harmonic mean respects the same structures (measurabilty, group invariance) as the operator P, which was used for proving *amenability* of Riemannian foliations and deck transformations groups in [CFW] and [LS], respectively.

We consider a more general situation, when the space of bounded harmonic functions of the operator P is not necessarily trivial, which means that the space (X, m) is "large enough" to provide the sample paths of the Markov chain with a non-trivial behaviour at infinity. In this case the time averages $\mathcal{T}f$ obtained by averaging f along sample paths of the Markov chain on X define a function on the *Poisson boundary* of the operator P, and the harmonic projection $\mathcal{S}f$ can be recovered from $\mathcal{T}f$ by the integral *Poisson formula* in the same way as any bounded harmonic function is represented as the Poisson integral of its boundary values. For example, if (X, m) is the hyperbolic plane, and P is the Markov operator of the Brownian motion, then the Poisson boundary coincides with the circle at infinity; see [Ka2] for other examples of Markov operators with a non-trivial Poisson boundary.

2. Markov operators and the Poisson formula

Let (X, m) be a Lebesgue space with a σ-finite measure m, and $P : L^\infty(X, m) \hookleftarrow$ be a *Markov operator* (e.g., see [Fo], [Re] for a definition). The operator P can be presented as

$$Pf(x) = \langle f, \pi_x \rangle , \tag{4}$$

where π_x is the measurable family of *transition probabilities* of the operator P. In order to avoid technical details we assume for a moment that almost all transition probabilities π_x are absolutely continuous with respect to m, i.e., $Pf(x) = \int f(y)p(x, y)\, dm(y)$, where $p(x, y) = d\pi_x/dm(y)$.

Let $X^{\mathbb{Z}_+} = \{\boldsymbol{x} = (x_0, x_1, x_2, \dots)\}$ be the *path space* of the associated Markov chain on X, and $\mathbf{P}_\theta$ be the Markov measure in the path space corresponding to an initial distribution θ on X (we shall also use the notation $\mathbf{P}_x$ if $\theta = \delta_x$ is just the delta-measure at a point $x \in X$). Then the one-dimensional distribution of the measure $\mathbf{P}_\theta$ at a time $n \geq 0$ is θP^n, where $\theta \mapsto \theta P$ is the adjoint operator of P acting in the space of measures on X. Powers of the operator P can be presented in terms of the measures $\mathbf{P}_x$ as

$$P^n f(x) = \int f(x_n)\, d\mathbf{P}_x(\boldsymbol{x}) \ . \tag{5}$$

The *Poisson boundary* ∂P of the operator P is defined as the space of ergodic components of the time shift in the path space $(X^{\mathbb{Z}_+}, \mathbf{P}_m)$, so that there exists a canonical map $\mathbf{bnd} : X^{\mathbb{Z}_+} \to \partial P$. The measure type $[\nu]$ on ∂P, which is the image of the type of the measure $\mathbf{P}_m$, is called the *harmonic measure type*. For any initial probability distribution θ on X the measure

$$\nu_\theta = \mathbf{bnd}\,(\mathbf{P}_\theta) \tag{6}$$

is called the *harmonic measure* corresponding to θ [Ka1].

The ergodic decomposition of the measure $\mathbf{P}_m$ with respect to the time shift in the path space gives rise to the family of *conditional measures* $\mathbf{P}_m^\gamma$ indexed by the points $\gamma \in \partial P$. These measures are Markov measures corresponding to the *conditional Markov operators* P^γ whose transition densities are $p^\gamma(x,y) = p(x,y) d\nu_y / d\nu_x(\gamma)$.

A function $f \in L^\infty(X,m)$ is called P*-harmonic* if $Pf = f$. The space of bounded harmonic functions $H^\infty(X,m,P) \subset L^\infty(X,m)$ is isometric to the space of bounded measurable functions $L^\infty(\partial P, [\nu])$ on the Poisson boundary. Namely, for any function $f \in H^\infty(X,m,P)$ there exists a limit $F(\mathbf{bnd}\,\boldsymbol{x}) = \lim f(x_n)$ along a.e. sample path $\boldsymbol{x} = (x_n)$, and, conversely, the harmonic function f can be recovered from its boundary values F by the *Poisson formula*

$$f(x) = \langle F, \nu_x \rangle = \mathcal{P}F(x) \ , \tag{7}$$

where $\mathcal{P} : L^\infty(\partial P, [\nu]) \to H^\infty(X,m,P)$ is the *Poisson operator* [Ka1].

All these constructions carry over to the situation when the Markov operator P does not have a kernel, i.e., its transition probabilities are not absolutely continuous with respect to the reference measure m. In this case transition probabilities π_x, harmonic measures ν_x, and conditional Markov operators P^γ are defined in terms of canonical systems of conditional measures in Lebesgue spaces [Ka1].

3. The main results

From now on we shall fix a Markov operator $P : L^\infty(X,m) \hookleftarrow$ on a Lebesgue space (X,m) and a measure-linear mean λ on $\mathbb{Z}_+$.

Theorem 1. *The map $\mathcal{S} = \mathcal{S}_\lambda$ defined as*

$$\mathcal{S}f = \lambda\{P^n f\} \tag{8}$$

is a positive norm 1 projection from $L^\infty(X,m)$ onto $H^\infty(X,m,P)$.

Proof. The fact that $\mathcal{S}f$ is a measurable function follows from universal measurability of λ. Clearly, the operator $\mathcal{S}$ is linear and it preserves harmonic functions (since it is positive, it is also immediately clear that it has norm 1 in $L^\infty(X,m)$). It remains to check that $\mathcal{S}f$ is harmonic. Indeed,

$$\begin{aligned} P\mathcal{S}f(x) &\overset{(4)}{=} \int \mathcal{S}f(y)\, d\pi_x(y) \overset{(8)}{=} \int \lambda\{P^n f\}(y)\, d\pi_x(y) \\ &\overset{(2)}{=} \lambda\left\{ \int P^n f(y)\, d\pi_x(y) \right\} \overset{(4)}{=} \lambda\{P^{n+1} f(x)\} \\ &\overset{(1)}{=} \lambda\{P^n(x)\} \overset{(8)}{=} \mathcal{S}f(x) \end{aligned}$$

□

Below we shall refer to $\mathcal{S} = \mathcal{S}_\lambda$ as the *harmonic projection* determined by the measure-linear mean λ. Since $P^n f(x) = \langle f, \delta_x P^n \rangle$, the operator $\mathcal{S}$ can be interpreted as asymptotic "space averaging" of the function f with respect to the n-step transition probabilities $\delta_x P^n$ of the operator P. If there are no non-constant bounded harmonic functions, i.e., if the Poisson boundary ∂P is trivial, $\mathcal{S}$ is a projection of $L^\infty(X,m)$ onto the space of constants, i.e., a mean on (X,m) called *harmonic*. Note that according to a zero-two law for Markov operators [Ka1, Theorem 2.3] triviality of the Poisson boundary is equivalent to independence of the states $\lambda\{\theta P^n\}$ of the choice of an initial distribution θ on X.

Remark. The construction of the harmonic projection in [LS] used continuity of the transition probabilities of the operator P. However, measure-linearity of λ allows us to get rid of this assumption and to construct the harmonic projection for an arbitrary Markov operator. Continuous time Markov processes (e.g., the Brownian motion) can be dealt with in the same way by using measure-linear invariant means on $\mathbb{R}_+$, cf. [Fi].

Applying λ to the values of a function $f \in L^\infty(X,m)$ along sample paths of the Markov chain on X we obtain a measurable function $\lambda\{f(x_n)\}$ on the path space $(X^{\mathbb{Z}_+}, \mathbf{P}_m)$. Since λ is an invariant mean, this function is invariant with respect to the shift T in the path space, so that it gives rise to a function

$$\mathcal{T} f(\mathbf{bnd}\,\boldsymbol{x}) = \lambda\{f(x_n)\} \tag{9}$$

on the Poisson boundary $(\partial P, [\nu])$ of the operator P. The operator $\mathcal{T} : L^\infty(X,m) \to L^\infty(\partial P, [\nu])$ corresponds to taking "time averages" of the function f along the sample paths of the Markov chain on X.

Theorem 2. *For any function $f \in L^\infty(X,m)$ its harmonic projection ("space average") $\mathcal{S}f$ is obtained by the Poisson formula from its "time averages" $\mathcal{T}f$, i.e., the diagram*

$$\begin{array}{ccc} & L^\infty(X,m) & \\ {}^{\mathcal{T}}\swarrow & & \searrow^{\mathcal{S}} \\ L^\infty(\partial P,[\nu]) & \underset{\mathcal{P}^{-1}}{\overset{\mathcal{P}}{\rightleftarrows}} & H^\infty(X,m,P) \end{array}$$

is commutative.

Proof.

$$\begin{aligned} \mathcal{S}f(x) &\overset{(8)}{=} \lambda\{P^n f(x)\} \overset{(5)}{=} \lambda\left\{\int f(x_n)\, d\mathbf{P}_x(\boldsymbol{x})\right\} \\ &\overset{(2)}{=} \int \lambda\{f(x_n)\}\, d\mathbf{P}_x(\boldsymbol{x})\} \overset{(9)}{=} \int \mathcal{T}f(\mathbf{bnd}\,\boldsymbol{x})\, d\mathbf{P}_x(\boldsymbol{x}) \overset{(6)}{=} \int \mathcal{T}f(\gamma)\, d\nu_x(\gamma) \overset{(7)}{=} \mathcal{P}\mathcal{T}f\,. \end{aligned}$$

□

Corollary. *If the Poisson boundary of the operator P is trivial, then the λ-averages of f along $\mathbf{P}_m$-a.e. sample path are the same and coincide with the space average $\mathcal{S}f$.*

The latter Corollary means that if the Poisson boundary is trivial, then the time averages along *random* sample paths coincide with the *deterministic* space averages. In fact, averages along sample paths can be recovered from space averages also in the case when the Poisson boundary is non-trivial. In order to do that, one has to apply the above Corollary to the ergodic components of the shift in the path space. By definition of the Poisson boundary as the space of ergodic components, all conditional measures $\mathbf{P}_m^\gamma$ are ergodic with respect to the time shift, so that the corresponding conditional Markov operators P^γ have trivial Poisson boundary. Denote by $\mathcal{S}^\gamma$ the corresponding conditional harmonic means. Then by the Corollary applied to the conditional measures we obtain

Theorem 3. *For any function $f \in L^\infty(X, m)$*

$$\mathcal{T}f(\gamma) = \mathcal{S}^\gamma f \ .$$

Remark. It would be interesting to find a measure-linear analogue of the Fatou theorem in the situation when the usual Fatou theorem for bounded harmonic functions holds (e.g., for Markov operators on hyperbolic spaces; see [An]), i.e., to replace the conditional harmonic means $\mathcal{S}^\gamma$ with means along "nice" sequences approaching the corresponding boundary point (geodesics in the hyperbolic case).

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