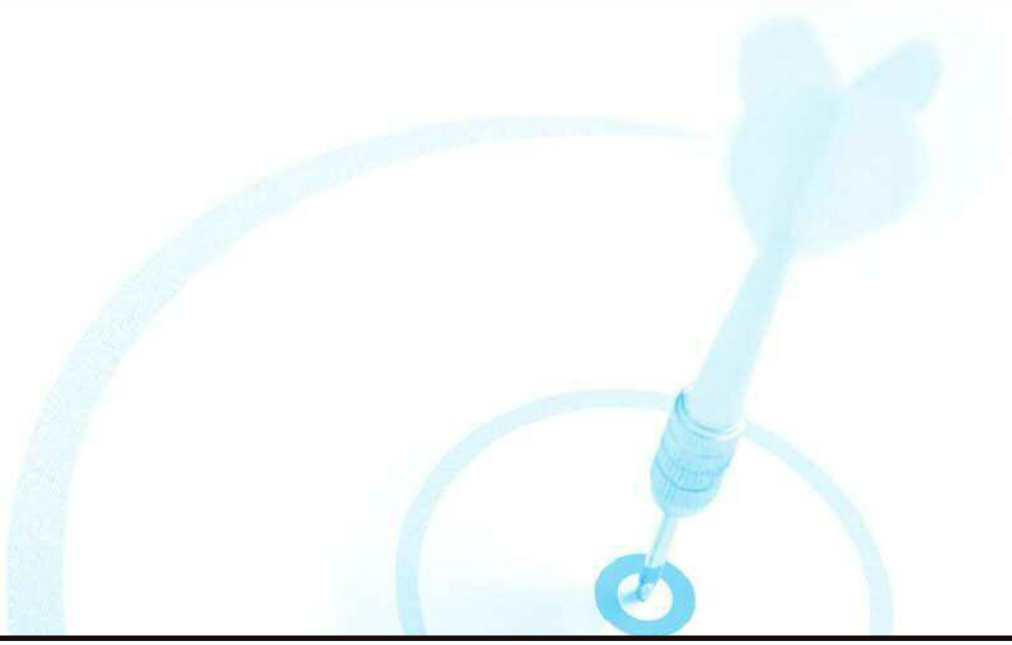
David Poole

Linear Algebra

A MODERN INTRODUCTION

4th edition



Linear Algebra

A MODERN INTRODUCTION

Fourth edition

David Poole

Trent University

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Linear Algebra
A Modern Introduction, 4th Edition
David Poole

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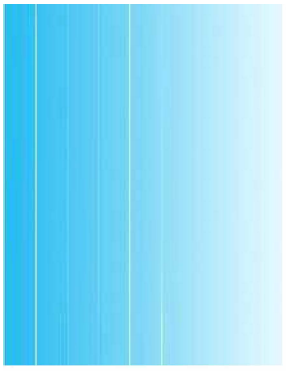

Dedicated to the memory of Peter Hilton, who was an exemplary mathematician, educator, and citizen—a unit vector in every sense.

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Preface

The last thing one knows when writing a book is what to put first.

—Blaise Pascal
Pensées, 1670

The fourth edition of *Linear Algebra: A Modern Introduction* preserves the approach and features that users found to be strengths of the previous editions. However, I have streamlined the text somewhat, added numerous clarifications, and freshened up the exercises.

I want students to see linear algebra as an exciting subject and to appreciate its tremendous usefulness. At the same time, I want to help them master the basic concepts and techniques of linear algebra that they will need in other courses, both in mathematics and in other disciplines. I also want students to appreciate the interplay of theoretical, applied, and numerical mathematics that pervades the subject.

This book is designed for use in an introductory one- or two-semester course sequence in linear algebra. First and foremost, it is intended for students, and I have tried my best to write the book so that students not only will find it readable but also will *want* to read it. As in the first three editions, I have taken into account the reality that students taking introductory linear algebra are likely to come from a variety of disciplines. In addition to mathematics majors, there are apt to be majors from engineering, physics, chemistry, computer science, biology, environmental science, geography, economics, psychology, business, and education, as well as other students taking the course as an elective or to fulfill degree requirements. Accordingly, the book balances theory and applications, is written in a conversational style yet is fully rigorous, and combines a traditional presentation with concern for student-centered learning.

There is no such thing as a universally best learning style. In any class, there will be some students who work well independently and others who work best in groups; some who prefer lecture-based learning and others who thrive in a workshop setting, doing explorations; some who enjoy algebraic manipulations, some who are adept at numerical calculations (with and without a computer), and some who exhibit strong geometric intuition. In this edition, I continue to present material in a variety of ways—*algebraically, geometrically, numerically*, and *verbally*—so that all types of learners can find a path to follow. I have also attempted to present the theoretical, computational, and applied topics in a flexible yet integrated way. In doing so, it is my hope that all students will be exposed to the many sides of linear algebra.

This book is compatible with the recommendations of the Linear Algebra Curriculum Study Group. From a pedagogical point of view, there is no doubt that for most students

For more on the recommendations of the Linear Algebra Curriculum Study Group, see *The College Mathematics Journal* **24** (1993), 41–46.

concrete examples should precede abstraction. I have taken this approach here. I also believe strongly that linear algebra is essentially about vectors and that students need to see vectors first (in a concrete setting) in order to gain some geometric insight. Moreover, introducing vectors early allows students to see how systems of linear equations arise naturally from geometric problems. Matrices then arise equally naturally as coefficient matrices of linear systems and as agents of change (linear transformations). This sets the stage for eigenvectors and orthogonal projections, both of which are best understood geometrically. The dart that appears on the cover of this book symbolizes a vector and reflects my conviction that geometric understanding should precede computational techniques.

I have tried to limit the number of theorems in the text. For the most part, results labeled as theorems either will be used later in the text or summarize preceding work. Interesting results that are not central to the book have been included as exercises or explorations. For example, the cross product of vectors is discussed only in explorations (in Chapters 1 and 4). Unlike most linear algebra textbooks, this book has no chapter on determinants. The essential results are all in Section 4.2, with other interesting material contained in an exploration. The book is, however, comprehensive for an introductory text. Wherever possible, I have included elementary and accessible proofs of theorems in order to avoid having to say, "The proof of this result is beyond the scope of this text." The result is, I hope, a work that is self-contained.

I have not been stingy with the applications: There are many more in the book than can be covered in a single course. However, it is important that students see the impressive range of problems to which linear algebra can be applied. I have included some modern material on finite linear algebra and coding theory that is not normally found in an introductory linear algebra text. There are also several impressive real-world applications of linear algebra and one item of historical, if not practical, interest; these applications are presented as self-contained "vignettes."

I hope that instructors will enjoy teaching from this book. More important, I hope that students using the book will come away with an appreciation of the beauty, power, and tremendous utility of linear algebra and that they will have fun along the way.

What's New in the Fourth Edition

The overall structure and style of *Linear Algebra: A Modern Introduction* remain the same in the fourth edition.

Here is a summary of what is new:

- The applications to coding theory have been moved to the new online Chapter 8.
- To further engage students, five writing projects have been added to the exercise sets. These projects give students a chance to research and write about aspects of the history and development of linear algebra. The explorations, vignettes, and many of the applications provide additional material for student projects.

See pages 49, 82, 283, 301, 443

- There are over 200 new or revised exercises. In response to reviewers' comments, there is now a full proof of the Cauchy-Schwarz Inequality in Chapter 1 in the form of a guided exercise.
- I have made numerous small changes in wording to improve the clarity or accuracy of the exposition. Also, several definitions have been made more explicit by giving them their own definition boxes and a few results have been highlighted by labeling them as theorems.
- All existing ancillaries have been updated.

Features

Clear Writing Style

The text is written is a simple, direct, conversational style. As much as possible, I have used "mathematical English" rather than relying excessively on mathematical notation. However, all proofs that are given are fully rigorous, and Appendix A contains an introduction to mathematical notation for those who wish to streamline their own writing. Concrete examples almost always precede theorems, which are then followed by further examples and applications. This flow—from specific to general and back again—is consistent throughout the book.

Key Concepts Introduced Early

Many students encounter difficulty in linear algebra when the course moves from the computational (solving systems of linear equations, manipulating vectors and matrices) to the theoretical (spanning sets, linear independence, subspaces, basis, and dimension). This book introduces all of the key concepts of linear algebra early, in a concrete setting, before revisiting them in full generality. Vector concepts such as dot product, length, orthogonality, and projection are first discussed in Chapter 1 in the concrete setting of $\mathbb{R}^2$ and $\mathbb{R}^3$ before the more general notions of inner product, norm, and orthogonal projection appear in Chapters 5 and 7. Similarly, spanning sets and linear independence are given a concrete treatment in Chapter 2 prior to their generalization to vector spaces in Chapter 6. The fundamental concepts of subspace, basis, and dimension appear first in Chapter 3 when the row, column, and null spaces of a matrix are introduced; it is not until Chapter 6 that these ideas are given a general treatment. In Chapter 4, eigenvalues and eigenvectors are introduced and explored for 2×2 matrices before their $n \times n$ counterparts appear. By the beginning of Chapter 4, all of the key concepts of linear algebra have been introduced, with concrete, computational examples to support them. When these ideas appear in full generality later in the book, students have had time to get used to them and, hence, are not so intimidated by them.

Emphasis on Vectors and Geometry

In keeping with the philosophy that linear algebra is primarily about vectors, this book stresses geometric intuition. Accordingly, the first chapter is about vectors, and it develops many concepts that will appear repeatedly throughout the text. Concepts such as orthogonality, projection, and linear combination are all found in Chapter 1, as is a comprehensive treatment of lines and planes in $\mathbb{R}^3$ that provides essential insight into the solution of systems of linear equations. This emphasis on vectors, geometry, and visualization is found throughout the text. Linear transformations are introduced as matrix transformations in Chapter 3, with many geometric examples, before general linear transformations are covered in Chapter 6. In Chapter 4, eigenvalues are introduced with "eigenpictures" as a visual aid. The proof of Perron's Theorem is given first heuristically and then formally, in both cases using a geometric argument. The geometry of linear dynamical systems reinforces and summarizes the material on eigenvalues and eigenvectors. In Chapter 5, orthogonal projections, orthogonal complements of subspaces, and the Gram-Schmidt Process are all presented in the concrete setting of $\mathbb{R}^3$ before being generalized to $\mathbb{R}^n$ and, in Chapter 7, to inner

product spaces. The nature of the singular value decomposition is also explained informally in Chapter 7 via a geometric argument. Of the more than 300 figures in the text, over 200 are devoted to fostering a geometric understanding of linear algebra.

Explorations

See pages 1, 136, 427, 529

See pages 32, 286, 460, 515, 543, 547

See pages 83, 84, 85, 396, 398

The introduction to each chapter is a guided exploration (Section 0) in which students are invited to discover, individually or in groups, some aspect of the upcoming chapter. For example, "The Racetrack Game" introduces vectors, "Matrices in Action" introduces matrix multiplication and linear transformations, "Fibonacci in (Vector) Space" touches on vector space concepts, and "Taxicab Geometry" sets up generalized norms and distance functions. Additional explorations found throughout the book include applications of vectors and determinants to geometry, an investigation of 3×3 magic squares, a study of symmetry via the tilings of M. C. Escher, an introduction to complex linear algebra, and optimization problems using geometric inequalities. There are also explorations that introduce important numerical considerations and the analysis of algorithms. Having students do some of these explorations is one way of encouraging them to become active learners and to give them "ownership" over a small part of the course.

Applications

See pages 623, 641

See pages 121, 226, 356, 607, 626

The book contains an abundant selection of applications chosen from a broad range of disciplines, including mathematics, computer science, physics, chemistry, engineering, biology, business, economics, psychology, geography, and sociology. Noteworthy among these is a strong treatment of coding theory, from error-detecting codes (such as International Standard Book Numbers) to sophisticated error-correcting codes (such as the Reed-Muller code that was used to transmit satellite photos from space). Additionally, there are five "vignettes" that briefly showcase some very modern applications of linear algebra: the Global Positioning System (GPS), robotics, Internet search engines, digital image compression, and the Codabar System.

Examples and Exercises

There are over 400 examples in this book, most worked in greater detail than is customary in an introductory linear algebra textbook. This level of detail is in keeping with the philosophy that students should want (and be able) to read a textbook. Accordingly, it is not intended that all of these examples be covered in class; many can be assigned for individual or group study, possibly as part of a project. Most examples have at least one counterpart exercise so that students can try out the skills covered in the example before exploring generalizations.

There are over 2000 exercises, more than in most textbooks at a similar level. Answers to most of the computational odd-numbered exercises can be found in the back of the book. Instructors will find an abundance of exercises from which to select homework assignments. The exercises in each section are graduated, progressing from the routine to the challenging. Exercises range from those intended for hand computation to those requiring the use of a calculator or computer algebra system, and from theoretical and numerical exercises to conceptual exercises. Many of the examples and exercises use actual data compiled from real-world situations. For example, there are problems on modeling the growth of caribou and seal populations, radiocarbon dating

See pages 248, 359, 526, 588

of the Stonehenge monument, and predicting major league baseball players' salaries. Working such problems reinforces the fact that linear algebra is a valuable tool for modeling real-life problems.

Additional exercises appear in the form of a review after each chapter. In each set, there are 10 true/false questions designed to test conceptual understanding, followed by 19 computational and theoretical exercises that summarize the main concepts and techniques of that chapter.

Biographical Sketches and Etymological Notes

It is important that students learn something about the history of mathematics and come to see it as a social and cultural endeavor as well as a scientific one. Accordingly, the text contains short biographical sketches about many of the mathematicians who contributed to the development of linear algebra. I hope that these will help to put a human face on the subject and give students another way of relating to the material.

I have found that many students feel alienated from mathematics because the terminology makes no sense to them—it is simply a collection of words to be learned. To help overcome this problem, I have included short etymological notes that give the origins of many of the terms used in linear algebra. (For example, why do we use the word *normal* to refer to a vector that is perpendicular to a plane?)

See page 34

Margin Icons

The margins of the book contain several icons whose purpose is to alert the reader in various ways. Calculus is not a prerequisite for this book, but linear algebra has many interesting and important applications to calculus. The $\frac{dy}{dx}$ icon denotes an example or exercise that requires calculus. (This material can be omitted if not everyone in the class has had at least one semester of calculus. Alternatively, this material can be assigned as projects.) The $a + bi$ icon denotes an example or exercise involving complex numbers. (For students unfamiliar with complex numbers, Appendix C contains all the background material that is needed.) The CAS icon indicates that a computer algebra system (such as Maple, Mathematica, or MATLAB) or a calculator with matrix capabilities (such as almost any graphing calculator) is required—or at least very useful—for solving the example or exercise.

In an effort to help students learn how to read and use this textbook most effectively, I have noted various places where the reader is advised to pause. These may be places where a calculation is needed, part of a proof must be supplied, a claim should be verified, or some extra thought is required. The icon appears in the margin at such places; the message is "Slow down. Get out your pencil. Think about this."

Technology

This book can be used successfully whether or not students have access to technology. However, calculators with matrix capabilities and computer algebra systems are now commonplace and, properly used, can enrich the learning experience as well as help with tedious calculations. In this text, I take the point of view that students need to master all of the basic techniques of linear algebra by solving by hand examples that are not too computationally difficult. Technology may then be used

(in whole or in part) to solve subsequent examples and applications and to apply techniques that rely on earlier ones. For example, when systems of linear equations are first introduced, detailed solutions are provided; later, solutions are simply given, and the reader is expected to verify them. This is a good place to use some form of technology. Likewise, when applications use data that make hand calculation impractical, use technology. All of the numerical methods that are discussed depend on the use of technology.

With the aid of technology, students can explore linear algebra in some exciting ways and discover much for themselves. For example, if one of the coefficients of a linear system is replaced by a parameter, how much variability is there in the solutions? How does changing a single entry of a matrix affect its eigenvalues? This book is not a tutorial on technology, and in places where technology can be used, I have not specified a particular type of technology. The student companion website that accompanies this book offers an online appendix called *Technology Bytes* that gives instructions for solving a selection of examples from each chapter using Maple, Mathematica, and MATLAB. By imitating these examples, students can do further calculations and explorations using whichever CAS they have and exploit the power of these systems to help with the exercises throughout the book, particularly those marked with the CAS icon. The website also contains data sets and computer code in Maple, Mathematica, and MATLAB formats keyed to many exercises and examples in the text. Students and instructors can import these directly into their CAS to save typing and eliminate errors.

Finite and Numerical Linear Algebra

The text covers two aspects of linear algebra that are scarcely ever mentioned together: finite linear algebra and numerical linear algebra. By introducing modular arithmetic early, I have been able to make finite linear algebra (more properly, "linear algebra over finite fields," although I do not use that phrase) a recurring theme throughout the book. This approach provides access to the material on coding theory in Chapter 8 (online). There is also an application to finite linear games in Section 2.4 that students really enjoy. In addition to being exposed to the applications of finite linear algebra, mathematics majors will benefit from seeing the material on finite fields, because they are likely to encounter it in such courses as discrete mathematics, abstract algebra, and number theory.

All students should be aware that in practice, it is impossible to arrive at exact solutions of large-scale problems in linear algebra. Exposure to some of the techniques of numerical linear algebra will provide an indication of how to obtain highly accurate approximate solutions. Some of the numerical topics included in the book are roundoff error and partial pivoting, iterative methods for solving linear systems and computing eigenvalues, the *LU* and *QR* factorizations, matrix norms and condition numbers, least squares approximation, and the singular value decomposition. The inclusion of numerical linear algebra also brings up some interesting and important issues that are completely absent from the *theory* of linear algebra, such as pivoting strategies, the condition of a linear system, and the convergence of iterative methods. This book not only raises these questions but also shows how one might approach them. Gerschgorin disks, matrix norms, and the singular values of a matrix, discussed in Chapters 4 and 7, are useful in this regard.

See pages 83, 84, 124, 180, 311, 392, 555, 561, 568, 590

See pages 319, 563, 600

Appendices

Appendix A contains an overview of mathematical notation and methods of proof, and Appendix B discusses mathematical induction. All students will benefit from these sections, but those with a mathematically oriented major may wish to pay particular attention to them. Some of the examples in these appendices are uncommon (for instance, Example B.6 in Appendix B) and underscore the power of the methods. Appendix C is an introduction to complex numbers. For students familiar with these results, this appendix can serve as a useful reference; for others, this section contains everything they need to know for those parts of the text that use complex numbers. Appendix D is about polynomials. I have found that many students require a refresher about these facts. Most students will be unfamiliar with Descartes's Rule of Signs; it is used in Chapter 4 to explain the behavior of the eigenvalues of Leslie matrices. Exercises to accompany the four appendices can be found on the book's website.

Short answers to most of the odd-numbered computational exercises are given at the end of the book. Exercise sets to accompany Appendixes A, B, C, and D are available on the companion website, along with their odd-numbered answers.

Ancillaries

For Instructors

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Complete Solutions Manual
The Complete Solutions Manual provides detailed solutions to all exercises in the text, including *Exploration* and *Chapter Review* exercises. The Complete Solutions Manual is available online.

Instructor's Guide
This online guide enhances the text with valuable teaching resources such as group work projects, teaching tips, interesting exam questions, examples and extra

material for lectures, and other items designed to reduce the instructor's preparation time and make linear algebra class an exciting and interactive experience. For each section of the text, the *Instructor's Guide* includes suggested time and emphasis, points to stress, questions for discussion, lecture materials and examples, technology tips, student projects, group work with solutions, sample assignments, and suggested test questions.

Solution Builder
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For Students

Student Solutions Manual (ISBN-13: 978-1-285-84195-3)
The Student Solutions Manual and Study Guide includes detailed solutions to all odd-numbered exercises and selected even-numbered exercises; section and chapter summaries of symbols, definitions, and theorems; and study tips and hints. Complex exercises are explored through a question-and-answer format designed to deepen understanding. Challenging and entertaining problems that further explore selected exercises are also included.

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Acknowledgments

The reviewers of the previous edition of this text contributed valuable and often insightful comments about the book. I am grateful for the time each of them took to do this. Their judgement and helpful suggestions have contributed greatly to the development and success of this book, and I would like to thank them personally:
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See page 284

See page 286

course" in determinants contains all the essential material students need, including an optional but elementary proof of the Laplace Expansion Theorem. The vignette "Lewis Carroll's Condensation Method" presents a historically interesting, alternative method of calculating determinants that students may find appealing. The exploration "Geometric Applications of Determinants" makes a nice project that contains several interesting and useful results. (Alternatively, instructors who wish to give more detailed coverage to determinants may choose to cover some of this exploration in class.) The basic theory of eigenvalues and eigenvectors is found in Section 4.3, and Section 4.4 deals with the important topic of diagonalization. Example 4.29 on powers of matrices is worth covering in class. The power method and its variants, discussed in Section 4.5, are optional, but all students should be aware of the method, and an applied course should cover it in detail. Gerschgorin's Disk Theorem can be covered independently of the rest of Section 4.5. Markov chains and the Leslie model of population growth reappear in Section 4.6. Although the proof of Perron's Theorem is optional, the theorem itself (like the stronger Perron-Frobenius Theorem) should at least be mentioned because it explains *why* we should expect a unique positive eigenvalue with a corresponding positive eigenvector in these applications. The applications on recurrence relations and differential equations connect linear algebra to discrete mathematics and calculus, respectively. The matrix exponential can be covered if your class has a good calculus background. The final topic of discrete linear dynamical systems revisits and summarizes many of the ideas in Chapter 4, looking at them in a new, geometric light. Students will enjoy reading how eigenvectors can be used to help rank sports teams and websites. This vignette can easily be extended to a project or enrichment activity.

See pages 325, 330

See page 348

Chapter 5: Orthogonality

See page 366

The introductory exploration, "Shadows on a Wall," is mathematics at its best: it takes a known concept (projection of a vector onto another vector) and generalizes it in a useful way (projection of a vector onto a subspace—a plane), while uncovering some previously unobserved properties. Section 5.1 contains the basic results about orthogonal and orthonormal sets of vectors that will be used repeatedly from here on. In particular, orthogonal matrices should be stressed. In Section 5.2, two concepts from Chapter 1 are generalized: the orthogonal complement of a subspace and the orthogonal projection of a vector onto a subspace. The Orthogonal Decomposition Theorem is important here and helps to set up the Gram-Schmidt Process. Also note the quick proof of the Rank Theorem. The Gram-Schmidt Process is detailed in Section 5.3, along with the extremely important QR factorization. The two explorations that follow outline how the QR factorization is computed in practice and how it can be used to approximate eigenvalues. Section 5.4 on orthogonal diagonalization of (real) symmetric matrices is needed for the applications that follow. It also contains the Spectral Theorem, one of the highlights of the theory of linear algebra. The applications in Section 5.5 are quadratic forms and graphing quadratic equations. I always include at least the second of these in my course because it extends what students already know about conic sections.

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Chapter 6: Vector Spaces

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The Fibonacci sequence reappears in Section 6.0, although it is not important that students have seen it before (Section 4.6). The purpose of this exploration is to show

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To the Instructor

"Would you tell me, please,
which way I ought to go from here?"
"That depends a good deal on where
you want to get to," said the Cat.

—Lewis Carroll
Alice's Adventures in Wonderland, 1865

This text was written with flexibility in mind. It is intended for use in a one- or two-semester course with 36 lectures per semester. The range of topics and applications makes it suitable for a variety of audiences and types of courses. However, there is more material in the book than can be covered in class, even in a two-semester course. After the following overview of the text are some brief suggestions for ways to use the book.

An Overview of the Text

Chapter 1: Vectors

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The racetrack game in Section 1.0 serves to introduce vectors in an informal way. (It's also quite a lot of fun to play!) Vectors are then formally introduced from both algebraic and geometric points of view. The operations of addition and scalar multiplication and their properties are first developed in the concrete settings of $\mathbb{R}^2$ and $\mathbb{R}^3$ before being generalized to $\mathbb{R}^n$. Modular arithmetic and finite linear algebra are also introduced. Section 1.2 defines the dot product of vectors and the related notions of length, angle, and orthogonality. The very important concept of (orthogonal) projection is developed here; it will reappear in Chapters 5 and 7. The exploration "Vectors and Geometry" shows how vector methods can be used to prove certain results in Euclidean geometry. Section 1.3 is a basic but thorough introduction to lines and planes in $\mathbb{R}^2$ and $\mathbb{R}^3$. This section is crucial for understanding the geometric significance of the solution of linear systems in Chapter 2. Note that the cross product of vectors in $\mathbb{R}^3$ is left as an exploration. The chapter concludes with an application to force vectors.

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Chapter 2: Systems of Linear Equations

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The introduction to this chapter serves to illustrate that there is more than one way to think of the solution to a system of linear equations. Sections 2.1 and 2.2 develop the

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main computational tool for solving linear systems: row reduction of matrices (Gaussian and Gauss-Jordan elimination). Nearly all subsequent computational methods in the book depend on this. The Rank Theorem appears here for the first time; it shows up again, in more generality, in Chapters 3, 5, and 6. Section 2.3 is very important; it introduces the fundamental notions of spanning sets and linear independence of vectors. Do not rush through this material. Section 2.4 contains six applications from which instructors can choose depending on the time available and the interests of the class. The vignette on the Global Positioning System provides another application that students will enjoy. The iterative methods in Section 2.5 will be optional for many courses but are essential for a course with an applied/numerical focus. The three explorations in this chapter are related in that they all deal with aspects of the use of computers to solve linear systems. All students should at least be made aware of these issues.

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Chapter 3: Matrices

This chapter contains some of the most important ideas in the book. It is a long chapter, but the early material can be covered fairly quickly, with extra time allowed for the crucial material in Section 3.5. Section 3.0 is an exploration that introduces the notion of a linear transformation: the idea that matrices are not just static objects but rather a type of function, transforming vectors into other vectors. All of the basic facts about matrices, matrix operations, and their properties are found in the first two sections. The material on partitioned matrices and the multiple representations of the matrix product is worth stressing, because it is used repeatedly in subsequent sections. The Fundamental Theorem of Invertible Matrices in Section 3.3 is very important and will appear several more times as new characterizations of invertibility are presented. Section 3.4 discusses the very important *LU* factorization of a matrix. If this topic is not covered in class, it is worth assigning as a project or discussing in a workshop. The point of Section 3.5 is to present many of the key concepts of linear algebra (subspace, basis, dimension, and rank) in the concrete setting of matrices before students see them in full generality. Although the examples in this section are all familiar, it is important that students get used to the new terminology and, in particular, understand what the notion of a basis means. The geometric treatment of linear transformations in Section 3.6 is intended to smooth the transition to general linear transformations in Chapter 6. The example of a projection is particularly important because it will reappear in Chapter 5. The vignette on robotic arms is a concrete demonstration of composition of linear (and affine) transformations. There are four applications from which to choose in Section 3.7. Either Markov chains or the Leslie model of population growth should be covered so that they can be used again in Chapter 4, where their behavior will be explained.

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Chapter 4: Eigenvalues and Eigenvectors

The introduction Section 4.0 presents an interesting dynamical system involving graphs. This exploration introduces the notion of an eigenvector and foreshadows the power method in Section 4.5. In keeping with the geometric emphasis of the book, Section 4.1 contains the novel feature of "eigenpictures" as a way of visualizing the eigenvectors of 2×2 matrices. Determinants appear in Section 4.2, motivated by their use in finding the characteristic polynomials of small matrices. This "crash

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that familiar vector space concepts (Section 3.5) can be used fruitfully in a new setting. Because all of the main ideas of vector spaces have already been introduced in Chapters 1–3, students should find Sections 6.1 and 6.2 fairly familiar. The emphasis here should be on using the vector space axioms to prove properties rather than relying on computational techniques. When discussing change of basis in Section 6.3, it is helpful to show students how to use the notation to remember how the construction works. Ultimately, the Gauss-Jordan method is the most efficient here. Sections 6.4 and 6.5 on linear transformations are important. The examples are related to previous results on matrices (and matrix transformations). In particular, it is important to stress that the kernel and range of a linear transformation generalize the null space and column space of a matrix. Section 6.6 puts forth the notion that (almost) all linear transformations are essentially matrix transformations. This builds on the information in Section 3.6, so students should not find it terribly surprising. However, the examples should be worked carefully. The connection between change of basis and similarity of matrices is noteworthy. The exploration "Tilings, Lattices, and the Crystallographic Restriction" is an impressive application of change of basis. The connection with the artwork of M. C. Escher makes it all the more interesting. The applications in Section 6.7 build on previous ones and can be included as time and interest permit.

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Chapter 7: Distance and Approximation

Section 7.0 opens with the entertaining "Taxicab Geometry" exploration. Its purpose is to set up the material on generalized norms and distance functions (metrics) that follows. Inner product spaces are discussed in Section 7.1; the emphasis here should be on the examples and using the axioms. The exploration "Vectors and Matrices with Complex Entries" shows how the concepts of dot product, symmetric matrix, orthogonal matrix, and orthogonal diagonalization can be extended from real to complex vector spaces. The following exploration, "Geometric Inequalities and Optimization Problems," is one that students typically enjoy. (They will have fun seeing how many "calculus" problems can be solved without using calculus at all!) Section 7.2 covers generalized vector and matrix norms and shows how the condition number of a matrix is related to the notion of ill-conditioned linear systems explored in Chapter 2. Least squares approximation (Section 7.3) is an important application of linear algebra in many other disciplines. The Best Approximation Theorem and the Least Squares Theorem are important, but their proofs are intuitively clear. Spend time here on the examples—a few should suffice. Section 7.4 presents the singular value decomposition, one of the most impressive applications of linear algebra. If your course gets this far, you will be amply rewarded. Not only does the SVD tie together many notions discussed previously; it also affords some new (and quite powerful) applications. If a CAS is available, the vignette on digital image compression is worth presenting; it is a visually impressive display of the power of linear algebra and a fitting culmination to the course. The further applications in Section 7.5 can be chosen according to the time available and the interests of the class.

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Chapter 8: Codes

This online chapter contains applications of linear algebra to the theory of codes. Section 8.1 begins with a discussion of how vectors can be used to design

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error-detecting codes such as the familiar Universal Product Code (UPC) and International Standard Book Number (ISBN). This topic only requires knowledge of Chapter 1. The vignette on the Codabar system used in credit and bank cards is an excellent classroom presentation that can even be used to introduce Section 8.1. Once students are familiar with matrix operations, Section 8.2 describes how codes can be designed to correct as well as detect errors. The Hamming codes introduced here are perhaps the most famous examples of such error-correcting codes. Dual codes, discussed in Section 8.3, are an important way of constructing new codes from old ones. The notion of orthogonal complement, introduced in Chapter 5, is the prerequisite concept here. The most important, and most widely used, class of codes is the class of linear codes that is defined in Section 8.4. The notions of subspace, basis, and dimension are key here. The powerful Reed-Muller codes used by NASA spacecraft are important examples of linear codes. Our discussion of codes concludes in Section 8.5 with the definition of the minimum distance of a code and the role it plays in determining the error-correcting capability of the code.

How to Use the Book

Students find the book easy to read, so I usually have them read a section before I cover the material in class. That way, I can spend class time highlighting the most important concepts, dealing with topics students find difficult, working examples, and discussing applications. I do not attempt to cover all of the material from the assigned reading in class. This approach enables me to keep the pace of the course fairly brisk, slowing down for those sections that students typically find challenging.

In a two-semester course, it is possible to cover the entire book, including a reasonable selection of applications. For extra flexibility, you might omit some of the topics (for example, give only a brief treatment of numerical linear algebra), thereby freeing up time for more in-depth coverage of the remaining topics, more applications, or some of the explorations. In an honors mathematics course that emphasizes proofs, much of the material in Chapters 1–3 can be covered quickly. Chapter 6 can then be covered in conjunction with Sections 3.5 and 3.6, and Chapter 7 can be integrated into Chapter 5. I would be sure to assign the explorations in Chapters 1, 4, 6, and 7 for such a class.

For a one-semester course, the nature of the course and the audience will determine which topics to include. Three possible courses are described below and on the following page. The basic course, described first, has fewer than 36 hours suggested, allowing time for extra topics, in-class review, and tests. The other two courses build on the basic course but are still quite flexible.

A Basic Course

A course designed for mathematics majors and students from other disciplines is outlined on the next page. This course does not mention general vector spaces at all (all concepts are treated in a concrete setting) and is very light on proofs. Still, it is a thorough introduction to linear algebra.

Section	Number of Lectures	Section	Number of Lectures
1.1	1	**3.6**	1–2
1.2	1–1.5	**4.1**	1
1.3	1–1.5	**4.2**	2
2.1	0.5–1	**4.3**	1
2.2	1–2	**4.4**	1–2
2.3	1–2	**5.1**	1–1.5
3.1	1–2	**5.2**	1–1.5
3.2	1	**5.3**	0.5
3.3	2	**5.4**	1
3.5	2	**7.3**	2

Total: 23–30 lectures

Because the students in a course such as this one represent a wide variety of disciplines, I would suggest using much of the remaining lecture time for applications. In my course, I do code vectors in Section 8.1, which students really seem to like, and at least one application from each of Chapters 2–5. Other applications can be assigned as projects, along with as many of the explorations as desired. There is also sufficient lecture time available to cover some of the theory in detail.

A Course with a Computational Emphasis

For a course with a computational emphasis, the basic course outlined on the previous page can be supplemented with the sections of the text dealing with numerical linear algebra. In such a course, I would cover part or all of Sections 2.5, 3.4, 4.5, 5.3, 7.2, and 7.4, ending with the singular value decomposition. The explorations in Chapters 2 and 5 are particularly well suited to such a course, as are almost any of the applications.

A Course for Students Who Have Already Studied Some Linear Algebra

Some courses will be aimed at students who have already encountered the basic principles of linear algebra in other courses. For example, a college algebra course will often include an introduction to systems of linear equations, matrices, and determinants; a multivariable calculus course will almost certainly contain material on vectors, lines, and planes. For students who have seen such topics already, much early material can be omitted and replaced with a quick review. Depending on the background of the class, it may be possible to skim over the material in the basic course up to Section 3.3 in about six lectures. If the class has a significant number of mathematics majors (and especially if this is the only linear algebra course they will take), I would be sure to cover Sections 6.1–6.5, 7.1, and 7.4 and as many applications as time permits. If the course has science majors (but not mathematics majors), I would cover Sections 6.1 and 7.1 and a broader selection of applications, being sure to include the material on differential equations and approximation of functions. If computer science students or engineers are prominently represented, I would try to do as much of the material on codes and numerical linear algebra as I could.

There are many other types of courses that can successfully use this text. I hope that you find it useful for your course and that you enjoy using it.

To the Student

"Where shall I begin, please your Majesty?" he asked.
"Begin at the beginning," the King said, gravely, "and go on till you come to the end: then stop."

—Lewis Carroll
Alice's Adventures in Wonderland, 1865

Linear algebra is an exciting subject. It is full of interesting results, applications to other disciplines, and connections to other areas of mathematics. The *Student Solutions Manual and Study Guide* contains detailed advice on how best to use this book; following are some general suggestions.

Linear algebra has several sides: There are *computational techniques, concepts,* and *applications*. One of the goals of this book is to help you master all of these facets of the subject and to see the interplay among them. Consequently, it is important that you read and understand each section of the text before you attempt the exercises in that section. If you read only examples that are related to exercises that have been assigned as homework, you will miss much. Make sure you understand the definitions of terms and the meaning of theorems. Don't worry if you have to read something more than once before you understand it. Have a pencil and calculator with you as you read. Stop to work out examples for yourself or to fill in missing calculations. The ⟶ icon in the margin indicates a place where you should pause and think over what you have read so far.

Answers to most odd-numbered computational exercises are in the back of the book. Resist the temptation to look up an answer before you have completed a question. And remember that even if your answer differs from the one in the back, you may still be right; there is more than one correct way to express some of the solutions. For example, a value of $1/\sqrt{2}$ can also be expressed as $\sqrt{2}/2$ and the set of all scalar multiples of the vector $\begin{bmatrix} 3 \\ 1/2 \end{bmatrix}$ is the same as the set of all scalar multiples of $\begin{bmatrix} 6 \\ 1 \end{bmatrix}$.

As you encounter new concepts, try to relate them to examples that you know. Write out proofs and solutions to exercises in a logical, connected way, using complete sentences. Read back what you have written to see whether it makes sense. Better yet, if you can, have a friend in the class read what you have written. If it doesn't make sense to another person, chances are that it doesn't make sense, period.

You will find that a calculator with matrix capabilities or a computer algebra system is useful. These tools can help you to check your own hand calculations and are indispensable for some problems involving tedious computations. Technology also

enables you to explore aspects of linear algebra on your own. You can play "what if?" games: What if I change one of the entries in this vector? What if this matrix is of a different size? Can I force the solution to be what I would like it to be by changing something? To signal places in the text or exercises where the use of technology is recommended, I have placed the icon CAS in the margin. The companion website that accompanies this book contains computer code working out selected exercises from the book using Maple, Mathematica, and MATLAB, as well as *Technology Bytes*, an appendix providing much additional advice about the use of technology in linear algebra.

You are about to embark on a journey through linear algebra. Think of this book as your travel guide. Are you ready? Let's go!