4 Eigenvalues and Eigenvectors

Almost every combination of the adjectives proper, latent, characteristic, eigen *and* secular, *with the nouns* root, number *and* value, *has been used in the literature for what we call a proper value.*

—Paul R. Halmos
Finite Dimensional Vector Spaces
(2nd edition)
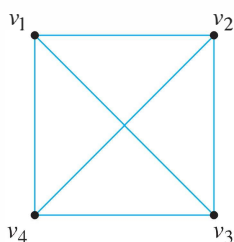Van Nostrand, 1958, p. 102

4.0 Introduction: A Dynamical System on Graphs

CAS

We saw in the last chapter that iterating matrix multiplication often produces interesting results. Both Markov chains and the Leslie model of population growth exhibit steady states in certain situations. One of the goals of this chapter is to help you understand such behavior. First we will look at another iterative process, or **dynamical system,** that uses matrices. (In the problems that follow, you will find it helpful to use a CAS or a calculator with matrix capabilities to facilitate the computations.)

Our example involves graphs (see Section 3.7). A **complete graph** is any graph in which every vertex is adjacent to every other vertex. If a complete graph has n vertices, it is denoted by K_n. For example, Figure 4.1 shows a representation of K_4.

Problem 1 Pick any vector $\mathbf{x}$ in $\mathbb{R}^4$ with nonnegative entries and label the vertices of K_4 with the components of $\mathbf{x}$, so that v_1 is labeled with x_1, and so on. Compute the adjacency matrix A of K_4 and relabel the vertices of the graph with the corresponding components of $A\mathbf{x}$. Try this for several vectors $\mathbf{x}$ and explain, in terms of the graph, how the new labels can be determined from the old labels.

Problem 2 Now *iterate* the process in Problem 1. That is, for a given choice of $\mathbf{x}$, relabel the vertices as described above and then apply A again (and again, and again) until a pattern emerges. Since components of the vectors themselves will get quite large, we will *scale* them by dividing each vector by its *largest component* after each iteration. Thus, if a computation results in the vector

$$\begin{bmatrix} 4 \\ 2 \\ 1 \\ 1 \end{bmatrix}$$

we will replace it by

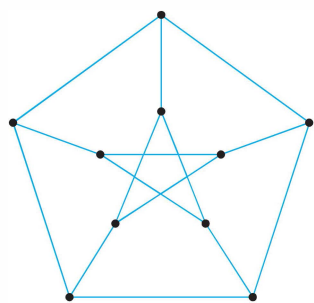$$\frac{1}{4}\begin{bmatrix} 4 \\ 2 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0.5 \\ 0.25 \\ 0.25 \end{bmatrix}$$
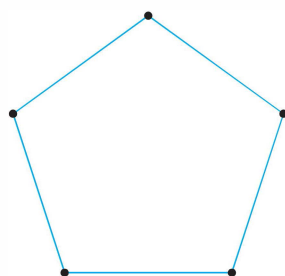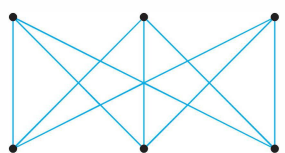

Figure 4.1
K_4

Figure 4.2

Figure 4.3

Figure 4.4

Note that this process guarantees that the largest component of each vector will now be 1. Do this for K_4, then K_3 and K_5. Use at least ten iterations and two-decimal-place accuracy. What appears to be happening?

Problem 3   You should have noticed that, in each case, the labeling vector is approaching a certain vector (a steady state label!). Label the vertices of the complete graphs with this steady state vector and apply the adjacency matrix A one more time (without scaling). What is the relationship between the new labels and the old ones?

Problem 4   Make a conjecture about the general case K_n. What is the steady state label? What happens if we label K_n with the steady state vector and apply the adjacency matrix A without scaling?

Problem 5   The ***Petersen graph*** is shown in Figure 4.2. Repeat the process in Problems 1 through 3 with this graph.

We will now explore the process with some other classes of graphs to see if they behave the same way. The ***cycle*** C_n is the graph with n vertices arranged in a cyclic fashion. For example, C_5 is the graph shown in Figure 4.3.

Problem 6   Repeat the process of Problems 1 through 3 with cycles C_n for various *odd* values of n and make a conjecture about the general case.

Problem 7   Repeat Problem 6 with *even* values of n. What happens?

A bipartite graph is a ***complete bipartite graph*** (see Exercises 74–78 in Section 3.7) if its vertices can be partitioned into sets U and V such that every vertex in U is adjacent to every vertex in V, and vice versa. If U and V each have n vertices, then the graph is denoted by $K_{n,n}$. For example, $K_{3,3}$ is the graph in Figure 4.4.

Problem 8   Repeat the process of Problems 1 through 3 with complete bipartite graphs $K_{n,n}$ for various values of n. What happens?

By the end of this chapter, you will be in a position to explain the observations you have made in this Introduction.

4.1 Introduction to Eigenvalues and Eigenvectors

In Chapter 3, we encountered the notion of a steady state vector in the context of two applications: Markov chains and the Leslie model of population growth. For a Markov chain with transition matrix P, a steady state vector $\mathbf{x}$ had the property that $P\mathbf{x} = \mathbf{x}$; for a Leslie matrix L, a steady state vector was a population vector $\mathbf{x}$ satisfying $L\mathbf{x} = r\mathbf{x}$, where r represented the steady state growth rate. For example, we saw that

$$\begin{bmatrix} 0.7 & 0.2 \\ 0.3 & 0.8 \end{bmatrix}\begin{bmatrix} 0.4 \\ 0.6 \end{bmatrix} = \begin{bmatrix} 0.4 \\ 0.6 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 0 & 4 & 3 \\ 0.5 & 0 & 0 \\ 0 & 0.25 & 0 \end{bmatrix}\begin{bmatrix} 18 \\ 6 \\ 1 \end{bmatrix} = 1.5 \begin{bmatrix} 18 \\ 6 \\ 1 \end{bmatrix}$$

In this chapter, we investigate this phenomenon more generally. That is, for a square matrix A, we ask whether there exist nonzero vectors $\mathbf{x}$ such that $A\mathbf{x}$ is just a scalar multiple of $\mathbf{x}$. This is the *eigenvalue problem,* and it is one of the most central problems in linear algebra. It has applications throughout mathematics and in many other fields as well.

The German adjective *eigen* means "own" or "characteristic of." *Eigenvalues* and *eigenvectors* are characteristic of a matrix in the sense that they contain important information about the nature of the matrix. The letter λ (lambda), the Greek equivalent of the English letter L, is used for eigenvalues because at one time they were also known as *latent values.* The prefix *eigen* is pronounced "EYE-gun."

Definition   Let A be an $n \times n$ matrix. A scalar λ is called an ***eigenvalue*** of A if there is a nonzero vector $\mathbf{x}$ such that $A\mathbf{x} = \lambda\mathbf{x}$. Such a vector $\mathbf{x}$ is called an ***eigenvector*** of A corresponding to λ.

Example 4.1

Show that $\mathbf{x} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is an eigenvector of $A = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$ and find the corresponding eigenvalue.

Solution   We compute

$$A\mathbf{x} = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 4 \end{bmatrix} = 4 \begin{bmatrix} 1 \\ 1 \end{bmatrix} = 4\mathbf{x}$$

from which it follows that $\mathbf{x}$ is an eigenvector of A corresponding to the eigenvalue 4.

Example 4.2

Show that 5 is an eigenvalue of $A = \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}$ and determine all eigenvectors corresponding to this eigenvalue.

Solution   We must show that there is a *nonzero* vector $\mathbf{x}$ such that $A\mathbf{x} = 5\mathbf{x}$. But this equation is equivalent to the equation $(A - 5I)\mathbf{x} = \mathbf{0}$, so we need to compute the null space of the matrix $A - 5I$. We find that

$$A - 5I = \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix} - \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix} = \begin{bmatrix} -4 & 2 \\ 4 & -2 \end{bmatrix}$$

Since the columns of this matrix are clearly linearly dependent, the Fundamental Theorem of Invertible Matrices implies that its null space is nonzero. Thus, $A\mathbf{x} = 5\mathbf{x}$ has a nontrivial solution, so 5 is an eigenvalue of A. We find its eigenvectors by computing the null space:

$$[A - 5I \,|\, \mathbf{0}] = \begin{bmatrix} -4 & 2 & | & 0 \\ 4 & -2 & | & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & -\frac{1}{2} & | & 0 \\ 0 & 0 & | & 0 \end{bmatrix}$$

Thus, if $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ is an eigenvector corresponding to the eigenvalue 5, it satisfies $x_1 - \frac{1}{2}x_2 = 0$, or $x_1 = \frac{1}{2}x_2$, so these eigenvectors are of the form

$$\mathbf{x} = \begin{bmatrix} \frac{1}{2}x_2 \\ x_2 \end{bmatrix}$$

That is, they are the nonzero multiples of $\begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix}$ (or, equivalently, the nonzero multiples of $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$).

The set of all eigenvectors corresponding to an eigenvalue λ of an $n \times n$ matrix A is just the set of *nonzero* vectors in the null space of $A - \lambda I$. It follows that this set of eigenvectors, together with the zero vector in $\mathbb{R}^n$, *is* the null space of $A - \lambda I$.

Definition   Let A be an $n \times n$ matrix and let λ be an eigenvalue of A. The collection of all eigenvectors corresponding to λ, together with the zero vector, is called the **eigenspace** of λ and is denoted by E_λ.

Therefore, in Example 4.2, $E_5 = \left\{ t \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\}$.

Example 4.3

Show that $\lambda = 6$ is an eigenvalue of $A = \begin{bmatrix} 7 & 1 & -2 \\ -3 & 3 & 6 \\ 2 & 2 & 2 \end{bmatrix}$ and find a basis for its eigenspace.

Solution   As in Example 4.2, we compute the null space of $A - 6I$. Row reduction produces

$$A - 6I = \begin{bmatrix} 1 & 1 & -2 \\ -3 & -3 & 6 \\ 2 & 2 & -4 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 1 & -2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

from which we see that the null space of $A - 6I$ is nonzero. Hence, 6 is an eigenvalue of A, and the eigenvectors corresponding to this eigenvalue satisfy $x_1 + x_2 - 2x_3 = 0$, or $x_1 = -x_2 + 2x_3$. It follows that

$$E_6 = \left\{ \begin{bmatrix} -x_2 + 2x_3 \\ x_2 \\ x_3 \end{bmatrix} \right\} = \left\{ x_2 \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} \right\} = \operatorname{span} \left(\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} \right)$$

In $\mathbb{R}^2$, we can give a geometric interpretation of the notion of an eigenvector. The equation $A\mathbf{x} = \lambda\mathbf{x}$ says that the vectors $A\mathbf{x}$ and $\mathbf{x}$ are parallel. Thus, $\mathbf{x}$ is an eigenvector of A if and only if A transforms $\mathbf{x}$ into a parallel vector [or, equivalently, if and only if $T_A(\mathbf{x})$ is parallel to $\mathbf{x}$, where T_A is the matrix transformation corresponding to A].

Example 4.4

Find the eigenvectors and eigenvalues of $A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ geometrically.

Solution   We recognize that A is the matrix of a reflection F in the x-axis (see Example 3.56). The only vectors that F maps parallel to themselves are vectors parallel to the y-axis (i.e., multiples of $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$), which are reversed (eigenvalue -1), and vectors parallel to the x-axis (i.e., multiples of $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$), which are sent to themselves (eigenvalue 1) (see Figure 4.5). Accordingly, $\lambda = -1$ and $\lambda = 1$ are the eigenvalues of A, and the corresponding eigenspaces are

$$E_{-1} = \operatorname{span} \left(\begin{bmatrix} 0 \\ 1 \end{bmatrix} \right) \quad \text{and} \quad E_1 = \operatorname{span} \left(\begin{bmatrix} 1 \\ 0 \end{bmatrix} \right)$$
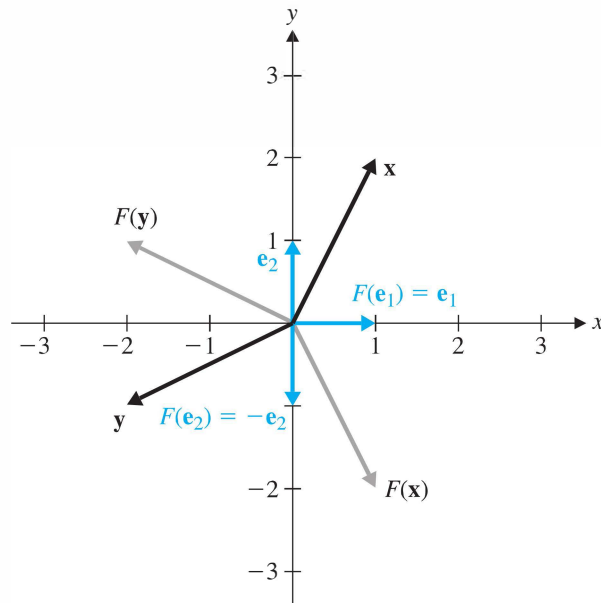
Figure 4.5
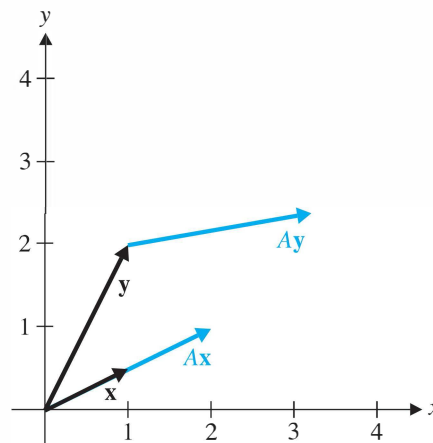The eigenvectors of a reflection

Figure 4.6

The discussion is based on the article "Eigenpictures: Picturing the Eigenvector Problem" by Steven Schonefeld in *The College Mathematics Journal 26* (1996), pp. 316–319.

Another way to think of eigenvectors geometrically is to draw **x** and A**x** head-to-tail. Then **x** will be an eigenvector of A if and only if **x** and A**x** are aligned in a straight line. In Figure 4.6, **x** is an eigenvector of A but **y** is not.

If **x** is an eigenvector of A corresponding to the eigenvalue λ, then so is any non-zero multiple of **x**. So, if we want to search for eigenvectors geometrically, we need only consider the effect of A on *unit* vectors. Figure 4.7(a) shows what happens when we transform unit vectors with the matrix $A = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$ of Example 4.1 and display the results head-to-tail, as in Figure 4.6. We can see that the vector $\mathbf{x} = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$ is an eigenvector, but we also notice that there appears to be an eigenvector in the second quadrant. Indeed, this is the case, and it turns out to be the vector $\begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$.
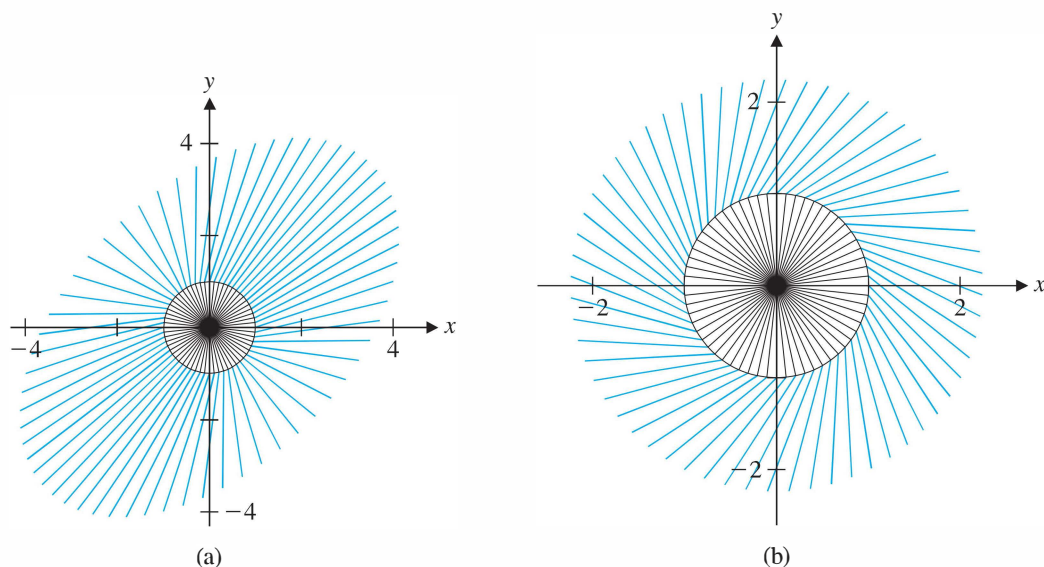
Figure 4.7

In Figure 4.7(b), we see what happens when we use the matrix $A = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$. There are no eigenvectors at all!

We now know how to find eigenvectors once we have the corresponding eigenvalues, and we have a geometric interpretation of them—but one question remains: How do we first find the eigenvalues of a given matrix? The key is the observation that λ is an eigenvalue of A if and only if the null space of $A - \lambda I$ is nontrivial.

Recall from Section 3.3 that the determinant of a 2×2 matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is the expression $\det A = ad - bc$, and A is invertible if and only if $\det A$ is nonzero. Furthermore, the Fundamental Theorem of Invertible Matrices guarantees that a matrix has a nontrivial null space if and only if it is noninvertible—hence, if and only if its determinant is zero. Putting these facts together, we see that (for 2×2 matrices at least) λ is an eigenvalue of A if and only if $\det(A - \lambda I) = 0$. This fact characterizes eigenvalues, and we will soon generalize it to square matrices of arbitrary size. For the moment, though, let's see how to use it with 2×2 matrices.

Example 4.5

Find all of the eigenvalues and corresponding eigenvectors of the matrix $A = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$ from Example 4.1.

Solution   The preceding remarks show that we must find all solutions λ of the equation $\det(A - \lambda I) = 0$. Since

$$\det(A - \lambda I) = \det \begin{bmatrix} 3 - \lambda & 1 \\ 1 & 3 - \lambda \end{bmatrix} = (3 - \lambda)(3 - \lambda) - 1 = \lambda^2 - 6\lambda + 8$$

we need to solve the quadratic equation $\lambda^2 - 6\lambda + 8 = 0$. The solutions to this equation are easily found to be $\lambda = 4$ and $\lambda = 2$. These are therefore the eigenvalues of A.

To find the eigenvectors corresponding to the eigenvalue $\lambda = 4$, we compute the null space of $A - 4I$. We find

$$[A - 4I \,|\, \mathbf{0}] = \begin{bmatrix} -1 & 1 & | & 0 \\ 1 & -1 & | & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -1 & | & 0 \\ 0 & 0 & | & 0 \end{bmatrix}$$

from which it follows that $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ is an eigenvector corresponding to $\lambda = 4$ if and only if $x_1 - x_2 = 0$ or $x_1 = x_2$. Hence, the eigenspace $E_4 = \left\{ \begin{bmatrix} x_2 \\ x_2 \end{bmatrix} \right\} = \left\{ x_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\} = \text{span}\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix} \right)$.

Similarly, for $\lambda = 2$, we have

$$[A - 2I \,|\, 0] = \begin{bmatrix} 1 & 1 & | & 0 \\ 1 & 1 & | & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 & | & 0 \\ 0 & 0 & | & 0 \end{bmatrix}$$

so $\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$ is an eigenvector corresponding to $\lambda = 2$ if and only if $y_1 + y_2 = 0$ or $y_1 = -y_2$. Thus, the eigenspace $E_2 = \left\{ \begin{bmatrix} -y_2 \\ y_2 \end{bmatrix} \right\} = \left\{ y_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right\} = \text{span}\left(\begin{bmatrix} -1 \\ 1 \end{bmatrix} \right)$.

Figure 4.8 shows graphically how the eigenvectors of A are transformed when multiplied by A: an eigenvector $\mathbf{x}$ in the eigenspace E_4 is transformed into $4\mathbf{x}$, and an eigenvector $\mathbf{y}$ in the eigenspace E_2 is transformed into $2\mathbf{y}$. As Figure 4.7(a) shows, the eigenvectors of A are the *only* vectors in $\mathbb{R}^2$ that are transformed into scalar multiples of themselves when multiplied by A.
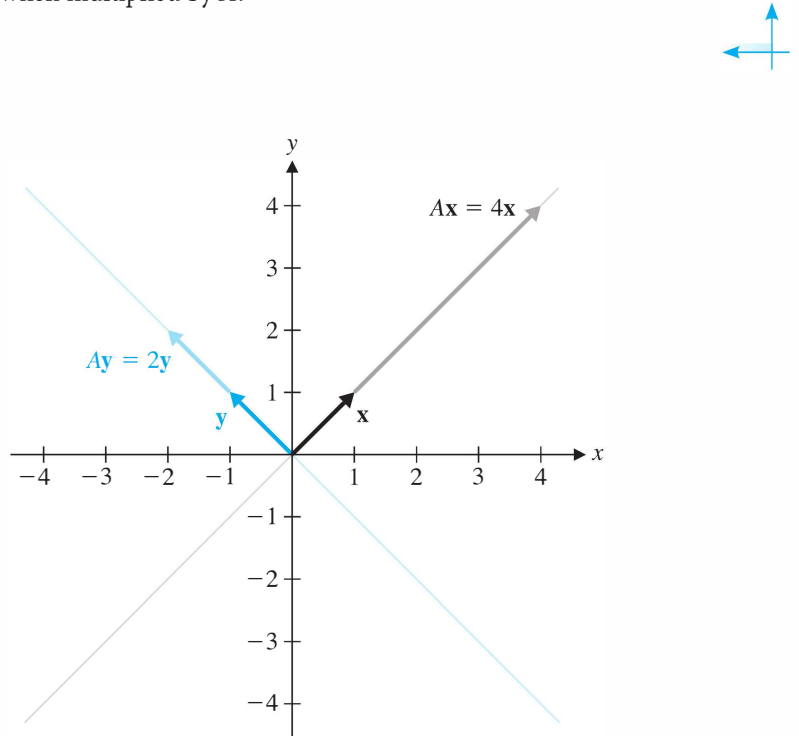

Figure 4.8
How A transforms eigenvectors

Remark You will recall that a polynomial equation with real coefficients (such as the quadratic equation in Example 4.5) need not have real roots; it may have *complex* roots. (See Appendix C.) It is also possible to compute eigenvalues and eigenvectors when the entries of a matrix come from $\mathbb{Z}_p$, where p is prime. Thus, it is important to specify the setting we intend to work in before we set out to compute the eigenvalues of a matrix. However, unless otherwise specified, the eigenvalues of a matrix whose entries are *real* numbers will be assumed to be real as well.

Example 4.6

Interpret the matrix in Example 4.5 as a matrix over $\mathbb{Z}_3$ and find its eigenvalues in that field.

Solution The solution proceeds exactly as above, except we work modulo 3. Hence, the quadratic equation $\lambda^2 - 6\lambda + 8 = 0$ becomes $\lambda^2 + 2 = 0$. This equation is the same as $\lambda^2 = -2 = 1$, giving $\lambda = 1$ and $\lambda = -1 = 2$ as the eigenvalues in $\mathbb{Z}_3$. (Check that the same answer would be obtained by *first* reducing A modulo 3 to obtain $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ and then working with this matrix.)

Example 4.7

Find the eigenvalues of $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ (a) over $\mathbb{R}$ and (b) over the complex numbers $\mathbb{C}$.

Solution We must solve the equation

$$0 = \det(A - \lambda I) = \det \begin{bmatrix} -\lambda & -1 \\ 1 & -\lambda \end{bmatrix} = \lambda^2 + 1$$

(a) Over $\mathbb{R}$, there are no solutions, so A has no real eigenvalues.

(b) Over $\mathbb{C}$, the solutions are $\lambda = i$ and $\lambda = -i$. (See Appendix C.)

In the next section, we will extend the notion of determinant from 2×2 to $n \times n$ matrices, which in turn will allow us to find the eigenvalues of arbitrary square matrices. (In fact, this isn't quite true—but we will at least be able to find a polynomial equation that the eigenvalues of a given matrix must satisfy.)

Exercises 4.1

In Exercises 1–6, show that **v** *is an eigenvector of A and find the corresponding eigenvalue.*

1. $A = \begin{bmatrix} 0 & 3 \\ 3 & 0 \end{bmatrix}$, $\mathbf{v} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

2. $A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$, $\mathbf{v} = \begin{bmatrix} 3 \\ -3 \end{bmatrix}$

3. $A = \begin{bmatrix} -1 & 1 \\ 6 & 0 \end{bmatrix}$, $\mathbf{v} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$

4. $A = \begin{bmatrix} 4 & -2 \\ 5 & -7 \end{bmatrix}$, $\mathbf{v} = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$

5. $A = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 1 & -2 \\ 1 & 0 & 1 \end{bmatrix}$, $\mathbf{v} = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$

6. $A = \begin{bmatrix} 0 & 1 & -1 \\ 1 & 1 & 1 \\ 1 & 2 & 0 \end{bmatrix}$, $\mathbf{v} = \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}$

In Exercises 7–12, show that λ is an eigenvalue of A and find one eigenvector corresponding to this eigenvalue.

7. $A = \begin{bmatrix} 2 & 2 \\ 2 & -1 \end{bmatrix}, \lambda = 3$

8. $A = \begin{bmatrix} 2 & 3 \\ 3 & 2 \end{bmatrix}, \lambda = -1$

9. $A = \begin{bmatrix} 0 & 4 \\ -1 & 5 \end{bmatrix}, \lambda = 1$

10. $A = \begin{bmatrix} 4 & -2 \\ 5 & -7 \end{bmatrix}, \lambda = -6$

11. $A = \begin{bmatrix} 1 & 0 & 2 \\ -1 & 1 & 1 \\ 2 & 0 & 1 \end{bmatrix}, \lambda = -1$

12. $A = \begin{bmatrix} 3 & 1 & -1 \\ 1 & 1 & 1 \\ 4 & 2 & 0 \end{bmatrix}, \lambda = 2$

In Exercises 13–18, find the eigenvalues and eigenvectors of A geometrically.

13. $A = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$ (reflection in the y-axis)

14. $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ (reflection in the line $y = x$)

**15.** $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ (projection onto the x-axis)

16. $A = \begin{bmatrix} \frac{16}{25} & \frac{12}{25} \\ \frac{12}{25} & \frac{9}{25} \end{bmatrix}$ (projection onto the line through the origin with direction vector $\begin{bmatrix} \frac{4}{5} \\ \frac{3}{5} \end{bmatrix}$)

17. $A = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$ (stretching by a factor of 2 horizontally and a factor of 3 vertically)
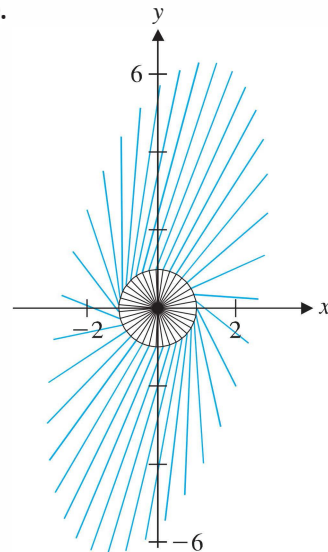
18. $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ (counterclockwise rotation of 90° about the origin)

In Exercises 19–22, the unit vectors $\mathbf{x}$ in $\mathbb{R}^2$ and their images $A\mathbf{x}$ under the action of a 2 × 2 matrix A are drawn head-to-tail, as in Figure 4.7. Estimate the eigenvectors and eigenvalues of A from each "eigenpicture."
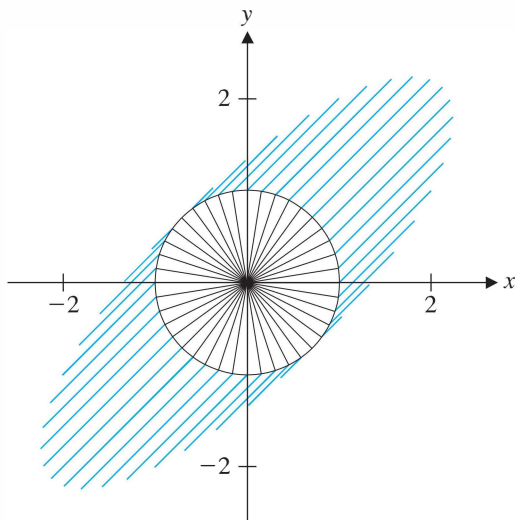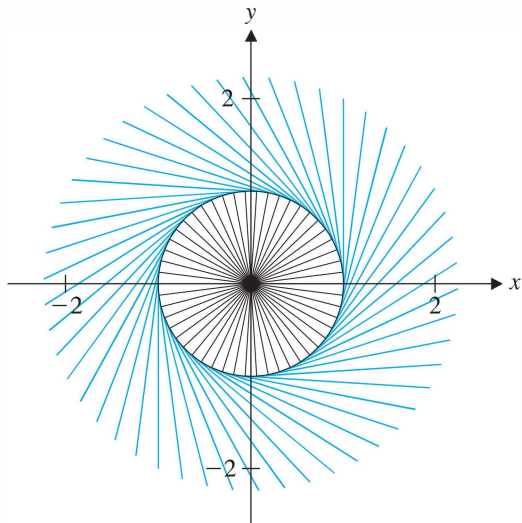
19.
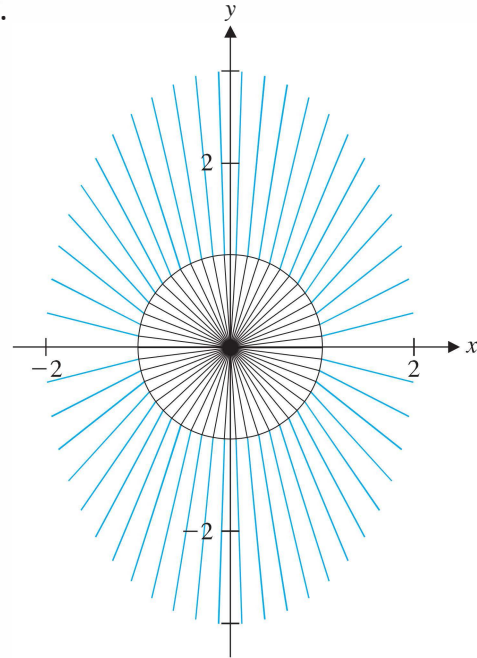
20.

21.

22.

In Exercises 23–26, use the method of Example 4.5 to find all of the eigenvalues of the matrix A. Give bases for each of the corresponding eigenspaces. Illustrate the eigenspaces and the effect of multiplying eigenvectors by A as in Figure 4.8.

23. $A = \begin{bmatrix} 4 & -1 \\ 2 & 1 \end{bmatrix}$ **24.** $A = \begin{bmatrix} 2 & 4 \\ 6 & 0 \end{bmatrix}$

25. $A = \begin{bmatrix} 2 & 5 \\ 0 & 2 \end{bmatrix}$ **26.** $A = \begin{bmatrix} 1 & 2 \\ -2 & 3 \end{bmatrix}$

⟨a + bi⟩

In Exercises 27–30, find all of the eigenvalues of the matrix A over the complex numbers $\mathbb{C}$. Give bases for each of the corresponding eigenspaces.

27. $A = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$ **28.** $A = \begin{bmatrix} 2 & -3 \\ 1 & 0 \end{bmatrix}$

29. $A = \begin{bmatrix} 1 & i \\ i & 1 \end{bmatrix}$ **30.** $A = \begin{bmatrix} 0 & 1+i \\ 1-i & 1 \end{bmatrix}$

In Exercises 31–34, find all of the eigenvalues of the matrix A over the indicated $\mathbb{Z}_p$.

31. $A = \begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix}$ over $\mathbb{Z}_3$ **32.** $A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$ over $\mathbb{Z}_3$

33. $A = \begin{bmatrix} 3 & 1 \\ 4 & 0 \end{bmatrix}$ over $\mathbb{Z}_5$ **34.** $A = \begin{bmatrix} 1 & 4 \\ 4 & 0 \end{bmatrix}$ over $\mathbb{Z}_5$

35. (a) Show that the eigenvalues of the 2 × 2 matrix

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

are the solutions of the quadratic equation $\lambda^2 - \text{tr}(A)\lambda + \det A = 0$, where $\text{tr}(A)$ is the trace of A. (See page 162.)

(b) Show that the eigenvalues of the matrix A in part (a) are

$$\lambda = \tfrac{1}{2}\left(a + d \pm \sqrt{(a-d)^2 + 4bc}\right)$$

(c) Show that the trace and determinant of the matrix A in part (a) are given by

$$\text{tr}(A) = \lambda_1 + \lambda_2 \quad \text{and} \quad \det A = \lambda_1\lambda_2$$

where λ_1 and λ_2 are the eigenvalues of A.

36. Consider again the matrix A in Exercise 35. Give conditions on a, b, c, and d such that A has

(a) two distinct real eigenvalues,
(b) one real eigenvalue, and
(c) no real eigenvalues.

37. Show that the eigenvalues of the upper triangular matrix

$$A = \begin{bmatrix} a & b \\ 0 & d \end{bmatrix}$$

are $\lambda = a$ and $\lambda = d$, and find the corresponding eigenspaces.

⟨a + bi⟩ **38.** Let a and b be real numbers. Find the eigenvalues and corresponding eigenspaces of

$$A = \begin{bmatrix} a & b \\ -b & a \end{bmatrix}$$

over the complex numbers.

4.2 Determinants

Historically, determinants preceded matrices—a curious fact in light of the way linear algebra is taught today, with matrices before determinants. Nevertheless, determinants arose independently of matrices in the solution of many practical problems, and the theory of determinants was well developed almost two centuries before matrices were deemed worthy of study in and of themselves. A snapshot of the history of determinants is presented at the end of this section.

Recall that the determinant of the 2×2 matrix $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ is

$$\det A = a_{11}a_{22} - a_{12}a_{21}$$

We first encountered this expression when we determined ways to compute the inverse of a matrix. In particular, we found that

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}^{-1} = \frac{1}{a_{11}a_{22} - a_{12}a_{21}} \begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix}$$

The determinant of a matrix A is sometimes also denoted by $|A|$, so for the 2×2 matrix $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ we may also write

$$|A| = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}$$

Warning   This notation for the determinant is reminiscent of absolute value notation. It is easy to mistake $\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$, the notation for determinant, for $\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$, the notation for the matrix itself. Do not confuse these. Fortunately, it will usually be clear from the context which is intended.

We define the determinant of a 1×1 matrix $A = [a]$ to be

$$\det A = |a| = a$$

(Note that we really have to be careful with notation here: $|a|$ does *not* denote the absolute value of a in this case.) How then should we define the determinant of a 3×3 matrix? If you ask your CAS for the inverse of

$$A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$$

the answer will be equivalent to

$$A^{-1} = \frac{1}{\Delta} \begin{bmatrix} ei - fh & ch - bi & bf - ce \\ fg - di & ai - cg & cd - af \\ dh - eg & bg - ah & ae - bd \end{bmatrix}$$

where $\Delta = aei - afh - bdi + bfg + cdh - ceg$. Observe that

$$\begin{aligned}
\Delta &= aei - afh - bdi + bfg + cdh - ceg \\
&= a(ei - fh) - b(di - fg) + c(dh - eg) \\
&= a \begin{vmatrix} e & f \\ h & i \end{vmatrix} - b \begin{vmatrix} d & f \\ g & i \end{vmatrix} + c \begin{vmatrix} d & e \\ g & h \end{vmatrix}
\end{aligned}$$

and that each of the entries in the matrix portion of A^{-1} appears to be the determinant of a 2×2 submatrix of A. In fact, this is true, and it is the basis of the definition of the determinant of a 3×3 matrix. The definition is *recursive* in the sense that the determinant of a 3×3 matrix is defined in terms of determinants of 2×2 matrices.

Definition   Let $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$. Then the **determinant** of A is the scalar

$$\det A = |A| = a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} \quad (1)$$

Notice that each of the 2×2 determinants is obtained by deleting the row and column of A that contain the entry the determinant is being multiplied by. For example, the first summand is a_{11} multiplied by the determinant of the submatrix obtained by deleting row 1 and column 1. Notice also that the plus and minus signs alternate in Equation (1). If we denote by A_{ij} the submatrix of a matrix A obtained by deleting row i and column j, then we may abbreviate Equation (1) as

$$\det A = a_{11} \det A_{11} - a_{12} \det A_{12} + a_{13} \det A_{13}$$

$$= \sum_{j=1}^{3} (-1)^{1+j} a_{1j} \det A_{1j}$$

For any square matrix A, $\det A_{ij}$ is called the **(i, j)-minor** of A.

Example 4.8   Compute the determinant of

$$A = \begin{bmatrix} 5 & -3 & 2 \\ 1 & 0 & 2 \\ 2 & -1 & 3 \end{bmatrix}$$

Solution   We compute

$$\det A = 5 \begin{vmatrix} 0 & 2 \\ -1 & 3 \end{vmatrix} - (-3) \begin{vmatrix} 1 & 2 \\ 2 & 3 \end{vmatrix} + 2 \begin{vmatrix} 1 & 0 \\ 2 & -1 \end{vmatrix}$$

$$= 5(0 - (-2)) + 3(3 - 4) + 2(-1 - 0)$$

$$= 5(2) + 3(-1) + 2(-1) = 5$$

With a little practice, you should find that you can easily work out 2×2 determinants in your head. Writing out the second line in the above solution is then unnecessary.

Another method for calculating the determinant of a 3×3 matrix is analogous to the method for calculating the determinant of a 2×2 matrix. Copy the first two columns of A to the right of the matrix and take the products of the elements on the six

diagonals shown below. Attach plus signs to the products from the downward-sloping diagonals and attach minus signs to the products from the upward-sloping diagonals.

$$
\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{matrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{matrix} \tag{2}
$$

This method gives

$$
a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{31}a_{22}a_{13} - a_{32}a_{23}a_{11} - a_{33}a_{21}a_{12}
$$

In Exercise 19, you are asked to check that this result agrees with that from Equation (1) for a 3×3 determinant.

Example 4.9

Calculate the determinant of the matrix in Example 4.8 using the method shown in (2).

Solution   We adjoin to A its first two columns and compute the six indicated products:

$$
\begin{bmatrix} 5 & -3 & 2 \\ 1 & 0 & 2 \\ 2 & -1 & 3 \end{bmatrix} \begin{matrix} 5 & -3 \\ 1 & 0 \\ 2 & -1 \end{matrix}
$$

Adding the three products at the bottom and subtracting the three products at the top gives

$$
\det A = 0 + (-12) + (-2) - 0 - (-10) - (-9) = 5
$$

as before.

Warning   We are about to define determinants for arbitrary square matrices. However, there is *no* analogue of the method in Example 4.9 for larger matrices. It is valid *only* for 3×3 matrices.

Determinants of $n \times n$ Matrices

The definition of the determinant of a 3×3 matrix extends naturally to arbitrary square matrices.

Definition   Let $A = [a_{ij}]$ be an $n \times n$ matrix, where $n \geq 2$. Then the ***determinant*** of A is the scalar

$$
\det A = |A| = a_{11} \det A_{11} - a_{12} \det A_{12} + \cdots + (-1)^{1+n}a_{1n} \det A_{1n}
$$

$$
= \sum_{j=1}^{n} (-1)^{1+j}a_{1j} \det A_{1j} \tag{3}
$$

It is convenient to combine a minor with its plus or minus sign. To this end, we define the *(i, j)-cofactor of A* to be

$$C_{ij} = (-1)^{i+j} \det A_{ij}$$

With this notation, definition (3) becomes

$$\det A = \sum_{j=1}^{n} a_{1j} C_{1j} \tag{4}$$

Exercise 20 asks you to check that this definition correctly gives the formula for the determinant of a 2×2 matrix when $n = 2$.

   Definition (4) is often referred to as *cofactor expansion along the first row.* It is an amazing fact that we get exactly the same result by expanding along *any* row (or even *any column*)! We summarize this fact as a theorem but defer the proof until the end of this section (since it is somewhat lengthy and would interrupt our discussion if we were to present it here).

Theorem 4.1

The Laplace Expansion Theorem

The determinant of an $n \times n$ matrix $A = [a_{ij}]$, where $n \geq 2$, can be computed as

$$\det A = a_{i1}C_{i1} + a_{i2}C_{i2} + \cdots + a_{in}C_{in}$$
$$= \sum_{j=1}^{n} a_{ij} C_{ij} \tag{5}$$

(which is the *cofactor expansion along the ith row*) and also as

$$\det A = a_{1j}C_{1j} + a_{2j}C_{2j} + \cdots + a_{nj}C_{nj}$$
$$= \sum_{i=1}^{n} a_{ij} C_{ij} \tag{6}$$

(the *cofactor expansion along the jth column*).

Since $C_{ij} = (-1)^{i+j} \det A_{ij}$, each cofactor is plus or minus the corresponding minor, with the correct sign given by the term $(-1)^{i+j}$. A quick way to determine whether the sign is $+$ or $-$ is to remember that the signs form a "checkerboard" pattern:

$$\begin{bmatrix} + & - & + & - & \cdots \\ - & + & - & + & \cdots \\ + & - & + & - & \cdots \\ - & + & - & + & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

Example 4.10

Compute the determinant of the matrix

$$A = \begin{bmatrix} 5 & -3 & 2 \\ 1 & 0 & 2 \\ 2 & -1 & 3 \end{bmatrix}$$

by (a) cofactor expansion along the third row and (b) cofactor expansion along the second column.

Solution

(a) We compute

$$\det A = a_{31}C_{31} + a_{32}C_{32} + a_{33}C_{33}$$

$$= 2 \begin{vmatrix} -3 & 2 \\ 0 & 2 \end{vmatrix} - (-1) \begin{vmatrix} 5 & 2 \\ 1 & 2 \end{vmatrix} + 3 \begin{vmatrix} 5 & -3 \\ 1 & 0 \end{vmatrix}$$

$$= 2(-6) + 8 + 3(3)$$

$$= 5$$

(b) In this case, we have

$$\det A = a_{12}C_{12} + a_{22}C_{22} + a_{32}C_{32}$$

$$= -(-3) \begin{vmatrix} 1 & 2 \\ 2 & 3 \end{vmatrix} + 0 \begin{vmatrix} 5 & 2 \\ 2 & 3 \end{vmatrix} - (-1) \begin{vmatrix} 5 & 2 \\ 1 & 2 \end{vmatrix}$$

$$= 3(-1) + 0 + 8$$

$$= 5$$

Pierre Simon Laplace (1749–1827) was born in Normandy, France, and was expected to become a clergyman until his mathematical talents were noticed at school. He made many important contributions to calculus, probability, and astronomy. He was an examiner of the young Napoleon Bonaparte at the Royal Artillery Corps and later, when Napoleon was in power, served briefly as Minister of the Interior and then Chancellor of the Senate. Laplace was granted the title of Count of the Empire in 1806 and received the title of Marquis de Laplace in 1817.

Notice that in part (b) of Example 4.10 we needed to do fewer calculations than in part (a) because we were expanding along a column that contained a zero entry—namely, a_{22}; therefore, we did not need to compute C_{22}. It follows that the Laplace Expansion Theorem is most useful when the matrix contains a row or column with lots of zeros, since, by choosing to expand along that row or column, we minimize the number of cofactors we need to compute.

Example 4.11

Compute the determinant of

$$A = \begin{bmatrix} 2 & -3 & 0 & 1 \\ 5 & 4 & 2 & 0 \\ 1 & -1 & 0 & 3 \\ -2 & 1 & 0 & 0 \end{bmatrix}$$

Solution First, notice that column 3 has only one nonzero entry; we should therefore expand along this column. Next, note that the $+/-$ pattern assigns a minus sign

to the entry $a_{23} = 2$. Thus, we have

$$\det A = a_{13}C_{13} + a_{23}C_{23} + a_{33}C_{33} + a_{43}C_{43}$$
$$= 0(C_{13}) + 2C_{23} + 0(C_{33}) + 0(C_{43})$$
$$= -2\begin{vmatrix} 2 & -3 & 1 \\ 1 & -1 & 3 \\ -2 & 1 & 0 \end{vmatrix}$$

We now continue by expanding along the third row of the determinant above (the third column would also be a good choice) to get

$$\det A = -2\left(-2\begin{vmatrix} -3 & 1 \\ -1 & 3 \end{vmatrix} - \begin{vmatrix} 2 & 1 \\ 1 & 3 \end{vmatrix} \right)$$
$$= -2(-2(-8) - 5)$$
$$= -2(11) = -22$$

(Note that the $+/-$ pattern for the 3×3 minor is not that of the original matrix but that of a 3×3 matrix in general.)

The Laplace expansion is particularly useful when the matrix is (upper or lower) triangular.

Example 4.12   Compute the determinant of

$$A = \begin{bmatrix} 2 & -3 & 1 & 0 & 4 \\ 0 & 3 & 2 & 5 & 7 \\ 0 & 0 & 1 & 6 & 0 \\ 0 & 0 & 0 & 5 & 2 \\ 0 & 0 & 0 & 0 & -1 \end{bmatrix}$$

Solution   We expand along the first column to get

$$\det A = 2\begin{vmatrix} 3 & 2 & 5 & 7 \\ 0 & 1 & 6 & 0 \\ 0 & 0 & 5 & 2 \\ 0 & 0 & 0 & -1 \end{vmatrix}$$

(We have omitted all cofactors corresponding to zero entries.) Now we expand along the first column again:

$$\det A = 2 \cdot 3\begin{vmatrix} 1 & 6 & 0 \\ 0 & 5 & 2 \\ 0 & 0 & -1 \end{vmatrix}$$

Continuing to expand along the first column, we complete the calculation:

$$\det A = 2 \cdot 3 \cdot 1\begin{vmatrix} 5 & 2 \\ 0 & -1 \end{vmatrix} = 2 \cdot 3 \cdot 1 \cdot (5(-1) - 2 \cdot 0) = 2 \cdot 3 \cdot 1 \cdot 5 \cdot (-1) = -30$$

Example 4.12 should convince you that the determinant of a triangular matrix is the product of its diagonal entries. You are asked to give a proof of this fact in Exercise 21. We record the result as a theorem.

Theorem 4.2

The determinant of a triangular matrix is the product of the entries on its main diagonal. Specifically, if $A = [a_{ij}]$ is an $n \times n$ triangular matrix, then

$$\det A = a_{11}a_{22}\cdots a_{nn}$$

Note   In general (that is, unless the matrix is triangular or has some other special form), computing a determinant by cofactor expansion is not efficient. For example, the determinant of a 3×3 matrix has $6 = 3!$ summands, each requiring two multiplications, and then five additions and subtractions are needed to finish off the calculations. For an $n \times n$ matrix, there will be $n!$ summands, each with $n - 1$ multiplications, and then $n! - 1$ additions and subtractions. The total number of operations is thus

$$T(n) = (n - 1)n! + n! - 1 > n!$$

Even the fastest of supercomputers cannot calculate the determinant of a moderately large matrix using cofactor expansion. To illustrate: Suppose we needed to calculate a 50×50 determinant. (Matrices *much* larger than 50×50 are used to store the data from digital images such as those transmitted over the Internet or taken by a digital camera.) To calculate the determinant directly would require, in general, more than $50!$ operations, and $50! \approx 3 \times 10^{64}$. If we had a computer that could perform a trillion (10^{12}) operations per second, it would take approximately 3×10^{52} seconds, or almost 10^{45} years, to finish the calculations. To put this in perspective, consider that astronomers estimate the age of the universe to be at least 10 billion (10^{10}) years. Thus, on even a very fast supercomputer, calculating a 50×50 determinant by cofactor expansion would take more than 10^{30} times the age of the universe!

Fortunately, there are better methods—and we now turn to developing more computationally effective means of finding determinants. First, we need to look at some of the properties of determinants.

Properties of Determinants

The most efficient way to compute determinants is to use row reduction. However, not every elementary row operation leaves the determinant of a matrix unchanged. The next theorem summarizes the main properties you need to understand in order to use row reduction effectively.

Theorem 4.3

Let $A = [a_{ij}]$ be a square matrix.

a. If A has a zero row (column), then $\det A = 0$.
b. If B is obtained by interchanging two rows (columns) of A, then $\det B = -\det A$.
c. If A has two identical rows (columns), then $\det A = 0$.
d. If B is obtained by multiplying a row (column) of A by k, then $\det B = k \det A$.
e. If A, B, and C are identical except that the ith row (column) of C is the sum of the ith rows (columns) of A and B, then $\det C = \det A + \det B$.
f. If B is obtained by adding a multiple of one row (column) of A to another row (column), then $\det B = \det A$.

Proof   We will prove (b) as Lemma 4.14 at the end of this section. The proofs of properties (a) and (f) are left as exercises. We will prove the remaining properties in terms of rows; the corresponding proofs for columns are analogous.

(c) If A has two identical rows, swap them to obtain the matrix B. Clearly, $B = A$, so $\det B = \det A$. On the other hand, by (b), $\det B = -\det A$. Therefore, $\det A = -\det A$, so $\det A = 0$.

(d) Suppose row i of A is multiplied by k to produce B; that is, $b_{ij} = ka_{ij}$ for $j = 1, \ldots, n$. Since the cofactors C_{ij} of the elements in the ith rows of A and B are identical (why?), expanding along the ith row of B gives

$$\det B = \sum_{j=1}^{n} b_{ij} C_{ij} = \sum_{j=1}^{n} ka_{ij} C_{ij} = k \sum_{j=1}^{n} a_{ij} C_{ij} = k \det A$$

(e) As in (d), the cofactors C_{ij} of the elements in the ith rows of A, B, and C are identical. Moreover, $c_{ij} = a_{ij} + b_{ij}$ for $j = 1, \ldots, n$. We expand along the ith row of C to obtain

$$\det C = \sum_{j=1}^{n} c_{ij} C_{ij} = \sum_{j=1}^{n} (a_{ij} + b_{ij}) C_{ij} = \sum_{j=1}^{n} a_{ij} C_{ij} + \sum_{j=1}^{n} b_{ij} C_{ij} = \det A + \det B$$

Notice that properties (b), (d), and (f) are related to elementary row operations. Since the echelon form of a square matrix is necessarily upper triangular, we can combine these properties with Theorem 2 to calculate determinants efficiently. (See Exploration: Counting Operations in Chapter 2, which shows that row reduction of an $n \times n$ matrix uses on the order of n^3 operations, far fewer than the $n!$ needed for cofactor expansion.) The next examples illustrate the computation of determinants using row reduction.

Example 4.13

Compute $\det A$ if

(a) $A = \begin{bmatrix} 2 & 3 & -1 \\ 0 & 5 & 3 \\ -4 & -6 & 2 \end{bmatrix}$

(b) $A = \begin{bmatrix} 0 & 2 & -4 & 5 \\ 3 & 0 & -3 & 6 \\ 2 & 4 & 5 & 7 \\ 5 & -1 & -3 & 1 \end{bmatrix}$

Solution

(a) Using property (f) and then property (a), we have

$$\det A = \begin{vmatrix} 2 & 3 & -1 \\ 0 & 5 & 3 \\ -4 & -6 & 2 \end{vmatrix} \overset{R_3 + 2R_1}{=} \begin{vmatrix} 2 & 3 & -1 \\ 0 & 5 & 3 \\ 0 & 0 & 0 \end{vmatrix} = 0$$

(b) We reduce A to echelon form as follows (there are other possible ways to do this):

$$
\det A = \begin{vmatrix} 0 & 2 & -4 & 5 \\ 3 & 0 & -3 & 6 \\ 2 & 4 & 5 & 7 \\ 5 & -1 & -3 & 1 \end{vmatrix} \overset{R_1 \leftrightarrow R_2}{=} -\begin{vmatrix} 3 & 0 & -3 & 6 \\ 0 & 2 & -4 & 5 \\ 2 & 4 & 5 & 7 \\ 5 & -1 & -3 & 1 \end{vmatrix} \overset{R_1/3}{=} -3\begin{vmatrix} 1 & 0 & -1 & 2 \\ 0 & 2 & -4 & 5 \\ 2 & 4 & 5 & 7 \\ 5 & -1 & -3 & 1 \end{vmatrix}
$$

$$
\overset{\substack{R_3-2R_1 \\ R_4-5R_1}}{=} -3\begin{vmatrix} 1 & 0 & -1 & 2 \\ 0 & 2 & -4 & 5 \\ 0 & 4 & 7 & 3 \\ 0 & -1 & 2 & -9 \end{vmatrix} \overset{R_2 \leftrightarrow R_4}{=} -(-3)\begin{vmatrix} 1 & 0 & -1 & 2 \\ 0 & -1 & 2 & -9 \\ 0 & 4 & 7 & 3 \\ 0 & 2 & -4 & 5 \end{vmatrix}
$$

$$
\overset{\substack{R_3+4R_2 \\ R_4+2R_2}}{=} 3\begin{vmatrix} 1 & 0 & -1 & 2 \\ 0 & -1 & 2 & -9 \\ 0 & 0 & 15 & -33 \\ 0 & 0 & 0 & -13 \end{vmatrix}
$$

$$
= 3 \cdot 1 \cdot (-1) \cdot 15 \cdot (-13) = 585
$$

Remark By Theorem 4.3, we can also use elementary column operations in the process of computing determinants, and we can "mix and match" elementary row and column operations. For example, in Example 4.13(a), we could have started by adding column 3 to column 1 to create a leading 1 in the upper left-hand corner. In fact, the method we used was faster, but in other examples column operations may speed up the calculations. Keep this in mind when you work determinants by hand.

Determinants of Elementary Matrices

Recall from Section 3.3 that an elementary matrix results from performing an elementary row operation on an identity matrix. Setting $A = I_n$ in Theorem 4.3 yields the following theorem.

Theorem 4.4

Let E be an $n \times n$ elementary matrix.

a. If E results from interchanging two rows of I_n, then $\det E = -1$.
b. If E results from multiplying one row of I_n by k, then $\det E = k$.
c. If E results from adding a multiple of one row of I_n to another row, then $\det E = 1$.

The word *lemma* is derived from the Greek verb *lambanein*, which means "to grasp." In mathematics, a lemma is a "helper theorem" that we "grasp hold of" and use to prove another, usually more important, theorem.

Proof Since $\det I_n = 1$, applying (b), (d), and (f) of Theorem 4.3 immediately gives (a), (b), and (c), respectively, of Theorem 4.4.

Next, recall that multiplying a matrix B by an elementary matrix *on the left* performs the corresponding elementary row operation on B. We can therefore rephrase (b), (d), and (f) of Theorem 4.3 succinctly as the following lemma, the proof of which is straightforward and is left as Exercise 43.

Lemma 4.5

Let B be an $n \times n$ matrix and let E be an $n \times n$ elementary matrix. Then

$$\det(EB) = (\det E)(\det B)$$

We can use Lemma 4.5 to prove the main theorem of this section: a characterization of invertibility in terms of determinants.

Theorem 4.6

A square matrix A is invertible if and only if $\det A \neq 0$.

Proof   Let A be an $n \times n$ matrix and let R be the reduced row echelon form of A. We will show first that $\det A \neq 0$ if and only if $\det R \neq 0$. Let $E_1, E_2, \ldots, E_r$ be the elementary matrices corresponding to the elementary row operations that reduce A to R. Then

$$E_r \cdots E_2 E_1 A = R$$

Taking determinants of both sides and repeatedly applying Lemma 4.5, we obtain

$$(\det E_r) \cdots (\det E_2)(\det E_1)(\det A) = \det R$$

By Theorem 4.4, the determinants of all the elementary matrices are nonzero. We conclude that $\det A \neq 0$ if and only if $\det R \neq 0$.

Now suppose that A is invertible. Then, by the Fundamental Theorem of Invertible Matrices, $R = I_n$, so $\det R = 1 \neq 0$. Hence, $\det A \neq 0$ also. Conversely, if $\det A \neq 0$, then $\det R \neq 0$, so R cannot contain a zero row, by Theorem 4.3(a). It follows that R must be I_n (why?), so A is invertible, by the Fundamental Theorem again.

Determinants and Matrix Operations

Let's now try to determine what relationship, if any, exists between determinants and some of the basic matrix operations. Specifically, we would like to find formulas for $\det(kA), \det(A + B), \det(AB), \det(A^{-1})$, and $\det(A^T)$ in terms of $\det A$ and $\det B$.

Theorem 4.3(d) does *not* say that $\det(kA) = k \det A$. The correct relationship between scalar multiplication and determinants is given by the following theorem.

Theorem 4.7

If A is an $n \times n$ matrix, then

$$\det(kA) = k^n \det A$$

You are asked to give a proof of this theorem in Exercise 44.

Unfortunately, there is no simple formula for $\det(A + B)$, and in general, $\det(A + B) \neq \det A + \det B$. (Find two 2×2 matrices that verify this.) It therefore comes as a pleasant surprise to find out that determinants are quite compatible with matrix multiplication. Indeed, we have the following nice formula due to Cauchy.

Augustin Louis Cauchy (1789–1857) was born in Paris and studied engineering but switched to mathematics because of poor health. A brilliant and prolific mathematician, he published over 700 papers, many on quite difficult problems. His name can be found on many theorems and definitions in differential equations, infinite series, probability theory, algebra, and physics. He is noted for introducing rigor into calculus, laying the foundation for the branch of mathematics known as analysis. Politically conservative, Cauchy was a royalist, and in 1830 he followed Charles X into exile. He returned to France in 1838 but did not return to his post at the Sorbonne until the university dropped its requirement that faculty swear an oath of loyalty to the new king.

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Theorem 4.8   If A and B are $n \times n$ matrices, then

$$\det(AB) = (\det A)(\det B)$$

Proof   We consider two cases: A invertible and A not invertible.

If A is invertible, then, by the Fundamental Theorem of Invertible Matrices, it can be written as a product of elementary matrices—say,

$$A = E_1 E_2 \cdots E_k$$

Then $AB = E_1 E_2 \cdots E_k B$, so k applications of Lemma 4.5 give

$$\det(AB) = \det(E_1 E_2 \cdots E_k B) = (\det E_1)(\det E_2) \cdots (\det E_k)(\det B)$$

Continuing to apply Lemma 4.5, we obtain

$$\det(AB) = \det(E_1 E_2 \cdots E_k)\det B = (\det A)(\det B)$$

On the other hand, if A is not invertible, then neither is AB, by Exercise 47 in Section 3.3. Thus, by Theorem 4.6, $\det A = 0$ and $\det(AB) = 0$. Consequently, $\det(AB) = (\det A)(\det B)$, since both sides are zero.

Example 4.14   Applying Theorem 4.8 to $A = \begin{bmatrix} 2 & 1 \\ 2 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} 5 & 1 \\ 2 & 1 \end{bmatrix}$, we find that

$$AB = \begin{bmatrix} 12 & 3 \\ 16 & 5 \end{bmatrix}$$

and that $\det A = 4$, $\det B = 3$, and $\det(AB) = 12 = 4 \cdot 3 = (\det A)(\det B)$, as claimed. (Check these assertions!)

The next theorem gives a nice relationship between the determinant of an invertible matrix and the determinant of its inverse.

Theorem 4.9   If A is invertible, then

$$\det(A^{-1}) = \frac{1}{\det A}$$

Proof   Since A is invertible, $AA^{-1} = I$, so $\det(AA^{-1}) = \det I = 1$. Hence, $(\det A)(\det A^{-1}) = 1$, by Theorem 4.8, and since $\det A \neq 0$ (why?), dividing by $\det A$ yields the result.

Example 4.15   Verify Theorem 4.9 for the matrix A of Example 4.14.

Solution   We compute

$$A^{-1} = \frac{1}{4}\begin{bmatrix} 3 & -1 \\ -2 & 2 \end{bmatrix} = \begin{bmatrix} \frac{3}{4} & -\frac{1}{4} \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

so

$$\det A^{-1} = \left(\frac{3}{4}\right)\left(\frac{1}{2}\right) - \left(-\frac{1}{4}\right)\left(-\frac{1}{2}\right) = \frac{3}{8} - \frac{1}{8} = \frac{1}{4} = \frac{1}{\det A}$$

Remark   The beauty of Theorem 4.9 is that sometimes we do not need to know *what* the inverse of a matrix is, but only that it *exists,* or to know what its determinant is. For the matrix A in the last two examples, once we know that $\det A = 4 \neq 0$, we immediately can deduce that A is invertible and that $\det A^{-1} = \frac{1}{4}$ without actually computing A^{-1}.

We now relate the determinant of a matrix A to that of its transpose A^T. Since the rows of A^T are just the columns of A, evaluating $\det A^T$ by expanding along the first row is identical to evaluating $\det A$ by expanding along its first column, which the Laplace Expansion Theorem allows us to do. Thus, we have the following result.

Theorem 4.10   For any square matrix A,

$$\det A = \det A^T$$

Cramer's Rule and the Adjoint

In this section, we derive two useful formulas relating determinants to the solution of linear systems and the inverse of a matrix. The first of these, **Cramer's Rule,** gives a formula for describing the solution of certain systems of n linear equations in n variables entirely in terms of determinants. While this result is of little practical use beyond 2×2 systems, it is of great theoretical importance.

We will need some new notation for this result and its proof. For an $n \times n$ matrix A and a vector $\mathbf{b}$ in $\mathbb{R}^n$, let $A_i(\mathbf{b})$ denote the matrix obtained by replacing the ith column of A by $\mathbf{b}$. That is,

$$\overset{\overset{\text{Column } i}{\downarrow}}{A_i(\mathbf{b})} = [\mathbf{a}_1 \cdots \mathbf{b} \cdots \mathbf{a}_n]$$

Theorem 1.11   **Cramer's Rule**

Let A be an invertible $n \times n$ matrix and let $\mathbf{b}$ be a vector in $\mathbb{R}^n$. Then the unique solution $\mathbf{x}$ of the system $A\mathbf{x} = \mathbf{b}$ is given by

$$x_i = \frac{\det(A_i(\mathbf{b}))}{\det A} \quad \text{for } i = 1, \ldots, n$$

Proof   The columns of the identity matrix $I = I_n$ are the standard unit vectors $\mathbf{e}_1$, $\mathbf{e}_2, \ldots, \mathbf{e}_n$. If $A\mathbf{x} = \mathbf{b}$, then

$$AI_i(\mathbf{x}) = A[\mathbf{e}_1 \;\; \cdots \;\; \mathbf{x} \;\; \cdots \;\; \mathbf{e}_n] = [A\mathbf{e}_1 \;\; \cdots \;\; A\mathbf{x} \;\; \cdots \;\; A\mathbf{e}_n]$$
$$= [\mathbf{a}_1 \;\; \cdots \;\; \mathbf{b} \;\; \cdots \;\; \mathbf{a}_n] = A_i(\mathbf{b})$$

Therefore, by Theorem 4.8,

$$(\det A)(\det I_i(\mathbf{x})) = \det(AI_i(\mathbf{x})) = \det(A_i(\mathbf{b}))$$

Now

$$\det I_i(\mathbf{x}) = \begin{vmatrix} 1 & 0 & \cdots & x_1 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & x_2 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & x_i & \cdots & 0 & 0 \\ \vdots & \vdots & & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & x_{n-1} & \cdots & 1 & 0 \\ 0 & 0 & \cdots & x_n & \cdots & 0 & 1 \end{vmatrix} = x_i$$

as can be seen by expanding along the ith row. Thus, $(\det A)x_i = \det(A_i(\mathbf{b}))$, and the result follows by dividing by $\det A$ (which is nonzero, since A is invertible).

Example 4.16   Use Cramer's Rule to solve the system

$$x_1 + 2x_2 = 2$$
$$-x_1 + 4x_2 = 1$$

Solution   We compute

$$\det A = \begin{vmatrix} 1 & 2 \\ -1 & 4 \end{vmatrix} = 6, \quad \det(A_1(\mathbf{b})) = \begin{vmatrix} 2 & 2 \\ 1 & 4 \end{vmatrix} = 6, \quad \text{and} \quad \det(A_2(\mathbf{b})) = \begin{vmatrix} 1 & 2 \\ -1 & 1 \end{vmatrix}$$
$$= 3$$

By Cramer's Rule,

$$x_1 = \frac{\det(A_1(\mathbf{b}))}{\det A} = \frac{6}{6} = 1 \quad \text{and} \quad x_2 = \frac{\det(A_2(\mathbf{b}))}{\det A} = \frac{3}{6} = \frac{1}{2}$$

Remark   As noted previously, Cramer's Rule is computationally inefficient for all but small systems of linear equations because it involves the calculation of many determinants. The effort expended to compute just one of these determinants, using even the most efficient method, would be better spent using Gaussian elimination to solve the system directly.

The final result of this section is a formula for the inverse of a matrix in terms of determinants. This formula was hinted at by the formula for the inverse of a 3×3 matrix, which was given without proof at the beginning of this section. Thus, we have come full circle.

Let's discover the formula for ourselves. If A is an invertible $n \times n$ matrix, its inverse is the (unique) matrix X that satisfies the equation $AX = I$. Solving for X one column at a time, let $\mathbf{x}_j$ be the jth column of X. That is,

$$\mathbf{x}_j = \begin{bmatrix} x_{1j} \\ \vdots \\ x_{ij} \\ \vdots \\ x_{nj} \end{bmatrix}$$

Therefore, $A\mathbf{x}_j = \mathbf{e}_j$, and by Cramer's Rule,

$$x_{ij} = \frac{\det(A_i(\mathbf{e}_j))}{\det A}$$

However,

$$\det(A_i(\mathbf{e}_j)) = \begin{vmatrix} a_{11} & a_{12} & \cdots & 0 & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & 0 & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ a_{j1} & a_{j2} & \cdots & 1 & \cdots & a_{jn} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & 0 & \cdots & a_{nn} \end{vmatrix} = (-1)^{j+i}\det A_{ji} = C_{ji}$$

*i*th column (↓)

which is the (j, i)-cofactor of A.

It follows that $x_{ij} = (1/\det A)C_{ji}$, so $A^{-1} = X = (1/\det A)[C_{ji}] = (1/\det A)[C_{ij}]^T$. In words, the inverse of A is the *transpose* of the matrix of cofactors of A, divided by the determinant of A.

The matrix

$$[C_{ji}] = [C_{ij}]^T = \begin{bmatrix} C_{11} & C_{21} & \cdots & C_{n1} \\ C_{12} & C_{22} & \cdots & C_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ C_{1n} & C_{2n} & \cdots & C_{nn} \end{bmatrix}$$

is called the **adjoint** (or **adjugate**) of A and is denoted by adj A. The result we have just proved can be stated as follows.

Theorem 4.12   Let A be an invertible $n \times n$ matrix. Then

$$A^{-1} = \frac{1}{\det A} \operatorname{adj} A$$

Example 4.17   Use the adjoint method to compute the inverse of

$$A = \begin{bmatrix} 1 & 2 & -1 \\ 2 & 2 & 4 \\ 1 & 3 & -3 \end{bmatrix}$$

Solution   We compute $\det A = -2$ and the nine cofactors

$$C_{11} = + \begin{vmatrix} 2 & 4 \\ 3 & -3 \end{vmatrix} = -18 \quad C_{12} = - \begin{vmatrix} 2 & 4 \\ 1 & -3 \end{vmatrix} = 10 \quad C_{13} = + \begin{vmatrix} 2 & 2 \\ 1 & 3 \end{vmatrix} = 4$$

$$C_{21} = - \begin{vmatrix} 2 & -1 \\ 3 & -3 \end{vmatrix} = 3 \quad C_{22} = + \begin{vmatrix} 1 & -1 \\ 1 & -3 \end{vmatrix} = -2 \quad C_{23} = - \begin{vmatrix} 1 & 2 \\ 1 & 3 \end{vmatrix} = -1$$

$$C_{31} = + \begin{vmatrix} 2 & -1 \\ 2 & 4 \end{vmatrix} = 10 \quad C_{32} = - \begin{vmatrix} 1 & -1 \\ 2 & 4 \end{vmatrix} = -6 \quad C_{33} = + \begin{vmatrix} 1 & 2 \\ 2 & 2 \end{vmatrix} = -2$$

The adjoint is the *transpose* of the matrix of cofactors—namely,

$$\operatorname{adj} A = \begin{bmatrix} -18 & 10 & 4 \\ 3 & -2 & -1 \\ 10 & -6 & -2 \end{bmatrix}^{T} = \begin{bmatrix} -18 & 3 & 10 \\ 10 & -2 & -6 \\ 4 & -1 & -2 \end{bmatrix}$$

Then

$$A^{-1} = \frac{1}{\det A} \operatorname{adj} A = -\frac{1}{2} \begin{bmatrix} -18 & 3 & 10 \\ 10 & -2 & -6 \\ 4 & -1 & -2 \end{bmatrix} = \begin{bmatrix} 9 & -\frac{3}{2} & -5 \\ -5 & 1 & 3 \\ -2 & \frac{1}{2} & 1 \end{bmatrix}$$

which is the same answer we obtained (with less work) in Example 3.30.

Proof of the Laplace Expansion Theorem

Unfortunately, there is no short, easy proof of the Laplace Expansion Theorem. The proof we give has the merit of being relatively straightforward. We break it down into several steps, the first of which is to prove that cofactor expansion along the first row of a matrix is the same as cofactor expansion along the first column.

Lemma 4.13   Let A be an $n \times n$ matrix. Then

$$a_{11}C_{11} + a_{12}C_{12} + \cdots + a_{1n}C_{1n} = \det A = a_{11}C_{11} + a_{21}C_{21} + \cdots + a_{n1}C_{n1} \quad (7)$$

Proof   We prove this lemma by induction on n. For $n = 1$, the result is trivial. Now assume that the result is true for $(n - 1) \times (n - 1)$ matrices; this is our induction hypothesis. Note that, by the definition of cofactor (or minor), all of the terms containing a_{11} are accounted for by the summand $a_{11}C_{11}$. We can therefore ignore terms containing a_{11}.

The ith summand on the right-hand side of Equation (7) is $a_{i1}C_{i1} = a_{i1}(-1)^{i+1}$ det A_{i1}. Now we expand det A_{i1} along the first row:

$$
\begin{vmatrix}
a_{12} & a_{13} & \cdots & a_{1j} & \cdots & a_{1n} \\
\vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\
a_{i-1,2} & a_{i-1,3} & \cdots & a_{i-1,j} & \cdots & a_{i-1,n} \\
a_{i+1,2} & a_{i+1,3} & \cdots & a_{i+1,j} & \cdots & a_{i+1,n} \\
\vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\
a_{n2} & a_{n3} & \cdots & a_{nj} & \cdots & a_{n,n}
\end{vmatrix}
$$

The jth term in this expansion of det A_{i1} is $a_{1j}(-1)^{1+j-1}$ det $A_{1i,1j}$, where the notation $A_{kl,rs}$ denotes the submatrix of A obtained by deleting rows k and l and columns r and s. Combining these, we see that the term containing $a_{i1}a_{1j}$ on the right-hand side of Equation (7) is

$$
a_{i1}(-1)^{i+1}a_{1j}(-1)^{1+j-1} \det A_{1i,1j} = (-1)^{i+j+1}a_{i1}a_{1j} \det A_{1i,1j}
$$

What is the term containing $a_{i1}a_{1j}$ on the left-hand side of Equation (7)? The factor a_{1j} occurs in the jth summand, $a_{1j}C_{1j} = a_{1j}(-1)^{1+j}$ det A_{1j}. By the induction hypothesis, we can expand det A_{1j} along its first column:

$$
\begin{vmatrix}
a_{21} & \cdots & a_{2,j-1} & a_{2,j+1} & \cdots & a_{2n} \\
a_{31} & \cdots & a_{3,j-1} & a_{3,j+1} & \cdots & a_{3n} \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
a_{i1} & \cdots & a_{i,j-1} & a_{i,j+1} & \cdots & a_{in} \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
a_{n1} & \cdots & a_{n,j-1} & a_{n,j+1} & \cdots & a_{nn}
\end{vmatrix}
$$

The ith term in this expansion of det A_{1j} is $a_{i1}(-1)^{(i-1)+1}$ det $A_{1i,1j}$, so the term containing $a_{i1}a_{1j}$ on the left-hand side of Equation (7) is

$$
a_{1j}(-1)^{1+j}a_{i1}(-1)^{(i-1)+1} \det A_{1i,1j} = (-1)^{i+j+1}a_{i1}a_{1j} \det A_{1i,1j}
$$

which establishes that the left- and right-hand sides of Equation (7) are equivalent.

Next, we prove property (b) of Theorem 4.3.

Lemma 4.14   Let A be an $n \times n$ matrix and let B be obtained by interchanging any two rows (columns) of A. Then

$$
\det B = -\det A
$$

Proof   Once again, the proof is by induction on n. The result can be easily checked when $n = 2$, so assume that it is true for $(n - 1) \times (n - 1)$ matrices. We will prove that the result is true for $n \times n$ matrices. First, we prove that it holds when two adjacent rows of A are interchanged—say, rows r and $r + 1$.

By Lemma 4.13, we can evaluate $\det B$ by cofactor expansion along its first column. The ith term in this expansion is $(-1)^{1+i} b_{i1} \det B_{i1}$. If $i \neq r$ and $i \neq r + 1$, then $b_{i1} = a_{i1}$ and B_{i1} is an $(n - 1) \times (n - 1)$ submatrix that is identical to A_{i1} except that two adjacent rows have been interchanged.

$$
\begin{vmatrix}
a_{11} & a_{12} & \cdots & a_{1n} \\
\vdots & \vdots & & \vdots \\
a_{i1} & a_{i2} & \cdots & a_{in} \\
\vdots & \vdots & & \\
a_{r+1,1} & a_{r+1,2} & \cdots & a_{r+1,n} \\
a_{r1} & a_{r2} & \cdots & a_{rn} \\
\vdots & \vdots & & \vdots \\
a_{n1} & a_{n2} & \cdots & a_{nn}
\end{vmatrix}
$$

Thus, by the induction hypothesis, $\det B_{i1} = -\det A_{i1}$ if $i \neq r$ and $i \neq r + 1$.

If $i = r$, then $b_{i1} = a_{r+1,1}$ and $B_{i1} = A_{r+1,1}$.

$$
\text{Row } i \rightarrow \begin{vmatrix}
a_{11} & a_{12} & \cdots & a_{1n} \\
\vdots & \vdots & & \vdots \\
a_{r+1,1} & a_{r+1,2} & \cdots & a_{r+1,n} \\
a_{r1} & a_{r2} & \cdots & a_{rn} \\
\vdots & \vdots & & \vdots \\
a_{n1} & a_{n2} & \cdots & a_{nn}
\end{vmatrix}
$$

Therefore, the rth summand in $\det B$ is

$$(-1)^{r+1} b_{r1} \det B_{r1} = (-1)^{r+1} a_{r+1,1} \det A_{r+1,1} = -(-1)^{(r+1)+1} a_{r+1,1} \det A_{r+1,1}$$

Similarly, if $i = r + 1$, then $b_{i1} = a_{r1}$, $B_{i1} = A_{r1}$, and the $(r + 1)$st summand in $\det B$ is

$$(-1)^{(r+1)+1} b_{r+1,1} \det B_{r+1,1} = (-1)^r a_{r1} \det A_{r1} = -(-1)^{r+1} a_{r1} \det A_{r1}$$

In other words, the rth and $(r + 1)$st terms in the first column cofactor expansion of $\det B$ are the *negatives* of the $(r + 1)$st and rth terms, respectively, in the first column cofactor expansion of $\det A$.

Substituting all of these results into $\det B$ and using Lemma 4.13 again, we obtain

$$\det B = \sum_{i=1}^{n} (-1)^{i+1} b_{i1} \det B_{i1}$$

$$= \sum_{\substack{i=1 \\ i \neq r, r+1}}^{n} (-1)^{i+1} b_{i1} \det B_{i1} + (-1)^{r+1} b_{r1} \det B_{r1} + (-1)^{(r+1)+1} b_{r+1,1} \det B_{r+1,1}$$

$$= \sum_{\substack{i=1 \\ i \neq r, r+1}}^{n} (-1)^{i+1} a_{i1}(-\det A_{i1}) - (-1)^{(r+1)+1} a_{r+1,1} \det A_{r+1,1} - (-1)^{r+1} a_{r1} \det A_{r1}$$

$$= -\sum_{i=1}^{n} (-1)^{i+1} a_{i1} \det A_{i1}$$

$$= -\det A$$

This proves the result for $n \times n$ matrices if adjacent rows are interchanged. To see that it holds for arbitrary row interchanges, we need only note that, for example, rows r and s, where $r < s$, can be swapped by performing $2(s - r) - 1$ interchanges of adjacent rows (see Exercise 67). Since the number of interchanges is *odd* and each one changes the sign of the determinant, the net effect is a change of sign, as desired.

The proof for column interchanges is analogous, except that we expand along row 1 instead of along column 1.

We can now prove the Laplace Expansion Theorem.

Proof of Theorem 4.1   Let B be the matrix obtained by moving row i of A to the top, using $i - 1$ interchanges of adjacent rows. By Lemma 4.14, $\det B = (-1)^{i-1} \det A$. But $b_{1j} = a_{ij}$ and $B_{1j} = A_{ij}$ for $j = 1, \ldots, n$.

$$
\det B = \begin{vmatrix}
a_{i1} & \cdots & a_{ij} & \cdots & a_{in} \\
a_{11} & \cdots & a_{1j} & \cdots & a_{1n} \\
\vdots & & \vdots & & \vdots \\
a_{i-1,1} & \cdots & a_{i-1,j} & \cdots & a_{i-1,n} \\
a_{i+1,1} & \cdots & a_{i+1,j} & \cdots & a_{i+1,n} \\
\vdots & & \vdots & & \vdots \\
a_{n1} & \cdots & a_{nj} & \cdots & a_{nn}
\end{vmatrix}
$$

Thus,

$$
\det A = (-1)^{i-1} \det B = (-1)^{i-1} \sum_{j=1}^{n} (-1)^{1+j} b_{1j} \det B_{1j}
$$

$$
= (-1)^{i-1} \sum_{j=1}^{n} (-1)^{1+j} a_{ij} \det A_{ij} = \sum_{j=1}^{n} (-1)^{i+j} a_{ij} \det A_{ij}
$$

which gives the formula for cofactor expansion along row i.

The proof for column expansion is similar, invoking Lemma 4.13 so that we can use column expansion instead of row expansion (see Exercise 68).

A Brief History of Determinants

A self-taught child prodigy, Takakazu Seki Kōwa (1642–1708) was descended from a family of samurai warriors. In addition to discovering determinants, he wrote about diophantine equations, magic squares, and Bernoulli numbers (before Bernoulli) and quite likely made discoveries in calculus.

As noted at the beginning of this section, the history of determinants predates that of matrices. Indeed, determinants were first introduced, independently, by Seki in 1683 and Leibniz in 1693. In 1748, determinants appeared in Maclaurin's *Treatise on Algebra*, which included a treatment of Cramer's Rule up to the 4×4 case. In 1750, Cramer himself proved the general case of his rule, applying it to curve fitting, and in 1772, Laplace gave a proof of his expansion theorem.

The term *determinant* was not coined until 1801, when it was used by Gauss. Cauchy made the first use of determinants in the modern sense in 1812. Cauchy, in fact, was responsible for developing much of the early theory of determinants, including several important results that we have mentioned: the product rule for determinants, the characteristic polynomial, and the notion of a diagonalizable matrix. Determinants did not become widely known until 1841, when Jacobi popularized them, albeit in the context of functions of several variables, such as are encountered in a multivariable calculus course. (These types of determinants were called "Jacobians" by Sylvester around 1850, a term that is still used today.)

Gottfried Wilhelm von Leibniz (1646–1716) was born in Leipzig and studied law, theology, philosophy, and mathematics. He is probably best known for developing (with Newton, independently) the main ideas of differential and integral calculus. However, his contributions to other branches of mathematics are also impressive. He developed the notion of a determinant, knew versions of Cramer's Rule and the Laplace Expansion Theorem before others were given credit for them, and laid the foundation for matrix theory through work he did on quadratic forms. Leibniz also was the first to develop the binary system of arithmetic. He believed in the importance of good notation and, along with the familiar notation for derivatives and integrals, introduced a form of subscript notation for the coefficients of a linear system that is essentially the notation we use today.

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By the late 19th century, the theory of determinants had developed to the stage that entire books were devoted to it, including Dodgson's *An Elementary Treatise on Determinants* in 1867 and Thomas Muir's monumental five-volume work, which appeared in the early 20th century. While their history is fascinating, today determinants are of theoretical more than practical interest. Cramer's Rule is a hopelessly inefficient method for solving a system of linear equations, and numerical methods have replaced any use of determinants in the computation of eigenvalues. Determinants are used, however, to give students an initial understanding of the characteristic polynomial (as in Sections 4.1 and 4.3).

Exercises 4.2

Compute the determinants in Exercises 1–6 using cofactor expansion along the first row and along the first column.

1. $\begin{vmatrix} 1 & 0 & 3 \\ 5 & 1 & 1 \\ 0 & 1 & 2 \end{vmatrix}$

2. $\begin{vmatrix} 0 & 1 & -1 \\ 2 & 3 & -2 \\ -1 & 3 & 0 \end{vmatrix}$

3. $\begin{vmatrix} 1 & -1 & 0 \\ -1 & 0 & 1 \\ 0 & 1 & -1 \end{vmatrix}$

4. $\begin{vmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{vmatrix}$

5. $\begin{vmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 3 & 1 & 2 \end{vmatrix}$

6. $\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix}$

Compute the determinants in Exercises 7–15 using cofactor expansion along any row or column that seems convenient.

7. $\begin{vmatrix} 5 & 2 & 2 \\ -1 & 1 & 2 \\ 3 & 0 & 0 \end{vmatrix}$

8. $\begin{vmatrix} 1 & 1 & -1 \\ 2 & 0 & 1 \\ 3 & -2 & 1 \end{vmatrix}$

9. $\begin{vmatrix} -4 & 1 & 3 \\ 2 & -2 & 4 \\ 1 & -1 & 0 \end{vmatrix}$

10. $\begin{vmatrix} \cos\theta & \sin\theta & \tan\theta \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{vmatrix}$

11. $\begin{vmatrix} a & b & 0 \\ 0 & a & b \\ a & 0 & b \end{vmatrix}$

12. $\begin{vmatrix} 0 & a & 0 \\ b & c & d \\ 0 & e & 0 \end{vmatrix}$

13. $\begin{vmatrix} 1 & -1 & 0 & 3 \\ 2 & 5 & 2 & 6 \\ 0 & 1 & 0 & 0 \\ 1 & 4 & 2 & 1 \end{vmatrix}$

14. $\begin{vmatrix} 2 & 0 & 3 & -1 \\ 1 & 0 & 2 & 2 \\ 0 & -1 & 1 & 4 \\ 2 & 0 & 1 & -3 \end{vmatrix}$

15. $\begin{vmatrix} 0 & 0 & 0 & a \\ 0 & 0 & b & c \\ 0 & d & e & f \\ g & h & i & j \end{vmatrix}$

In Exercises 16–18, compute the indicated 3×3 determinants using the method of Example 4.9.

16. The determinant in Exercise 6

17. The determinant in Exercise 8

18. The determinant in Exercise 11

19. Verify that the method indicated in (2) agrees with Equation (1) for a 3×3 determinant.

20. Verify that definition (4) agrees with the definition of a 2×2 determinant when $n = 2$.

21. Prove Theorem 4.2. [*Hint:* A proof by induction would be appropriate here.]

In Exercises 22–25, evaluate the given determinant using elementary row and/or column operations and Theorem 4.3 to reduce the matrix to row echelon form.

22. The determinant in Exercise 1

23. The determinant in Exercise 9

24. The determinant in Exercise 13

25. The determinant in Exercise 14

In Exercises 26–34, use properties of determinants to evaluate the given determinant by inspection. Explain your reasoning.

26. $\begin{vmatrix} 1 & 1 & 1 \\ 3 & 0 & -2 \\ 2 & 2 & 2 \end{vmatrix}$

27. $\begin{vmatrix} 3 & 1 & 0 \\ 0 & -2 & 5 \\ 0 & 0 & 4 \end{vmatrix}$

28. $\begin{vmatrix} 0 & 0 & 1 \\ 0 & 5 & 2 \\ 3 & -1 & 4 \end{vmatrix}$

29. $\begin{vmatrix} 2 & 3 & -4 \\ 1 & -3 & -2 \\ -1 & 5 & 2 \end{vmatrix}$

30. $\begin{vmatrix} 1 & 2 & 3 \\ 0 & 4 & 1 \\ 1 & 6 & 4 \end{vmatrix}$

31. $\begin{vmatrix} 4 & 1 & 3 \\ -2 & 0 & -2 \\ 5 & 4 & 1 \end{vmatrix}$

32. $\begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix}$

33. $\begin{vmatrix} 0 & 2 & 0 & 0 \\ -3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 4 \\ 0 & 0 & 1 & 0 \end{vmatrix}$

34. $\begin{vmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{vmatrix}$

Find the determinants in Exercises 35–40, assuming that

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = 4$$

35. $\begin{vmatrix} 2a & 2b & 2c \\ d & e & f \\ g & h & i \end{vmatrix}$

36. $\begin{vmatrix} 2a & b/3 & -c \\ 2d & e/3 & -f \\ 2g & h/3 & -i \end{vmatrix}$

37. $\begin{vmatrix} d & e & f \\ a & b & c \\ g & h & i \end{vmatrix}$

38. $\begin{vmatrix} a-c & b & c \\ d-f & e & f \\ g-i & h & i \end{vmatrix}$

39. $\begin{vmatrix} 2c & b & a \\ 2f & e & d \\ 2i & h & g \end{vmatrix}$

40. $\begin{vmatrix} a+2g & b+2h & c+2i \\ 3d+2g & 3e+2h & 3f+2i \\ g & h & i \end{vmatrix}$

41. Prove Theorem 4.3(a).   **42.** Prove Theorem 4.3(f).

43. Prove Lemma 4.5.   **44.** Prove Theorem 4.7.

In Exercises 45 and 46, use Theorem 4.6 to find all values of k for which A is invertible.

45. $A = \begin{bmatrix} k & -k & 3 \\ 0 & k+1 & 1 \\ k & -8 & k-1 \end{bmatrix}$

46. $A = \begin{bmatrix} k & k & 0 \\ k^2 & 2 & k \\ 0 & k & k \end{bmatrix}$

In Exercises 47–52, assume that A and B are $n \times n$ matrices with $\det A = 3$ and $\det B = -2$. Find the indicated determinants.

47. $\det(AB)$ **48.** $\det(A^2)$ **49.** $\det(B^{-1}A)$

50. $\det(2A)$ **51.** $\det(3B^T)$ **52.** $\det(AA^T)$

In Exercises 53–56, A and B are $n \times n$ matrices.

53. Prove that $\det(AB) = \det(BA)$.

54. If B is invertible, prove that $\det(B^{-1}AB) = \det(A)$.

55. If A is idempotent (that is, $A^2 = A$), find all possible values of $\det(A)$.

56. A square matrix A is called **nilpotent** if $A^m = O$ for some $m > 1$. (The word *nilpotent* comes from the Latin *nil*, meaning "nothing," and *potere*, meaning "to have power." A nilpotent matrix is thus one that

becomes "nothing"—that is, the zero matrix—when raised to some power.) Find all possible values of $\det(A)$ if A is nilpotent.

In Exercises 57–60, use Cramer's Rule to solve the given linear system.

57. $x + y = 1$
$\quad x - y = 2$

58. $2x - y = 5$
$\quad x + 3y = -1$

59. $2x + y + 3z = 1$
$\quad\quad y + z = 1$
$\quad\quad\quad\quad z = 1$

60. $\quad x + y - z = 1$
$\quad\quad x + y + z = 2$
$\quad\quad x - y = 3$

In Exercises 61–64, use Theorem 4.12 to compute the inverse of the coefficient matrix for the given exercise.

61. Exercise 57

62. Exercise 58

63. Exercise 59

64. Exercise 60

65. If A is an invertible $n \times n$ matrix, show that adj A is also invertible and that

$$(\text{adj } A)^{-1} = \frac{1}{\det A} A = \text{adj } (A^{-1})$$

66. If A is an $n \times n$ matrix, prove that

$$\det(\text{adj } A) = (\det A)^{n-1}$$

67. Verify that if $r < s$, then rows r and s of a matrix can be interchanged by performing $2(s - r) - 1$ interchanges of adjacent rows.

68. Prove that the Laplace Expansion Theorem holds for column expansion along the jth column.

69. Let A be a square matrix that can be partitioned as

$$A = \left[\begin{array}{c|c} P & Q \\ \hline O & S \end{array}\right]$$

where P and S are square matrices. Such a matrix is said to be in **block (upper) triangular form.** Prove that

$$\det A = (\det P)(\det S)$$

[*Hint:* Try a proof by induction on the number of rows of P.]

70. (a) Give an example to show that if A can be partitioned as

$$A = \left[\begin{array}{c|c} P & Q \\ \hline R & S \end{array}\right]$$

where P, Q, R, and S are all square, then it is not necessarily true that

$$\det A = (\det P)(\det S) - (\det Q)(\det R)$$

(b) Assume that A is partitioned as in part (a) and that P is invertible. Let

$$B = \left[\begin{array}{c|c} P^{-1} & O \\ \hline -RP^{-1} & I \end{array}\right]$$

Compute $\det(BA)$ using Exercise 69 and use the result to show that

$$\det A = \det P \det(S - RP^{-1}Q)$$

[The matrix $S - RP^{-1}Q$ is called the **Schur complement** of P in A, after Issai Schur (1875–1941), who was born in Belarus but spent most of his life in Germany. He is known mainly for his fundamental work on the representation theory of groups, but he also worked in number theory, analysis, and other areas.]

(c) Assume that A is partitioned as in part (a), that P is invertible, and that $PR = RP$. Prove that

$$\det A = \det(PS - RQ)$$

Writing Project — Which Came First: The Matrix or the Determinant?

The way in which matrices and determinants are taught today—matrices before determinants—bears little resemblance to the way these topics developed historically. There is a brief history of determinants at the end of Section 4.2.

Write a report on the history of matrices and determinants. How did the notations used for each evolve over time? Who were some of the key mathematicians involved and what were their contributions?

1. Florian Cajori, *A History of Mathematical Notations* (New York: Dover, 1993).
2. Howard Eves, *An Introduction to the History of Mathematics* (Sixth Edition) (Philadelphia: Saunders College Publishing, 1990).
3. Victor J. Katz, *A History of Mathematics: An Introduction* (Third Edition) (Reading, MA: Addison Wesley Longman, 2008).
4. Eberhard Knobloch, Determinants, in Ivor Grattan-Guinness, ed., *Companion Encyclopedia of the History and Philosophy of the Mathematical Sciences* (London: Routledge, 2013).

Vignette

Lewis Carroll's Condensation Method

In 1866, Charles Dodgson—better known by his pseudonym Lewis Carroll—published his only mathematical research paper. In it, he described a "new and brief method" for computing determinants, which he called "condensation." Although not well known today and rendered obsolete by numerical methods for evaluating determinants, the condensation method is very useful for hand calculation. When calculators or computer algebra systems are not available, many students find condensation to be their method of choice. It requires only the ability to compute 2×2 determinants.

We require the following terminology.

Definition   If A is an $n \times n$ matrix with $n \geq 3$, the **interior** of A, denoted int(A), is the $(n - 2) \times (n - 2)$ matrix obtained by deleting the first row, last row, first column, and last column of A.

We will illustrate the condensation method for the 5×5 matrix

$$A = \begin{bmatrix} 2 & 3 & -1 & 2 & 0 \\ 1 & 2 & 3 & 1 & -4 \\ 2 & -1 & 2 & 1 & 1 \\ 3 & 1 & -1 & 2 & -2 \\ -4 & 1 & 0 & 1 & 2 \end{bmatrix}$$

Begin by setting A_0 equal to the 6×6 matrix all of whose entries are 1. Then, we set $A_1 = A$. It is useful to imagine A_0 as the base of a pyramid with A_1 centered on top of A_0. We are going to add successively smaller and smaller layers to the pyramid until we reach a 1×1 matrix at the top—this will contain det A. (Figure 4.9)
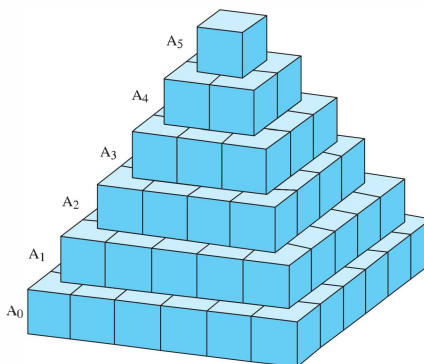
Charles Lutwidge Dodgson (1832–1898) is much better known by his pen name, Lewis Carroll, under which he wrote *Alice's Adventures in Wonderland* and *Through the Looking Glass*. He also wrote several mathematics books and collections of logic puzzles.

This vignette is based on the article "Lewis Carroll's Condensation Method for Evaluating Determinants" by Adrian Rice and Eve Torrence in *Math Horizons*, November 2006, pp. 12–15. For further details of the condensation method, see David M. Bressoud, *Proofs and Confirmations: The Story of the Alternating Sign Matrix Conjecture*, MAA Spectrum Series (Cambridge University Press, 1999).

Figure 4.9

Next, we "condense" A_1 into a 4×4 matrix A_2' whose entries are the determinants of all 2×2 submatrices of A_1:

$$
A_2' = \begin{bmatrix}
\begin{vmatrix} 2 & 3 \\ 1 & 2 \end{vmatrix} & \begin{vmatrix} 3 & -1 \\ 2 & 3 \end{vmatrix} & \begin{vmatrix} -1 & 2 \\ 3 & 1 \end{vmatrix} & \begin{vmatrix} 2 & 0 \\ 1 & -4 \end{vmatrix} \\[6pt]
\begin{vmatrix} 1 & 2 \\ 2 & -1 \end{vmatrix} & \begin{vmatrix} 2 & 3 \\ -1 & 2 \end{vmatrix} & \begin{vmatrix} 3 & 1 \\ 2 & 1 \end{vmatrix} & \begin{vmatrix} 1 & -4 \\ 1 & 1 \end{vmatrix} \\[6pt]
\begin{vmatrix} 2 & -1 \\ 3 & 1 \end{vmatrix} & \begin{vmatrix} -1 & 2 \\ 1 & -1 \end{vmatrix} & \begin{vmatrix} 2 & 1 \\ -1 & 2 \end{vmatrix} & \begin{vmatrix} 1 & 1 \\ 2 & -2 \end{vmatrix} \\[6pt]
\begin{vmatrix} 3 & 1 \\ -4 & 1 \end{vmatrix} & \begin{vmatrix} 1 & -1 \\ 1 & 0 \end{vmatrix} & \begin{vmatrix} -1 & 2 \\ 0 & 1 \end{vmatrix} & \begin{vmatrix} 2 & -2 \\ 1 & 2 \end{vmatrix}
\end{bmatrix}
= \begin{bmatrix}
1 & 11 & -7 & -8 \\
-5 & 7 & 1 & 5 \\
5 & -1 & 5 & -4 \\
7 & 1 & -1 & 6
\end{bmatrix}
$$

Now we divide each entry of A_2' by the corresponding entry of $\text{int}(A_0)$ to get matrix A_2. Since A_0 is all 1s, this means $A_2 = A_2'$.

We repeat the procedure, constructing A_3' from the 2×2 submatrices of A_2 and then dividing each entry of A_3' by the corresponding entry of $\text{int}(A_1)$, and so on. We obtain:

$$
A_3' = \begin{bmatrix}
\begin{vmatrix} 1 & 11 \\ -5 & 7 \end{vmatrix} & \begin{vmatrix} 11 & -7 \\ 7 & 1 \end{vmatrix} & \begin{vmatrix} -7 & -8 \\ 1 & 5 \end{vmatrix} \\[6pt]
\begin{vmatrix} -5 & 7 \\ 5 & -1 \end{vmatrix} & \begin{vmatrix} 7 & 1 \\ -1 & 5 \end{vmatrix} & \begin{vmatrix} 1 & 5 \\ 5 & -4 \end{vmatrix} \\[6pt]
\begin{vmatrix} 5 & -1 \\ 7 & 1 \end{vmatrix} & \begin{vmatrix} -1 & 5 \\ 1 & -1 \end{vmatrix} & \begin{vmatrix} 5 & -4 \\ -1 & 6 \end{vmatrix}
\end{bmatrix}
= \begin{bmatrix}
62 & 60 & -27 \\
-30 & 36 & -29 \\
12 & -4 & 26
\end{bmatrix},
$$

$$
A_3 = \begin{bmatrix}
62/2 & 60/3 & -27/1 \\
-30/-1 & 36/2 & -29/1 \\
12/1 & -4/-1 & 26/2
\end{bmatrix}
= \begin{bmatrix}
31 & 20 & -27 \\
30 & 18 & -29 \\
12 & 4 & 13
\end{bmatrix},
$$

$$
A_4' = \begin{bmatrix}
\begin{vmatrix} 31 & 20 \\ 30 & 18 \end{vmatrix} & \begin{vmatrix} 20 & -27 \\ 18 & -29 \end{vmatrix} \\[6pt]
\begin{vmatrix} 30 & 18 \\ 12 & 4 \end{vmatrix} & \begin{vmatrix} 18 & -29 \\ 4 & 13 \end{vmatrix}
\end{bmatrix}
= \begin{bmatrix}
-42 & -94 \\
-96 & 350
\end{bmatrix},
$$

$$
A_4 = \begin{bmatrix}
-42/7 & -94/1 \\
-96/(-1) & 350/5
\end{bmatrix}
= \begin{bmatrix}
-6 & -94 \\
96 & 70
\end{bmatrix}
$$

$$
A_5' = \begin{bmatrix} \begin{vmatrix} -6 & -94 \\ 96 & 70 \end{vmatrix} \end{bmatrix} = [8604],
$$

$$
A_5 = [8604/18] = [478]
$$

As can be checked by other methods, $\det A = 478$. In general, for an $n \times n$ matrix A, the condensation method will produce a 1×1 matrix A_n containing $\det A$.

Clearly, the method breaks down if the interior of any of the A_i's contains a zero, since we would then be trying to divide by zero to construct A_{i+1}. However, careful use of elementary row and column operations can be used to eliminate the zeros so that we can proceed.

Exploration

Geometric Applications of Determinants

This exploration will reveal some of the amazing applications of determinants to geometry. In particular, we will see that determinants are closely related to area and volume formulas and can be used to produce the equations of lines, planes, and certain other curves. Most of these ideas arose when the theory of determinants was being developed as a subject in its own right.

The Cross Product

Recall from Exploration: The Cross Product in Chapter 1 that the cross product

of $\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$ is the vector $\mathbf{u} \times \mathbf{v}$ defined by

$$\mathbf{u} \times \mathbf{v} = \begin{bmatrix} u_2 v_3 - u_3 v_2 \\ u_3 v_1 - u_1 v_3 \\ u_1 v_2 - u_2 v_1 \end{bmatrix}$$

If we write this cross product as $(u_2 v_3 - u_3 v_2)\mathbf{e}_1 - (u_1 v_3 - u_3 v_1)\mathbf{e}_2 + (u_1 v_2 - u_2 v_1)\mathbf{e}_3$, where $\mathbf{e}_1$, $\mathbf{e}_2$, and $\mathbf{e}_3$ are the standard basis vectors, then we see that the *form* of this formula is

$$\mathbf{u} \times \mathbf{v} = \det \begin{bmatrix} \mathbf{e}_1 & u_1 & v_1 \\ \mathbf{e}_2 & u_2 & v_2 \\ \mathbf{e}_3 & u_3 & v_3 \end{bmatrix}$$

if we expand along the first column. (This is not a proper determinant, of course, since $\mathbf{e}_1$, $\mathbf{e}_2$, and $\mathbf{e}_3$ are vectors, not scalars; however, it gives a useful way of remembering the somewhat awkward cross product formula. It also lets us use properties of determinants to verify some of the properties of the cross product.)

Now let's revisit some of the exercises from Chapter 1.

1. Use the determinant version of the cross product to compute $\mathbf{u} \times \mathbf{v}$.

(a) $\mathbf{u} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \mathbf{v} = \begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix}$ (b) $\mathbf{u} = \begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix}, \mathbf{v} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$

(c) $\mathbf{u} = \begin{bmatrix} -1 \\ 2 \\ 3 \end{bmatrix}, \mathbf{v} = \begin{bmatrix} 2 \\ -4 \\ -6 \end{bmatrix}$ (d) $\mathbf{u} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \mathbf{v} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$

2. If $\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}, \mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$, and $\mathbf{w} = \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix}$, show that

$$\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = \det \begin{bmatrix} u_1 & v_1 & w_1 \\ u_2 & v_2 & w_2 \\ u_3 & v_3 & w_3 \end{bmatrix}$$

3. Use properties of determinants (and Problem 2 above, if necessary) to prove the given property of the cross product.

(a) $\mathbf{v} \times \mathbf{u} = -(\mathbf{u} \times \mathbf{v})$ (b) $\mathbf{u} \times \mathbf{0} = \mathbf{0}$

(c) $\mathbf{u} \times \mathbf{u} = \mathbf{0}$ (d) $\mathbf{u} \times k\mathbf{v} = k(\mathbf{u} \times \mathbf{v})$

(e) $\mathbf{u} \times (\mathbf{v} + \mathbf{w}) = \mathbf{u} \times \mathbf{v} + \mathbf{u} \times \mathbf{w}$ (f) $\mathbf{u} \cdot (\mathbf{u} \times \mathbf{v}) = 0$ and $\mathbf{v} \cdot (\mathbf{u} \times \mathbf{v}) = 0$

(g) $\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = (\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w}$ (the *triple scalar product identity*)

Area and Volume

We can now give a geometric interpretation of the determinants of 2×2 and 3×3 matrices. Recall that if $\mathbf{u}$ and $\mathbf{v}$ are vectors in $\mathbb{R}^3$, then the area $\mathcal{A}$ of the parallelogram determined by these vectors is given by $\mathcal{A} = \|\mathbf{u} \times \mathbf{v}\|$. (See Exploration: The Cross Product in Chapter 1.)

4. Let $\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$. Show that the area $\mathcal{A}$ of the parallelogram determined by $\mathbf{u}$ and $\mathbf{v}$ is given by

$$\mathcal{A} = \left| \det \begin{bmatrix} u_1 & v_1 \\ u_2 & v_2 \end{bmatrix} \right|$$

[*Hint:* Write $\mathbf{u}$ and $\mathbf{v}$ as $\begin{bmatrix} u_1 \\ u_2 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} v_1 \\ v_2 \\ 0 \end{bmatrix}$.]
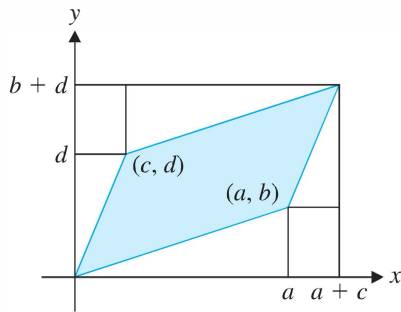

Figure 4.10

5. Derive the area formula in Problem 4 geometrically, using Figure 4.10 as a guide. [*Hint:* Subtract areas from the large rectangle until the parallelogram remains.] Where does the absolute value sign come from in this case?
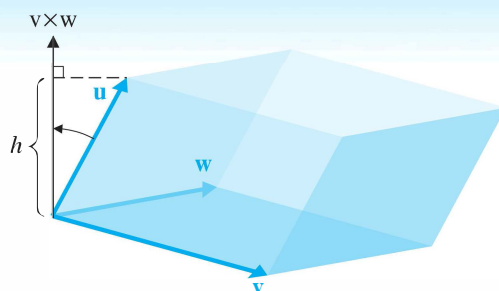
Figure 4.11

6. Find the area of the parallelogram determined by **u** and **v**.

(a) $\mathbf{u} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$, $\mathbf{v} = \begin{bmatrix} -1 \\ 4 \end{bmatrix}$ (b) $\mathbf{u} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$, $\mathbf{v} = \begin{bmatrix} 5 \\ 5 \end{bmatrix}$

Generalizing from Problems 4–6, consider a ***parallelepiped,*** a three-dimensional solid resembling a "slanted" brick, whose six faces are all parallelograms with opposite faces parallel and congruent (Figure 4.11). Its volume is given by the area of its base times its height.

7. Prove that the volume $\mathcal{V}$ of the parallelepiped determined by **u**, **v**, and **w** is given by the absolute value of the determinant of the 3×3 matrix $[\mathbf{u} \quad \mathbf{v} \quad \mathbf{w}]$ with **u**, **v**, and **w** as its columns. [*Hint:* From Figure 4.11 you can see that the height h can be expressed as $h = \|\mathbf{u}\| \cos \theta$, where θ is the angle between **u** and $\mathbf{v} \times \mathbf{w}$. Use this fact to show that $\mathcal{V} = |\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})|$ and apply the result of Problem 2.]

8. Show that the volume $\mathcal{V}$ of the tetrahedron determined by **u**, **v**, and **w** (Figure 4.12) is given by

Figure 4.12

$$\mathcal{V} = \tfrac{1}{6} |\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})|$$

[*Hint:* From geometry, we know that the volume of such a solid is $\mathcal{V} = \tfrac{1}{3}$ (area of the base)(height).]

Now let's view these geometric interpretations from a transformational point of view. Let A be a 2×2 matrix and let P be the parallelogram determined by the vectors **u** and **v**. We will consider the effect of the matrix transformation T_A on the area of P. Let $T_A(P)$ denote the parallelogram determined by $T_A(\mathbf{u}) = A\mathbf{u}$ and $T_A(\mathbf{v}) = A\mathbf{v}$.

9. Prove that the area of $T_A(P)$ is given by $|\det A|$ (area of P).

10. Let A be a 3×3 matrix and let P be the parallelepiped determined by the vectors **u**, **v**, and **w**. Let $T_A(P)$ denote the parallelepiped determined by $T_A(\mathbf{u}) = A\mathbf{u}$, $T_A(\mathbf{v}) = A\mathbf{v}$, and $T_A(\mathbf{w}) = A\mathbf{w}$. Prove that the volume of $T_A(P)$ is given by $|\det A|$ (volume of P).

The preceding problems illustrate that the determinant of a matrix captures what the corresponding matrix transformation does to the area or volume of figures upon which the transformation acts. (Although we have considered only certain types of figures, the result is perfectly general and can be made rigorous. We will not do so here.)
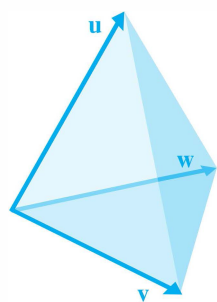
Lines and Planes

Suppose we are given two distinct points (x_1, y_1) and (x_2, y_2) in the plane. There is a unique line passing through these points, and its equation is of the form

$$ax + by + c = 0$$

Since the two given points are on this line, their coordinates satisfy this equation. Thus,

$$ax_1 + by_1 + c = 0$$
$$ax_2 + by_2 + c = 0$$

The three equations together can be viewed as a system of linear equations in the variables a, b, and c. Since there *is* a nontrivial solution (i.e., the line exists), the coefficient matrix

$$\begin{bmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{bmatrix}$$

cannot be invertible, by the Fundamental Theorem of Invertible Matrices. Consequently, its determinant must be zero, by Theorem 4.6. Expanding this determinant gives the equation of the line.

The equation of the line through the points (x_1, y_1) and (x_2, y_2) is given by

$$\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} = 0$$

---

11. Use the method described above to find the equation of the line through the given points.

(a) $(2, 3)$ and $(-1, 0)$ (b) $(1, 2)$ and $(4, 3)$

12. Prove that the three points (x_1, y_1), (x_2, y_2), and (x_3, y_3) are collinear (lie on the same line) if and only if

$$\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 0$$

13. Show that the equation of the plane through the three noncollinear points (x_1, y_1, z_1), (x_2, y_2, z_2), and (x_3, y_3, z_3) is given by

$$\begin{vmatrix} x & y & z & 1 \\ x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \end{vmatrix} = 0$$

What happens if the three points *are* collinear? [*Hint:* Explain what happens when row reduction is used to evaluate the determinant.]

14. Prove that the four points (x_1, y_1, z_1), (x_2, y_2, z_2), (x_3, y_3, z_3), and (x_4, y_4, z_4) are coplanar (lie in the same plane) if and only if

$$\begin{vmatrix} x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \\ x_4 & y_4 & z_4 & 1 \end{vmatrix} = 0$$

Curve Fitting

When data arising from experimentation take the form of points (x, y) that can be plotted in the plane, it is often of interest to find a relationship between the variables x and y. Ideally, we would like to find a function whose graph passes through all of the points. Sometimes all we want is an approximation (see Section 7.3), but exact results are also possible in certain situations.

15. From Figure 4.13 it appears as though we may be able to find a parabola passing through the points $A(-1, 10)$, $B(0, 5)$, and $C(3, 2)$. The equation of such a parabola is of the form $y = a + bx + cx^2$. By substituting the given points into this equation, set up a system of three linear equations in the variables a, b, and c. *Without solving the system,* use Theorem 4.6 to argue that it must have a unique solution. Then solve the system to find the equation of the parabola in Figure 4.13.

16. Use the method of Problem 15 to find the polynomials of degree at most 2 that pass through the following sets of points.

(a) $A(1, -1)$, $B(2, 4)$, $C(3, 3)$ (b) $A(-1, -3)$, $B(1, -1)$, $C(3, 1)$

17. Generalizing from Problems 15 and 16, suppose a_1, a_2, and a_3 are distinct real numbers. For any real numbers b_1, b_2, and b_3, we want to show that there is a unique quadratic with equation of the form $y = a + bx + cx^2$ passing through the points (a_1, b_1), (a_2, b_2), and (a_3, b_3). Do this by demonstrating that the coefficient matrix of the associated linear system has the determinant

$$\begin{vmatrix} 1 & a_1 & a_1^2 \\ 1 & a_2 & a_2^2 \\ 1 & a_3 & a_3^2 \end{vmatrix} = (a_2 - a_1)(a_3 - a_1)(a_3 - a_2)$$

which is necessarily nonzero. (Why?)

18. Let a_1, a_2, a_3, and a_4 be distinct real numbers. Show that

$$\begin{vmatrix} 1 & a_1 & a_1^2 & a_1^3 \\ 1 & a_2 & a_2^2 & a_2^3 \\ 1 & a_3 & a_3^2 & a_3^3 \\ 1 & a_4 & a_4^2 & a_4^3 \end{vmatrix} = (a_2 - a_1)(a_3 - a_1)(a_4 - a_1)(a_3 - a_2)(a_4 - a_2)(a_4 - a_3) \neq 0$$

For any real numbers b_1, b_2, b_3, and b_4, use this result to prove that there is a unique cubic with equation $y = a + bx + cx^2 + dx^3$ passing through the four points (a_1, b_1), (a_2, b_2), (a_3, b_3), and (a_4, b_4). (Do *not* actually solve for a, b, c, and d.)
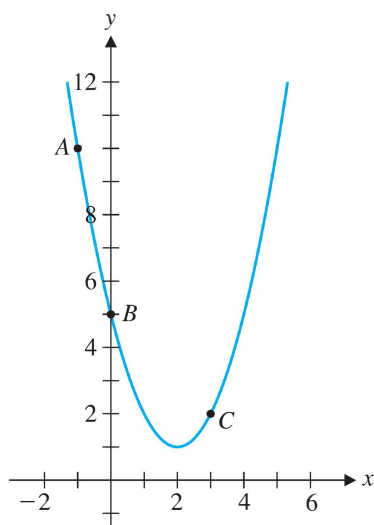
Figure 4.13

19. Let $a_1, a_2, \ldots, a_n$ be n real numbers. Prove that

$$\begin{vmatrix} 1 & a_1 & a_1^2 & \cdots & a_1^{n-1} \\ 1 & a_2 & a_2^2 & \cdots & a_2^{n-1} \\ 1 & a_3 & a_3^2 & \cdots & a_3^{n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & a_n & a_n^2 & \cdots & a_n^{n-1} \end{vmatrix} = \prod_{1 \le i < j \le n} (a_j - a_i)$$

where $\prod_{1 \le i < j \le n} (a_j - a_i)$ means the product of all terms of the form $(a_j - a_i)$, where $i < j$ and both i and j are between 1 and n. [The determinant of a matrix of this form (or its transpose) is called a **Vandermonde determinant**, named after the French mathematician A. T. Vandermonde (1735–1796).]

Deduce that for any n points in the plane whose x-coordinates are all distinct, there is a unique polynomial of degree $n - 1$ whose graph passes through the given points.

4.3

Eigenvalues and Eigenvectors of $n \times n$ Matrices

Now that we have defined the determinant of an $n \times n$ matrix, we can continue our discussion of eigenvalues and eigenvectors in a general context. Recall from Section 4.1 that λ is an eigenvalue of A if and only if $A - \lambda I$ is noninvertible. By Theorem 4.6, this is true if and only if $\det(A - \lambda I) = 0$. To summarize:

> The eigenvalues of a square matrix A are precisely the solutions λ of the equation
> $$\det(A - \lambda I) = 0$$

When we expand $\det(A - \lambda I)$, we get a polynomial in λ, called the ***characteristic polynomial*** of A. The equation $\det(A - \lambda I) = 0$ is called the ***characteristic equation*** of A. For example, if $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, its characteristic polynomial is

$$\det(A - \lambda I) = \begin{vmatrix} a - \lambda & b \\ c & d - \lambda \end{vmatrix} = (a - \lambda)(d - \lambda) - bc = \lambda^2 - (a + d)\lambda + (ad - bc)$$

If A is $n \times n$, its characteristic polynomial will be of degree n. According to the Fundamental Theorem of Algebra (see Appendix D), a polynomial of degree n with real or complex coefficients has at most n distinct roots. Applying this fact to the characteristic polynomial, we see that an $n \times n$ matrix with real or complex entries has at most n distinct eigenvalues.

Let's summarize the procedure we will follow (for now) to find the eigenvalues and eigenvectors (eigenspaces) of a matrix.

> Let A be an $n \times n$ matrix.
>
> 1. Compute the characteristic polynomial $\det(A - \lambda I)$ of A.
> 2. Find the eigenvalues of A by solving the characteristic equation $\det(A - \lambda I) = 0$ for λ.
> 3. For each eigenvalue λ, find the null space of the matrix $A - \lambda I$. This is the eigenspace E_λ, the nonzero vectors of which are the eigenvectors of A corresponding to λ.
> 4. Find a basis for each eigenspace.

Example 4.18

Find the eigenvalues and the corresponding eigenspaces of

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 2 & -5 & 4 \end{bmatrix}$$

Solution We follow the procedure outlined previously. The characteristic polynomial is

$$\det(A - \lambda I) = \begin{vmatrix} -\lambda & 1 & 0 \\ 0 & -\lambda & 1 \\ 2 & -5 & 4 - \lambda \end{vmatrix}$$

$$= -\lambda \begin{vmatrix} -\lambda & 1 \\ -5 & 4 - \lambda \end{vmatrix} - \begin{vmatrix} 0 & 1 \\ 2 & 4 - \lambda \end{vmatrix}$$

$$= -\lambda(\lambda^2 - 4\lambda + 5) - (-2)$$

$$= -\lambda^3 + 4\lambda^2 - 5\lambda + 2$$

To find the eigenvalues, we need to solve the characteristic equation $\det(A - \lambda I) = 0$ for λ. The characteristic polynomial factors as $-(\lambda - 1)^2(\lambda - 2)$. (The Factor Theorem is helpful here; see Appendix D.) Thus, the characteristic equation is $-(\lambda - 1)^2(\lambda - 2) = 0$, which clearly has solutions $\lambda = 1$ and $\lambda = 2$. Since $\lambda = 1$ is a multiple root and $\lambda = 2$ is a simple root, let us label them $\lambda_1 = \lambda_2 = 1$ and $\lambda_3 = 2$.

To find the eigenvectors corresponding to $\lambda_1 = \lambda_2 = 1$, we find the null space of

$$A - 1I = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 2 & -5 & 4 - 1 \end{bmatrix} = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 2 & -5 & 3 \end{bmatrix}$$

Row reduction produces

$$[A - I \,|\, 0] = \begin{bmatrix} -1 & 1 & 0 & | & 0 \\ 0 & -1 & 1 & | & 0 \\ 2 & -5 & 3 & | & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & -1 & | & 0 \\ 0 & 1 & -1 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{bmatrix}$$

(We knew in advance that we *must* get at least one zero row. Why?) Thus, $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ is in the eigenspace E_1 if and only if $x_1 - x_3 = 0$ and $x_2 - x_3 = 0$. Setting the free variable $x_3 = t$, we see that $x_1 = t$ and $x_2 = t$, from which it follows that

$$E_1 = \left\{ \begin{bmatrix} t \\ t \\ t \end{bmatrix} \right\} = \left\{ t \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\} = \text{span}\left(\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right)$$

To find the eigenvectors corresponding to $\lambda_3 = 2$, we find the null space of $A - 2I$ by row reduction:

$$[A - 2I \,|\, 0] = \begin{bmatrix} -2 & 1 & 0 & | & 0 \\ 0 & -2 & 1 & | & 0 \\ 2 & -5 & 2 & | & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & -\frac{1}{4} & | & 0 \\ 0 & 1 & -\frac{1}{2} & | & 0 \\ 0 & 0 & 0 & | & 0 \end{bmatrix}$$

So $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ is in the eigenspace E_2 if and only if $x_1 = \frac{1}{4}x_3$ and $x_2 = \frac{1}{2}x_3$. Setting the free variable $x_3 = t$, we have

$$E_2 = \left\{ \begin{bmatrix} \frac{1}{4}t \\ \frac{1}{2}t \\ t \end{bmatrix} \right\} = \left\{ t \begin{bmatrix} \frac{1}{4} \\ \frac{1}{2} \\ 1 \end{bmatrix} \right\} = \text{span}\left(\begin{bmatrix} \frac{1}{4} \\ \frac{1}{2} \\ 1 \end{bmatrix} \right) = \text{span}\left(\begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix} \right)$$

where we have cleared denominators in the basis by multiplying through by the least common denominator 4. (Why is this permissible?)

Remark   Notice that in Example 4.18, A is a 3×3 matrix but has only two distinct eigenvalues. However, if we count multiplicities, A has exactly three eigenvalues ($\lambda = 1$ twice and $\lambda = 2$ once). This is what the Fundamental Theorem of Algebra guarantees. Let us define the **algebraic multiplicity** of an eigenvalue to be its multiplicity as a root of the characteristic equation. Thus, $\lambda = 1$ has algebraic multiplicity 2 and $\lambda = 2$ has algebraic multiplicity 1.

Next notice that each eigenspace has a basis consisting of just one vector. In other words, dim $E_1 = $ dim $E_2 = 1$. Let us define the **geometric multiplicity** of an eigenvalue λ to be dim E_λ, the dimension of its corresponding eigenspace. As you will see in Section 4.4, a comparison of these two notions of multiplicity is important.

Example 4.19

Find the eigenvalues and the corresponding eigenspaces of

$$A = \begin{bmatrix} -1 & 0 & 1 \\ 3 & 0 & -3 \\ 1 & 0 & -1 \end{bmatrix}$$

Solution   The characteristic equation is

$$0 = \det(A - \lambda I) = \begin{vmatrix} -1-\lambda & 0 & 1 \\ 3 & -\lambda & -3 \\ 1 & 0 & -1-\lambda \end{vmatrix} = -\lambda \begin{vmatrix} -1-\lambda & 1 \\ 1 & -1-\lambda \end{vmatrix}$$

$$= -\lambda(\lambda^2 + 2\lambda) = -\lambda^2(\lambda + 2)$$

Hence, the eigenvalues are $\lambda_1 = \lambda_2 = 0$ and $\lambda_3 = -2$. Thus, the eigenvalue 0 has algebraic multiplicity 2 and the eigenvalue -2 has algebraic multiplicity 1.

For $\lambda_1 = \lambda_2 = 0$, we compute

$$[A - 0I \,|\, 0] = [A \,|\, 0] = \begin{bmatrix} -1 & 0 & 1 & | & 0 \\ 3 & 0 & -3 & | & 0 \\ 1 & 0 & -1 & | & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & -1 & | & 0 \\ 0 & 0 & 0 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{bmatrix}$$

from which it follows that an eigenvector $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ in E_0 satisfies $x_1 = x_3$. Therefore,

both x_2 and x_3 are free. Setting $x_2 = s$ and $x_3 = t$, we have

$$E_0 = \left\{ \begin{bmatrix} t \\ s \\ t \end{bmatrix} \right\} = \left\{ s\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\} = \text{span}\left(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right)$$

For $\lambda_3 = -2$,

$$[A - (-2)I \,|\, 0] = [A + 2I \,|\, 0] = \begin{bmatrix} 1 & 0 & 1 & | & 0 \\ 3 & 2 & -3 & | & 0 \\ 1 & 0 & 1 & | & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 1 & | & 0 \\ 0 & 1 & -3 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{bmatrix}$$

so $x_3 = t$ is free and $x_1 = -x_3 = -t$ and $x_2 = 3x_3 = 3t$. Consequently,

$$E_{-2} = \left\{ \begin{bmatrix} -t \\ 3t \\ t \end{bmatrix} \right\} = \left\{ t \begin{bmatrix} -1 \\ 3 \\ 1 \end{bmatrix} \right\} = \text{span}\left(\begin{bmatrix} -1 \\ 3 \\ 1 \end{bmatrix} \right)$$

It follows that $\lambda_1 = \lambda_2 = 0$ has geometric multiplicity 2 and $\lambda_3 = -2$ has geometric multiplicity 1. (Note that the algebraic multiplicity equals the geometric multiplicity for each eigenvalue.)

In some situations, the eigenvalues of a matrix are very easy to find. If A is a triangular matrix, then so is $A - \lambda I$, and Theorem 4.2 says that $\det(A - \lambda I)$ is just the product of the diagonal entries. This implies that the characteristic equation of a triangular matrix is

$$(a_{11} - \lambda)(a_{22} - \lambda) \cdots (a_{nn} - \lambda) = 0$$

from which it follows immediately that the eigenvalues are $\lambda_1 = a_{11}$, $\lambda_2 = a_{22}$, $\ldots$, $\lambda_n = a_{nn}$. We summarize this result as a theorem and illustrate it with an example.

| **Theorem 4.15** | The eigenvalues of a triangular matrix are the entries on its main diagonal. |

| **Example 4.20** | The eigenvalues of |

$$A = \begin{bmatrix} 2 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 3 & 0 & 3 & 0 \\ 5 & 7 & 4 & -2 \end{bmatrix}$$

are $\lambda_1 = 2$, $\lambda_2 = 1$, $\lambda_3 = 3$, and $\lambda_4 = -2$, by Theorem 4.15. Indeed, the characteristic polynomial is just $(2 - \lambda)(1 - \lambda)(3 - \lambda)(-2 - \lambda)$.

Note that diagonal matrices are a special case of Theorem 4.15. In fact, a diagonal matrix is both upper and lower triangular.

Eigenvalues capture much important information about the behavior of a matrix. Once we know the eigenvalues of a matrix, we can deduce a great many things without doing any more work. The next theorem is one of the most important in this regard.

| **Theorem 4.16** | A square matrix A is invertible if and only if 0 is *not* an eigenvalue of A. |

Proof   Let A be a square matrix. By Theorem 4.6, A is invertible if and only if $\det A \neq 0$. But $\det A \neq 0$ is equivalent to $\det(A - 0I) \neq 0$, which says that 0 is not a root of the characteristic equation of A (i.e., 0 is not an eigenvalue of A).

We can now extend the Fundamental Theorem of Invertible Matrices to include results we have proved in this chapter.

Theorem 4.17

The Fundamental Theorem of Invertible Matrices: Version 3

Let A be an $n \times n$ matrix. The following statements are equivalent:

a. A is invertible.
b. $A\mathbf{x} = \mathbf{b}$ has a unique solution for every $\mathbf{b}$ in $\mathbb{R}^n$.
c. $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.
d. The reduced row echelon form of A is I_n.
e. A is a product of elementary matrices.
f. rank$(A) = n$
g. nullity$(A) = 0$
h. The column vectors of A are linearly independent.
i. The column vectors of A span $\mathbb{R}^n$.
j. The column vectors of A form a basis for $\mathbb{R}^n$.
k. The row vectors of A are linearly independent.
l. The row vectors of A span $\mathbb{R}^n$.
m. The row vectors of A form a basis for $\mathbb{R}^n$.
n. det $A \neq 0$
o. 0 is not an eigenvalue of A.

Proof   The equivalence (a) $\Leftrightarrow$ (n) is Theorem 4.6, and we just proved (a) $\Leftrightarrow$ (o) in Theorem 4.16.

There are nice formulas for the eigenvalues of the powers and inverses of a matrix.

Theorem 4.18

Let A be a square matrix with eigenvalue λ and corresponding eigenvector $\mathbf{x}$.

a. For any positive integer n, λ^n is an eigenvalue of A^n with corresponding eigenvector $\mathbf{x}$.
b. If A is invertible, then $1/\lambda$ is an eigenvalue of A^{-1} with corresponding eigenvector $\mathbf{x}$.
c. If A is invertible, then for any integer n, λ^n is an eigenvalue of A^n with corresponding eigenvector $\mathbf{x}$.

Proof   We are given that $A\mathbf{x} = \lambda\mathbf{x}$.

(a) We proceed by induction on n. For $n = 1$, the result is just what has been given. Assume the result is true for $n = k$. That is, assume that, for some positive integer k, $A^k\mathbf{x} = \lambda^k\mathbf{x}$. We must now prove the result for $n = k + 1$. But

$$A^{k+1}\mathbf{x} = A(A^k\mathbf{x}) = A(\lambda^k\mathbf{x})$$

by the induction hypothesis. Using property (d) of Theorem 3.3, we have

$$A(\lambda^k\mathbf{x}) = \lambda^k(A\mathbf{x}) = \lambda^k(\lambda\mathbf{x}) = \lambda^{k+1}\mathbf{x}$$

Thus, $A^{k+1}\mathbf{x} = \lambda^{k+1}\mathbf{x}$, as required. By induction, the result is true for all integers $n \geq 1$.

(b) You are asked to prove this property in Exercise 13.

(c) You are asked to prove this property in Exercise 14.

The next example shows one application of this theorem.

Example 4.21

Compute $\begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix}^{10} \begin{bmatrix} 5 \\ 1 \end{bmatrix}$.

Solution   Let $A = \begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix}$ and $\mathbf{x} = \begin{bmatrix} 5 \\ 1 \end{bmatrix}$; then what we want to find is $A^{10}\mathbf{x}$. The eigenvalues of A are $\lambda_1 = -1$ and $\lambda_2 = 2$, with corresponding eigenvectors $\mathbf{v}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$. That is,

$$A\mathbf{v}_1 = -\mathbf{v}_1 \quad \text{and} \quad A\mathbf{v}_2 = 2\mathbf{v}_2$$

(Check this.) Since $\{\mathbf{v}_1, \mathbf{v}_2\}$ forms a basis for $\mathbb{R}^2$ (why?), we can write $\mathbf{x}$ as a linear combination of $\mathbf{v}_1$ and $\mathbf{v}_2$. Indeed, as is easily checked, $\mathbf{x} = 3\mathbf{v}_1 + 2\mathbf{v}_2$.

Therefore, using Theorem 4.18(a), we have

$$A^{10}\mathbf{x} = A^{10}(3\mathbf{v}_1 + 2\mathbf{v}_2) = 3(A^{10}\mathbf{v}_1) + 2(A^{10}\mathbf{v}_2)$$

$$= 3(\lambda_1^{10})\mathbf{v}_1 + 2(\lambda_2^{10})\mathbf{v}_2$$

$$= 3(-1)^{10}\begin{bmatrix} 1 \\ -1 \end{bmatrix} + 2(2^{10})\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 3 + 2^{11} \\ -3 + 2^{12} \end{bmatrix} = \begin{bmatrix} 2051 \\ 4093 \end{bmatrix}$$

This is certainly a lot easier than computing A^{10} first; in fact, there are no matrix multiplications at all!

When it can be used, the method of Example 4.21 is quite general. We summarize it as the following theorem, which you are asked to prove in Exercise 42.

Theorem 4.19

Suppose the $n \times n$ matrix A has eigenvectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m$ with corresponding eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_m$. If $\mathbf{x}$ is a vector in $\mathbb{R}^n$ that can be expressed as a linear combination of these eigenvectors—say,

$$\mathbf{x} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_m\mathbf{v}_m$$

then, for any integer k,

$$A^k\mathbf{x} = c_1\lambda_1^k\mathbf{v}_1 + c_2\lambda_2^k\mathbf{v}_2 + \cdots + c_m\lambda_m^k\mathbf{v}_m$$

Warning   The catch here is the "if" in the second sentence. There is absolutely no guarantee that such a linear combination is possible. The best possible situation would be if there were a basis of $\mathbb{R}^n$ consisting of eigenvectors of A; we will explore this possibility further in the next section. As a step in that direction, however, we have the following theorem, which states that eigenvectors corresponding to *distinct* eigenvalues are linearly independent.

Theorem 4.20

Let A be an $n \times n$ matrix and let $\lambda_1, \lambda_2, \ldots, \lambda_m$ be distinct eigenvalues of A with corresponding eigenvectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m$. Then $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m$ are linearly independent.

Proof   The proof is indirect. We will assume that $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m$ are linearly *dependent* and show that this assumption leads to a contradiction.

If $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m$ are linearly dependent, then one of these vectors must be expressible as a linear combination of the previous ones. Let $\mathbf{v}_{k+1}$ be the first of the vectors $\mathbf{v}_i$ that can be so expressed. In other words, $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ are linearly independent, but there are scalars $c_1, c_2, \ldots, c_k$ such that

$$\mathbf{v}_{k+1} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_k\mathbf{v}_k \tag{1}$$

Multiplying both sides of Equation (1) by A from the left and using the fact that $A\mathbf{v}_i = \lambda_i\mathbf{v}_i$ for each i, we have

$$\begin{aligned}
\lambda_{k+1}\mathbf{v}_{k+1} = A\mathbf{v}_{k+1} &= A(c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_k\mathbf{v}_k) \\
&= c_1 A\mathbf{v}_1 + c_2 A\mathbf{v}_2 + \cdots + c_k A\mathbf{v}_k \\
&= c_1\lambda_1\mathbf{v}_1 + c_2\lambda_2\mathbf{v}_2 + \cdots + c_k\lambda_k\mathbf{v}_k
\end{aligned} \tag{2}$$

Now we multiply both sides of Equation (1) by λ_{k+1} to get

$$\lambda_{k+1}\mathbf{v}_{k+1} = c_1\lambda_{k+1}\mathbf{v}_1 + c_2\lambda_{k+1}\mathbf{v}_2 + \cdots + c_k\lambda_{k+1}\mathbf{v}_k \tag{3}$$

When we subtract Equation (3) from Equation (2), we obtain

$$0 = c_1(\lambda_1 - \lambda_{k+1})\mathbf{v}_1 + c_2(\lambda_2 - \lambda_{k+1})\mathbf{v}_2 + \cdots + c_k(\lambda_k - \lambda_{k+1})\mathbf{v}_k$$

The linear independence of $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ implies that

$$c_1(\lambda_1 - \lambda_{k+1}) = c_2(\lambda_2 - \lambda_{k+1}) = \cdots = c_k(\lambda_k - \lambda_{k+1}) = 0$$

Since the eigenvalues λ_i are all distinct, the terms in parentheses $(\lambda_i - \lambda_{k+1})$, $i = 1, \ldots, k$, are all nonzero. Hence, $c_1 = c_2 = \cdots = c_k = 0$. This implies that

$$\mathbf{v}_{k+1} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_k\mathbf{v}_k = 0\mathbf{v}_1 + 0\mathbf{v}_2 + \cdots + 0\mathbf{v}_k = 0$$

which is impossible, since the eigenvector $\mathbf{v}_{k+1}$ cannot be zero. Thus, we have a contradiction, which means that our assumption that $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m$ are linearly dependent is false. It follows that $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m$ must be linearly independent.

Exercises 4.3

In Exercises 1–12, compute (a) the characteristic polynomial of A, (b) the eigenvalues of A, (c) a basis for each eigenspace of A, and (d) the algebraic and geometric multiplicity of each eigenvalue.

1. $A = \begin{bmatrix} 1 & 3 \\ -2 & 6 \end{bmatrix}$ **2.** $A = \begin{bmatrix} 2 & 1 \\ -1 & 0 \end{bmatrix}$

3. $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & -2 & 1 \\ 0 & 0 & 3 \end{bmatrix}$ **4.** $A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}$

5. $A = \begin{bmatrix} 1 & 2 & 0 \\ -1 & -1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$ **6.** $A = \begin{bmatrix} 1 & 0 & 2 \\ 3 & -1 & 3 \\ 2 & 0 & 1 \end{bmatrix}$

7. $A = \begin{bmatrix} 4 & 0 & 1 \\ 2 & 3 & 2 \\ -1 & 0 & 2 \end{bmatrix}$ **8.** $A = \begin{bmatrix} 1 & -1 & -1 \\ 0 & 2 & 0 \\ -1 & -1 & 1 \end{bmatrix}$

9. $A = \begin{bmatrix} 3 & 1 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 1 & 1 \end{bmatrix}$ **10.** $A = \begin{bmatrix} 2 & 1 & 1 & 0 \\ 0 & 1 & 4 & 5 \\ 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & 2 \end{bmatrix}$

11. $A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 1 & 3 & 0 \\ -2 & 1 & 2 & -1 \end{bmatrix}$ **12.** $A = \begin{bmatrix} 4 & 0 & 1 & 0 \\ 0 & 4 & 1 & 1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 3 & 0 \end{bmatrix}$

13. Prove Theorem 4.18(b).

14. Prove Theorem 4.18(c). [*Hint:* Combine the proofs of parts (a) and (b) and see the fourth Remark following Theorem 3.9 (page 169).]

In Exercises 15 and 16, A is a 2 × 2 matrix with eigenvectors $\mathbf{v}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ *and* $\mathbf{v}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ *corresponding to eigenvalues* $\lambda_1 = \frac{1}{2}$ *and* $\lambda_2 = 2$, *respectively, and* $\mathbf{x} = \begin{bmatrix} 5 \\ 1 \end{bmatrix}$.

15. Find $A^{10}\mathbf{x}$.

16. Find $A^k\mathbf{x}$. What happens as k becomes large (i.e., $k \to \infty$)?

In Exercises 17 and 18, A is a 3 × 3 matrix with eigenvectors $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$, *and* $\mathbf{v}_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ *corresponding to eigenvalues* $\lambda_1 = -\frac{1}{3}$, $\lambda_2 = \frac{1}{3}$, *and* $\lambda_3 = 1$, *respectively, and* $\mathbf{x} = \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}$.

17. Find $A^{20}\mathbf{x}$.

18. Find $A^k\mathbf{x}$. What happens as k becomes large (i.e., $k \to \infty$)?

19. (a) Show that, for any square matrix A, A^T and A have the same characteristic polynomial and hence the same eigenvalues.

  (b) Give an example of a 2 × 2 matrix A for which A^T and A have different eigenspaces.

20. Let A be a nilpotent matrix (that is, $A^m = O$ for some $m > 1$). Show that $\lambda = 0$ is the only eigenvalue of A.

21. Let A be an idempotent matrix (that is, $A^2 = A$). Show that $\lambda = 0$ and $\lambda = 1$ are the only possible eigenvalues of A.

22. If $\mathbf{v}$ is an eigenvector of A with corresponding eigenvalue λ and c is a scalar, show that $\mathbf{v}$ is an eigenvector of $A - cI$ with corresponding eigenvalue $\lambda - c$.

23. (a) Find the eigenvalues and eigenspaces of
$$A = \begin{bmatrix} 3 & 2 \\ 5 & 0 \end{bmatrix}$$

  (b) Using Theorem 4.18 and Exercise 22, find the eigenvalues and eigenspaces of $A^{-1}, A - 2I$, and $A + 2I$.

24. Let A and B be $n \times n$ matrices with eigenvalues λ and μ, respectively.

  (a) Give an example to show that $\lambda + \mu$ need not be an eigenvalue of $A + B$.

  (b) Give an example to show that $\lambda\mu$ need not be an eigenvalue of AB.

  (c) Suppose λ and μ correspond to the *same* eigenvector $\mathbf{x}$. Show that, in this case, $\lambda + \mu$ is an eigenvalue of $A + B$ and $\lambda\mu$ is an eigenvalue of AB.

25. If A and B are two row equivalent matrices, do they necessarily have the same eigenvalues? Either prove that they do or give a counterexample.

Let p(x) be the polynomial
$$p(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0$$

The **companion matrix** *of p(x) is the n × n matrix*

$$C(p) = \begin{bmatrix} -a_{n-1} & -a_{n-2} & \cdots & -a_1 & -a_0 \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix} \qquad (4)$$

26. Find the companion matrix of $p(x) = x^2 - 7x + 12$ and then find the characteristic polynomial of $C(p)$.

27. Find the companion matrix of $p(x) = x^3 + 3x^2 - 4x + 12$ and then find the characteristic polynomial of $C(p)$.

28. (a) Show that the companion matrix $C(p)$ of $p(x) = x^2 + ax + b$ has characteristic polynomial $\lambda^2 + a\lambda + b$.

  (b) Show that if λ is an eigenvalue of the companion matrix $C(p)$ in part (a), then $\begin{bmatrix} \lambda \\ 1 \end{bmatrix}$ is an eigenvector of $C(p)$ corresponding to λ.

29. (a) Show that the companion matrix $C(p)$ of $p(x) = x^3 + ax^2 + bx + c$ has characteristic polynomial $-(\lambda^3 + a\lambda^2 + b\lambda + c)$.

  (b) Show that if λ is an eigenvalue of the companion matrix $C(p)$ in part (a), then $\begin{bmatrix} \lambda^2 \\ \lambda \\ 1 \end{bmatrix}$ is an eigenvector of $C(p)$ corresponding to λ.

30. Construct a nontriangular 2 × 2 matrix with eigenvalues 2 and 5. [*Hint:* Use Exercise 28.]

31. Construct a nontriangular 3 × 3 matrix with eigenvalues -2, 1, and 3. [*Hint:* Use Exercise 29.]

32. (a) Use mathematical induction to prove that, for $n \geq 2$, the companion matrix $C(p)$ of $p(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0$ has characteristic polynomial $(-1)^n p(\lambda)$. [*Hint:* Expand by cofactors along the last column. You may find it helpful to introduce the polynomial $q(x) = (p(x) - a_0)/x$.]

(b) Show that if λ is an eigenvalue of the companion matrix $C(p)$ in Equation (4), then an eigenvector corresponding to λ is given by

$$\begin{bmatrix} \lambda^{n-1} \\ \lambda^{n-2} \\ \vdots \\ \lambda \\ 1 \end{bmatrix}$$

If $p(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ and A is a square matrix, we can define a square matrix $p(A)$ by

$$p(A) = A^n + a_{n-1}A^{n-1} + \cdots + a_1 A + a_0 I$$

An important theorem in advanced linear algebra says that if $c_A(\lambda)$ is the characteristic polynomial of the matrix A, then $c_A(A) = O$ (in words, every matrix satisfies its characteristic equation). This is the celebrated **Cayley-Hamilton Theorem,** named after Arthur Cayley (1821–1895), pictured below, and Sir William Rowan Hamilton (see page 2). Cayley proved this theorem in 1858. Hamilton discovered it, independently, in his work on quaternions, a generalization of the complex numbers.
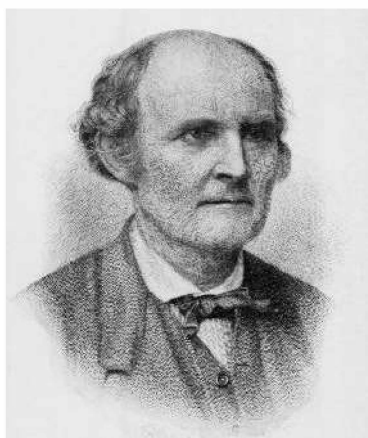
Bettmann/CORBIS

33. Verify the Cayley-Hamilton Theorem for $A = \begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix}$.

That is, find the characteristic polynomial $c_A(\lambda)$ of A and show that $c_A(A) = O$.

34. Verify the Cayley-Hamilton Theorem for

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}.$$

The Cayley-Hamilton Theorem can be used to calculate powers and inverses of matrices. For example, if A is a 2×2 matrix with characteristic polynomial $c_A(\lambda) = \lambda^2 + a\lambda + b$, then $A^2 + aA + bI = O$, so

$$A^2 = -aA - bI$$

and
$$\begin{aligned} A^3 = AA^2 &= A(-aA - bI) \\ &= -aA^2 - bA \\ &= -a(-aA - bI) - bA \\ &= (a^2 - b)A + abI \end{aligned}$$

It is easy to see that by continuing in this fashion we can express any positive power of A as a linear combination of I and A. From $A^2 + aA + bI = O$, we also obtain $A(A + aI) = -bI$, so

$$A^{-1} = -\frac{1}{b}A - \frac{a}{b}I$$

provided $b \neq 0$.

35. For the matrix A in Exercise 33, use the Cayley-Hamilton Theorem to compute A^2, A^3, and A^4 by expressing each as a linear combination of I and A.

36. For the matrix A in Exercise 34, use the Cayley-Hamilton Theorem to compute A^3 and A^4 by expressing each as a linear combination of I, A, and A^2.

37. For the matrix A in Exercise 33, use the Cayley-Hamilton Theorem to compute A^{-1} and A^{-2} by expressing each as a linear combination of I and A.

38. For the matrix A in Exercise 34, use the Cayley-Hamilton Theorem to compute A^{-1} and A^{-2} by expressing each as a linear combination of I, A, and A^2.

39. Show that if the square matrix A can be partitioned as

$$A = \begin{bmatrix} P & Q \\ \hline O & S \end{bmatrix}$$

where P and S are square matrices, then the characteristic polynomial of A is $c_A(\lambda) = c_P(\lambda)c_S(\lambda)$. [*Hint:* Use Exercise 69 in Section 4.2.]

40. Let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be a complete set of eigenvalues (repetitions included) of the $n \times n$ matrix A. Prove that

$\det(A) = \lambda_1 \lambda_2 \cdots \lambda_n$ and
$\text{tr}(A) = \lambda_1 + \lambda_2 + \cdots + \lambda_n$

[*Hint:* The characteristic polynomial of A factors as

$\det(A - \lambda I) = (-1)^n(\lambda - \lambda_1)(\lambda - \lambda_2)\cdots(\lambda - \lambda_n)$

Find the constant term and the coefficient of λ^{n-1} on the left and right sides of this equation.]

41. Let A and B be $n \times n$ matrices. Prove that the sum of *all* the eigenvalues of $A + B$ is the sum of all the eigenvalues of A and B individually. Prove that the product of *all* the eigenvalues of AB is the product of all the eigenvalues of A and B individually. (Compare this exercise with Exercise 24.)

42. Prove Theorem 4.19.

Writing Project   **The History of Eigenvalues**

Like much of linear algebra, the way the topic of eigenvalues is taught today does not correspond to its historical development. Eigenvalues arose out of problems in systems of differential equations before the concept of a matrix was even formulated.

Write a report on the historical development of eigenvalues. Describe the types of mathematical problems in which they originally arose. Who were some of the key mathematicians involved with these problems? How did the terminology for eigenvalues change over time?

1. Thomas Hawkins, Cauchy and the Spectral Theory of Matrices, *Historia Mathematica 2* (1975), pp. 1–29.
2. Victor J. Katz, *A History of Mathematics: An Introduction* (Third Edition) (Reading, MA: Addison Wesley Longman, 2008).
3. Morris Kline, *Mathematical Thought from Ancient to Modern Times* (Oxford: Oxford University Press, 1972).

4.4   Similarity and Diagonalization

As you saw in the last section, triangular and diagonal matrices are nice in the sense that their eigenvalues are transparently displayed. It would be pleasant if we could relate a given square matrix to a triangular or diagonal one in such a way that they had exactly the same eigenvalues. Of course, we already know one procedure for converting a square matrix into triangular form—namely, Gaussian elimination. Unfortunately, this process does not preserve the eigenvalues of the matrix. In this section, we consider a different sort of transformation of a matrix that does behave well with respect to eigenvalues.

Similar Matrices

Definition   Let A and B be $n \times n$ matrices. We say that A ***is similar to*** B if there is an invertible $n \times n$ matrix P such that $P^{-1}AP = B$. If A is similar to B, we write $A \sim B$.

Remarks
- If $A \sim B$, we can write, equivalently, that $A = PBP^{-1}$ or $AP = PB$.
- Similarity is a *relation* on square matrices in the same sense that "less than or equal to" is a relation on the integers. Note that there is a *direction* (or *order*) implicit in the definition. Just as $a \le b$ does not necessarily imply $b \le a$, we should not assume that $A \sim B$ implies $B \sim A$. (In fact, this *is* true, as we will prove in the next theorem, but it does not follow immediately from the definition.)
- The matrix P depends on A and B. It is not unique for a given pair of similar matrices A and B. To see this, simply take $A = B = I$, in which case $I \sim I$, since $P^{-1}IP = I$ for *any* invertible matrix P.

Example 4.22

Let $A = \begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 \\ -2 & -1 \end{bmatrix}$. Then $A \sim B$, since

$$\begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 1 \\ -1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ -2 & -1 \end{bmatrix}$$

Thus, $AP = PB$ with $P = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$. (Note that it is not necessary to compute P^{-1}. See the first Remark before Example 4.22.)

Theorem 4.21

Let A, B, and C be $n \times n$ matrices.

a. $A \sim A$
b. If $A \sim B$, then $B \sim A$.
c. If $A \sim B$ and $B \sim C$, then $A \sim C$.

Proof (a) This property follows from the fact that $I^{-1}AI = A$.

(b) If $A \sim B$, then $P^{-1}AP = B$ for some invertible matrix P. As noted in the first Remark on the previous page, this is equivalent to $PBP^{-1} = A$. Setting $Q = P^{-1}$, we have $Q^{-1}BQ = (P^{-1})^{-1}BP^{-1} = PBP^{-1} = A$. Therefore, by definition, $B \sim A$.

(c) You are asked to prove property (c) in Exercise 30.

Remark Any relation satisfying the three properties of Theorem 4.21 is called an **equivalence relation.** Equivalence relations arise frequently in mathematics, and objects that are related via an equivalence relation usually share important properties. We are about to see that this is true of similar matrices.

Theorem 4.22

Let A and B be $n \times n$ matrices with $A \sim B$. Then

a. $\det A = \det B$
b. A is invertible if and only if B is invertible.
c. A and B have the same rank.
d. A and B have the same characteristic polynomial.
e. A and B have the same eigenvalues.
f. $A^m \sim B^m$ for all integers $m \geq 0$.
g. If A is invertible, then $A^m \sim B^m$ for all integers m.

Proof We prove (a) and (d) and leave the remaining properties as exercises. If $A \sim B$, then $P^{-1}AP = B$ for some invertible matrix P.

(a) Taking determinants of both sides, we have

$$\det B = \det(P^{-1}AP) = (\det P^{-1})(\det A)(\det P)$$
$$= \left(\frac{1}{\det P}\right)(\det A)(\det P) = \det A$$

(d) The characteristic polynomial of B is

$$\det(B - \lambda I) = \det(P^{-1}AP - \lambda I)$$
$$= \det(P^{-1}AP - \lambda P^{-1}IP)$$

$$= \det(P^{-1}AP - P^{-1}(\lambda I)P)$$
$$= \det(P^{-1}(A - \lambda I)P) = \det(A - \lambda I)$$

with the last step following as in (a). Thus, $\det(B - \lambda I) = \det(A - \lambda I)$; that is, the characteristic polynomials of B and A are the same.

Remark Two matrices may have properties (a) through (e) (and more) in common and yet still not be similar. For example, $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ both have determinant 1 and rank 2, are invertible, and have characteristic polynomial $(1 - \lambda)^2$ and eigenvalues $\lambda_1 = \lambda_2 = 1$. But A is not similar to B, since $P^{-1}AP = P^{-1}IP = I \neq B$ for any invertible matrix P.

Theorem 4.22 is most useful in showing that two matrices are *not* similar, since A and B cannot be similar if any of properties (a) through (e) fails.

Example 4.23

(a) $A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$ are not similar, since $\det A = -3$ but $\det B = 3$.

(b) $A = \begin{bmatrix} 1 & 3 \\ 2 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 1 \\ 3 & -1 \end{bmatrix}$ are not similar, since the characteristic polynomial of A is $\lambda^2 - 3\lambda - 4$ while that of B is $\lambda^2 - 4$. (Check this.) Note that A and B do have the same determinant and rank, however.

Diagonalization

The best possible situation is when a square matrix is similar to a diagonal matrix. As you are about to see, whether a matrix is diagonalizable is closely related to the eigenvalues and eigenvectors of the matrix.

Definition An $n \times n$ matrix A is *diagonalizable* if there is a diagonal matrix D such that A is similar to D—that is, if there is an invertible $n \times n$ matrix P such that $P^{-1}AP = D$.

Example 4.24

$A = \begin{bmatrix} 1 & 3 \\ 2 & 2 \end{bmatrix}$ is diagonalizable since, if $P = \begin{bmatrix} 1 & 3 \\ 1 & -2 \end{bmatrix}$ and $D = \begin{bmatrix} 4 & 0 \\ 0 & -1 \end{bmatrix}$, then $P^{-1}AP = D$, as can be easily checked. (Actually, it is faster to check the equivalent statement $AP = PD$, since it does not require finding P^{-1}.)

Example 4.24 begs the question of where matrices P and D came from. Observe that the diagonal entries 4 and -1 of D are the eigenvalues of A, since they are the roots of its characteristic polynomial, which we found in Example 4.23(b). The origin of matrix P is less obvious, but, as we are about to demonstrate, its entries are obtained from the eigenvectors of A. Theorem 4.23 makes this connection precise.

Theorem 4.23

Let A be an $n \times n$ matrix. Then A is diagonalizable if and only if A has n linearly independent eigenvectors.

More precisely, there exist an invertible matrix P and a diagonal matrix D such that $P^{-1}AP = D$ if and only if the columns of P are n linearly independent eigenvectors of A and the diagonal entries of D are the eigenvalues of A corresponding to the eigenvectors in P in the same order.

Proof   Suppose first that A is similar to the diagonal matrix D via $P^{-1}AP = D$ or, equivalently, $AP = PD$. Let the columns of P be $\mathbf{p}_1, \mathbf{p}_2, \ldots, \mathbf{p}_n$ and let the diagonal entries of D be $\lambda_1, \lambda_2, \ldots, \lambda_n$. Then

$$A\begin{bmatrix} \mathbf{p}_1 & \mathbf{p}_2 & \cdots & \mathbf{p}_n \end{bmatrix} = \begin{bmatrix} \mathbf{p}_1 & \mathbf{p}_2 & \cdots & \mathbf{p}_n \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix} \tag{1}$$

or
$$\begin{bmatrix} A\mathbf{p}_1 & A\mathbf{p}_2 & \cdots & A\mathbf{p}_n \end{bmatrix} = \begin{bmatrix} \lambda_1\mathbf{p}_1 & \lambda_2\mathbf{p}_2 & \cdots & \lambda_n\mathbf{p}_n \end{bmatrix} \tag{2}$$

where the right-hand side is just the column-row representation of the product PD. Equating columns, we have

$$A\mathbf{p}_1 = \lambda_1\mathbf{p}_1, A\mathbf{p}_2 = \lambda_2\mathbf{p}_2, \ldots, A\mathbf{p}_n = \lambda_n\mathbf{p}_n$$

which proves that the column vectors of P are eigenvectors of A whose corresponding eigenvalues are the diagonal entries of D in the same order. Since P is invertible, its columns are linearly independent, by the Fundamental Theorem of Invertible Matrices.

Conversely, if A has n linearly independent eigenvectors $\mathbf{p}_1, \mathbf{p}_2, \ldots, \mathbf{p}_n$ with corresponding eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$, respectively, then

$$A\mathbf{p}_1 = \lambda_1\mathbf{p}_1, A\mathbf{p}_2 = \lambda_2\mathbf{p}_2, \ldots, A\mathbf{p}_n = \lambda_n\mathbf{p}_n$$

This implies Equation (2) above, which is equivalent to Equation (1). Consequently, if we take P to be the $n \times n$ matrix with columns $\mathbf{p}_1, \mathbf{p}_2, \ldots, \mathbf{p}_n$, then Equation (1) becomes $AP = PD$. Since the columns of P are linearly independent, the Fundamental Theorem of Invertible Matrices implies that P is invertible, so $P^{-1}AP = D$; that is, A is diagonalizable.

Example 4.25

If possible, find a matrix P that diagonalizes

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 2 & -5 & 4 \end{bmatrix}$$

Solution   We studied this matrix in Example 4.18, where we discovered that it has eigenvalues $\lambda_1 = \lambda_2 = 1$ and $\lambda_3 = 2$. The eigenspaces have the following bases:

$$\text{For } \lambda_1 = \lambda_2 = 1, E_1 \text{ has basis } \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

$$\text{For } \lambda_3 = 2, E_2 \text{ has basis } \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}.$$

Since all other eigenvectors are just multiples of one of these two basis vectors, there cannot be three linearly independent eigenvectors. By Theorem 4.23, therefore, A is not diagonalizable.

Example 4.26

If possible, find a matrix P that diagonalizes

$$A = \begin{bmatrix} -1 & 0 & 1 \\ 3 & 0 & -3 \\ 1 & 0 & -1 \end{bmatrix}$$

Solution   This is the matrix of Example 4.19. There, we found that the eigenvalues of A are $\lambda_1 = \lambda_2 = 0$ and $\lambda_3 = -2$, with the following bases for the eigenspaces:

$$\text{For } \lambda_1 = \lambda_2 = 0, E_0 \text{ has basis } \mathbf{p_1} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \text{ and } \mathbf{p_2} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}.$$

$$\text{For } \lambda_3 = -2, E_{-2} \text{ has basis } \mathbf{p_3} = \begin{bmatrix} -1 \\ 3 \\ 1 \end{bmatrix}.$$

It is straightforward to check that these three vectors are linearly independent. Thus, if we take

$$P = [\mathbf{p_1} \ \mathbf{p_2} \ \mathbf{p_3}] = \begin{bmatrix} 0 & 1 & -1 \\ 1 & 0 & 3 \\ 0 & 1 & 1 \end{bmatrix}$$

then P is invertible. Furthermore,

$$P^{-1}AP = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -2 \end{bmatrix} = D$$

as can be easily checked. (If you are checking by hand, it is much easier to check the equivalent equation $AP = PD$.)

Remarks
- When there are enough eigenvectors, they can be placed into the columns of P in any order. However, the eigenvalues will come up on the diagonal of D in the same order as their corresponding eigenvectors in P. For example, if we had chosen

$$P = [\mathbf{p_1} \ \mathbf{p_3} \ \mathbf{p_2}] = \begin{bmatrix} 0 & -1 & 1 \\ 1 & 3 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

then we would have found

$$P^{-1}AP = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

• In Example 4.26, you were asked to check that the eigenvectors $\mathbf{p}_1$, $\mathbf{p}_2$, and $\mathbf{p}_3$ were linearly independent. Was it necessary to check this? We knew that $\{\mathbf{p}_1, \mathbf{p}_2\}$ was linearly independent, since it was a basis for the eigenspace E_0. We also knew that the sets $\{\mathbf{p}_1, \mathbf{p}_3\}$ and $\{\mathbf{p}_2, \mathbf{p}_3\}$ were linearly independent, by Theorem 4.20. But we could *not* conclude from this information that $\{\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3\}$ was linearly independent. The next theorem, however, guarantees that linear independence is preserved when the bases of different eigenspaces are combined.

Theorem 4.24   Let A be an $n \times n$ matrix and let $\lambda_1, \lambda_2, \ldots, \lambda_k$ be distinct eigenvalues of A. If $\mathcal{B}_i$ is a basis for the eigenspace E_{λ_i}, then $\mathcal{B} = \mathcal{B}_1 \cup \mathcal{B}_2 \cup \cdots \cup \mathcal{B}_k$ (i.e., the total collection of basis vectors for all of the eigenspaces) is linearly independent.

Proof   Let $\mathcal{B}_i = \{\mathbf{v}_{i1}, \mathbf{v}_{i2}, \ldots, \mathbf{v}_{in_i}\}$ for $i = 1, \ldots, k$. We have to show that

$$\mathcal{B} = \{\mathbf{v}_{11}, \mathbf{v}_{12}, \ldots, \mathbf{v}_{1n_1}, \mathbf{v}_{21}, \mathbf{v}_{22}, \ldots, \mathbf{v}_{2n_2}, \ldots, \mathbf{v}_{k1}, \mathbf{v}_{k2}, \ldots, \mathbf{v}_{kn_k}\}$$

is linearly independent. Suppose some nontrivial linear combination of these vectors is the zero vector—say,

$$(c_{11}\mathbf{v}_{11} + \cdots + c_{1n_1}\mathbf{v}_{1n_1}) + (c_{21}\mathbf{v}_{21} + \cdots + c_{2n_2}\mathbf{v}_{2n_2}) + \cdots + (c_{k1}\mathbf{v}_{k1} + \cdots + c_{kn_k}\mathbf{v}_{kn_k}) = 0 \tag{3}$$

Denoting the sums in parentheses by $\mathbf{x}_1, \mathbf{x}_2, \ldots \mathbf{x}_k$, we can write Equation (3) as

$$\mathbf{x}_1 + \mathbf{x}_2 + \cdots + \mathbf{x}_k = 0 \tag{4}$$

Now each $\mathbf{x}_i$ is in E_{λ_i} (why?) and so either is an eigenvector corresponding to λ_i or is $\mathbf{0}$. But, since the eigenvalues λ_i are distinct, if *any* of the factors $\mathbf{x}_i$ is an eigenvector, they are linearly independent, by Theorem 4.20. Yet Equation (4) is a linear *dependence* relationship; this is a contradiction. We conclude that Equation (3) must be trivial; that is, all of its coefficients are zero. Hence, $\mathcal{B}$ is linearly independent.

There is one case in which diagonalizability is automatic: an $n \times n$ matrix with n distinct eigenvalues.

Theorem 4.25   If A is an $n \times n$ matrix with n distinct eigenvalues, then A is diagonalizable.

Proof   Let $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ be eigenvectors corresponding to the n distinct eigenvalues of A. (Why could there not be *more* than n such eigenvectors?) By Theorem 4.20, $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ are linearly independent, so, by Theorem 4.23, A is diagonalizable.

Example 4.27   The matrix

$$A = \begin{bmatrix} 2 & -3 & 7 \\ 0 & 5 & 1 \\ 0 & 0 & -1 \end{bmatrix}$$

has eigenvalues $\lambda_1 = 2$, $\lambda_2 = 5$, and $\lambda_3 = -1$, by Theorem 4.15. Since these are three distinct eigenvalues for a 3×3 matrix, A is diagonalizable, by Theorem 4.25. (If we actually require a matrix P such that $P^{-1}AP$ is diagonal, we must still compute bases for the eigenspaces, as in Example 4.19 and Example 4.26 above.)

The final theorem of this section is an important result that characterizes diagonalizable matrices in terms of the two notions of multiplicity that were introduced following Example 4.18. It gives precise conditions under which an $n \times n$ matrix can be diagonalized, even when it has fewer than n eigenvalues, as in Example 4.26. We first prove a lemma that holds whether or not a matrix is diagonalizable.

Lemma 4.26

If A is an $n \times n$ matrix, then the geometric multiplicity of each eigenvalue is less than or equal to its algebraic multiplicity.

Proof   Suppose λ_1 is an eigenvalue of A with geometric multiplicity p; that is, $\dim E_{\lambda_1} = p$. Specifically, let E_{λ_1} have basis $\mathcal{B}_1 = \{v_1, v_2, \ldots, v_p\}$. Let Q be any invertible $n \times n$ matrix having $v_1, v_2, \ldots, v_p$ as its first p columns — say,

$$Q = [v_1 \quad \cdots \quad v_p \quad v_{p+1} \quad \cdots \quad v_n]$$

or, as a partitioned matrix,

$$Q = [U \mid V]$$

Let

$$Q^{-1} = \left[\frac{C}{D}\right]$$

where C is $p \times n$.

Since the columns of U are eigenvectors corresponding to λ_1, $AU = \lambda_1 U$. We also have

$$\left[\begin{array}{c|c} I_p & O \\ \hline O & I_{n-p} \end{array}\right] = I_n = Q^{-1}Q = \left[\frac{C}{D}\right][U \mid V] = \left[\begin{array}{c|c} CU & CV \\ \hline DU & DV \end{array}\right]$$

from which we obtain $CU = I_p$, $CV = O$, $DU = O$, and $DV = I_{n-p}$. Therefore,

$$Q^{-1}AQ = \left[\frac{C}{D}\right]A[U \mid V] = \left[\begin{array}{c|c} CAU & CAV \\ \hline DAU & DAV \end{array}\right] = \left[\begin{array}{c|c} \lambda_1 CU & CAV \\ \hline \lambda_1 DU & DAV \end{array}\right] = \left[\begin{array}{c|c} \lambda_1 I_p & CAV \\ \hline O & DAV \end{array}\right]$$

By Exercise 69 in Section 4.2, it follows that

$$\det(Q^{-1}AQ - \lambda I) = (\lambda_1 - \lambda)^p \det(DAV - \lambda I) \tag{5}$$

But $\det(Q^{-1}AQ - \lambda I)$ is the characteristic polynomial of $Q^{-1}AQ$, which is the same as the characteristic polynomial of A, by Theorem 4.22(d). Thus, Equation (5) implies that the algebraic multiplicity of λ_1 is at least p, its geometric multiplicity.

Theorem 4.27

The Diagonalization Theorem

Let A be an $n \times n$ matrix whose distinct eigenvalues are $\lambda_1, \lambda_2, \ldots, \lambda_k$. The following statements are equivalent:

a. A is diagonalizable.
b. The union $\mathcal{B}$ of the bases of the eigenspaces of A (as in Theorem 4.24) contains n vectors.
c. The algebraic multiplicity of each eigenvalue equals its geometric multiplicity.

Proof   (a) $\Rightarrow$ (b) If A is diagonalizable, then it has n linearly independent eigenvectors, by Theorem 4.23. If n_i of these eigenvectors correspond to the eigenvalue λ_i, then $\mathcal{B}_i$ contains at least n_i vectors. (We already know that these n_i vectors are linearly independent; the only thing that might prevent them from being a basis for E_{λ_i} is that they might not span it.) Thus, $\mathcal{B}$ contains at least n vectors. But, by Theorem 4.24, $\mathcal{B}$ is a linearly independent set in $\mathbb{R}^n$; hence, it contains exactly n vectors.

(b) $\Rightarrow$ (c) Let the geometric multiplicity of λ_i be $d_i = \dim E_{\lambda_i}$ and let the algebraic multiplicity of λ_i be m_i. By Lemma 4.26, $d_i \le m_i$ for $i = 1, \ldots, k$. Now assume that property (b) holds. Then we also have

$$n = d_1 + d_2 + \cdots + d_k \le m_1 + m_2 + \cdots + m_k$$

But $m_1 + m_2 + \cdots + m_k = n$, since the sum of the algebraic multiplicities of the eigenvalues of A is just the degree of the characteristic polynomial of A—namely, n.

It follows that $d_1 + d_2 + \cdots + d_k = m_1 + m_2 + \cdots + m_k$, which implies that

$$(m_1 - d_1) + (m_2 - d_2) + \cdots + (m_k - d_k) = 0 \tag{6}$$

Using Lemma 4.26 again, we know that $m_i - d_i \ge 0$ for $i = 1, \ldots, k$, from which we can deduce that each summand in Equation (6) is zero; that is, $m_i = d_i$ for $i = 1, \ldots, k$.

(c) $\Rightarrow$ (a) If the algebraic multiplicity m_i and the geometric multiplicity d_i are equal for each eigenvalue λ_i of A, then $\mathcal{B}$ has $d_1 + d_2 + \cdots + d_k = m_1 + m_2 + \cdots + m_k = n$ vectors, which are linearly independent, by Theorem 4.24. Thus, these are n linearly independent eigenvectors of A, and A is diagonalizable, by Theorem 4.23.

Example 4.28

(a) The matrix $A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 2 & -5 & 4 \end{bmatrix}$ from Example 4.18 has two distinct eigenvalues, $\lambda_1 = \lambda_2 = 1$ and $\lambda_3 = 2$. Since the eigenvalue $\lambda_1 = \lambda_2 = 1$ has algebraic multiplicity 2 but geometric multiplicity 1, A is not diagonalizable, by the Diagonalization Theorem. (See also Example 4.25.)

(b) The matrix $A = \begin{bmatrix} -1 & 0 & 1 \\ 3 & 0 & -3 \\ 1 & 0 & -1 \end{bmatrix}$ from Example 4.19 also has two distinct eigenvalues, $\lambda_1 = \lambda_2 = 0$ and $\lambda_3 = -2$. The eigenvalue 0 has algebraic and geometric multiplicity 2, and the eigenvalue -2 has algebraic and geometric multiplicity 1. Thus, this matrix *is* diagonalizable, by the Diagonalization Theorem. (This agrees with our findings in Example 4.26.)

We conclude this section with an application of diagonalization to the computation of the powers of a matrix.

Example 4.29

Compute A^{10} if $A = \begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix}$.

Solution   In Example 4.21, we found that this matrix has eigenvalues $\lambda_1 = -1$ and $\lambda_2 = 2$, with corresponding eigenvectors $\mathbf{v}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$. It follows

(from any one of a number of theorems in this section) that A is diagonalizable and $P^{-1}AP = D$, where

$$P = [\mathbf{v}_1 \quad \mathbf{v}_2] = \begin{bmatrix} 1 & 1 \\ -1 & 2 \end{bmatrix} \quad \text{and} \quad D = \begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix}$$

Solving for A, we have $A = PDP^{-1}$ and, by Theorem 4.22(f), $A^n = PD^nP^{-1}$ for all $n \geq 1$.

Since

$$D^n = \begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix}^n = \begin{bmatrix} (-1)^n & 0 \\ 0 & 2^n \end{bmatrix}$$

we have

$$A^n = PD^nP^{-1} = \begin{bmatrix} 1 & 1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} (-1)^n & 0 \\ 0 & 2^n \end{bmatrix}\begin{bmatrix} 1 & 1 \\ -1 & 2 \end{bmatrix}^{-1}$$

$$= \begin{bmatrix} 1 & 1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} (-1)^n & 0 \\ 0 & 2^n \end{bmatrix}\begin{bmatrix} \frac{2}{3} & -\frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} \end{bmatrix}$$

$$= \begin{bmatrix} \dfrac{2(-1)^n + 2^n}{3} & \dfrac{(-1)^{n+1} + 2^n}{3} \\ \dfrac{2(-1)^{n+1} + 2^{n+1}}{3} & \dfrac{(-1)^{n+2} + 2^{n+1}}{3} \end{bmatrix}$$

Since we were only asked for A^{10}, this is more than we needed. But now we can simply set $n = 10$ to find

$$A^{10} = \begin{bmatrix} \dfrac{2(-1)^{10} + 2^{10}}{3} & \dfrac{(-1)^{11} + 2^{10}}{3} \\ \dfrac{2(-1)^{11} + 2^{11}}{3} & \dfrac{(-1)^{12} + 2^{11}}{3} \end{bmatrix} = \begin{bmatrix} 342 & 341 \\ 682 & 683 \end{bmatrix}$$

Exercises 4.4

In Exercises 1–4, show that A and B are not similar matrices.

1. $A = \begin{bmatrix} 4 & 1 \\ 3 & 1 \end{bmatrix}, B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

2. $A = \begin{bmatrix} 2 & 1 \\ -4 & 6 \end{bmatrix}, B = \begin{bmatrix} 3 & -1 \\ -5 & 7 \end{bmatrix}$

3. $A = \begin{bmatrix} 2 & 1 & 4 \\ 0 & 2 & 3 \\ 0 & 0 & 4 \end{bmatrix}, B = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 4 & 0 \\ 2 & 3 & 4 \end{bmatrix}$

4. $A = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{bmatrix}, B = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}$

In Exercises 5–7, a diagonalization of the matrix A is given in the form $P^{-1}AP = D$. List the eigenvalues of A and bases for the corresponding eigenspaces.

5. $\begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 5 & -1 \\ 2 & 2 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 4 & 0 \\ 0 & 3 \end{bmatrix}$

6. $\begin{bmatrix} \frac{1}{6} & \frac{1}{6} & \frac{1}{6} \\ \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\ \frac{1}{3} & \frac{1}{3} & -\frac{2}{3} \end{bmatrix}\begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}\begin{bmatrix} 3 & 1 & 0 \\ 1 & -1 & 1 \\ 2 & 0 & -1 \end{bmatrix}$

$$= \begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

7. $\begin{bmatrix} \frac{1}{8} & \frac{1}{8} & \frac{1}{8} \\ -\frac{1}{4} & \frac{3}{4} & -\frac{1}{4} \\ \frac{5}{8} & -\frac{3}{8} & -\frac{3}{8} \end{bmatrix} \begin{bmatrix} 1 & 3 & 3 \\ 2 & 0 & 2 \\ 3 & 3 & 1 \end{bmatrix} \begin{bmatrix} 3 & 0 & 1 \\ 2 & 1 & 0 \\ 3 & -1 & -1 \end{bmatrix}$

$= \begin{bmatrix} 6 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{bmatrix}$

In Exercises 8–15, determine whether A is diagonalizable and, if so, find an invertible matrix P and a diagonal matrix D such that $P^{-1}AP = D$.

8. $A = \begin{bmatrix} 5 & 2 \\ 2 & 5 \end{bmatrix}$ **9.** $A = \begin{bmatrix} -3 & 4 \\ -1 & 1 \end{bmatrix}$

10. $A = \begin{bmatrix} 3 & 1 & 0 \\ 0 & 3 & 1 \\ 0 & 0 & 3 \end{bmatrix}$ **11.** $A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}$

12. $A = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 2 & 1 \\ 3 & 0 & 1 \end{bmatrix}$ **13.** $A = \begin{bmatrix} 1 & 2 & 1 \\ -1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$

14. $A = \begin{bmatrix} 2 & 0 & 0 & 2 \\ 0 & 3 & 2 & 1 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ **15.** $A = \begin{bmatrix} 2 & 0 & 0 & 4 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & -2 \end{bmatrix}$

In Exercises 16–23, use the method of Example 4.29 to compute the indicated power of the matrix.

16. $\begin{bmatrix} -4 & 6 \\ -3 & 5 \end{bmatrix}^9$ **17.** $\begin{bmatrix} -1 & 6 \\ 1 & 0 \end{bmatrix}^{10}$

18. $\begin{bmatrix} 4 & -3 \\ -1 & 2 \end{bmatrix}^{-6}$ **19.** $\begin{bmatrix} 0 & 3 \\ 1 & 2 \end{bmatrix}^k$

20. $\begin{bmatrix} 2 & 1 & 2 \\ 2 & 1 & 2 \\ 2 & 1 & 2 \end{bmatrix}^8$ **21.** $\begin{bmatrix} 1 & 1 & 1 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}^{2015}$

22. $\begin{bmatrix} 2 & 0 & 1 \\ 1 & 1 & 1 \\ 1 & 0 & 2 \end{bmatrix}^k$ **23.** $\begin{bmatrix} 1 & 1 & 0 \\ 2 & -2 & 2 \\ 0 & 1 & 1 \end{bmatrix}^k$

In Exercises 24–29, find all (real) values of k for which A is diagonalizable.

24. $A = \begin{bmatrix} 1 & 1 \\ 0 & k \end{bmatrix}$ **25.** $A = \begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}$

26. $A = \begin{bmatrix} k & 1 \\ 1 & 0 \end{bmatrix}$ **27.** $A = \begin{bmatrix} 1 & 0 & k \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

28. $A = \begin{bmatrix} 1 & k & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ **29.** $A = \begin{bmatrix} 1 & 1 & k \\ 1 & 1 & k \\ 1 & 1 & k \end{bmatrix}$

30. Prove Theorem 4.21(c).

31. Prove Theorem 4.22(b).

32. Prove Theorem 4.22(c).

33. Prove Theorem 4.22(e).

34. Prove Theorem 4.22(f).

35. Prove Theorem 4.22(g).

36. If A and B are invertible matrices, show that AB and BA are similar.

37. Prove that if A and B are similar matrices, then $\text{tr}(A) = \text{tr}(B)$. [*Hint:* Find a way to use Exercise 45 from Section 3.2.]

In general, it is difficult to show that two matrices are similar. However, if two similar matrices are diagonalizable, the task becomes easier. In Exercises 38–41, show that A and B are similar by showing that they are similar to the same diagonal matrix. Then find an invertible matrix P such that $P^{-1}AP = B$.

38. $A = \begin{bmatrix} 3 & 1 \\ 0 & -1 \end{bmatrix}, B = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$

39. $A = \begin{bmatrix} 5 & -3 \\ 4 & -2 \end{bmatrix}, B = \begin{bmatrix} -1 & 1 \\ -6 & 4 \end{bmatrix}$

40. $A = \begin{bmatrix} 2 & 1 & 0 \\ 0 & -2 & 1 \\ 0 & 0 & 1 \end{bmatrix}, B = \begin{bmatrix} 3 & 2 & -5 \\ 1 & 2 & -1 \\ 2 & 2 & -4 \end{bmatrix}$

41. $A = \begin{bmatrix} 1 & 0 & 2 \\ 1 & -1 & 1 \\ 2 & 0 & 1 \end{bmatrix}, B = \begin{bmatrix} -3 & -2 & 0 \\ 6 & 5 & 0 \\ 4 & 4 & -1 \end{bmatrix}$

42. Prove that if A is similar to B, then A^T is similar to B^T.

43. Prove that if A is diagonalizable, so is A^T.

44. Let A be an invertible matrix. Prove that if A is diagonalizable, so is A^{-1}.

45. Prove that if A is a diagonalizable matrix with only one eigenvalue λ, then A is of the form $A = \lambda I$. (Such a matrix is called a **scalar matrix**.)

46. Let A and B be $n \times n$ matrices, each with n distinct eigenvalues. Prove that A and B have the same eigenvectors if and only if $AB = BA$.

47. Let A and B be similar matrices. Prove that the algebraic multiplicities of the eigenvalues of A and B are the same.

48. Let A and B be similar matrices. Prove that the geometric multiplicities of the eigenvalues of A and B are the same. [*Hint:* Show that, if $B = P^{-1}AP$, then every eigenvector of B is of the form $P^{-1}\mathbf{v}$ for some eigenvector $\mathbf{v}$ of A.]

49. Prove that if A is a diagonalizable matrix such that every eigenvalue of A is either 0 or 1, then A is idempotent (that is, $A^2 = A$).

50. Let A be a nilpotent matrix (that is, $A^m = O$ for some $m > 1$). Prove that if A is diagonalizable, then A must be the zero matrix.

51. Suppose that A is a 6×6 matrix with characteristic polynomial $c_A(\lambda) = (1 + \lambda)(1 - \lambda)^2(2 - \lambda)^3$.

(a) Prove that it is not possible to find three linearly independent vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ in $\mathbb{R}^6$ such that $A\mathbf{v}_1 = \mathbf{v}_1, A\mathbf{v}_2 = \mathbf{v}_2$, and $A\mathbf{v}_3 = \mathbf{v}_3$.

(b) If A is diagonalizable, what are the dimensions of the eigenspaces E_{-1}, E_1, and E_2?

52. Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$.

(a) Prove that A is diagonalizable if $(a - d)^2 + 4bc > 0$ and is not diagonalizable if $(a - d)^2 + 4bc < 0$.

(b) Find two examples to demonstrate that if $(a - d)^2 + 4bc = 0$, then A may or may not be diagonalizable.

4.5   Iterative Methods for Computing Eigenvalues

At this point, the only method we have for computing the eigenvalues of a matrix is to solve the characteristic equation. However, there are several problems with this method that render it impractical in all but small examples. The first problem is that it depends on the computation of a determinant, which is a very time-consuming process for large matrices. The second problem is that the characteristic equation is a polynomial equation, and there are no formulas for solving polynomial equations of degree higher than 4 (polynomials of degrees 2, 3, and 4 can be solved using the quadratic formula and its analogues). Thus, we are forced to *approximate* eigenvalues in most practical problems. Unfortunately, methods for approximating the roots of a polynomial are quite sensitive to roundoff error and are therefore unreliable.

Instead, we bypass the characteristic polynomial altogether and take a different approach, approximating an eigenvector first and then using this eigenvector to find the corresponding eigenvalue. In this section, we will explore several variations on one such method that is based on a simple iterative technique.

In 1824, the Norwegian mathematician Niels Henrik Abel (1802–1829) proved that a general fifth-degree (quintic) polynomial equation is not *solvable by radicals;* that is, there is no formula for its roots in terms of its coefficients that uses only the operations of addition, subtraction, multiplication, division, and taking nth roots. In a paper written in 1830 and published posthumously in 1846, the French mathematician Evariste Galois (1811–1832) gave a more complete theory that established conditions under which an arbitrary polynomial equation can be solved by radicals. Galois's work was instrumental in establishing the branch of algebra called *group theory*; his approach to polynomial equations is now known as *Galois theory*.

The Power Method

The power method applies to an $n \times n$ matrix that has a ***dominant eigenvalue*** λ_1—that is, an eigenvalue that is larger in absolute value than all of the other eigenvalues. For example, if a matrix has eigenvalues $-4, -3, 1$, and 3, then -4 is the dominant eigenvalue, since $4 = |-4| > |-3| \geq |3| \geq |1|$. On the other hand, a matrix with eigenvalues $-4, -3, 3$, and 4 has no dominant eigenvalue.

The power method proceeds iteratively to produce a sequence of scalars that converges to λ_1 and a sequence of vectors that converges to the corresponding eigenvector $\mathbf{v}_1$, the ***dominant eigenvector.*** For simplicity, we will assume that the matrix A is diagonalizable. The following theorem is the basis for the power method.

Theorem 4.28   Let A be an $n \times n$ diagonalizable matrix with dominant eigenvalue λ_1. Then there exists a nonzero vector $\mathbf{x}_0$ such that the sequence of vectors $\mathbf{x}_k$ defined by

$$\mathbf{x}_1 = A\mathbf{x}_0, \mathbf{x}_2 = A\mathbf{x}_1, \mathbf{x}_3 = A\mathbf{x}_2, \ldots, \mathbf{x}_k = A\mathbf{x}_{k-1}, \ldots$$

approaches a dominant eigenvector of A.

Proof   We may assume that the eigenvalues of A have been labeled so that

$$|\lambda_1| > |\lambda_2| \geq |\lambda_3| \geq \cdots \geq |\lambda_n|$$

Let $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ be the corresponding eigenvectors. Since $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ are linearly independent (why?), they form a basis for $\mathbb{R}^n$. Consequently, we can write $\mathbf{x}_0$ as a linear combination of these eigenvectors—say,

$$\mathbf{x}_0 = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_n\mathbf{v}_n$$

Now $\mathbf{x}_1 = A\mathbf{x}_0, \mathbf{x}_2 = A\mathbf{x}_1 = A(A\mathbf{x}_0) = A^2\mathbf{x}_0, \mathbf{x}_3 = A\mathbf{x}_2 = A(A^2\mathbf{x}_0) = A^3\mathbf{x}_0$, and, generally,

$$\mathbf{x}_k = A^k\mathbf{x}_0 \quad \text{for } k \geq 1$$

As we saw in Example 4.21,

$$A^k\mathbf{x}_0 = c_1\lambda_1^k\mathbf{v}_1 + c_2\lambda_2^k\mathbf{v}_2 + \cdots + c_n\lambda_n^k\mathbf{v}_n$$

$$= \lambda_1^k\left(c_1\mathbf{v}_1 + c_2\left(\frac{\lambda_2}{\lambda_1}\right)^k\mathbf{v}_2 + \cdots + c_n\left(\frac{\lambda_n}{\lambda_1}\right)^k\mathbf{v}_n\right) \tag{1}$$

where we have used the fact that $\lambda_1 \neq 0$.

The fact that λ_1 is the dominant eigenvalue means that each of the fractions $\lambda_2/\lambda_1, \lambda_3/\lambda_1, \ldots, \lambda_n/\lambda_1$, is less than 1 in absolute value. Thus,

$$\left(\frac{\lambda_2}{\lambda_1}\right)^k, \left(\frac{\lambda_3}{\lambda_1}\right)^k, \ldots, \left(\frac{\lambda_n}{\lambda_1}\right)^k$$

all go to zero as $k \to \infty$. It follows that

$$\mathbf{x}_k = A^k\mathbf{x}_0 \to \lambda_1^k c_1\mathbf{v}_1 \quad \text{as } k \to \infty \tag{2}$$

Now, since $\lambda_1 \neq 0$ and $\mathbf{v}_1 \neq \mathbf{0}$, $\mathbf{x}_k$ is approaching a *nonzero* multiple of $\mathbf{v}_1$ (that is, an eigenvector corresponding to λ_1) *provided* $c_1 \neq 0$. (This is the required condition on the initial vector $\mathbf{x}_0$: It must have a nonzero component c_1 in the direction of the dominant eigenvector $\mathbf{v}_1$.)

Example 4.30

Approximate the dominant eigenvector of $A = \begin{bmatrix} 1 & 1 \\ 2 & 0 \end{bmatrix}$ using the method of Theorem 4.28.

Solution   We will take $\mathbf{x}_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ as the initial vector. Then

$$\mathbf{x}_1 = A\mathbf{x}_0 = \begin{bmatrix} 1 & 1 \\ 2 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$\mathbf{x}_2 = A\mathbf{x}_1 = \begin{bmatrix} 1 & 1 \\ 2 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

We continue in this fashion to obtain the values of $\mathbf{x}_k$ in Table 4.1.

Table 4.1

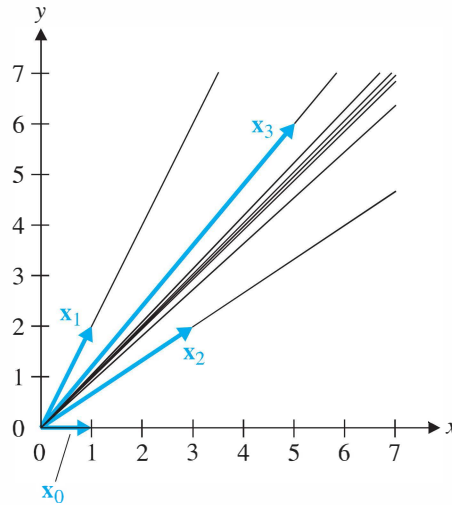| k	0	1	2	3	4	5	6	7	8
$\mathbf{x}_k$	$\begin{bmatrix} 1 \\ 0 \end{bmatrix}$	$\begin{bmatrix} 1 \\ 2 \end{bmatrix}$	$\begin{bmatrix} 3 \\ 2 \end{bmatrix}$	$\begin{bmatrix} 5 \\ 6 \end{bmatrix}$	$\begin{bmatrix} 11 \\ 10 \end{bmatrix}$	$\begin{bmatrix} 21 \\ 22 \end{bmatrix}$	$\begin{bmatrix} 43 \\ 42 \end{bmatrix}$	$\begin{bmatrix} 85 \\ 86 \end{bmatrix}$	$\begin{bmatrix} 171 \\ 170 \end{bmatrix}$
r_k	—	0.50	1.50	0.83	1.10	0.95	1.02	0.99	1.01
l_k	—	1.00	3.00	1.67	2.20	1.91	2.05	1.98	2.01

Figure 4.14

Figure 4.14 shows what is happening geometrically. We know that the eigenspace for the dominant eigenvector will have dimension 1. (Why? See Exercise 46.) Therefore, it is a line through the origin in $\mathbb{R}^2$. The first few iterates $\mathbf{x}_k$ are shown along with the directions they determine. It appears as though the iterates are converging on the line whose direction vector is $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$. To confirm that this is the dominant eigenvector we seek, we need only observe that the ratio r_k of the first to the second component of $\mathbf{x}_k$ gets very close to 1 as k increases. The second line in the body of Table 4.1 gives these values, and you can see clearly that r_k is indeed approaching 1. We deduce that a dominant eigenvector of A is $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

Once we have found a dominant eigenvector, how can we find the corresponding dominant eigenvalue? One approach is to observe that if an $\mathbf{x}_k$ is approximately a dominant eigenvector of A for the dominant eigenvalue λ_1, then

$$\mathbf{x}_{k+1} = A\mathbf{x}_k \approx \lambda_1 \mathbf{x}_k$$

It follows that the ratio l_k of the first component of $\mathbf{x}_{k+1}$ to that of $\mathbf{x}_k$ will approach λ_1 as k increases. Table 4.1 gives the values of l_k, and you can see that they are approaching 2, which is the dominant eigenvalue.

There is a drawback to the method of Example 4.30: The components of the iterates $\mathbf{x}_k$ get very large very quickly and can cause significant roundoff errors. To avoid this drawback, we can multiply each iterate by some scalar that reduces the magnitude of its components. Since scalar multiples of the iterates $\mathbf{x}_k$ will still converge to a dominant eigenvector, this approach is acceptable. There are various ways to accomplish it. One is to normalize each $\mathbf{x}_k$ by dividing it by $\|\mathbf{x}_k\|$ (i.e., to make each iterate a *unit* vector). An easier method—and the one we will use—is to divide each $\mathbf{x}_k$ by the component with the maximum absolute value, so that the largest component is now 1. This method is called **scaling**. Thus, if m_k denotes the component of $\mathbf{x}_k$ with the maximum absolute value, we will replace $\mathbf{x}_k$ by $\mathbf{y}_k = (1/m_k)\mathbf{x}_k$.

We illustrate this approach with the calculations from Example 4.30. For $\mathbf{x}_0$, there is nothing to do, since $m_0 = 1$. Hence,

$$\mathbf{y}_0 = \mathbf{x}_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

We then compute $\mathbf{x}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ as before, but now we scale with $m_1 = 2$ to get

$$\mathbf{y}_1 = \left(\frac{1}{2}\right)\mathbf{x}_1 = \left(\frac{1}{2}\right)\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 0.5 \\ 1 \end{bmatrix}$$

Now the calculations change. We take

$$\mathbf{x}_2 = A\mathbf{y}_1 = \begin{bmatrix} 1 & 1 \\ 2 & 0 \end{bmatrix}\begin{bmatrix} 0.5 \\ 1 \end{bmatrix} = \begin{bmatrix} 1.5 \\ 1 \end{bmatrix}$$

and scale to get

$$\mathbf{y}_2 = \left(\frac{1}{1.5}\right)\begin{bmatrix} 1.5 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0.67 \end{bmatrix}$$

The next few calculations are summarized in Table 4.2.

You can now see clearly that the sequence of vectors $\mathbf{y}_k$ is converging to $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$, a dominant eigenvector. Moreover, the sequence of scalars m_k converges to the corresponding dominant eigenvalue $\lambda_1 = 2$.

Table 4.2

k	0	1	2	3	4	5	6	7	8
$\mathbf{x}_k$	$\begin{bmatrix} 1 \\ 0 \end{bmatrix}$	$\begin{bmatrix} 1 \\ 2 \end{bmatrix}$	$\begin{bmatrix} 1.5 \\ 1 \end{bmatrix}$	$\begin{bmatrix} 1.67 \\ 2 \end{bmatrix}$	$\begin{bmatrix} 1.83 \\ 1.67 \end{bmatrix}$	$\begin{bmatrix} 1.91 \\ 2 \end{bmatrix}$	$\begin{bmatrix} 1.95 \\ 1.91 \end{bmatrix}$	$\begin{bmatrix} 1.98 \\ 2 \end{bmatrix}$	$\begin{bmatrix} 1.99 \\ 1.98 \end{bmatrix}$
$\mathbf{y}_k$	$\begin{bmatrix} 1 \\ 0 \end{bmatrix}$	$\begin{bmatrix} 0.5 \\ 1 \end{bmatrix}$	$\begin{bmatrix} 1 \\ 0.67 \end{bmatrix}$	$\begin{bmatrix} 0.83 \\ 1 \end{bmatrix}$	$\begin{bmatrix} 1 \\ 0.91 \end{bmatrix}$	$\begin{bmatrix} 0.95 \\ 1 \end{bmatrix}$	$\begin{bmatrix} 1 \\ 0.98 \end{bmatrix}$	$\begin{bmatrix} 0.99 \\ 1 \end{bmatrix}$	$\begin{bmatrix} 1 \\ 0.99 \end{bmatrix}$
m_k	1	2	1.5	2	1.83	2	1.95	2	1.99

This method, called the ***power method,*** is summarized below.

The Power Method

Let A be a diagonalizable $n \times n$ matrix with a corresponding dominant eigenvalue λ_1.

1. Let $\mathbf{x}_0 = \mathbf{y}_0$ be any initial vector in $\mathbb{R}^n$ whose largest component is 1.
2. Repeat the following steps for $k = 1, 2, \ldots$:
   (a) Compute $\mathbf{x}_k = A\mathbf{y}_{k-1}$.
   (b) Let m_k be the component of $\mathbf{x}_k$ with the largest absolute value.
   (c) Set $\mathbf{y}_k = (1/m_k)\mathbf{x}_k$.

For most choices of $\mathbf{x}_0$, m_k converges to the dominant eigenvalue λ_1 and $\mathbf{y}_k$ converges to a dominant eigenvector.

Example 4.31

Use the power method to approximate the dominant eigenvalue and a dominant eigenvector of

$$A = \begin{bmatrix} 0 & 5 & -6 \\ -4 & 12 & -12 \\ -2 & -2 & 10 \end{bmatrix}$$

Solution   Taking as our initial vector

$$\mathbf{x}_0 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

we compute the entries in Table 4.3.

You can see that the vectors $\mathbf{y}_k$ are approaching $\begin{bmatrix} 0.50 \\ 1 \\ -0.50 \end{bmatrix}$ and the scalars m_k are approaching 16. This suggests that they are, respectively, a dominant eigenvector and the dominant eigenvalue of A.

Remarks

- If the initial vector $\mathbf{x}_0$ has a zero component in the direction of the dominant eigenvector $\mathbf{v}_1$ (i.e., if $c = 0$ in the proof of Theorem 4.28), then the power method

Table 4.3

k	0	1	2	3	4	5	6	7
$\mathbf{x}_k$	$\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$	$\begin{bmatrix} -1 \\ -4 \\ 6 \end{bmatrix}$	$\begin{bmatrix} -9.33 \\ -19.33 \\ 11.67 \end{bmatrix}$	$\begin{bmatrix} 8.62 \\ 17.31 \\ -9.00 \end{bmatrix}$	$\begin{bmatrix} 8.12 \\ 16.25 \\ -8.20 \end{bmatrix}$	$\begin{bmatrix} 8.03 \\ 16.05 \\ -8.04 \end{bmatrix}$	$\begin{bmatrix} 8.01 \\ 16.01 \\ -8.01 \end{bmatrix}$	$\begin{bmatrix} 8.00 \\ 16.00 \\ -8.00 \end{bmatrix}$
$\mathbf{y}_k$	$\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$	$\begin{bmatrix} -0.17 \\ -0.67 \\ 1 \end{bmatrix}$	$\begin{bmatrix} 0.48 \\ 1 \\ -0.60 \end{bmatrix}$	$\begin{bmatrix} 0.50 \\ 1 \\ -0.52 \end{bmatrix}$	$\begin{bmatrix} 0.50 \\ 1 \\ -0.50 \end{bmatrix}$	$\begin{bmatrix} 0.50 \\ 1 \\ -0.50 \end{bmatrix}$	$\begin{bmatrix} 0.50 \\ 1 \\ -0.50 \end{bmatrix}$	$\begin{bmatrix} 0.50 \\ 1 \\ -0.50 \end{bmatrix}$
m_k	1	6	-19.33	17.31	16.25	16.05	16.01	16.00

will not converge to a dominant eigenvector. However, it is quite likely that during the calculation of the subsequent iterates, at some point roundoff error will produce an $\mathbf{x}_k$ with a nonzero component in the direction of $\mathbf{v}_1$. The power method will then start to converge to a multiple of $\mathbf{v}_1$. (This is one instance where roundoff errors actually help!)

- The power method still works when there is a *repeated* dominant eigenvalue, or even when the matrix is not diagonalizable, under certain conditions. Details may be found in most modern textbooks on numerical analysis. (See Exercises 21–24.)

- For some matrices the power method converges rapidly to a dominant eigenvector, while for others the convergence may be quite slow. A careful look at the proof of Theorem 4.28 reveals why. Since $|\lambda_2/\lambda_1| \geq |\lambda_3/\lambda_1| \geq \cdots \geq |\lambda_n/\lambda_1|$, if $|\lambda_2/\lambda_1|$ is close to zero, then $(\lambda_2/\lambda_1)^k, \ldots, (\lambda_n/\lambda_1)^k$ will all approach zero rapidly. Equation (2) then shows that $\mathbf{x}_k = A^k\mathbf{x}_0$ will approach $\lambda_1^k c_1 \mathbf{v}_1$ rapidly too.

As an illustration, consider Example 4.31. The eigenvalues are 16, 4, and 2, so $\lambda_2/\lambda_1 = 4/16 = 0.25$. Since $0.25^7 \approx 0.00006$, by the seventh iteration we should have close to four-decimal-place accuracy. This is exactly what we saw.

- There is an alternative way to estimate the dominant eigenvalue λ_1 of a matrix A in conjunction with the power method. First, observe that if $A\mathbf{x} = \lambda_1\mathbf{x}$, then

$$\frac{(A\mathbf{x}) \cdot \mathbf{x}}{\mathbf{x} \cdot \mathbf{x}} = \frac{(\lambda_1\mathbf{x}) \cdot \mathbf{x}}{\mathbf{x} \cdot \mathbf{x}} = \frac{\lambda_1(\mathbf{x} \cdot \mathbf{x})}{\mathbf{x} \cdot \mathbf{x}} = \lambda_1$$

The expression $R(\mathbf{x}) = ((A\mathbf{x}) \cdot \mathbf{x})/(\mathbf{x} \cdot \mathbf{x})$ is called a ***Rayleigh quotient.*** As we compute the iterates $\mathbf{x}_k$, the successive Rayleigh quotients $R(\mathbf{x}_k)$ should approach λ_1. In fact, for symmetric matrices, the Rayleigh quotient method is about twice as fast as the scaling factor method. (See Exercises 17–20.)

John William Strutt (1842–1919), Baron Rayleigh, was a British physicist who made major contributions to the fields of acoustics and optics. In 1871, he gave the first correct explanation of why the sky is blue, and in 1895, he discovered the inert gas argon, for which discovery he received the Nobel Prize in 1904. Rayleigh was president of the Royal Society from 1905 to 1908 and became chancellor of Cambridge University in 1908. He used Rayleigh quotients in an 1873 paper on vibrating systems and later in his book *The Theory of Sound.*

The Shifted Power Method and the Inverse Power Method

The power method can help us approximate the *dominant* eigenvalue of a matrix, but what should we do if we want the other eigenvalues? Fortunately, there are several variations of the power method that can be applied.

The ***shifted power method*** uses the observation that, if λ is an eigenvalue of A, then $\lambda - \alpha$ is an eigenvalue of $A - \alpha I$ for any scalar α (Exercise 22 in Section 4.3). Thus, if λ_1 is the dominant eigenvalue of A, the eigenvalues of $A - \lambda_1 I$ will be 0, $\lambda_2 - \lambda_1, \lambda_3 - \lambda_1, \ldots, \lambda_n - \lambda_1$. We can then apply the power method to compute $\lambda_2 - \lambda_1$, and from this value we can find λ_2. Repeating this process will allow us to compute all of the eigenvalues.

Example 4.32   Use the shifted power method to compute the second eigenvalue of the matrix $A = \begin{bmatrix} 1 & 1 \\ 2 & 0 \end{bmatrix}$ from Example 4.30.

Solution   In Example 4.30, we found that $\lambda_1 = 2$. To find λ_2, we apply the power method to

$$A - 2I = \begin{bmatrix} -1 & 1 \\ 2 & -2 \end{bmatrix}$$

We take $\mathbf{x}_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, but other choices will also work. The calculations are summarized in Table 4.4.

Table 4.4

k	0	1	2	3	4
$\mathbf{x}_k$	$\begin{bmatrix} 1 \\ 0 \end{bmatrix}$	$\begin{bmatrix} -1 \\ 2 \end{bmatrix}$	$\begin{bmatrix} 1.5 \\ -3 \end{bmatrix}$	$\begin{bmatrix} 1.5 \\ -3 \end{bmatrix}$	$\begin{bmatrix} 1.5 \\ -3 \end{bmatrix}$
$\mathbf{y}_k$	$\begin{bmatrix} 1 \\ 0 \end{bmatrix}$	$\begin{bmatrix} -0.5 \\ 1 \end{bmatrix}$	$\begin{bmatrix} -0.5 \\ 1 \end{bmatrix}$	$\begin{bmatrix} -0.5 \\ 1 \end{bmatrix}$	$\begin{bmatrix} -0.5 \\ 1 \end{bmatrix}$
m_k	1	2	-3	-3	-3

Our choice of $\mathbf{x}_0$ has produced the eigenvalue -3 after only two iterations. Therefore, $\lambda_2 - \lambda_1 = -3$, so $\lambda_2 = \lambda_1 - 3 = 2 - 3 = -1$ is the second eigenvalue of A.

Recall from property (b) of Theorem 4.18 that if A is invertible with eigenvalue λ, then A^{-1} has eigenvalue $1/\lambda$. Therefore, if we apply the power method to A^{-1}, its dominant eigenvalue will be the *reciprocal of the smallest* (in magnitude) eigenvalue of A. To use this **inverse power method,** we follow the same steps as in the power method, except that in step 2(a) we compute $\mathbf{x}_k = A^{-1} \mathbf{y}_{k-1}$. (In practice, we don't actually compute A^{-1} explicitly; instead, we solve the equivalent equation $A\mathbf{x}_k = \mathbf{y}_{k-1}$ for $\mathbf{x}_k$ using Gaussian elimination. This turns out to be faster.)

Example 4.33

Use the inverse power method to compute the second eigenvalue of the matrix $A = \begin{bmatrix} 1 & 1 \\ 2 & 0 \end{bmatrix}$ from Example 4.30.

Solution   We start, as in Example 4.30, with $\mathbf{x}_0 = \mathbf{y}_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. To solve $A\mathbf{x}_1 = \mathbf{y}_0$, we use row reduction:

$$[A \mid \mathbf{y}_0] = \begin{bmatrix} 1 & 1 & | & 1 \\ 2 & 0 & | & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & | & 0 \\ 0 & 1 & | & 1 \end{bmatrix}$$

Thus, $\mathbf{x}_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, so $\mathbf{y}_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. Then we get $\mathbf{x}_2$ from $A\mathbf{x}_2 = \mathbf{y}_1$:

$$[A \mid \mathbf{y}_1] = \begin{bmatrix} 1 & 1 & | & 0 \\ 2 & 0 & | & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & | & 0.5 \\ 0 & 1 & | & -0.5 \end{bmatrix}$$

Hence, $\mathbf{x}_2 = \begin{bmatrix} 0.5 \\ -0.5 \end{bmatrix}$, and, by scaling, we get $\mathbf{y}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$. Continuing, we get the values shown in Table 4.5, where the values m_k are converging to -1. Thus, the smallest eigenvalue of A is the reciprocal of -1 (which is also -1). This agrees with our previous finding in Example 4.32.

Table 4.5

k	0	1	2	3	4	5	6	7	8	9
$\mathbf{x}_k$	$\begin{bmatrix} 1 \\ 0 \end{bmatrix}$	$\begin{bmatrix} 0 \\ 1 \end{bmatrix}$	$\begin{bmatrix} 0.5 \\ -0.5 \end{bmatrix}$	$\begin{bmatrix} -0.5 \\ 1.5 \end{bmatrix}$	$\begin{bmatrix} 0.5 \\ -0.83 \end{bmatrix}$	$\begin{bmatrix} 0.5 \\ -1.1 \end{bmatrix}$	$\begin{bmatrix} 0.5 \\ -0.95 \end{bmatrix}$	$\begin{bmatrix} 0.5 \\ -1.02 \end{bmatrix}$	$\begin{bmatrix} 0.5 \\ -0.99 \end{bmatrix}$	$\begin{bmatrix} 0.5 \\ -1.01 \end{bmatrix}$
$\mathbf{y}_k$	$\begin{bmatrix} 0 \\ 1 \end{bmatrix}$	$\begin{bmatrix} 1 \\ 0 \end{bmatrix}$	$\begin{bmatrix} 1 \\ -1 \end{bmatrix}$	$\begin{bmatrix} -0.33 \\ 1 \end{bmatrix}$	$\begin{bmatrix} -0.6 \\ 1 \end{bmatrix}$	$\begin{bmatrix} -0.45 \\ 1 \end{bmatrix}$	$\begin{bmatrix} -0.52 \\ 1 \end{bmatrix}$	$\begin{bmatrix} -0.49 \\ 1 \end{bmatrix}$	$\begin{bmatrix} -0.51 \\ 1 \end{bmatrix}$	$\begin{bmatrix} -0.50 \\ 1 \end{bmatrix}$
m_k	1	1	0.5	1.5	-0.83	-1.1	-0.95	-1.02	-0.99	-1.01

The Shifted Inverse Power Method

The most versatile of the variants of the power method is one that combines the two just mentioned. It can be used to find an approximation for *any* eigenvalue, provided we have a close approximation to that eigenvalue. In other words, if a scalar α is given, the ***shifted inverse power method*** will find the eigenvalue λ of A that is closest to α.

If λ is an eigenvalue of A and $\alpha \neq \lambda$, then $A - \alpha I$ is invertible if α is not an eigenvalue of A and $1/(\lambda - \alpha)$ is an eigenvalue of $(A - \alpha I)^{-1}$. (See Exercise 45.) If α is close to λ, then $1/(\lambda - \alpha)$ will be a dominant eigenvalue of $(A - \alpha I)^{-1}$. In fact, if α is *very* close to λ, then $1/(\lambda - \alpha)$ will be *much* bigger in magnitude than the next eigenvalue, so (as noted in the third Remark following Example 4.31) the convergence will be very rapid.

Example 4.34

Use the shifted inverse power method to approximate the eigenvalue of

$$A = \begin{bmatrix} 0 & 5 & -6 \\ -4 & 12 & -12 \\ -2 & -2 & 10 \end{bmatrix}$$

that is closest to 5.

Solution   Shifting, we have

$$A - 5I = \begin{bmatrix} -5 & 5 & -6 \\ -4 & 7 & -12 \\ -2 & -2 & 5 \end{bmatrix}$$

Now we apply the inverse power method with

$$\mathbf{x}_0 = \mathbf{y}_0 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

We solve $(A - 5I)\mathbf{x}_1 = \mathbf{y}_0$ for $\mathbf{x}_1$:

$$[A - 5I \mid \mathbf{y}_0] = \begin{bmatrix} -5 & 5 & -6 & 1 \\ -4 & 7 & -12 & 1 \\ -2 & -2 & 5 & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 0 & -0.61 \\ 0 & 1 & 0 & -0.88 \\ 0 & 0 & 1 & -0.39 \end{bmatrix}$$

Table 4.6

k	0	1	2	3	4	5	6	7
$\mathbf{x}_k$	$\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$	$\begin{bmatrix} -0.61 \\ -0.88 \\ -0.39 \end{bmatrix}$	$\begin{bmatrix} -0.41 \\ -0.69 \\ -0.35 \end{bmatrix}$	$\begin{bmatrix} -0.47 \\ -0.89 \\ -0.44 \end{bmatrix}$	$\begin{bmatrix} -0.49 \\ -0.95 \\ -0.48 \end{bmatrix}$	$\begin{bmatrix} -0.50 \\ -0.98 \\ -0.49 \end{bmatrix}$	$\begin{bmatrix} -0.50 \\ -0.99 \\ -0.50 \end{bmatrix}$	$\begin{bmatrix} -0.50 \\ -1.00 \\ -0.50 \end{bmatrix}$
$\mathbf{y}_k$	$\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$	$\begin{bmatrix} 0.69 \\ 1.00 \\ 0.45 \end{bmatrix}$	$\begin{bmatrix} 0.59 \\ 1.00 \\ 0.51 \end{bmatrix}$	$\begin{bmatrix} 0.53 \\ 1.00 \\ 0.50 \end{bmatrix}$	$\begin{bmatrix} 0.51 \\ 1.00 \\ 0.50 \end{bmatrix}$	$\begin{bmatrix} 0.50 \\ 1.00 \\ 0.50 \end{bmatrix}$	$\begin{bmatrix} 0.50 \\ 1.00 \\ 0.50 \end{bmatrix}$	$\begin{bmatrix} 0.50 \\ 1.00 \\ 0.50 \end{bmatrix}$
m_k	1	−0.88	−0.69	−0.89	−0.95	−0.98	−0.99	−1.00

This gives

$$\mathbf{x}_1 = \begin{bmatrix} -0.61 \\ -0.88 \\ -0.39 \end{bmatrix}, \quad m_1 = -0.88, \quad \text{and} \quad \mathbf{y}_1 = \frac{1}{m_1}\mathbf{x}_1 = -\frac{1}{0.88}\begin{bmatrix} -0.61 \\ -0.88 \\ -0.39 \end{bmatrix} = \begin{bmatrix} 0.69 \\ 1 \\ 0.45 \end{bmatrix}$$

We continue in this fashion to obtain the values in Table 4.6, from which we deduce that the eigenvalue of A closest to 5 is approximately $5 + 1/m_7 \approx 5 + 1/(-1) = 4$, which, in fact, is exact.

The power method and its variants represent only one approach to the computation of eigenvalues. In Chapter 5, we will discuss another method based on the QR factorization of a matrix. For a more complete treatment of this topic, you can consult almost any textbook on numerical methods.

/ $a + bi$ / ## Gerschgorin's Theorem

In this section, we have discussed several variations on the power method for approximating the eigenvalues of a matrix. All of these methods are iterative, and the speed with which they converge depends on the choice of initial vector. If only we had some "inside information" about the location of the eigenvalues of a given matrix, then we could make a judicious choice of the initial vector and perhaps speed up the convergence of the iterative process.

We owe this theorem to the Russian mathematician S. Gerschgorin (1901–1933), who stated it in 1931. It did not receive much attention until 1949, when it was resurrected by Olga Taussky-Todd in a note she published in the *American Mathematical Monthly*.

Fortunately, there is a way to estimate the location of the eigenvalues of any matrix. **Gerschgorin's Disk Theorem** states that the eigenvalues of a (real or complex) $n \times n$ matrix all lie inside the union of n circular disks in the complex plane.

Definition   Let $A = [a_{ij}]$ be a (real or complex) $n \times n$ matrix, and let r_i denote the sum of the absolute values of the off-diagonal entries in the ith row of A; that is, $r_i = \sum_{j \neq i} |a_{ij}|$. The *$i$th Gerschgorin disk* is the circular disk D_i in the complex plane with center a_{ii} and radius r_i. That is,

$$D_i = \{z \text{ in } \mathbb{C} : |z - a_{ii}| \leq r_i\}$$

Olga Taussky-Todd (1906–1995) was born in Olmütz in the Austro-Hungarian Empire (now Olmuac in the Czech Republic). She received her doctorate in number theory from the University of Vienna in 1930. During World War II, she worked for the National Physical Laboratory in London, where she investigated the problem of flutter in the wings of super-sonic aircraft. Although the problem involved differential equations, the stability of an aircraft depended on the eigenvalues of a related matrix. Taussky-Todd remembered Gerschgorin's Theorem from her graduate studies in Vienna and was able to use it to simplify the otherwise laborious computations needed to determine the eigenvalues relevant to the flutter problem.

Taussky-Todd moved to the United States in 1947, and ten years later she became the first woman appointed to the California Institute of Technology. In her career, she produced over 200 publications and received numerous awards. She was instrumental in the development of the branch of mathematics now known as matrix theory.

Example 4.35

Sketch the Gerschgorin disks and the eigenvalues for the following matrices:

(a) $A = \begin{bmatrix} 2 & 1 \\ 2 & -3 \end{bmatrix}$ (b) $A = \begin{bmatrix} 1 & -3 \\ 2 & 3 \end{bmatrix}$

Solution (a) The two Gerschgorin disks are centered at 2 and -3 with radii 1 and 2, respectively. The characteristic polynomial of A is $\lambda^2 + \lambda - 8$, so the eigenvalues are

$$\lambda = (-1 \pm \sqrt{1^2 - 4(-8)})/2 \approx 2.37, -3.37$$

Figure 4.15 shows that the eigenvalues are contained within the two Gerschgorin disks.

(b) The two Gerschgorin disks are centered at 1 and 3 with radii $|-3| = 3$ and 2, respectively. The characteristic polynomial of A is $\lambda^2 - 4\lambda + 9$, so the eigenvalues are

$$\lambda = (4 \pm \sqrt{(-4)^2 - 4(9)})/2 = 2 \pm i\sqrt{5} \approx 2 + 2.23i, 2 - 2.23i$$

Figure 4.16 plots the location of the eigenvalues relative to the Gerschgorin disks.
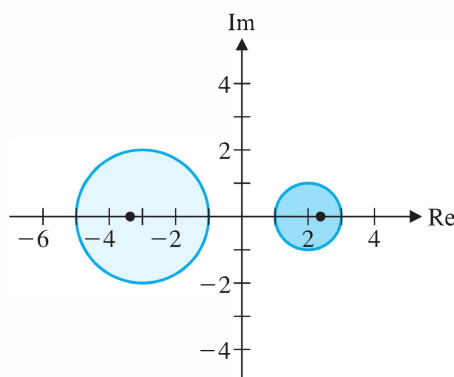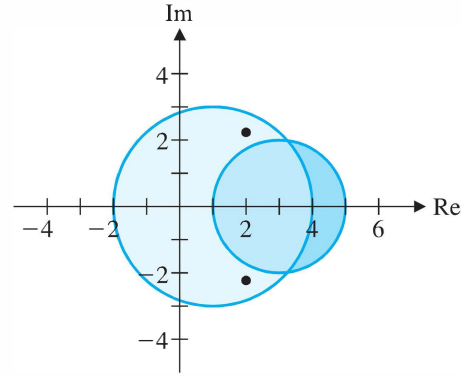

Figure 4.15

Figure 4.16

As Example 4.35 suggests, the eigenvalues of a matrix are contained within its Gerschgorin disks. The next theorem verifies that this is so.

Theorem 4.29   **Gerschgorin's Disk Theorem**

Let A be an $n \times n$ (real or complex) matrix. Then every eigenvalue of A is contained within a Gerschgorin disk.

Proof   Let λ be an eigenvalue of A with corresponding eigenvector $\mathbf{x}$. Let x_i be the entry of $\mathbf{x}$ with the largest absolute value—and hence nonzero. (Why?) Then $A\mathbf{x} = \lambda\mathbf{x}$, the ith row of which is

$$\begin{bmatrix} a_{i1} & a_{i2} & \cdots & a_{in} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \lambda x_i \quad \text{or} \quad \sum_{j=1}^{n} a_{ij}x_j = \lambda x_i$$

Rearranging, we have

$$(\lambda - a_{ii})x_i = \sum_{j \neq i} a_{ij}x_j \quad \text{or} \quad \lambda - a_{ii} = \frac{\sum_{j \neq i} a_{ij}x_j}{x_i}$$

because $x_i \neq 0$. Taking absolute values and using properties of absolute value (see Appendix C), we obtain

$$|\lambda - a_{ii}| = \left| \frac{\sum_{j \neq i} a_{ij}x_j}{x_i} \right| = \frac{\left| \sum_{j \neq i} a_{ij}x_j \right|}{|x_i|} \leq \frac{\sum_{j \neq i} |a_{ij}x_j|}{|x_i|} = \frac{\sum_{j \neq i} |a_{ij}||x_j|}{|x_i|} \leq \sum_{j \neq i} |a_{ij}| = r_i$$

because $|x_j| \leq |x_i|$ for $j \neq i$.

This establishes that the eigenvalue λ is contained within the Gerschgorin disk centered at a_{ii} with radius r_i.

Remarks

- There is a corresponding version of the preceding theorem for Gerschgorin disks whose radii are the sum of the off-diagonal entries in the *i*th *column* of *A*.

- It can be shown that if *k* of the Gerschgorin disks are disjoint from the other disks, then exactly *k* eigenvalues are contained within the union of these *k* disks. In particular, if a single disk is disjoint from the other disks, then it must contain exactly one eigenvalue of the matrix. Example 4.35(a) illustrates this.

- Note that in Example 4.35(a), 0 is not contained in a Gerschgorin disk; that is, 0 is not an eigenvalue of *A*. Hence, without any further computation, we can deduce that the matrix *A* is invertible by Theorem 4.16. This observation is particularly useful when applied to larger matrices, because the Gerschgorin disks can be determined directly from the entries of the matrix.

Example 4.36

Consider the matrix $A = \begin{bmatrix} 2 & 1 & 0 \\ \frac{1}{2} & 6 & \frac{1}{2} \\ 2 & 0 & 8 \end{bmatrix}$. Gerschgorin's Theorem tells us that the eigenvalues of *A* are contained within three disks centered at 2, 6, and 8 with radii 1, 1, and 2, respectively. See Figure 4.17(a). Because the first disk is disjoint from the other two, it must contain exactly one eigenvalue, by the second Remark after Theorem 4.29. Because the characteristic polynomial of *A* has real coefficients, if it has complex roots (i.e., eigenvalues of *A*), they must occur in conjugate pairs. (See Appendix D.) Hence there is a unique real eigenvalue between 1 and 3, and the union of the other two disks contains two (possibly complex) eigenvalues whose real parts lie between 5 and 10.

On the other hand, the first Remark after Theorem 4.29 tells us that the same three eigenvalues of *A* are contained in disks centered at 2, 6, and 8 with radii $\frac{5}{2}$, 1, and $\frac{1}{2}$, respectively. See Figure 4.17(b). These disks are mutually disjoint, so each contains a single (and hence real) eigenvalue. Combining these results, we deduce that *A* has three real eigenvalues, one in each of the intervals [1, 3], [5, 7], and [7.5, 8.5]. (Compute the actual eigenvalues of *A* to verify this.)

(a)   (b)
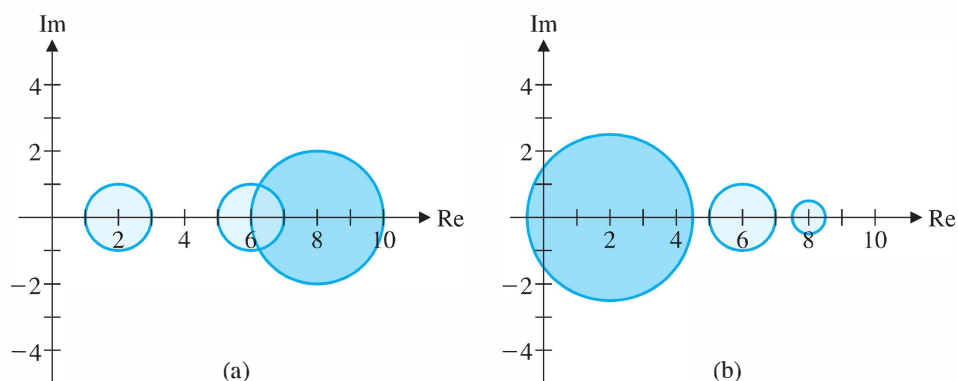
Figure 4.17

Exercises 4.5

In Exercises 1–4, a matrix A is given along with an iterate
$\mathbf{x}_5$, *produced as in Example 4.30.*
(a) Use these data to approximate a dominant eigenvector
whose first component is 1 and a corresponding dominant
eigenvalue. (Use three-decimal-place accuracy.)
(b) Compare your approximate eigenvalue in part (a) with
the actual dominant eigenvalue.

1. $A = \begin{bmatrix} 1 & 2 \\ 5 & 4 \end{bmatrix}$, $\mathbf{x}_5 = \begin{bmatrix} 4443 \\ 11109 \end{bmatrix}$

2. $A = \begin{bmatrix} 7 & 4 \\ -3 & -1 \end{bmatrix}$, $\mathbf{x}_5 = \begin{bmatrix} 7811 \\ -3904 \end{bmatrix}$

3. $A = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$, $\mathbf{x}_5 = \begin{bmatrix} 144 \\ 89 \end{bmatrix}$

4. $A = \begin{bmatrix} 1.5 & 0.5 \\ 2.0 & 3.0 \end{bmatrix}$, $\mathbf{x}_5 = \begin{bmatrix} 60.625 \\ 239.500 \end{bmatrix}$

In Exercises 5–8, a matrix A is given along with an iterate
$\mathbf{x}_k$, *produced using the power method, as in Example 4.31.*
(a) Approximate the dominant eigenvalue and eigenvector
by computing the corresponding m_k and $\mathbf{y}_k$. (b) Verify that
you have approximated an eigenvalue and an eigenvector
of A by comparing $A\mathbf{y}_k$ with $m_k\mathbf{y}_k$.

5. $A = \begin{bmatrix} 2 & -3 \\ -3 & 10 \end{bmatrix}$, $\mathbf{x}_5 = \begin{bmatrix} -3.667 \\ 11.001 \end{bmatrix}$

6. $A = \begin{bmatrix} 5 & 2 \\ 2 & -2 \end{bmatrix}$, $\mathbf{x}_{10} = \begin{bmatrix} 5.530 \\ 1.470 \end{bmatrix}$

7. $A = \begin{bmatrix} 4 & 0 & 6 \\ -1 & 3 & 1 \\ 6 & 0 & 4 \end{bmatrix}$, $\mathbf{x}_8 = \begin{bmatrix} 10.000 \\ 0.001 \\ 10.000 \end{bmatrix}$

8. $A = \begin{bmatrix} 1 & 2 & -2 \\ 1 & 1 & -3 \\ 0 & -1 & 1 \end{bmatrix}$, $\mathbf{x}_{10} = \begin{bmatrix} 3.415 \\ 2.914 \\ -1.207 \end{bmatrix}$

In Exercises 9–14, use the power method to approximate
the dominant eigenvalue and eigenvector of A. Use the given
initial vector $\mathbf{x}_0$, the specified number of iterations k, and
three-decimal-place accuracy.

9. $A = \begin{bmatrix} 14 & 12 \\ 5 & 3 \end{bmatrix}$, $\mathbf{x}_0 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $k = 5$

10. $A = \begin{bmatrix} -6 & 4 \\ 8 & -2 \end{bmatrix}$, $\mathbf{x}_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $k = 6$

11. $A = \begin{bmatrix} 7 & 2 \\ 2 & 3 \end{bmatrix}$, $\mathbf{x}_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $k = 6$

12. $A = \begin{bmatrix} 3.5 & 1.5 \\ 1.5 & -0.5 \end{bmatrix}$, $\mathbf{x}_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $k = 6$

13. $A = \begin{bmatrix} 9 & 4 & 8 \\ 4 & 15 & -4 \\ 8 & -4 & 9 \end{bmatrix}$, $\mathbf{x}_0 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, $k = 5$

14. $A = \begin{bmatrix} 3 & 1 & 0 \\ 1 & 3 & 1 \\ 0 & 1 & 3 \end{bmatrix}$, $\mathbf{x}_0 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, $k = 6$

In Exercises 15 and 16, use the power method to approxi-
mate the dominant eigenvalue and eigenvector of A to
two-decimal-place accuracy. Choose any initial vector you
like (but keep the first Remark after Example 4.31 in mind!)
and apply the method until the digit in the second decimal
place of the iterates stops changing.

15. $A = \begin{bmatrix} 4 & 1 & 3 \\ 0 & 2 & 0 \\ 1 & 1 & 2 \end{bmatrix}$ **16.** $A = \begin{bmatrix} 12 & 6 & -6 \\ 2 & 0 & -2 \\ -6 & 6 & 12 \end{bmatrix}$

Rayleigh quotients are described on page 316. In Exer-
cises 17–20, to see how the Rayleigh quotient method ap-
proximates the dominant eigenvalue more rapidly than the
ordinary power method, compute the successive Rayleigh
quotients $R(\mathbf{x}_i)$ for $i = 1, \ldots, k$ for the matrix A in the given
exercise.

17. Exercise 11   **18.** Exercise 12

19. Exercise 13   **20.** Exercise 14

The matrices in Exercises 21–24 either are not diagonaliz-
able or do not have a dominant eigenvalue (or both). Apply
the power method anyway with the given initial vector $\mathbf{x}_0$,
performing eight iterations in each case. Compute the exact
eigenvalues and eigenvectors and explain what is happening.

21. $A = \begin{bmatrix} 4 & 1 \\ 0 & 4 \end{bmatrix}$, $\mathbf{x}_0 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ **22.** $A = \begin{bmatrix} 3 & 1 \\ -1 & 1 \end{bmatrix}$, $\mathbf{x}_0 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

23. $A = \begin{bmatrix} 4 & 0 & 1 \\ 0 & 4 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $\mathbf{x}_0 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

24. $A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 5 & 1 \\ 0 & 0 & 5 \end{bmatrix}$, $\mathbf{x}_0 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

In Exercises 25–28, the power method does not converge to the dominant eigenvalue and eigenvector. Verify this, using the given initial vector $\mathbf{x}_0$. Compute the exact eigenvalues and eigenvectors and explain what is happening.

25. $A = \begin{bmatrix} -1 & 2 \\ -1 & 1 \end{bmatrix}$, $\mathbf{x}_0 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

26. $A = \begin{bmatrix} 2 & 1 \\ -2 & 5 \end{bmatrix}$, $\mathbf{x}_0 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

27. $A = \begin{bmatrix} -5 & 1 & 7 \\ 0 & 4 & 0 \\ 7 & 1 & -5 \end{bmatrix}$, $\mathbf{x}_0 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

28. $A = \begin{bmatrix} 1 & -1 & 0 \\ 1 & 1 & 0 \\ 1 & -1 & 1 \end{bmatrix}$, $\mathbf{x}_0 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

In Exercises 29–32, apply the shifted power method to approximate the second eigenvalue of the matrix A in the given exercise. Use the given initial vector $\mathbf{x}_0$, k iterations, and three-decimal-place accuracy.

29. Exercise 9   **30.** Exercise 10

31. Exercise 13   **32.** Exercise 14

In Exercises 33–36, apply the inverse power method to approximate, for the matrix A in the given exercise, the eigenvalue that is smallest in magnitude. Use the given initial vector $\mathbf{x}_0$, k iterations, and three-decimal-place accuracy.

33. Exercise 9   **34.** Exercise 10

35. Exercise 7, $\mathbf{x}_0 = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$, $k = 5$

36. Exercise 14

In Exercises 37–40, use the shifted inverse power method to approximate, for the matrix A in the given exercise, the eigenvalue closest to α.

37. Exercise 9, $\alpha = 0$ **38.** Exercise 12, $\alpha = 0$

39. Exercise 7, $\alpha = 5$ **40.** Exercise 13, $\alpha = -2$

Exercise 32 in Section 4.3 demonstrates that every polynomial is (plus or minus) the characteristic polynomial of its own companion matrix. Therefore, the roots of a polynomial p are the eigenvalues of $C(p)$. Hence, we can use the methods of this section to approximate the roots of any polynomial when exact results are not readily available. In Exercises 41–44, apply the shifted inverse power method to the companion matrix $C(p)$ of p to approximate the root of p closest to α to three decimal places.

41. $p(x) = x^2 + 2x - 2$, $\alpha = 0$

42. $p(x) = x^2 - x - 3$, $\alpha = 2$

43. $p(x) = x^3 - 2x^2 + 1$, $\alpha = 0$

44. $p(x) = x^3 - 5x^2 + x + 1$, $\alpha = 5$

45. Let λ be an eigenvalue of A with corresponding eigenvector $\mathbf{x}$. If $\alpha \neq \lambda$ and α is not an eigenvalue of A, show that $1/(\lambda - \alpha)$ is an eigenvalue of $(A - \alpha I)^{-1}$ with corresponding eigenvector $\mathbf{x}$. (Why must $A - \alpha I$ be invertible?)

46. If A has a dominant eigenvalue λ_1, prove that the eigenspace E_{λ_1} is one-dimensional.

$\boxed{a + bi}$ *In Exercises 47–50, draw the Gerschgorin disks for the given matrix.*

47. $\begin{bmatrix} 1 & 1 & 0 \\ \frac{1}{2} & 4 & \frac{1}{2} \\ 1 & 0 & 5 \end{bmatrix}$

48. $\begin{bmatrix} 2 & -i & 0 \\ 1 & 2i & 1+i \\ 0 & 1 & -2i \end{bmatrix}$

49. $\begin{bmatrix} 4 - 3i & i & 2 & -2 \\ i & -1+i & 0 & 0 \\ 1+i & -i & 5+6i & 2i \\ 1 & -2i & 2i & -5-5i \end{bmatrix}$

50. $\begin{bmatrix} 2 & \frac{1}{2} & 0 & 0 \\ \frac{1}{4} & 4 & \frac{1}{4} & 0 \\ 0 & \frac{1}{6} & 6 & \frac{1}{6} \\ 0 & 0 & \frac{1}{8} & 8 \end{bmatrix}$

51. A square matrix is ***strictly diagonally dominant*** if the absolute value of each diagonal entry is greater than the sum of the absolute values of the remaining entries in that row. (See Section 2.5.) Use Gerschgorin's Disk Theorem to prove that a strictly diagonally dominant matrix must be invertible. [*Hint:* See the third Remark after Theorem 4.29.]

52. If A is an $n \times n$ matrix, let $\|A\|$ denote the maximum of the sums of the absolute values of the rows of A; that is,
$$\|A\| = \max_{1 \le i \le n} \left(\sum_{j=1}^{n} |a_{ij}| \right).$$
(See Section 7.2.) Prove that if λ is an eigenvalue of A, then $|\lambda| \le \|A\|$.

53. Let λ be an eigenvalue of a stochastic matrix A (see Section 3.7). Prove that $|\lambda| \le 1$. [*Hint:* Apply Exercise 52 to A^T.]

54. Prove that the eigenvalues of $A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 2 & 5 & 0 & 0 \\ \frac{1}{2} & 0 & 3 & \frac{1}{2} \\ 0 & 0 & \frac{3}{4} & 7 \end{bmatrix}$ are all real, and locate each of these eigenvalues within a closed interval on the real line.

4.6 Applications and the Perron-Frobenius Theorem

In this section, we will explore several applications of eigenvalues and eigenvectors. We begin by revisiting some applications from previous chapters.

Markov Chains

Section 3.7 introduced Markov chains and made several observations about the transition (stochastic) matrices associated with them. In particular, we observed that if P is the transition matrix of a Markov chain, then P has a steady state vector $\mathbf{x}$. That is, there is a vector $\mathbf{x}$ such that $P\mathbf{x} = \mathbf{x}$. This is equivalent to saying that P has 1 as an eigenvalue. We are now in a position to prove this fact.

Theorem 4.30   If P is the $n \times n$ transition matrix of a Markov chain, then 1 is an eigenvalue of P.

Proof   Recall that every transition matrix is stochastic; hence, each of its columns sums to 1. Therefore, if $\mathbf{j}$ is a row vector consisting of n 1s, then $\mathbf{j}P = \mathbf{j}$. (See Exercise 13 in Section 3.7.) Taking transposes, we have

$$P^T\mathbf{j}^T = (\mathbf{j}P)^T = \mathbf{j}^T$$

which implies that $\mathbf{j}^T$ is an eigenvector of P^T with corresponding eigenvalue 1. By Exercise 19 in Section 4.3, P and P^T have the same eigenvalues, so 1 is also an eigenvalue of P.

In fact, much more is true. For most transition matrices, *every* eigenvalue λ satisfies $|\lambda| \leq 1$ and the eigenvalue 1 is *dominant*; that is, if $\lambda \neq 1$, then $|\lambda| < 1$. We need the following two definitions: A matrix is called ***positive*** if all of its entries are positive, and a square matrix is called ***regular*** if some power of it is positive. For example, $A = \begin{bmatrix} 3 & 1 \\ 2 & 2 \end{bmatrix}$ is positive but $B = \begin{bmatrix} 3 & 1 \\ 2 & 0 \end{bmatrix}$ is not. However, B is regular, since $B^2 = \begin{bmatrix} 11 & 3 \\ 6 & 2 \end{bmatrix}$ is positive.

Theorem 4.31   Let P be an $n \times n$ transition matrix with eigenvalue λ.

a. $|\lambda| \leq 1$
b. If P is regular and $\lambda \neq 1$, then $|\lambda| < 1$.

Proof   As in Theorem 4.30, the trick to proving this theorem is to use the fact that P^T has the same eigenvalues as P.

(a) Let $\mathbf{x}$ be an eigenvector of P^T corresponding to λ and let x_k be the component of $\mathbf{x}$ with the largest absolute value m. Then $|x_i| \leq |x_k| = m$ for $i = 1, 2, \ldots, n$. Comparing the kth components of the equation $P^T\mathbf{x} = \lambda\mathbf{x}$, we have

$$p_{1k}x_1 + p_{2k}x_2 + \cdots + p_{nk}x_n = \lambda x_k$$

(Remember that the rows of P^T are the columns of P.) Taking absolute values, we obtain

$$
\begin{aligned}
|\lambda|m = |\lambda||x_k| = |\lambda x_k| &= |p_{1k}x_1 + p_{2k}x_2 + \cdots + p_{nk}x_n| \\
&\leq |p_{1k}x_1| + |p_{2k}x_2| + \cdots + |p_{nk}x_n| \\
&= p_{1k}|x_1| + p_{2k}|x_2| + \cdots + p_{nk}|x_n| \qquad (1) \\
&\leq p_{1k}m + p_{2k}m + \cdots + p_{nk}m \\
&= (p_{1k} + p_{2k} + \cdots + p_{nk})m = m
\end{aligned}
$$

The first inequality follows from the Triangle Inequality in $\mathbb{R}$, and the last equality comes from the fact that the rows of P^T sum to 1. Thus, $|\lambda|m \leq m$. After dividing by m, we have $|\lambda| \leq 1$, as desired.

(b) We will prove the equivalent implication: If $|\lambda| = 1$, then $\lambda = 1$. First, we show that it is true when P (and therefore P^T) is a positive matrix. If $|\lambda| = 1$, then all of the inequalities in Equations (1) are actually equalities. In particular,

$$
p_{1k}|x_1| + p_{2k}|x_2| + \cdots + p_{nk}|x_n| = p_{1k}m + p_{2k}m + \cdots + p_{nk}m
$$

Equivalently,

$$
p_{1k}(m - |x_1|) + p_{2k}(m - |x_2|) + \cdots + p_{nk}(m - |x_n|) = 0 \qquad (2)
$$

Now, since P is positive, $p_{ik} > 0$ for $i = 1, 2, \ldots, n$. Also, $m - |x_i| \geq 0$ for $i = 1, 2, \ldots, n$. Therefore, each summand in Equation (2) must be zero, and this can happen only if $|x_i| = m$ for $i = 1, 2, \ldots, n$. Furthermore, we get equality in the Triangle Inequality in $\mathbb{R}$ if and only if all of the summands are positive or all are negative; in other words, the $p_{ik}x_i$'s all have the same sign. This implies that

$$
\mathbf{x} = \begin{bmatrix} m \\ m \\ \vdots \\ m \end{bmatrix} = m\mathbf{j}^T \quad \text{or} \quad \mathbf{x} = \begin{bmatrix} -m \\ -m \\ \vdots \\ -m \end{bmatrix} = -m\mathbf{j}^T
$$

where $\mathbf{j}$ is a row vector of n 1s, as in Theorem 4.30. Thus, in either case, the eigenspace of P^T corresponding to λ is $E_\lambda = \text{span}(\mathbf{j}^T)$.

But, using the proof of Theorem 4.30, we see that $\mathbf{j}^T = P^T\mathbf{j}^T = \lambda\mathbf{j}^T$, and, comparing components, we find that $\lambda = 1$. This handles the case where P is positive.

If P is regular, then some power of P is positive—say, P^k. It follows that P^{k+1} must also be positive. (Why?) Since λ^k and λ^{k+1} are eigenvalues of P^k and P^{k+1}, respectively, by Theorem 4.18, we have just proved that $\lambda^k = \lambda^{k+1} = 1$. Therefore, $\lambda^k(\lambda - 1) = 0$, which implies that $\lambda = 1$, since $\lambda = 0$ is impossible if $|\lambda| = 1$.

We can now explain some of the behavior of Markov chains that we observed in Chapter 3. In Example 3.64, we saw that for the transition matrix

$$
P = \begin{bmatrix} 0.7 & 0.2 \\ 0.3 & 0.8 \end{bmatrix}
$$

and initial state vector $\mathbf{x}_0 = \begin{bmatrix} 0.6 \\ 0.4 \end{bmatrix}$, the state vectors $\mathbf{x}_k$ converge to the vector $\mathbf{x} = \begin{bmatrix} 0.4 \\ 0.6 \end{bmatrix}$, a steady state vector for P (i.e., $P\mathbf{x} = \mathbf{x}$). We are going to prove that for regular

Markov chains, this always happens. Indeed, we will prove much more. Recall that the state vectors $\mathbf{x}_k$ satisfy $\mathbf{x}_k = P^k \mathbf{x}_0$. Let's investigate what happens to the powers P^k as P becomes large.

Example 4.37

The transition matrix $P = \begin{bmatrix} 0.7 & 0.2 \\ 0.3 & 0.8 \end{bmatrix}$ has characteristic equation

$$0 = \det(P - \lambda I) = \begin{vmatrix} 0.7 - \lambda & 0.2 \\ 0.3 & 0.8 - \lambda \end{vmatrix} = \lambda^2 - 1.5\lambda + 0.5 = (\lambda - 1)(\lambda - 0.5)$$

so its eigenvalues are $\lambda_1 = 1$ and $\lambda_2 = 0.5$. (Note that, thanks to Theorems 4.30 and 4.31, we knew in advance that 1 would be an eigenvalue and the other eigenvalue would be less than 1 in absolute value. However, we still needed to compute λ_2.) The eigenspaces are

$$E_1 = \text{span}\left(\begin{bmatrix} 2 \\ 3 \end{bmatrix} \right) \quad \text{and} \quad E_{0.5} = \text{span}\left(\begin{bmatrix} 1 \\ -1 \end{bmatrix} \right)$$

So, taking $Q = \begin{bmatrix} 2 & 1 \\ 3 & -1 \end{bmatrix}$, we know that $Q^{-1}PQ = \begin{bmatrix} 1 & 0 \\ 0 & 0.5 \end{bmatrix} = D$. From the method used in Example 4.29 in Section 4.4, we have

$$P^k = QD^kQ^{-1} = \begin{bmatrix} 2 & 1 \\ 3 & -1 \end{bmatrix} \begin{bmatrix} 1^k & 0 \\ 0 & (0.5)^k \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 3 & -1 \end{bmatrix}^{-1}$$

Now, as $k \to \infty$, $(0.5)^k \to 0$, so

$$D^k \to \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad P^k \to \begin{bmatrix} 2 & 1 \\ 3 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 3 & -1 \end{bmatrix}^{-1} = \begin{bmatrix} 0.4 & 0.4 \\ 0.6 & 0.6 \end{bmatrix}$$

(Observe that the columns of this "limit matrix" are identical and each is a steady state vector for P.) Now let $\mathbf{x}_0 = \begin{bmatrix} a \\ b \end{bmatrix}$ be any initial probability vector (i.e., $a + b = 1$). Then

$$\mathbf{x}_k = P^k \mathbf{x}_0 \to \begin{bmatrix} 0.4 & 0.4 \\ 0.6 & 0.6 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0.4a + 0.4b \\ 0.6a + 0.6b \end{bmatrix} = \begin{bmatrix} 0.4 \\ 0.6 \end{bmatrix}$$

Not only does this explain what we saw in Example 3.64, it also tells us that the state vectors $\mathbf{x}_k$ will converge to the steady state vector $\mathbf{x} = \begin{bmatrix} 0.4 \\ 0.6 \end{bmatrix}$ for *any* choice of $\mathbf{x}_0$!

There is nothing special about Example 4.37. The next theorem shows that this type of behavior *always* occurs with regular transition matrices. Before we can present the theorem, we need the following lemma.

Lemma 4.32

Let P be a regular $n \times n$ transition matrix. If P is diagonalizable, then the dominant eigenvalue $\lambda_1 = 1$ has algebraic multiplicity 1.

Proof   The eigenvalues of P and P^T are the same. From the proof of Theorem 4.31(b), $\lambda_1 = 1$ has geometric multiplicity 1 as an eigenvalue of P^T. Since P is diagonalizable, so is P^T, by Exercise 41 in Section 4.4. Therefore, the eigenvalue $\lambda_1 = 1$ has algebraic multiplicity 1, by the Diagonalization Theorem.

Theorem 4.33

Let P be a regular $n \times n$ transition matrix. Then as $k \to \infty$, P^k approaches an $n \times n$ matrix L whose columns are identical, each equal to the same vector $\mathbf{x}$. This vector $\mathbf{x}$ is a steady state probability vector for P.

See *Finite Markov Chains* by J. G. Kemeny and J. L. Snell (New York: Springer-Verlag, 1976).

Proof   To simplify the proof, we will consider only the case where P is diagonalizable. The theorem is true, however, without this assumption.

We diagonalize P as $Q^{-1}PQ = D$ or, equivalently, $P = QDQ^{-1}$, where

$$D = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix}$$

From Theorems 4.30 and 4.31, we know that each eigenvalue λ_i either is 1 or satisfies $|\lambda_i| < 1$. Hence, as $k \to \infty$, λ_i^k approaches 1 or 0 for $i = 1, \ldots, n$. It follows that D^k approaches a diagonal matrix—say, D^*—each of whose diagonal entries is 1 or 0. Thus, $P^k = QD^kQ^{-1}$ approaches $L = QD^*Q^{-1}$. We write

$$\lim_{k \to \infty} P^k = L$$

We are taking some liberties with the notion of a limit. Nevertheless, these steps should be intuitively clear. Rigorous proofs follow from the properties of limits, which you may have encountered in a calculus course. Rather than get sidetracked with a discussion of matrix limits, we will omit the proofs.

Observe that

$$PL = P \lim_{k \to \infty} P^k = \lim_{k \to \infty} PP^k = \lim_{k \to \infty} P^{k+1} = L$$

Therefore, each column of L is an eigenvector of P corresponding to $\lambda_1 = 1$. To see that each of these columns is a *probability* vector (i.e., L is a stochastic matrix), we need only observe that, if $\mathbf{j}$ is the row vector with n 1s, then

$$\mathbf{j}L = \mathbf{j} \lim_{k \to \infty} P^k = \lim_{k \to \infty} \mathbf{j}P^k = \lim_{k \to \infty} \mathbf{j} = \mathbf{j}$$

since P^k is a stochastic matrix, by Exercise 14 in Section 3.7. Exercise 13 in Section 3.7 now implies that L is stochastic.

We need only show that the columns of L are identical. The ith column of L is just $L\mathbf{e}_i$, where $\mathbf{e}_i$ is the ith standard basis vector. Let $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ be eigenvectors of P forming a basis of $\mathbb{R}^n$, with $\mathbf{v}_1$ corresponding to $\lambda_1 = 1$. Write

$$\mathbf{e}_i = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_n\mathbf{v}_n$$

for scalars $c_1, c_2, \ldots, c_n$. Then, by Theorem 4.19,

$$P^k\mathbf{e}_i = c_1 1^k\mathbf{v}_1 + c_2\lambda_2^k\mathbf{v}_2 + \cdots + c_n\lambda_n^k\mathbf{v}_n$$

By Lemma 4.32, $\lambda_j \neq 1$ for $j \neq 1$, so, by Theorem 4.31(b), $|\lambda_j| < 1$ for $j \neq 1$. Hence, $\lambda_j^k \to 0$ as $k \to \infty$, for $j \neq 1$. It follows that

$$L\mathbf{e}_i = \lim_{k \to \infty} P^k\mathbf{e}_i = c_1\mathbf{v}_1$$

In other words, column i of L is an eigenvector corresponding to $\lambda_1 = 1$. But we have shown that the columns of L are probability vectors, so $L\mathbf{e}_i$ is the *unique* multiple $\mathbf{x}$ of $\mathbf{v}_1$ whose components sum to 1. Since this is true for each column of L, it implies that all of the columns of L are identical, each equal to this vector $\mathbf{x}$.

Remark   Since L is a stochastic matrix, we can interpret it as the ***long range transition matrix*** of the Markov chain. That is, L_{ij} represents the probability of being in state i, having started from state j, *if the transitions were to continue indefinitely*. The fact that the columns of L are identical says that the *starting state does not matter,* as the next example illustrates.

Example 4.38

Recall the rat in a box from Example 3.65. The transition matrix was

$$P = \begin{bmatrix} 0 & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{2} & 0 & \frac{2}{3} \\ \frac{1}{2} & \frac{2}{3} & 0 \end{bmatrix}$$

We determined that the steady state probability vector was

$$\mathbf{x} = \begin{bmatrix} \frac{1}{4} \\ \frac{3}{8} \\ \frac{3}{8} \end{bmatrix}$$

Hence, the powers of P approach

$$L = \begin{bmatrix} \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ \frac{3}{8} & \frac{3}{8} & \frac{3}{8} \\ \frac{3}{8} & \frac{3}{8} & \frac{3}{8} \end{bmatrix} = \begin{bmatrix} 0.250 & 0.250 & 0.250 \\ 0.375 & 0.375 & 0.375 \\ 0.375 & 0.375 & 0.375 \end{bmatrix}$$

from which we can see that the rat will *eventually* spend 25% of its time in compartment 1 and 37.5% of its time in each of the other two compartments.

We conclude our discussion of regular Markov chains by proving that the steady state vector $\mathbf{x}$ is independent of the initial state. The proof is easily adapted to cover the case of state vectors whose components sum to an arbitrary constant—say, s. In the exercises, you are asked to prove some other properties of regular Markov chains.

Theorem 4.34

Let P be a regular $n \times n$ transition matrix, with $\mathbf{x}$ the steady state probability vector for P, as in Theorem 4.33. Then, for any initial probability vector $\mathbf{x}_0$, the sequence of iterates $\mathbf{x}_k$ approaches $\mathbf{x}$.

Proof   Let

$$\mathbf{x}_0 = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

where $x_1 + x_2 + \cdots + x_n = 1$. Since $\mathbf{x}_k = P^k \mathbf{x}_0$, we must show that $\lim_{k \to \infty} P^k \mathbf{x}_0 = \mathbf{x}$. Now, by Theorem 4.33, the long range transition matrix is $L = [\mathbf{x} \quad \mathbf{x} \quad \cdots \quad \mathbf{x}]$ and $\lim_{k \to \infty} P^k = L$. Therefore,

$$\lim_{k \to \infty} P^k \mathbf{x}_0 = (\lim_{k \to \infty} P^k)\mathbf{x}_0 = L\mathbf{x}_0$$

$$= [\mathbf{x} \quad \mathbf{x} \quad \cdots \quad \mathbf{x}] \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

$$= x_1\mathbf{x} + x_2\mathbf{x} + \cdots + x_n\mathbf{x}$$

$$= (x_1 + x_2 + \cdots + x_n)\mathbf{x} = \mathbf{x}$$

Population Growth

We return to the Leslie model of population growth, which we first explored in Section 3.7. In Example 3.67 in that section, we saw that for the Leslie matrix

$$L = \begin{bmatrix} 0 & 4 & 3 \\ 0.5 & 0 & 0 \\ 0 & 0.25 & 0 \end{bmatrix}$$

iterates of the population vectors began to approach a multiple of the vector

$$\mathbf{x} = \begin{bmatrix} 18 \\ 6 \\ 1 \end{bmatrix}$$

In other words, the three age classes of this population eventually ended up in the ratio $18:6:1$. Moreover, once this state is reached, it is stable, since the ratios for the following year are given by

$$L\mathbf{x} = \begin{bmatrix} 0 & 4 & 3 \\ 0.5 & 0 & 0 \\ 0 & 0.25 & 0 \end{bmatrix} \begin{bmatrix} 18 \\ 6 \\ 1 \end{bmatrix} = \begin{bmatrix} 27 \\ 9 \\ 1.5 \end{bmatrix} = 1.5\mathbf{x}$$

and the components are still in the ratio $27:9:1.5 = 18:6:1$. Observe that 1.5 represents the *growth rate* of this population when it has reached its steady state.

We can now recognize that $\mathbf{x}$ is an eigenvector of L corresponding to the eigenvalue $\lambda = 1.5$. Thus, the steady state growth rate is a *positive* eigenvalue of L, and an eigenvector corresponding to this eigenvalue represents the *relative* sizes of the age classes when the steady state has been reached. We can compute these directly, without having to iterate as we did before.

Example 4.39   Find the steady state growth rate and the corresponding ratios between the age classes for the Leslie matrix L above.

Solution   We need to find all positive eigenvalues and corresponding eigenvectors of L. The characteristic polynomial of L is

$$\det(L - \lambda I) = \begin{vmatrix} -\lambda & 4 & 3 \\ 0.5 & -\lambda & 0 \\ 0 & 0.25 & -\lambda \end{vmatrix} = -\lambda^3 + 2\lambda + 0.375$$

so we must solve $-\lambda^3 + 2\lambda + 0.375 = 0$ or, equivalently, $8\lambda^3 - 16\lambda - 3 = 0$. Factoring, we have

$$(2\lambda - 3)(4\lambda^2 + 6\lambda + 1) = 0$$

(See Appendix D.) Since the second factor has only the roots $(-3 + \sqrt{5})/4 \approx -0.19$ and $(-3 - \sqrt{5})/4 \approx -1.31$, the only positive root of this equation is $\lambda = \frac{3}{2} = 1.5$. The corresponding eigenvectors are in the null space of $L - 1.5I$, which we find by row reduction:

$$[L - 1.5I \mid 0] = \begin{bmatrix} -1.5 & 4 & 3 & | & 0 \\ 0.5 & -1.5 & 0 & | & 0 \\ 0 & 0.25 & -1.5 & | & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & -18 & | & 0 \\ 0 & 1 & -6 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{bmatrix}$$

Thus, if $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ is an eigenvector corresponding to $\lambda = 1.5$, it satisfies $x_1 = 18x_3$ and $x_2 = 6x_3$. That is,

$$E_{1.5} = \left\{ \begin{bmatrix} 18x_3 \\ 6x_3 \\ x_3 \end{bmatrix} \right\} = \text{span} \left(\begin{bmatrix} 18 \\ 6 \\ 1 \end{bmatrix} \right)$$

Hence, the steady state growth rate is 1.5, and when this rate has been reached, the age classes are in the ratio $18:6:1$, as we saw before.

In Example 4.39, there was only one candidate for the steady state growth rate: the unique positive eigenvalue of L. But what would we have done if L had had more than one positive eigenvalue or none? We were also apparently fortunate that there was a corresponding eigenvector all of whose components were positive, which allowed us to relate these components to the size of the population. We can prove that this situation is not accidental; that is, *every* Leslie matrix has exactly one positive eigenvalue and a corresponding eigenvector with positive components.

Recall that the form of a Leslie matrix is

$$L = \begin{bmatrix} b_1 & b_2 & b_3 & \cdots & b_{n-1} & b_n \\ s_1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & s_2 & 0 & \cdots & 0 & 0 \\ 0 & 0 & s_3 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & s_{n-1} & 0 \end{bmatrix} \tag{3}$$

Since the entries s_j represent survival probabilities, we will assume that they are all nonzero (otherwise, the population would rapidly die out). We will also assume that at least one of the birth parameters b_i is nonzero (otherwise, there would be no births and, again, the population would die out). With these standing assumptions, we can now prove the assertion we made above as a theorem.

Theorem 4.35   Every Leslie matrix has a unique positive eigenvalue and a corresponding eigenvector with positive components.

Proof   Let L be as in Equation (3). The characteristic polynomial of L is

$$c_L(\lambda) = \det(L - \lambda I)$$

$$= (-1)^n(\lambda^n - b_1\lambda^{n-1} - b_2s_1\lambda^{n-2} - b_3s_1s_2\lambda^{n-3} - \cdots - b_ns_1s_2\cdots s_{n-1})$$

$$= (-1)^n f(\lambda)$$

(You are asked to prove this in Exercise 16.) The eigenvalues of L are therefore the roots of $f(\lambda)$. Since at least one of the birth parameters b_i is positive and all of the survival probabilities s_j are positive, the coefficients of $f(\lambda)$ change sign exactly once. By Descartes's Rule of Signs (Appendix D), therefore, $f(\lambda)$ has exactly one positive root. Let us call it λ_1.

By direct calculation, we can check that an eigenvector corresponding to λ_1 is

$$\mathbf{x}_1 = \begin{bmatrix} 1 \\ s_1/\lambda_1 \\ s_1s_2/\lambda_1^2 \\ s_1s_2s_3/\lambda_1^3 \\ \vdots \\ s_1s_2s_3\cdots s_{n-1}/\lambda_1^{n-1} \end{bmatrix}$$

(You are asked to prove this in Exercise 18.) Clearly, all of the components of $\mathbf{x}_1$ are positive.

In fact, more is true. With the additional requirement that *two consecutive* birth parameters b_i and b_{i+1} are positive, it turns out that the unique positive eigenvalue λ_1 of L is *dominant*; that is, every other (real or complex) eigenvalue λ of L satisfies $|\lambda| < \lambda_1$. (It is beyond the scope of this book to prove this result, but a partial proof is outlined in Exercise 27 for readers who are familiar with the algebra of complex numbers.) This explains why we get convergence to a steady state vector when we iterate the population vectors: It is just the power method working for us!

The Perron-Frobenius Theorem

In the previous two applications, Markov chains and Leslie matrices, we saw that the eigenvalue of interest was positive and dominant. Moreover, there was a corresponding eigenvector with positive components. It turns out that a remarkable theorem guarantees that this will be the case for a large class of matrices, including many of the ones we have been considering. The first version of this theorem is for positive matrices.

Oskar Perron (1880–1975) was a German mathematician who did work in many fields of mathematics, including analysis, differential equations, algebra, geometry, and number theory. Perron's Theorem was published in 1907 in a paper on continued fractions.

First, we need some terminology and notation. Let's agree to refer to a vector as *positive* if all of its components are positive. For two $m \times n$ matrices $A = [a_{ij}]$ and $B = [b_{ij}]$, we will write $A \geq B$ if $a_{ij} \geq b_{ij}$ for all i and j. (Similar definitions will apply for $A > B$, $A \leq B$, and so on.) Thus, a positive vector $\mathbf{x}$ satisfies $\mathbf{x} > \mathbf{0}$. Let us define $|A| = [|a_{ij}|]$ to be the matrix of the absolute values of the entries of A.

Theorem 4.36   **Perron's Theorem**

Let A be a positive $n \times n$ matrix. Then A has a real eigenvalue λ_1 with the following properties:

a. $\lambda_1 > 0$
b. λ_1 has a corresponding positive eigenvector.
c. If λ is any other eigenvalue of A, then $|\lambda| \leq \lambda_1$.

Intuitively, we can see why the first two statements should be true. Consider the case of a 2×2 positive matrix A. The corresponding matrix transformation maps the first quadrant of the plane properly into itself, since all components are positive. If we repeatedly allow A to act on the images we get, they necessarily converge toward some ray in the first quadrant (Figure 4.18). A direction vector for this ray will be a positive vector $\mathbf{x}$, which must be mapped into some positive multiple of itself (say, λ_1), since A leaves the ray fixed. In other words, $A\mathbf{x} = \lambda_1\mathbf{x}$, with $\mathbf{x}$ and λ_1 both positive.

Proof   For some nonzero vectors $\mathbf{x}$, $A\mathbf{x} \geq \lambda\mathbf{x}$ for some scalar λ. When this happens, then $A(k\mathbf{x}) \geq \lambda(k\mathbf{x})$ for all $k > 0$; thus, we need only consider *unit* vectors $\mathbf{x}$. In Chapter 7, we will see that A maps the set of all unit vectors in $\mathbb{R}^n$ (the *unit sphere*) into a "generalized ellipsoid." So, as $\mathbf{x}$ ranges over the nonnegative vectors on this unit sphere, there will be a maximum value of λ such that $A\mathbf{x} \geq \lambda\mathbf{x}$. (See Figure 4.19.) Denote this number by λ_1 and the corresponding unit vector by $\mathbf{x}_1$.
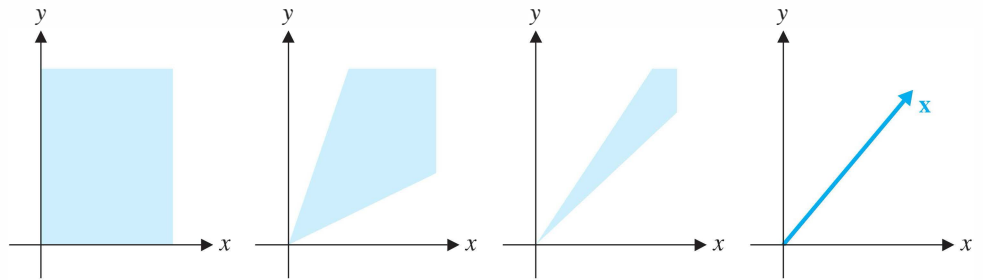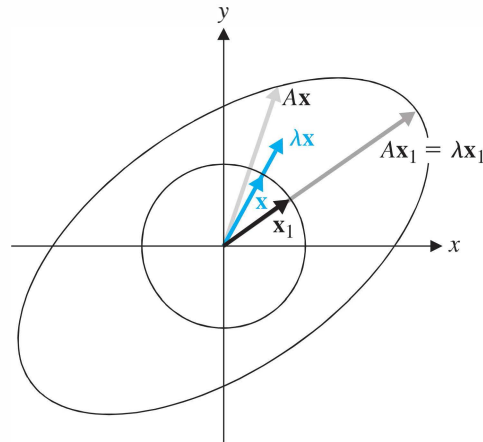

Figure 4.18

Figure 4.19

We now show that $A\mathbf{x}_1 = \lambda_1\mathbf{x}_1$. If not, then $A\mathbf{x}_1 > \lambda_1\mathbf{x}_1$, and, applying A again, we obtain

$$A(A\mathbf{x}_1) > A(\lambda_1\mathbf{x}_1) = \lambda_1(A\mathbf{x}_1)$$

where the inequality is preserved, since A is positive. (See Exercise 40 and Section 3.7 Exercise 36.) But then $\mathbf{y} = (1/\|A\mathbf{x}_1\|)A\mathbf{x}_1$ is a unit vector that satisfies $A\mathbf{y} > \lambda_1\mathbf{y}$, so there will be some $\lambda_2 > \lambda_1$ such that $A\mathbf{y} \geq \lambda_2\mathbf{y}$. This contradicts the fact that λ_1 was the maximum value with this property. Consequently, it must be the case that $A\mathbf{x}_1 = \lambda_1\mathbf{x}_1$; that is, λ_1 is an eigenvalue of A.

Now A is positive and $\mathbf{x}_1$ is positive, so $\lambda_1\mathbf{x}_1 = A\mathbf{x}_1 > \mathbf{0}$. This means that $\lambda_1 > 0$ and $\mathbf{x}_1 > \mathbf{0}$, which completes the proof of (a) and (b).

To prove (c), suppose λ is any other (real or complex) eigenvalue of A with corresponding eigenvector $\mathbf{z}$. Then $A\mathbf{z} = \lambda\mathbf{z}$, and, taking absolute values, we have

$$A|\mathbf{z}| = |A||\mathbf{z}| \geq |A\mathbf{z}| = |\lambda\mathbf{z}| = |\lambda||\mathbf{z}| \tag{4}$$

where the middle inequality follows from the Triangle Inequality. (See Exercise 40.) Since $|\mathbf{z}| > \mathbf{0}$, the unit vector $\mathbf{u}$ in the direction of $|\mathbf{z}|$ is also positive and satisfies $A\mathbf{u} \geq |\lambda|\mathbf{u}$. By the maximality of λ_1 from the first part of this proof, we must have $|\lambda| \leq \lambda_1$.

In fact, more is true. It turns out that λ_1 is dominant, so $|\lambda| < \lambda_1$ for any eigenvalue $\lambda \neq \lambda_1$. It is also the case that λ_1 has algebraic, and hence geometric, multiplicity 1. We will not prove these facts.

Perron's Theorem can be generalized from positive to certain nonnegative matrices. Frobenius did so in 1912. The result requires a technical condition on the matrix. A square matrix A is called ***reducible*** if, subject to some permutation of the rows and the same permutation of the columns, A can be written in block form as

$$\begin{bmatrix} B & C \\ O & D \end{bmatrix}$$

where B and D are square. Equivalently, A is reducible if there is some permutation matrix P such that

$$PAP^T = \begin{bmatrix} B & C \\ O & D \end{bmatrix}$$

(See page 187.) For example, the matrix

$$A = \begin{bmatrix} 2 & 0 & 0 & 1 & 3 \\ 4 & 2 & 1 & 5 & 5 \\ 1 & 2 & 7 & 3 & 0 \\ 6 & 0 & 0 & 2 & 1 \\ 1 & 0 & 0 & 7 & 2 \end{bmatrix}$$

is reducible, since interchanging rows 1 and 3 and then columns 1 and 3 produces

$$\begin{bmatrix} 7 & 2 & 1 & 3 & 0 \\ 1 & 2 & 4 & 5 & 5 \\ 0 & 0 & 2 & 1 & 3 \\ 0 & 0 & 6 & 2 & 1 \\ 0 & 0 & 1 & 7 & 2 \end{bmatrix}$$

(This is just PAP^T, where

$$P = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Check this!)

A square matrix A that is not reducible is called ***irreducible.*** If $A^k > O$ for some k, then A is called ***primitive.*** For example, every regular Markov chain has a primitive transition matrix, by definition. It is not hard to show that every primitive matrix is irreducible. (Do you see why? Try showing the contrapositive of this.)

Theorem 4.37   **The Perron-Frobenius Theorem**

Let A be an irreducible nonnegative $n \times n$ matrix. Then A has a real eigenvalue λ_1 with the following properties:

a. $\lambda_1 > 0$
b. λ_1 has a corresponding positive eigenvector.
c. If λ is any other eigenvalue of A, then $|\lambda| \leq \lambda_1$. If A is primitive, then this inequality is strict.
d. If λ is an eigenvalue of A such that $|\lambda| = \lambda_1$, then λ is a (complex) root of the equation $\lambda^n - \lambda_1^n = 0$.
e. λ_1 has algebraic multiplicity 1.

See *Matrix Analysis* by R. A. Horn and C. R. Johnson (Cambridge, England: Cambridge University Press, 1985).

The interested reader can find a proof of the Perron-Frobenius Theorem in many texts on nonnegative matrices or matrix analysis. The eigenvalue λ_1 is often called the ***Perron root*** of A, and a corresponding *probability* eigenvector (which is necessarily unique) is called the ***Perron eigenvector*** of A.

Linear Recurrence Relations

The ***Fibonacci numbers*** are the numbers in the sequence 0, 1, 1, 2, 3, 5, 8, 13, 21, . . . , where, after the first two terms, each new term is obtained by summing the two terms preceding it. If we denote the nth Fibonacci number by f_n, then this sequence is completely defined by the equations $f_0 = 0, f_1 = 1$, and, for $n \geq 2$,

$$f_n = f_{n-1} + f_{n-2}$$

This last equation is an example of a linear recurrence relation. We will return to the Fibonacci numbers, but first we will consider linear recurrence relations somewhat more generally.

Leonardo of Pisa (1170–1250), pictured left, is better known by his nickname, Fibonacci, which means "son of Bonaccio." He wrote a number of important books, many of which have survived, including *Liber abaci* and *Liber quadratorum*. The Fibonacci sequence appears as the solution to a problem in *Liber abaci*: "A certain man put a pair of rabbits in a place surrounded on all sides by a wall. How many pairs of rabbits can be produced from that pair in a year if it is supposed that every month each pair begets a new pair which from the second month on becomes productive?" The name *Fibonacci numbers* was given to the terms of this sequence by the French mathematician Edouard Lucas (1842–1891).

Definition   Let $(x_n) = (x_0, x_1, x_2, \ldots)$ be a sequence of numbers that is defined as follows:

1. $x_0 = a_0, x_1 = a_1, \ldots, x_{k-1} = a_{k-1}$, where $a_0, a_1, \ldots, a_{k-1}$ are scalars.
2. For all $n \geq k$, $x_n = c_1 x_{n-1} + c_2 x_{n-2} + \cdots + c_k x_{n-k}$, where $c_1, c_2, \ldots, c_k$ are scalars.

If $c_k \neq 0$, the equation in (2) is called a ***linear recurrence relation of order k.*** The equations in (1) are referred to as the ***initial conditions*** of the recurrence.

Thus, the Fibonacci numbers satisfy a linear recurrence relation of order 2.

Remarks

- If, in order to define the nth term in a recurrence relation, we require the $(n - k)$th term but no term before it, then the recurrence relation has order k.
- The number of initial conditions is the order of the recurrence relation.
- It is not necessary that the first term of the sequence be called x_0. We could start at x_1 or anywhere else.
- It is possible to have even more general linear recurrence relations by allowing the coefficients c_i to be *functions* rather than scalars and by allowing an extra, isolated coefficient, which may also be a function. An example would be the recurrence

$$x_n = 2x_{n-1} - n^2 x_{n-2} + \frac{1}{n} x_{n-3} + n$$

We will not consider such recurrences here.

Example 4.40

Consider the sequence (x_n) defined by the initial conditions $x_1 = 1$, $x_2 = 5$ and the recurrence relation $x_n = 5x_{n-1} - 6x_{n-2}$ for $n \geq 2$. Write out the first five terms of this sequence.

Solution   We are given the first two terms. We use the recurrence relation to calculate the next three terms. We have

$$x_3 = 5x_2 - 6x_1 = 5 \cdot 5 - 6 \cdot 1 = 19$$
$$x_4 = 5x_3 - 6x_2 = 5 \cdot 19 - 6 \cdot 5 = 65$$
$$x_5 = 5x_4 - 6x_3 = 5 \cdot 65 - 6 \cdot 19 = 211$$

so the sequence begins $1, 5, 19, 65, 211, \ldots$.

Clearly, if we were interested in, say, the 100th term of the sequence in Example 4.40, then the approach used there would be rather tedious, since we would have to apply the recurrence relation 98 times. It would be nice if we could find an *explicit* formula for x_n as a function of n. We refer to finding such a formula as **solving** the recurrence relation. We will illustrate the process with the sequence from Example 4.40.

To begin, we rewrite the recurrence relation as a matrix equation. Let

$$A = \begin{bmatrix} 5 & -6 \\ 1 & 0 \end{bmatrix}$$

and introduce vectors $\mathbf{x}_n = \begin{bmatrix} x_n \\ x_{n-1} \end{bmatrix}$ for $n \geq 2$. Thus, $\mathbf{x}_2 = \begin{bmatrix} x_2 \\ x_1 \end{bmatrix} = \begin{bmatrix} 5 \\ 1 \end{bmatrix}$, $\mathbf{x}_3 = \begin{bmatrix} x_3 \\ x_2 \end{bmatrix} = \begin{bmatrix} 19 \\ 5 \end{bmatrix}$, $\mathbf{x}_4 = \begin{bmatrix} x_4 \\ x_3 \end{bmatrix} = \begin{bmatrix} 65 \\ 19 \end{bmatrix}$, and so on. Now observe that, for $n \geq 2$, we have

$$A\mathbf{x}_{n-1} = \begin{bmatrix} 5 & -6 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} x_{n-1} \\ x_{n-2} \end{bmatrix} = \begin{bmatrix} 5x_{n-1} - 6x_{n-2} \\ x_{n-1} \end{bmatrix} = \begin{bmatrix} x_n \\ x_{n-1} \end{bmatrix} = \mathbf{x}_n$$

Notice that this is the same type of equation we encountered with Markov chains and Leslie matrices. As in those cases, we can write

$$\mathbf{x}_n = A\mathbf{x}_{n-1} = A^2\mathbf{x}_{n-2} = \cdots = A^{n-2}\mathbf{x}_2$$

We now use the technique of Example 4.29 to compute the powers of A.

The characteristic equation of A is

$$\lambda^2 - 5\lambda + 6 = 0$$

from which we find that the eigenvalues are $\lambda_1 = 3$ and $\lambda_2 = 2$. (Notice that the form of the characteristic equation follows that of the recurrence relation. If we write the recurrence as $x_n - 5x_{n-1} + 6x_{n-2} = 0$, it is apparent that the coefficients are exactly the same!) The corresponding eigenspaces are

$$E_3 = \text{span}\left(\begin{bmatrix} 3 \\ 1 \end{bmatrix}\right) \quad \text{and} \quad E_2 = \text{span}\left(\begin{bmatrix} 2 \\ 1 \end{bmatrix}\right)$$

Setting $P = \begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix}$, we know that $P^{-1}AP = D = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}$. Then $A = PDP^{-1}$ and

$$A^k = PD^kP^{-1} = \begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 3^k & 0 \\ 0 & 2^k \end{bmatrix}\begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix}^{-1}$$

$$= \begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 3^k & 0 \\ 0 & 2^k \end{bmatrix}\begin{bmatrix} 1 & -2 \\ -1 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 3^{k+1} - 2^{k+1} & -2(3^{k+1}) + 3(2^{k+1}) \\ 3^k - 2^k & -2(3^k) + 3(2^k) \end{bmatrix}$$

It now follows that

$$\begin{bmatrix} x_n \\ x_{n-1} \end{bmatrix} = \mathbf{x}_n = A^{n-2}\mathbf{x}_2 = \begin{bmatrix} 3^{n-1} - 2^{n-1} & -2(3^{n-1}) + 3(2^{n-1}) \\ 3^{n-2} - 2^{n-2} & -2(3^{n-2}) + 3(2^{n-2}) \end{bmatrix}\begin{bmatrix} 5 \\ 1 \end{bmatrix} = \begin{bmatrix} 3^n - 2^n \\ 3^{n-1} - 2^{n-1} \end{bmatrix}$$

from which we read off the solution $x_n = 3^n - 2^n$. (To check our work, we could plug in $n = 1, 2, \ldots, 5$ to verify that this formula gives the same terms that we calculated using the recurrence relation. Try it!)

Observe that x_n is a linear combination of powers of the eigenvalues. This is necessarily the case as long as the eigenvalues are distinct [as Theorem 4.38(a) will make explicit]. Using this observation, we can save ourselves some work. Once we have computed the eigenvalues $\lambda_1 = 3$ and $\lambda_2 = 2$, we can immediately write

$$x_n = c_1 3^n + c_2 2^n$$

where c_1 and c_2 are to be determined. Using the initial conditions, we have

$$1 = x_1 = c_1 3^1 + c_2 2^1 = 3c_1 + 2c_2$$

when $n = 1$ and

$$5 = x_2 = c_1 3^2 + c_2 2^2 = 9c_1 + 4c_2$$

when $n = 2$. We now solve the system

$$3c_1 + 2c_2 = 1$$
$$9c_1 + 4c_2 = 5$$

for c_1 and c_2 to obtain $c_1 = 1$ and $c_2 = -1$. Thus, $x_n = 3^n - 2^n$, as before.

This is the method we will use in practice. We now illustrate its use to find an explicit formula for the Fibonacci numbers.

Example 4.41   Solve the Fibonacci recurrence $f_0 = 0, f_1 = 1$, and $f_n = f_{n-1} + f_{n-2}$ for $n \geq 2$.

Solution   Writing the recurrence as $f_n - f_{n-1} - f_{n-2} = 0$, we see that the characteristic equation is $\lambda^2 - \lambda - 1 = 0$, so the eigenvalues are

$$\lambda_1 = \frac{1 + \sqrt{5}}{2} \quad \text{and} \quad \lambda_2 = \frac{1 - \sqrt{5}}{2}$$

It follows from the discussion above that the solution to the recurrence relation has the form

$$f_n = c_1 \lambda_1^n + c_2 \lambda_2^n = c_1 \left(\frac{1 + \sqrt{5}}{2} \right)^n + c_2 \left(\frac{1 - \sqrt{5}}{2} \right)^n$$

for some scalars c_1 and c_2.

Using the initial conditions, we find

$$0 = f_0 = c_1 \lambda_1^0 + c_2 \lambda_2^0 = c_1 + c_2$$

and

$$1 = f_1 = c_1 \lambda_1^1 + c_2 \lambda_2^1 = c_1 \left(\frac{1 + \sqrt{5}}{2} \right) + c_2 \left(\frac{1 - \sqrt{5}}{2} \right)$$

Solving for c_1 and c_2, we obtain $c_1 = 1/\sqrt{5}$ and $c_2 = -1/\sqrt{5}$. Hence, an explicit formula for the nth Fibonacci number is

$$f_n = \frac{1}{\sqrt{5}} \left(\frac{1 + \sqrt{5}}{2} \right)^n - \frac{1}{\sqrt{5}} \left(\frac{1 - \sqrt{5}}{2} \right)^n \tag{5}$$

Jacques Binet (1786–1856) made contributions to matrix theory, number theory, physics, and astronomy. He discovered the rule for matrix multiplication in 1812. Binet's formula for the Fibonacci numbers is actually due to Euler, who published it in 1765; however, it was forgotten until Binet published his version in 1843. Like Cauchy, Binet was a royalist, and he lost his university position when Charles X abdicated in 1830. He received many honors for his work, including his election, in 1843, to the Académie des Sciences.

Formula (5) is a remarkable formula, because it is defined in terms of the *irrational* number $\sqrt{5}$ yet the Fibonacci numbers are all integers! Try plugging in a few values for n to see how the $\sqrt{5}$ terms cancel out to leave the integer values f_n. Formula (5) is known as **Binet's formula.**

The method we have just outlined works for any second order linear recurrence relation whose associated eigenvalues are all distinct. When there is a repeated eigenvalue, the technique must be modified, since the diagonalization method we used may no longer work. The next theorem summarizes both situations.

Theorem 4.38

Let $x_n = ax_{n-1} + bx_{n-2}$ be a recurrence relation that is satisfied by a sequence (x_n). Let λ_1 and λ_2 be the eigenvalues of the associated characteristic equation $\lambda^2 - a\lambda - b = 0$.

a. If $\lambda_1 \neq \lambda_2$, then $x_n = c_1\lambda_1^n + c_2\lambda_2^n$ for some scalars c_1 and c_2.
b. If $\lambda_1 = \lambda_2 = \lambda$, then $x_n = c_1\lambda^n + c_2 n\lambda^n$ for some scalars c_1 and c_2.

In either case, c_1 and c_2 can be determined using the initial conditions.

Proof   (a) Generalizing our discussion above, we can write the recurrence as $\mathbf{x}_n = A\mathbf{x}_{n-1}$, where

$$\mathbf{x}_n = \begin{bmatrix} x_n \\ x_{n-1} \end{bmatrix} \quad \text{and} \quad A = \begin{bmatrix} a & b \\ 1 & 0 \end{bmatrix}$$

Since A has distinct eigenvalues, it can be diagonalized. The rest of the details are left for Exercise 53.

(b) We will show that $x_n = c_1\lambda^n + c_2 n\lambda^n$ satisfies the recurrence relation $x_n = ax_{n-1} + bx_{n-2}$ or, equivalently,

$$x_n - ax_{n-1} - bx_{n-2} = 0 \tag{6}$$

if $\lambda^2 - a\lambda - b = 0$. Since

$$x_{n-1} = c_1\lambda^{n-1} + c_2(n-1)\lambda^{n-1} \quad \text{and} \quad x_{n-2} = c_1\lambda^{n-2} + c_2(n-2)\lambda^{n-2}$$

substitution into Equation (6) yields

$$\begin{aligned}
x_n - ax_{n-1} - bx_{n-2} &= (c_1\lambda^n + c_2 n\lambda^n) - a(c_1\lambda^{n-1} + c_2(n-1)\lambda^{n-1}) \\
&\quad - b(c_1\lambda^{n-2} + c_2(n-2)\lambda^{n-2}) \\
&= c_1(\lambda^n - a\lambda^{n-1} - b\lambda^{n-2}) + c_2(n\lambda^n - a(n-1)\lambda^{n-1} \\
&\quad - b(n-2)\lambda^{n-2}) \\
&= c_1\lambda^{n-2}(\lambda^2 - a\lambda - b) + c_2 n\lambda^{n-2}(\lambda^2 - a\lambda - b) + c_2\lambda^{n-2}(a\lambda + 2b) \\
&= c_1\lambda^{n-2}(0) + c_2 n\lambda^{n-2}(0) + c_2\lambda^{n-2}(a\lambda + 2b) \\
&= c_2\lambda^{n-2}(a\lambda + 2b)
\end{aligned}$$

But, since λ is a double root of $\lambda^2 - a\lambda - b = 0$, we must have $a^2 + 4b = 0$ and $\lambda = a/2$, using the quadratic formula. Consequently, $a\lambda + 2b = a^2/2 + 2b = -4b/2 + 2b = 0$, so

$$x_n - ax_{n-1} - bx_{n-2} = c_2\lambda^{n-2}(a\lambda + 2b) = c_2\lambda^{n-2}(0) = 0$$

Suppose the initial conditions are $x_0 = r$ and $x_1 = s$. Then, in either (a) or (b) there is a unique solution for c_1 and c_2. (See Exercise 54.)

Example 4.42   Solve the recurrence relation $x_0 = 1$, $x_1 = 6$, and $x_n = 6x_{n-1} - 9x_{n-2}$ for $n \geq 2$.

Solution   The characteristic equation is $\lambda^2 - 6\lambda + 9 = 0$, which has $\lambda = 3$ as a double root. By Theorem 4.38(b), we must have $x_n = c_1 3^n + c_2 n 3^n = (c_1 + c_2 n)3^n$. Since $1 = x_0 = c_1$ and $6 = x_1 = (c_1 + c_2)3$, we find that $c_2 = 1$, so

$$x_n = (1 + n)3^n$$

The techniques outlined in Theorem 4.38 can be extended to higher order recurrence relations. We state, without proof, the general result.

Theorem 4.39   Let $x_n = a_{m-1}x_{n-1} + a_{m-2}x_{n-2} + \cdots + a_0 x_{n-m}$ be a recurrence relation of order m that is satisfied by a sequence (x_n). Suppose the associated characteristic polynomial

$$\lambda^m - a_{m-1}\lambda^{m-1} - a_{m-2}\lambda^{m-2} - \cdots - a_0$$

factors as $(\lambda - \lambda_1)^{m_1}(\lambda - \lambda_2)^{m_2}\cdots(\lambda - \lambda_k)^{m_k}$, where $m_1 + m_2 + \cdots + m_k = m$. Then x_n has the form

$$x_n = (c_{11}\lambda_1^n + c_{12}n\lambda_1^n + c_{13}n^2\lambda_1^n + \cdots + c_{1m_1}n^{m_1-1}\lambda_1^n) + \cdots$$
$$+ (c_{k1}\lambda_k^n + c_{k2}n\lambda_k^n + c_{k3}n^2\lambda_k^n + \cdots + c_{km_k}n^{m_k-1}\lambda_k^n)$$

$\dfrac{dy}{dx}$

Systems of Linear Differential Equations

In calculus, you learn that if $x = x(t)$ is a differentiable function satisfying a differential equation of the form $x' = kx$, where k is a constant, then the general solution is $x = Ce^{kt}$, where C is a constant. If an initial condition $x(0) = x_0$ is specified, then, by substituting $t = 0$ in the general solution, we find that $C = x_0$. Hence, the unique solution to the differential equation that satisfies the initial condition is

$$x = x_0 e^{kt}$$

Suppose we have n differentiable functions of t—say, $x_1, x_2, \ldots, x_n$—that satisfy a *system of differential equations*

$$x_1' = a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n$$
$$x_2' = a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n$$
$$\vdots$$
$$x_n' = a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n$$

We can write this system in matrix form as $\mathbf{x}' = A\mathbf{x}$, where

$$\mathbf{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{bmatrix}, \quad \mathbf{x}'(t) = \begin{bmatrix} x_1'(t) \\ x_2'(t) \\ \vdots \\ x_n'(t) \end{bmatrix}, \quad \text{and} \quad A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$$

Now we can use matrix methods to help us find the solution.

First, we make a useful observation. Suppose we want to solve the following system of differential equations:

$$x_1' = 2x_1$$
$$x_2' = 5x_2$$

Each equation can be solved separately, as above, to give

$$x_1 = C_1 e^{2t}$$
$$x_2 = C_2 e^{5t}$$

where C_1 and C_2 are constants. Notice that, in matrix form, our equation $\mathbf{x}' = A\mathbf{x}$ has a *diagonal* coefficient matrix

$$A = \begin{bmatrix} 2 & 0 \\ 0 & 5 \end{bmatrix}$$

and the eigenvalues 2 and 5 occur in the exponentials e^{2t} and e^{5t} of the solution. This suggests that, for an arbitrary system, we should start by diagonalizing the coefficient matrix, if possible.

Example 4.43

Solve the following system of differential equations:

$$x_1' = x_1 + 2x_2$$
$$x_2' = 3x_1 + 2x_2$$

Solution   Here the coefficient matrix is $A = \begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix}$, and we find that the eigenvalues are $\lambda_1 = 4$ and $\lambda_2 = -1$, with corresponding eigenvectors $\mathbf{v}_1 = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$, respectively. Therefore, A is diagonalizable, and the matrix P that does the job is

$$P = [\mathbf{v}_1 \quad \mathbf{v}_2] = \begin{bmatrix} 2 & -1 \\ 3 & 1 \end{bmatrix}$$

We know that

$$P^{-1}AP = \begin{bmatrix} 4 & 0 \\ 0 & -1 \end{bmatrix} = D$$

Let $\mathbf{x} = P\mathbf{y}$ (so that $\mathbf{x}' = P\mathbf{y}'$) and substitute these results into the original equation $\mathbf{x}' = A\mathbf{x}$ to get $P\mathbf{y}' = AP\mathbf{y}$ or, equivalently,

$$\mathbf{y}' = P^{-1}AP\mathbf{y} = D\mathbf{y}$$

This is just the system

$$y_1' = 4y_1$$
$$y_2' = -y_2$$

whose general solution is

$$y_1 = C_1 e^{4t} \qquad \text{or} \qquad \mathbf{y} = \begin{bmatrix} C_1 e^{4t} \\ C_2 e^{-t} \end{bmatrix}$$
$$y_2 = C_2 e^{-t}$$

To find **x**, we just compute

$$\mathbf{x} = P\mathbf{y} = \begin{bmatrix} 2 & -1 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} C_1 e^{4t} \\ C_2 e^{-t} \end{bmatrix} = \begin{bmatrix} 2C_1 e^{4t} - C_2 e^{-t} \\ 3C_1 e^{4t} + C_2 e^{-t} \end{bmatrix}$$

so $x_1 = 2C_1 e^{4t} - C_2 e^{-t}$ and $x_2 = 3C_1 e^{4t} + C_2 e^{-t}$. (Check that these values satisfy the given system.)

Remark Observe that we could also express the solution in Example 4.43 as

$$\mathbf{x} = C_1 e^{4t} \begin{bmatrix} 2 \\ 3 \end{bmatrix} + C_2 e^{-t} \begin{bmatrix} -1 \\ 1 \end{bmatrix} = C_1 e^{4t} \mathbf{v}_1 + C_2 e^{-t} \mathbf{v}_2$$

This technique generalizes easily to $n \times n$ systems where the coefficient matrix is diagonalizable. The next theorem, whose proof is left as an exercise, summarizes the situation.

Theorem 4.40   Let A be an $n \times n$ diagonalizable matrix and let $P = [\mathbf{v}_1 \quad \mathbf{v}_2 \quad \cdots \quad \mathbf{v}_n]$ be such that

$$P^{-1}AP = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix}$$

Then the general solution to the system $\mathbf{x}' = A\mathbf{x}$ is

$$\mathbf{x} = C_1 e^{\lambda_1 t} \mathbf{v}_1 + C_2 e^{\lambda_2 t} \mathbf{v}_2 + \cdots + C_n e^{\lambda_n t} \mathbf{v}_n$$

The next example involves a biological model in which two species live in the same ecosystem. It is reasonable to assume that the growth rate of each species depends on the sizes of *both* populations. (Of course, there are other factors that govern growth, but we will keep our model simple by ignoring these.)

If $x_1(t)$ and $x_2(t)$ denote the sizes of the two populations at time t, then $x_1'(t)$ and $x_2'(t)$ are their rates of growth at time t. Our model is of the form

$$x_1'(t) = a x_1(t) + b x_2(t)$$
$$x_2'(t) = c x_1(t) + d x_2(t)$$

where the coefficients a, b, c, and d depend on the conditions.

Example 4.44   Raccoons and squirrels inhabit the same ecosystem and compete with each other for food, water, and space. Let the raccoon and squirrel populations at time t years be given by $r(t)$ and $s(t)$, respectively. In the absence of squirrels, the raccoon growth rate is $r'(t) = 2.5r(t)$, but when squirrels are present, the competition slows the raccoon growth rate to $r'(t) = 2.5r(t) - s(t)$. The squirrel population is similarly affected by the raccoons. In the absence of raccoons, the growth rate of the squirrel population is $s'(t) = 2.5s(t)$, and the population growth rate for squirrels when they are sharing the ecosystem with raccoons is $s'(t) = -0.25r(t) + 2.5s(t)$. Suppose that initially

there are 60 raccoons and 60 squirrels in the ecosystem. Determine what happens to these two populations.

Solution   Our system is $\mathbf{x'} = A\mathbf{x}$, where

$$\mathbf{x} = \mathbf{x}(t) = \begin{bmatrix} r(t) \\ s(t) \end{bmatrix} \quad \text{and} \quad A = \begin{bmatrix} 2.5 & -1.0 \\ -0.25 & 2.5 \end{bmatrix}$$

The eigenvalues of A are $\lambda_1 = 3$ and $\lambda_2 = 2$, with corresponding eigenvectors $\mathbf{v}_1 = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$. By Theorem 4.40, the general solution to our system is

$$\mathbf{x}(t) = C_1 e^{3t} \mathbf{v}_1 + C_2 e^{2t} \mathbf{v}_2 = C_1 e^{3t} \begin{bmatrix} -2 \\ 1 \end{bmatrix} + C_2 e^{2t} \begin{bmatrix} 2 \\ 1 \end{bmatrix} \tag{7}$$

The initial population vector is $\mathbf{x}(0) = \begin{bmatrix} r(0) \\ s(0) \end{bmatrix} = \begin{bmatrix} 60 \\ 60 \end{bmatrix}$, so, setting $t = 0$ in Equation (7), we have

$$C_1 \begin{bmatrix} -2 \\ 1 \end{bmatrix} + C_2 \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 60 \\ 60 \end{bmatrix}$$

Solving this equation, we find $C_1 = 15$ and $C_2 = 45$. Hence,

$$\mathbf{x}(t) = 15 e^{3t} \begin{bmatrix} -2 \\ 1 \end{bmatrix} + 45 e^{2t} \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

from which we find $r(t) = -30 e^{3t} + 90 e^{2t}$ and $s(t) = 15 e^{3t} + 45 e^{2t}$. Figure 4.20 shows the graphs of these two functions, and you can see clearly that the raccoon population dies out after a little more than 1 year. (Can you determine *exactly* when it dies out?)

We now consider a similar example, in which one species is a source of food for the other. Such a model is called a **predator-prey model.** Once again, our model will be drastically oversimplified in order to illustrate its main features.
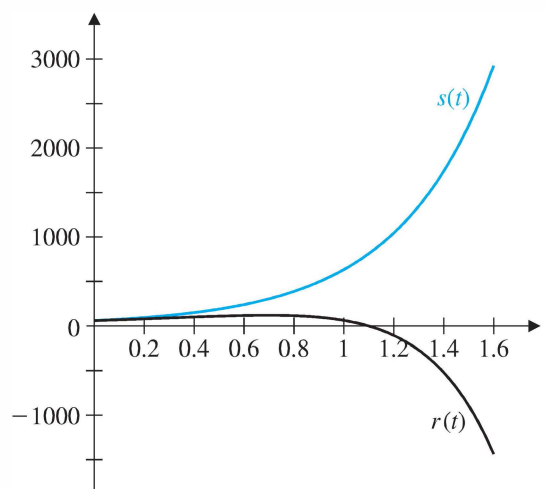

Figure 4.20
Raccoon and squirrel populations

/ a + bi /   **Example 4.45**   Robins and worms cohabit an ecosystem. The robins eat the worms, which are their only source of food. The robin and worm populations at time t years are denoted by $r(t)$ and $w(t)$, respectively, and the equations governing the growth of the two populations are

$$r'(t) = w(t) - 12$$
$$w'(t) = -r(t) + 10 \tag{8}$$

If initially 6 robins and 20 worms occupy the ecosystem, determine the behavior of the two populations over time.

Solution   The first thing we notice about this example is the presence of the extra constants, -12 and 10, in the two equations. Fortunately, we can get rid of them with a simple change of variables. If we let $r(t) = x(t) + 10$ and $w(t) = y(t) + 12$, then $r'(t) = x'(t)$ and $w'(t) = y'(t)$. Substituting into Equations (8), we have

$$x'(t) = y(t)$$
$$y'(t) = -x(t) \tag{9}$$

which is easier to work with. Equations (9) have the form $\mathbf{x}' = A\mathbf{x}$, where $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$. Our new initial conditions are

$$x(0) = r(0) - 10 = 6 - 10 = -4 \quad \text{and} \quad y(0) = w(0) - 12 = 20 - 12 = 8$$

so $\mathbf{x}(0) = \begin{bmatrix} -4 \\ 8 \end{bmatrix}$.

Proceeding as in the last example, we find the eigenvalues and eigenvectors of A. The characteristic polynomial is $\lambda^2 + 1$, which has no real roots. What should we do? We have no choice but to use the complex roots, which are $\lambda_1 = i$ and $\lambda_2 = -i$. The corresponding eigenvectors are also complex—namely, $\mathbf{v}_1 = \begin{bmatrix} 1 \\ i \end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix} 1 \\ -i \end{bmatrix}$. By Theorem 4.40, our solution has the form

$$\mathbf{x}(t) = C_1 e^{it}\mathbf{v}_1 + C_2 e^{-it}\mathbf{v}_2 = C_1 e^{it}\begin{bmatrix} 1 \\ i \end{bmatrix} + C_2 e^{-it}\begin{bmatrix} 1 \\ -i \end{bmatrix}$$

From $\mathbf{x}(0) = \begin{bmatrix} -4 \\ 8 \end{bmatrix}$, we get

$$C_1\begin{bmatrix} 1 \\ i \end{bmatrix} + C_2\begin{bmatrix} 1 \\ -i \end{bmatrix} = \begin{bmatrix} -4 \\ 8 \end{bmatrix}$$

whose solution is $C_1 = -2 - 4i$ and $C_2 = -2 + 4i$. So the solution to system (9) is

$$\mathbf{x}(t) = (-2 - 4i)e^{it}\begin{bmatrix} 1 \\ i \end{bmatrix} + (-2 + 4i)e^{-it}\begin{bmatrix} 1 \\ -i \end{bmatrix}$$

What are we to make of this solution? Robins and worms inhabit a real world—yet our solution involves complex numbers! Fearlessly proceeding, we apply Euler's formula

$$e^{it} = \cos t + i \sin t$$
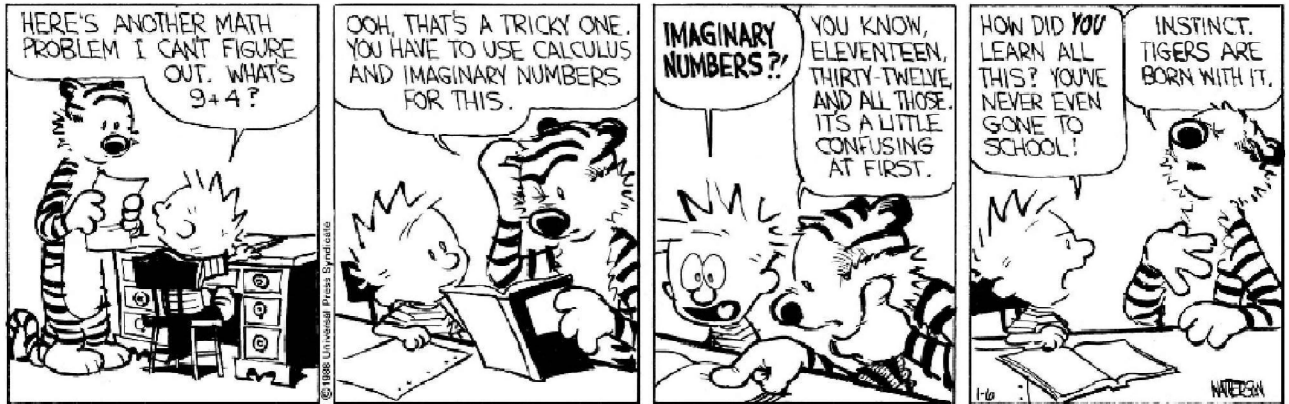
(Appendix C) to get $e^{-it} = \cos(-t) + i\sin(-t) = \cos t - i\sin t$. Substituting, we have

$$\mathbf{x}(t) = (-2 - 4i)(\cos t + i\sin t)\begin{bmatrix} 1 \\ i \end{bmatrix} + (-2 + 4i)(\cos t - i\sin t)\begin{bmatrix} 1 \\ -i \end{bmatrix}$$

$$= \begin{bmatrix} (-2\cos t + 4\sin t) + i(-4\cos t - 2\sin t) \\ (4\cos t + 2\sin t) + i(-2\cos t + 4\sin t) \end{bmatrix}$$

$$+ \begin{bmatrix} (-2\cos t + 4\sin t) + i(4\cos t + 2\sin t) \\ (4\cos t + 2\sin t) + i(2\cos t - 4\sin t) \end{bmatrix}$$

$$= \begin{bmatrix} -4\cos t + 8\sin t \\ 8\cos t + 4\sin t \end{bmatrix}$$

This gives $x(t) = -4\cos t + 8\sin t$ and $y(t) = 8\cos t + 4\sin t$. Putting everything in terms of our original variables, we conclude that

$$r(t) = x(t) + 10 = -4\cos t + 8\sin t + 10$$

and
$$w(t) = y(t) + 12 = 8\cos t + 4\sin t + 12$$
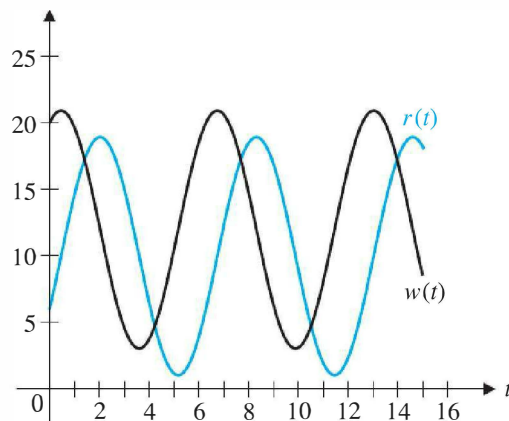
Figure 4.21
Robin and worm populations
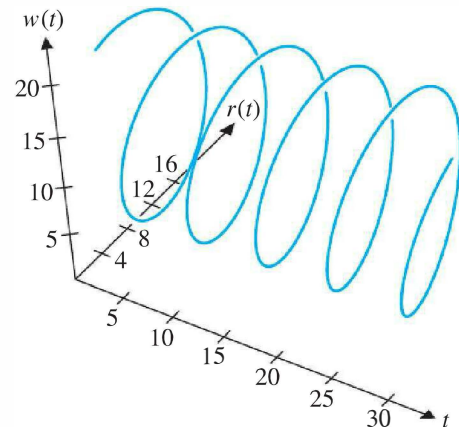
Figure 4.22

So our solution is real after all! The graphs of $r(t)$ and $w(t)$ in Figure 4.21 show that the two populations oscillate periodically. As the robin population increases, the worm population starts to decrease, but as the robins' only food source diminishes, their numbers start to decline as well. As the predators disappear, the worm population begins to recover. As its food supply increases, so does the robin population, and the cycle repeats itself. This oscillation is typical of examples in which the eigenvalues are complex.

Plotting robins, worms, and time on separate axes, as in Figure 4.22, clearly reveals the cyclic nature of the two populations.

We conclude this section by looking at what we have done from a different point of view. If $x = x(t)$ is a differentiable function of t, then the general solution of the ordinary differential equation $x' = ax$ is $x = ce^{at}$, where c is a scalar. The systems of linear differential equations we have been considering have the form $\mathbf{x}' = A\mathbf{x}$, so if we simply plowed ahead without thinking, we might be tempted to deduce that the solution would be $\mathbf{x} = \mathbf{c}e^{At}$, where $\mathbf{c}$ is a vector. But what on earth could this mean? On the right-hand side, we have the *number* e raised to the power of a *matrix*. This appears to be nonsense, yet you will see that there is a way to make sense of it.

Let's start by considering the expression e^A. In calculus, you learn that the function e^x has a power series expansion

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

that converges for every real number x. By analogy, let us define

$$e^A = I + A + \frac{A^2}{2!} + \frac{A^3}{3!} + \cdots$$

The right-hand side is just defined in terms of powers of A, and it can be shown that it converges for any real matrix A. So now e^A is a matrix, called the **exponential** of A. But how can we compute e^A or e^{At}? For diagonal matrices, it is easy.

Example 4.46

Compute e^{Dt} for $D = \begin{bmatrix} 4 & 0 \\ 0 & -1 \end{bmatrix}$.

Solution   From the definition, we have

$$e^{Dt} = I + Dt + \frac{(Dt)^2}{2!} + \frac{(Dt)^3}{3!} + \cdots$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 4t & 0 \\ 0 & -t \end{bmatrix} + \frac{1}{2!}\begin{bmatrix} (4t)^2 & 0 \\ 0 & (-t)^2 \end{bmatrix} + \frac{1}{3!}\begin{bmatrix} (4t)^3 & 0 \\ 0 & (-t)^3 \end{bmatrix} + \cdots$$

$$= \begin{bmatrix} 1 + (4t) + \frac{1}{2!}(4t)^2 + \frac{1}{3!}(4t)^3 + \cdots & 0 \\ 0 & 1 + (-t) + \frac{1}{2!}(-t)^2 + \frac{1}{3!}(-t)^3 + \cdots \end{bmatrix}$$

$$= \begin{bmatrix} e^{4t} & 0 \\ 0 & e^{-t} \end{bmatrix}$$

The matrix exponential is also nice if A is diagonalizable.

Example 4.47

Compute e^A for $A = \begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix}$.

Solution In Example 4.43, we found the eigenvalues of A to be $\lambda_1 = 4$ and $\lambda_2 = -1$, with corresponding eigenvectors $\mathbf{v}_1 = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$, respectively. Hence, with $P = [\mathbf{v}_1 \quad \mathbf{v}_2] = \begin{bmatrix} 2 & -1 \\ 3 & 1 \end{bmatrix}$, we have $P^{-1}AP = D = \begin{bmatrix} 4 & 0 \\ 0 & -1 \end{bmatrix}$. Since $A = PDP^{-1}$, we have $A^k = PD^kP^{-1}$, so

$$
\begin{aligned}
e^A &= I + A + \frac{A^2}{2!} + \frac{A^3}{3!} + \cdots \\
&= PIP^{-1} + PDP^{-1} + \frac{1}{2!}PD^2P^{-1} + \frac{1}{3!}PD^3P^{-1} + \cdots \\
&= P\left(I + D + \frac{D^2}{2!} + \frac{D^3}{3!} + \cdots\right)P^{-1} \\
&= Pe^DP^{-1} \\
&= \begin{bmatrix} 2 & -1 \\ 3 & 1 \end{bmatrix}\begin{bmatrix} e^4 & 0 \\ 0 & e^{-1} \end{bmatrix}\begin{bmatrix} 2 & -1 \\ 3 & 1 \end{bmatrix}^{-1} \\
&= \frac{1}{5}\begin{bmatrix} 2e^4 + 3e^{-1} & 2e^4 - 2e^{-1} \\ 3e^4 - 3e^{-1} & 3e^4 + 2e^{-1} \end{bmatrix}
\end{aligned}
$$

We are now in a position to show that our bold (and seemingly foolish) guess at an "exponential" solution of $\mathbf{x}' = A\mathbf{x}$ was not so far off after all!

Theorem 4.41

Let A be an $n \times n$ diagonalizable matrix with eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$. Then the general solution to the system $\mathbf{x}' = A\mathbf{x}$ is $\mathbf{x} = e^{At}\mathbf{c}$, where $\mathbf{c}$ is an arbitrary constant vector. If an initial condition $\mathbf{x}(0)$ is specified, then $\mathbf{c} = \mathbf{x}(0)$.

Proof Let P diagonalize A. Then $A = PDP^{-1}$, and, as in Example 4.47,

$$e^{At} = Pe^{Dt}P^{-1}$$

Hence, we need to check that $\mathbf{x}' = A\mathbf{x}$ is satisfied by $\mathbf{x} = Pe^{Dt}P^{-1}\mathbf{c}$. Now, everything is constant except for e^{Dt}, so

$$\mathbf{x}' = \frac{d\mathbf{x}}{dt} = \frac{d}{dt}(Pe^{Dt}P^{-1}\mathbf{c}) = P\frac{d}{dt}(e^{Dt})P^{-1}\mathbf{c} \tag{10}$$

If

$$D = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix}$$

then

$$
e^{Dt} = \begin{bmatrix}
e^{\lambda_1 t} & 0 & \cdots & 0 \\
0 & e^{\lambda_2 t} & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & e^{\lambda_n t}
\end{bmatrix}
$$

Taking derivatives, we have

$$
\frac{d}{dt}(e^{Dt}) = \begin{bmatrix}
\dfrac{d}{dt}(e^{\lambda_1 t}) & 0 & \cdots & 0 \\
0 & \dfrac{d}{dt}(e^{\lambda_2 t}) & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & \dfrac{d}{dt}(e^{\lambda_n t})
\end{bmatrix}
$$

$$
= \begin{bmatrix}
\lambda_1 e^{\lambda_1 t} & 0 & \cdots & 0 \\
0 & \lambda_2 e^{\lambda_2 t} & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & \lambda_n e^{\lambda_n t}
\end{bmatrix}
$$

$$
= \begin{bmatrix}
\lambda_1 & 0 & \cdots & 0 \\
0 & \lambda_2 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & \lambda_n
\end{bmatrix}
\begin{bmatrix}
e^{\lambda_1 t} & 0 & \cdots & 0 \\
0 & e^{\lambda_2 t} & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & e^{\lambda_n t}
\end{bmatrix}
$$

$$
= De^{Dt}
$$

Substituting this result into Equation (10), we obtain

$$
\mathbf{x}' = PDe^{Dt}P^{-1}\mathbf{c} = PDP^{-1}Pe^{Dt}P^{-1}\mathbf{c} = (PDP^{-1})(Pe^{Dt}P^{-1})\mathbf{c} = Ae^{At}\mathbf{c} = A\mathbf{x}
$$

as required.

The last statement follows easily from the fact that if $\mathbf{x} = \mathbf{x}(t) = e^{At}\mathbf{c}$, then

$$
\mathbf{x}(0) = e^{A \cdot 0}\mathbf{c} = e^O\mathbf{c} = I\mathbf{c} = \mathbf{c}
$$

since $e^O = I$. (Why?)

In fact, Theorem 4.41 is true even if A is not diagonalizable, but we will not prove this. Computation of matrix exponentials for nondiagonalizable matrices requires the *Jordan normal form* of a matrix, a topic that may be found in more advanced linear algebra texts.

For example, see *Linear Algebra* by S. H. Friedberg, A. J. Insel, and L. E. Spence (Englewood Cliffs, NJ: Prentice-Hall, 1979).

Ideally, this short digression has served to illustrate the power of mathematics to generalize and the value of creative thinking. Matrix exponentials turn out to be very important tools in many applications of linear algebra, both theoretical and applied.

Discrete Linear Dynamical Systems

We conclude this chapter as we began it—by looking at dynamical systems. Markov chains and the Leslie model of population growth are examples of **discrete linear dynamical systems.** Each can be described by a matrix equation of the form

$$
\mathbf{x}_{k+1} = A\mathbf{x}_k
$$

where the vector $\mathbf{x}_k$ records the state of the system at "time" k and A is a square matrix. As we have seen, the long-term behavior of these systems is related to the eigenvalues and eigenvectors of the matrix A. The power method exploits the iterative nature of such dynamical systems to approximate eigenvalues and eigenvectors, and the Perron-Frobenius Theorem gives specialized information about the long-term behavior of a discrete linear dynamical system whose coefficient matrix A is nonnegative.

When A is a 2 × 2 matrix, we can describe the evolution of a dynamical system geometrically. The equation $\mathbf{x}_{k+1} = A\mathbf{x}_k$ is really an infinite collection of equations. Beginning with an initial vector $\mathbf{x}_0$, we have:

$$\mathbf{x}_1 = A\mathbf{x}_0$$
$$\mathbf{x}_2 = A\mathbf{x}_1$$
$$\mathbf{x}_3 = A\mathbf{x}_2$$
$$\vdots$$

The set $\{\mathbf{x}_0, \mathbf{x}_1, \mathbf{x}_2, \ldots\}$ is called a ***trajectory*** of the system. (For graphical purposes, we will identify each vector in a trajectory with its head so that we can plot it as a point.) Note that $\mathbf{x}_k = A^k \mathbf{x}_0$.

Example 4.48

Let $A = \begin{bmatrix} 0.5 & 0 \\ 0 & 0.8 \end{bmatrix}$. For the dynamical system $\mathbf{x}_{k+1} = A\mathbf{x}_k$, plot the first five points in the trajectories with the following initial vectors:

(a) $\mathbf{x}_0 = \begin{bmatrix} 5 \\ 0 \end{bmatrix}$ (b) $\mathbf{x}_0 = \begin{bmatrix} 0 \\ -5 \end{bmatrix}$ (c) $\mathbf{x}_0 = \begin{bmatrix} 4 \\ 4 \end{bmatrix}$ (d) $\mathbf{x}_0 = \begin{bmatrix} -2 \\ 4 \end{bmatrix}$

Solution   (a) We compute $\mathbf{x}_1 = A\mathbf{x}_0 = \begin{bmatrix} 2.5 \\ 0 \end{bmatrix}$, $\mathbf{x}_2 = A\mathbf{x}_1 = \begin{bmatrix} 1.25 \\ 0 \end{bmatrix}$, $\mathbf{x}_3 = A\mathbf{x}_2 = \begin{bmatrix} 0.625 \\ 0 \end{bmatrix}$, $\mathbf{x}_4 = A\mathbf{x}_3 = \begin{bmatrix} 0.3125 \\ 0 \end{bmatrix}$. These are plotted in Figure 4.23, and the points are connected to highlight the trajectory. Similar calculations produce the trajectories marked (*b*), (*c*), and (*d*) in Figure 4.23.
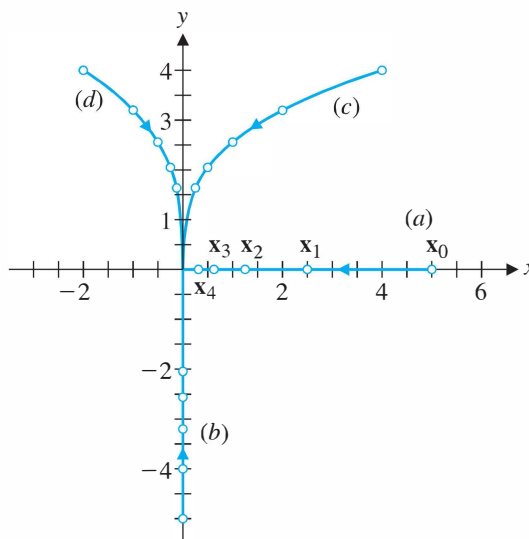

Figure 4.23

In Example 4.48, every trajectory converges to $\mathbf{0}$. The origin is called an **attractor** in this case. We can understand why this is so from Theorem 4.19. The matrix A in Example 4.48 has eigenvectors $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ corresponding to its eigenvalues 0.5 and 0.8, respectively. (Check this.) Accordingly, for any initial vector

$$\mathbf{x}_0 = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

we have

$$\mathbf{x}_k = A^k \mathbf{x}_0 = c_1 (0.5)^k \begin{bmatrix} 1 \\ 0 \end{bmatrix} + c_2 (0.8)^k \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Because both $(0.5)^k$ and $(0.8)^k$ approach zero as k gets large, $\mathbf{x}_k$ approaches $\mathbf{0}$ for any choice of $\mathbf{x}_0$. In addition, we know from Theorem 4.28 that because 0.8 is the dominant eigenvalue of A, $\mathbf{x}_k$ will approach a multiple of the corresponding eigenvector $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ as long as $c_2 \neq 0$ (the coefficient of $\mathbf{x}_0$ corresponding to $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$). In other words, all trajectories except those that begin on the x-axis (where $c_2 = 0$) will approach the y-axis, as Figure 4.23 shows.

Example 4.49

Discuss the behavior of the dynamical system $\mathbf{x}_{k+1} = A\mathbf{x}_k$ corresponding to the matrix $A = \begin{bmatrix} 0.65 & -0.15 \\ -0.15 & 0.65 \end{bmatrix}$.

Solution   The eigenvalues of A are 0.5 and 0.8 with corresponding eigenvectors $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$, respectively. (Check this.) Hence for an initial vector $\mathbf{x}_0 = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix}$, we have

$$\mathbf{x}_k = A^k \mathbf{x}_0 = c_1 (0.5)^k \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 (0.8)^k \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

Once again the origin is an attractor, because $\mathbf{x}_k$ approaches $\mathbf{0}$ for any choice of $\mathbf{x}_0$. If $c_2 \neq 0$, the trajectory will approach the line through the origin with direction vector $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$. Several such trajectories are shown in Figure 4.24. The vectors $\mathbf{x}_0$ where $c_2 = 0$ are on the line through the origin with direction vector $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$, and the corresponding trajectory in this case follows this line into the origin.
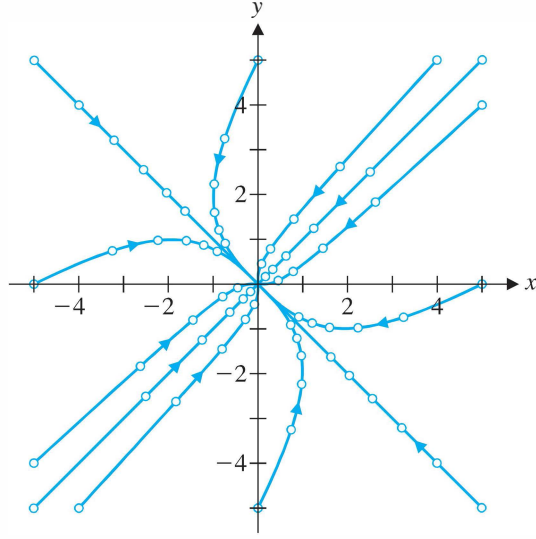
Figure 4.24

Example 4.50

Discuss the behavior of the dynamical systems $\mathbf{x}_{k+1} = A\mathbf{x}_k$ corresponding to the following matrices:

(a) $A = \begin{bmatrix} 4 & 1 \\ 1 & 4 \end{bmatrix}$ (b) $A = \begin{bmatrix} 1 & 0.5 \\ 0.5 & 1 \end{bmatrix}$

Solution   (a) The eigenvalues of A are 5 and 3 with corresponding eigenvectors $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$, respectively. Hence for an initial vector $\mathbf{x}_0 = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix}$, we have

$$\mathbf{x}_k = A^k \mathbf{x}_0 = c_1 5^k \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 3^k \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

As k becomes large, so do both 5^k and 3^k. Hence, $\mathbf{x}_k$ tends away from the origin. Because the dominant eigenvalue of 5 has corresponding eigenvector $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$, all trajectories for which $c_1 \neq 0$ will eventually end up in the first or the third quadrant. Trajectories with $c_1 = 0$ start and stay on the line $y = -x$ whose direction vector is $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$. See Figure 4.25(a).

(b) In this example, the eigenvalues are 1.5 and 0.5 with corresponding eigenvectors $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$, respectively. Hence,

$$\mathbf{x}_k = c_1 (1.5)^k \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 (0.5)^k \begin{bmatrix} -1 \\ 1 \end{bmatrix} \quad \text{if} \quad \mathbf{x}_0 = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$
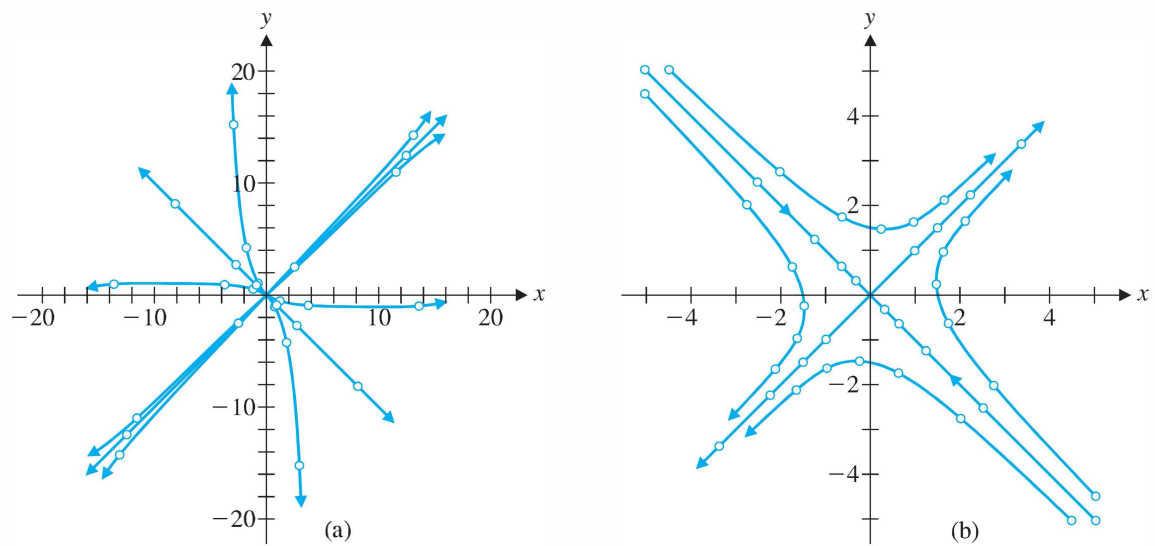
Figure 4.25

If $c_1 = 0$, then $\mathbf{x}_k = c_2(0.5)^k \begin{bmatrix} -1 \\ 1 \end{bmatrix} \to \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ as $k \to \infty$. But if $c_1 \neq 0$, then

$$\mathbf{x}_k = c_1(1.5)^k \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2(0.5)^k \begin{bmatrix} -1 \\ 1 \end{bmatrix} \approx c_1(1.5)^k \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad \text{as} \quad k \to \infty$$

and such trajectories asymptotically approach the line $y = x$. See Figure 4.25(b).

In Example 4.50(a), all points that start out near the origin become increasingly large in magnitude because $|\lambda| > 1$ for both eigenvalues; **0** is called a **repeller.** In Example 4.50(b), **0** is called a **saddle point** because the origin attracts points in some directions and repels points in other directions. In this case, $|\lambda_1| < 1$ and $|\lambda_2| > 1$.

The next example shows what can happen when the eigenvalues of a real 2×2 matrix are complex (and hence conjugates of one another).

Example 4.51

Plot the trajectory beginning with $\mathbf{x}_0 = \begin{bmatrix} 4 \\ 4 \end{bmatrix}$ for the dynamical systems $\mathbf{x}_{k+1} = A\mathbf{x}_k$ corresponding to the following matrices:

(a) $A = \begin{bmatrix} 0.5 & -0.5 \\ 0.5 & 0.5 \end{bmatrix}$ (b) $A = \begin{bmatrix} 0.2 & -1.2 \\ 0.6 & 1.4 \end{bmatrix}$

Solution   The trajectories are shown in Figure 4.26(a) and (b), respectively. Note that (a) is a trajectory spiraling into the origin, whereas (b) appears to follow an elliptical orbit.
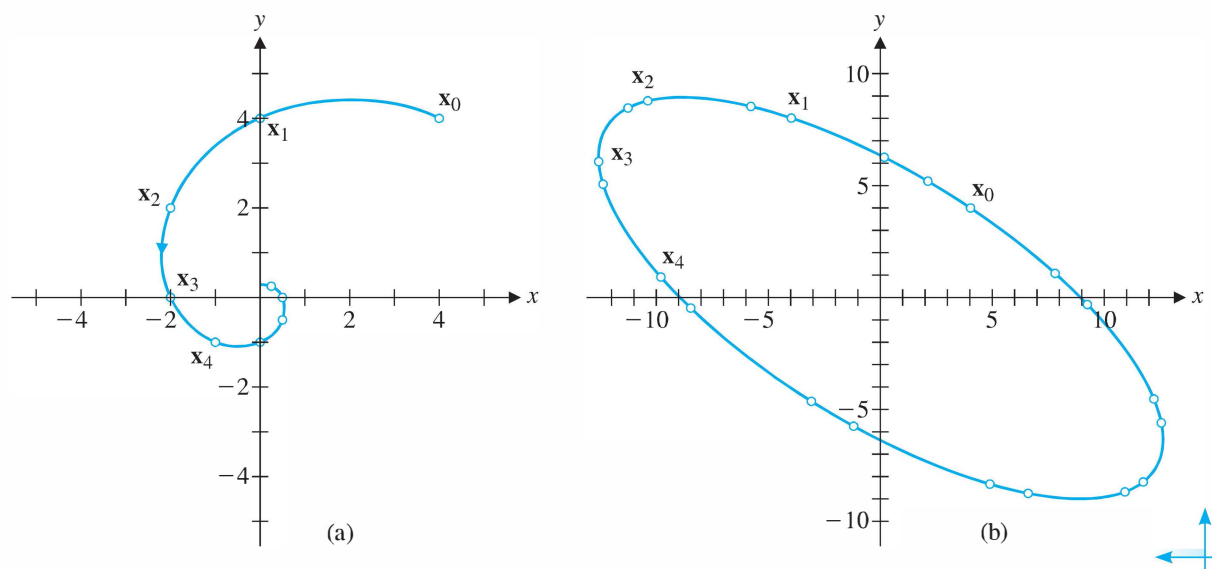
Figure 4.26

The following theorem explains the spiral behavior of the trajectory in Example 4.51(a).

Theorem 4.42

Let $A = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$. The eigenvalues of A are $\lambda = a \pm bi$, and if a and b are not both zero, then A can be factored as

$$A = \begin{bmatrix} a & -b \\ b & a \end{bmatrix} = \begin{bmatrix} r & 0 \\ 0 & r \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

where $r = |\lambda| = \sqrt{a^2 + b^2}$ and θ is the principal argument of $a + bi$.

Proof The eigenvalues of A are

$$\lambda = \tfrac{1}{2}(2a \pm \sqrt{4(-b^2)}) = \tfrac{1}{2}(2a \pm 2\sqrt{b^2}\sqrt{-1}) = a \pm |b|i = a \pm bi$$

by Exercise 35(b) in Section 4.1. Figure 4.27 displays $a + bi$, r, and θ. It follows that

$$A = \begin{bmatrix} a & -b \\ b & a \end{bmatrix} = r\begin{bmatrix} a/r & -b/r \\ b/r & a/r \end{bmatrix} = \begin{bmatrix} r & 0 \\ 0 & r \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

Remark Geometrically, Theorem 4.42 implies that when $A = \begin{bmatrix} a & -b \\ b & a \end{bmatrix} \neq O$, the linear transformation $T(\mathbf{x}) = A\mathbf{x}$ is the composition of a rotation $R = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ through the angle θ followed by a scaling $S = \begin{bmatrix} r & 0 \\ 0 & r \end{bmatrix}$ with factor r (Figure 4.28). In Example 4.51(a), the eigenvalues are $\lambda = 0.5 \pm 0.5i$ so $r = |\lambda| = \sqrt{2}/2 \approx 0.707 < 1$, and hence the trajectories all spiral inward toward $\mathbf{0}$.

The next theorem shows that, in general, when a real 2×2 matrix has complex eigenvalues, it is similar to a matrix of the form $\begin{bmatrix} a & -b \\ b & a \end{bmatrix}$. For a complex vector

$$\mathbf{x} = \begin{bmatrix} z \\ w \end{bmatrix} = \begin{bmatrix} a + bi \\ c + di \end{bmatrix} = \begin{bmatrix} a \\ b \end{bmatrix} + \begin{bmatrix} c \\ d \end{bmatrix} i$$
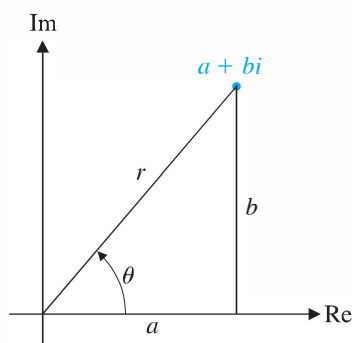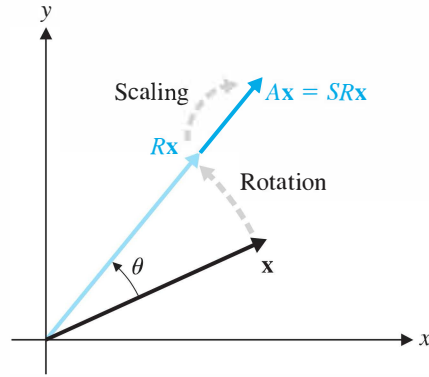

Figure 4.27

Figure 4.28
A rotation followed by a scaling

we define the real part, Re **x,** and the imaginary part, Im **x,** of **x** to be

$$\text{Re } \mathbf{x} = \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} \text{Re}z \\ \text{Re}w \end{bmatrix} \quad \text{and} \quad \text{Im } \mathbf{x} = \begin{bmatrix} c \\ d \end{bmatrix} = \begin{bmatrix} \text{Im}z \\ \text{Im}w \end{bmatrix}$$

Theorem 4.43   Let A be a real 2×2 matrix with a complex eigenvalue $\lambda = a - bi$ (where $b \neq 0$) and corresponding eigenvector **x**. Then the matrix $P = [\text{Re } \mathbf{x} \quad \text{Im } \mathbf{x}]$ is invertible and

$$A = P \begin{bmatrix} a & -b \\ b & a \end{bmatrix} P^{-1}$$

Proof   Let $\mathbf{x} = \mathbf{u} + \mathbf{v}i$ so that Re **x** = **u** and Im **x** = **v**. From $A\mathbf{x} = \lambda\mathbf{x}$, we have

$$A\mathbf{u} + A\mathbf{v}i = A\mathbf{x} = \lambda\mathbf{x} = (a - bi)(\mathbf{u} + \mathbf{v}i)$$
$$= a\mathbf{u} + a\mathbf{v}i - b\mathbf{u}i + b\mathbf{v} = (a\mathbf{u} + b\mathbf{v}) + (-b\mathbf{u} + a\mathbf{v})i$$

Equating real and imaginary parts, we obtain

$$A\mathbf{u} = a\mathbf{u} + b\mathbf{v} \quad \text{and} \quad A\mathbf{v} = -b\mathbf{u} + a\mathbf{v}$$

Now $P = [\mathbf{u} | \mathbf{v}]$, so

$$P \begin{bmatrix} a & -b \\ b & a \end{bmatrix} = [\mathbf{u} | \mathbf{v}] \begin{bmatrix} a & -b \\ b & a \end{bmatrix} = [a\mathbf{u} + b\mathbf{v} | -b\mathbf{u} + a\mathbf{v}] = [A\mathbf{u} | A\mathbf{v}] = A[\mathbf{u} | \mathbf{v}]$$
$$= AP$$

To show that P is invertible, it is enough to show that **u** and **v** are linearly independent. If **u** and **v** were not linearly independent, then it would follow that $\mathbf{v} = k\mathbf{u}$ for some (nonzero complex) scalar k, because neither **u** nor **v** is **0**. Thus,

$$\mathbf{x} = \mathbf{u} + \mathbf{v}i = \mathbf{u} + k\mathbf{u}i = (1 + ki)\mathbf{u}$$

Now, because A is real, $A\mathbf{x} = \lambda\mathbf{x}$ implies that

$$A\overline{\mathbf{x}} = \overline{A}\overline{\mathbf{x}} = \overline{A\mathbf{x}} = \overline{\lambda\mathbf{x}} = \overline{\lambda}\overline{\mathbf{x}}$$

so $\overline{\mathbf{x}} = \mathbf{u} - \mathbf{v}i$ is an eigenvector corresponding to the other eigenvalue $\overline{\lambda} = a + bi$. But

$$\overline{\mathbf{x}} = \overline{(1 + ki)\mathbf{u}} = (1 - \overline{k}i)\mathbf{u}$$

because **u** is a real vector. Hence, the eigenvectors **x** and $\bar{\mathbf{x}}$ of A are both nonzero multiples of **u** and therefore are multiples of one another. This is impossible because eigenvectors corresponding to distinct eigenvalues must be linearly independent by Theorem 4.20. (This theorem is valid over the complex numbers as well as the real numbers.)

This contradiction implies that **u** and **v** are linearly independent and hence P is invertible. It now follows that

$$A = P\begin{bmatrix} a & -b \\ b & a \end{bmatrix}P^{-1}$$

Theorem 4.43 serves to explain Example 4.51(b). The eigenvalues of $A = \begin{bmatrix} 0.2 & -1.2 \\ 0.6 & 1.4 \end{bmatrix}$ are $0.8 \pm 0.6i$. For $\lambda = 0.8 - 0.6i$, a corresponding eigenvector is

$$\mathbf{x} = \begin{bmatrix} -1-i \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix} + \begin{bmatrix} -1 \\ 0 \end{bmatrix}i$$

From Theorem 4.43, it follows that for $P = \begin{bmatrix} -1 & -1 \\ 1 & 0 \end{bmatrix}$ and $C = \begin{bmatrix} 0.8 & -0.6 \\ 0.6 & 0.8 \end{bmatrix}$, we have

$$A = PCP^{-1} \quad \text{and} \quad P^{-1}AP = C$$

For the given dynamical system $\mathbf{x}_{k+1} = A\mathbf{x}_k$, we perform a change of variable. Let

$$\mathbf{x}_k = P\mathbf{y}_k \quad \text{(or, equivalently, } \mathbf{y}_k = P^{-1}\mathbf{x}_k)$$

Then

$$P\mathbf{y}_{k+1} = \mathbf{x}_{k+1} = A\mathbf{x}_k = AP\mathbf{y}_k$$

so

$$\mathbf{y}_{k+1} = \mathbf{x}_{k+1} = A\mathbf{x}_k = P^{-1}AP\mathbf{y}_k = C\mathbf{y}_k$$

Now C has the same eigenvalues as A (why?) and $|0.8 \pm 0.6i| = 1$. Thus, the dynamical system $\mathbf{y}_{k+1} = C\mathbf{y}_k$ simply rotates the points in every trajectory in a circle about the origin by Theorem 4.42.

To determine a trajectory of the dynamical system in Example 4.51(b), we iteratively apply the linear transformation $T(\mathbf{x}) = A\mathbf{x} = PCP^{-1}\mathbf{x}$. The transformation can be thought of as the composition of a change of variable (**x** to **y**), followed by the rotation determined by C, followed by the reverse change of variable (**y** back to **x**). We will encounter this idea again in the application to graphing quadratic equations in Section 5.5 and, more generally, as "change of basis" in Section 6.3. In Exercise 74 of Section 5.5, you will show that the trajectory in Example 4.51(b) is indeed an ellipse, as it appears to be from Figure 4.26(b).

To summarize then: If a real 2×2 matrix A has complex eigenvalues $\lambda = a \pm bi$, then the trajectories of the dynamical system $\mathbf{x}_{k+1} = A\mathbf{x}_k$ spiral inward if $|\lambda| < 1$ (**0** is a ***spiral attractor***), spiral outward if $|\lambda| > 1$ (**0** is a ***spiral repeller***), and lie on a closed orbit if $|\lambda| = 1$ (**0** is an ***orbital center***).

Vignette

Ranking Sports Teams and Searching the Internet

In any competitive sports league, it is not necessarily a straightforward process to rank the players or teams. Counting wins and losses alone overlooks the possibility that one team may accumulate a large number of victories against weak teams, while another team may have fewer victories but all of them against strong teams. Which of these teams is better? How should we compare two teams that never play one another? Should points scored be taken into account? Points against?

Despite these complexities, the ranking of athletes and sports teams has become a commonplace and much-anticipated feature in the media. For example, there are various annual rankings of U.S. college football and basketball teams, and golfers and tennis players are also ranked internationally. There are many copyrighted schemes used to produce such rankings, but we can gain some insight into how to approach the problem by using the ideas from this chapter.

To establish the basic idea, let's revisit Example 3.69. Five tennis players play one another in a round-robin tournament. Wins and losses are recorded in the form of a digraph in which a directed edge from i to j indicates that player i defeats player j. The corresponding adjacency matrix A therefore has $a_{ij} = 1$ if player i defeats player j and has $a_{ij} = 0$ otherwise.
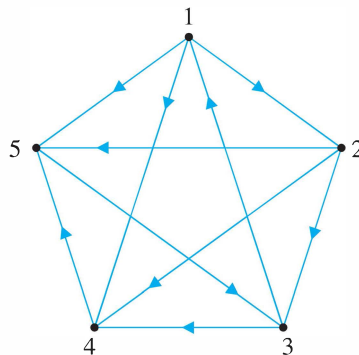
$$A = \begin{bmatrix} 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$

We would like to associate a ranking r_i with player i in such a way that $r_i > r_j$ indicates that player i is ranked more highly than player j. For this

purpose, let's require that the r_i's be probabilities (that is, $0 \le r_i \le 1$ for all i, and $r_1 + r_2 + r_3 + r_4 + r_5 = 1$) and then organize the rankings in a *ranking vector*

$$\mathbf{r} = \begin{bmatrix} r_1 \\ r_2 \\ r_3 \\ r_4 \\ r_5 \end{bmatrix}$$

Furthermore, let's insist that player i's ranking should be proportional to the sum of the rankings of the players defeated by player i. For example, player 1 defeated players 2, 4, and 5, so we want

$$r_1 = \alpha(r_2 + r_4 + r_5)$$

where α is the constant of proportionality. Writing out similar equations for the other players produces the following system:

$$\begin{aligned} r_1 &= \alpha(r_2 + r_4 + r_5) \\ r_2 &= \alpha(r_3 + r_4 + r_5) \\ r_3 &= \alpha(r_1 + r_4) \\ r_4 &= \alpha r_5 \\ r_5 &= \alpha r_3 \end{aligned}$$

Observe that we can write this system in matrix form as

$$\begin{bmatrix} r_1 \\ r_2 \\ r_3 \\ r_4 \\ r_5 \end{bmatrix} = \alpha \begin{bmatrix} 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} r_1 \\ r_2 \\ r_3 \\ r_4 \\ r_5 \end{bmatrix} \qquad \text{or} \qquad \mathbf{r} = \alpha A \mathbf{r}$$

Equivalently, we see that the ranking vector $\mathbf{r}$ must satisfy $A\mathbf{r} = \dfrac{1}{\alpha}\mathbf{r}$. In other words, $\mathbf{r}$ is an eigenvector corresponding to the matrix A!

Furthermore, A is a primitive nonnegative matrix, so the Perron-Frobenius Theorem guarantees that there is a *unique* ranking vector $\mathbf{r}$. In this example, the ranking vector turns out to be

$$\mathbf{r} = \begin{bmatrix} 0.29 \\ 0.27 \\ 0.22 \\ 0.08 \\ 0.14 \end{bmatrix}$$

so we would rank the players in the order 1, 2, 3, 5, 4.

By modifying the matrix A, it is possible to take into account many of the complexities mentioned in the opening paragraph. However, this simple example has served to indicate one useful approach to the problem of ranking teams.

The same idea can be used to understand how an Internet search engine such as Google works. Older search engines used to return the results of a search *unordered*. Useful sites would often be buried among irrelevant ones. Much scrolling was often needed to uncover what you were looking for. By contrast, Google returns search results *ordered* according to their likely relevance. Thus, a method for ranking websites is needed.

Instead of teams playing one another, we now have websites linking to one another. We can once again use a digraph to model the situation, only now an edge from i to j indicates that website i links to (or refers to) website j. So whereas for the sports team digraph, incoming directed edges are bad (they indicate losses), for the Internet digraph, incoming directed edges are good (they indicate links from other sites). In this setting, we want the ranking of website i to be proportional to sum of the rankings of all the websites that link to i.

Using the digraph on page 356 to represent just five websites, we have

$$r_4 = \alpha(r_1 + r_2 + r_3)$$

for example. It is easy to see that we now want to use the *transpose* of the adjacency matrix of the digraph. Therefore, the ranking vector $\mathbf{r}$ must satisfy $A^T \mathbf{r} = \dfrac{1}{a}\mathbf{r}$ and will thus be the Perron eigenvector of A^T. In this example, we obtain

$$A^T = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 \end{bmatrix} \quad \text{and} \quad \mathbf{r} = \begin{bmatrix} 0.14 \\ 0.08 \\ 0.22 \\ 0.27 \\ 0.29 \end{bmatrix}$$

so a search that turns up these five sites would list them in the order 5, 4, 3, 1, 2.

Google actually uses a variant of the method described here and computes the ranking vector via an iterative method very similar to the power method (Section 4.5).

Exercises 4.6

Markov Chains

Which of the stochastic matrices in Exercises 1–6 are regular?

1. $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

2. $\begin{bmatrix} 1 & \frac{1}{2} \\ 0 & \frac{1}{2} \end{bmatrix}$

3. $\begin{bmatrix} \frac{1}{3} & 1 \\ \frac{2}{3} & 0 \end{bmatrix}$

4. $\begin{bmatrix} \frac{1}{2} & 0 & 1 \\ \frac{1}{2} & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$

5. $\begin{bmatrix} 0.1 & 0 & 0.5 \\ 0.5 & 1 & 0 \\ 0.4 & 0 & 0.5 \end{bmatrix}$

6. $\begin{bmatrix} 0.5 & 1 & 0 \\ 0.5 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

In Exercises 7–9, P is the transition matrix of a regular Markov chain. Find the long range transition matrix L of P.

7. $P = \begin{bmatrix} \frac{1}{3} & \frac{1}{6} \\ \frac{2}{3} & \frac{5}{6} \end{bmatrix}$

8. $P = \begin{bmatrix} \frac{1}{2} & \frac{1}{3} & \frac{1}{6} \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{3} \\ 0 & \frac{1}{6} & \frac{1}{2} \end{bmatrix}$

9. $P = \begin{bmatrix} 0.2 & 0.3 & 0.4 \\ 0.6 & 0.1 & 0.4 \\ 0.2 & 0.6 & 0.2 \end{bmatrix}$

10. Prove that the steady state probability vector of a regular Markov chain is unique. [*Hint:* Use Theorem 4.33 or Theorem 4.34.]

Population Growth

In Exercises 11–14, calculate the positive eigenvalue and a corresponding positive eigenvector of the Leslie matrix L.

11. $L = \begin{bmatrix} 0 & 2 \\ 0.5 & 0 \end{bmatrix}$

12. $L = \begin{bmatrix} 1 & 1.5 \\ 0.5 & 0 \end{bmatrix}$

13. $L = \begin{bmatrix} 0 & 7 & 4 \\ 0.5 & 0 & 0 \\ 0 & 0.5 & 0 \end{bmatrix}$

14. $L = \begin{bmatrix} 1 & 5 & 3 \\ \frac{1}{3} & 0 & 0 \\ 0 & \frac{2}{3} & 0 \end{bmatrix}$

15. If a Leslie matrix has a unique positive eigenvalue λ_1, what is the significance for the population if $\lambda_1 > 1$? $\lambda_1 < 1$? $\lambda_1 = 1$?

16. Verify that the characteristic polynomial of the Leslie matrix L in Equation (3) is

$$c_L(\lambda) = (-1)^n(\lambda^n - b_1\lambda^{n-1} - b_2 s_1 \lambda^{n-2} - b_3 s_1 s_2 \lambda^{n-3}$$
$$- \cdots - b_n s_1 s_2 \cdots s_{n-1})$$

[*Hint:* Use mathematical induction and expand $\det(L - \lambda I)$ along the last column.]

17. If all of the survival rates s_i are nonzero, let

$$P = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & s_1 & 0 & \cdots & 0 \\ 0 & 0 & s_1 s_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & s_1 s_2 \cdots s_{n-1} \end{bmatrix}$$

Compute $P^{-1}LP$ and use it to find the characteristic polynomial of L. [*Hint:* Refer to Exercise 32 in Section 4.3.]

18. Verify that an eigenvector of L corresponding to λ_1 is

$$\mathbf{x}_1 = \begin{bmatrix} 1 \\ s_1/\lambda_1 \\ s_1 s_2/\lambda_1^2 \\ s_1 s_2 s_3/\lambda_1^3 \\ \vdots \\ s_1 s_2 s_3 \cdots s_{n-1}/\lambda_1^{n-1} \end{bmatrix}$$

[*Hint:* Combine Exercise 17 above with Exercise 32 in Section 4.3 and Exercise 46 in Section 4.4.]

CAS *In Exercises 19–21, compute the steady state growth rate of the population with the Leslie matrix L from the given exercise. Then use Exercise 18 to help find the corresponding distribution of the age classes.*

19. Exercise 39 in Section 3.7

20. Exercise 40 in Section 3.7

21. Exercise 44 in Section 3.7

CAS **22.** Many species of seal have suffered from commercial hunting. They have been killed for their skin, blubber, and meat. The fur trade, in particular, reduced some seal populations to the point of extinction. Today, the greatest threats to seal populations are decline of fish stocks due to overfishing, pollution, disturbance of habitat, entanglement in marine debris, and culling by fishery owners. Some seals have been declared endangered species; other species are carefully managed. Table 4.7 gives the birth and survival rates for the northern fur seal, divided into 2-year age classes. [The data are based on A. E. York and J. R. Hartley, "Pup Production Following Harvest of Female Northern Fur Seals," *Canadian Journal of Fisheries and Aquatic Science, 38* (1981), pp. 84–90.]

Table 4.7

Age (years)	Birth Rate	Survival Rate
0–2	0.00	0.91
2–4	0.02	0.88
4–6	0.70	0.85
6–8	1.53	0.80
8–10	1.67	0.74
10–12	1.65	0.67
12–14	1.56	0.59
14–16	1.45	0.49
16–18	1.22	0.38
18–20	0.91	0.27
20–22	0.70	0.17
22–24	0.22	0.15
24–26	0.00	0.00

(a) Construct the Leslie matrix L for these data and compute the positive eigenvalue and a corresponding positive eigenvector.

(b) In the long run, what percentage of seals will be in each age class and what will the growth rate be?

Exercise 23 shows that the long-run behavior of a population can be determined directly from the entries of its Leslie matrix.

23. The *net reproduction rate* of a population is defined as

$$r = b_1 + b_2 s_1 + b_3 s_1 s_2 + \cdots + b_n s_1 s_2 \cdots s_{n-1}$$

where the b_i are the birth rates and the s_j are the survival rates for the population.

(a) Explain why r can be interpreted as the average number of daughters born to a single female over her lifetime.

(b) Show that $r = 1$ if and only if $\lambda_1 = 1$. (This represents *zero population growth*.) [*Hint:* Let

$$g(\lambda) = \frac{b_1}{\lambda} + \frac{b_2 s_1}{\lambda^2} + \frac{b_3 s_1 s_2}{\lambda^3} + \cdots + \frac{b_n s_1 s_2 \cdots s_{n-1}}{\lambda^n}$$

Show that λ is an eigenvalue of L if and only if $g(\lambda) = 1$.]

(c) Assuming that there is a unique positive eigenvalue λ_1, show that $r < 1$ if and only if the population is decreasing and $r > 1$ if and only if the population is increasing.

A sustainable harvesting policy is a procedure that allows a certain fraction of a population (represented by a population distribution vector x *) to be harvested so that the population returns to* x *after one time interval (where a time interval is the length of one age class). If h is the fraction of each age class that is harvested, then we can express the harvesting procedure mathematically as follows: If we start with a population vector* x, *after one time interval we have* Lx; *harvesting removes* hLx, *leaving*

$$Lx - hLx = (1 - h)Lx$$

Sustainability requires that

$$(1 - h)Lx = x$$

24. If λ_1 is the unique positive eigenvalue of a Leslie matrix L and h is the sustainable harvest ratio, prove that $h = 1 - 1/\lambda_1$.

25. (a) Find the sustainable harvest ratio for the woodland caribou in Exercise 44 in Section 3.7.

(b) Using the data in Exercise 44 in Section 3.7, reduce the caribou herd according to your answer to part (a). Verify that the population returns to its original level after one time interval.

26. Find the sustainable harvest ratio for the seal in Exercise 22. (Conservationists have had to harvest seal populations when overfishing has reduced the available food supply to the point where the seals are in danger of starvation.)

27. Let L be a Leslie matrix with a unique positive eigenvalue λ_1. Show that if λ is any other (real or complex) eigenvalue of L, then $|\lambda| \leq \lambda_1$. [*Hint:* Write $\lambda = r(\cos\theta + i\sin\theta)$ and substitute it into the equation $g(\lambda) = 1$, as in part (b) of Exercise 23. Use De Moivre's Theorem and then take the real part of both sides. The Triangle Inequality should prove useful.]

The Perron-Frobenius Theorem

In Exercises 28–31, find the Perron root and the corresponding Perron eigenvector of A.

28. $A = \begin{bmatrix} 2 & 0 \\ 1 & 1 \end{bmatrix}$ **29.** $A = \begin{bmatrix} 1 & 3 \\ 2 & 0 \end{bmatrix}$

30. $A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$ **31.** $A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$

It can be shown that a nonnegative $n \times n$ matrix is irreducible if and only if $(I + A)^{n-1} > O$. In Exercises 32–35, use this criterion to determine whether the matrix A is irreducible. If A is reducible, find a permutation of its rows and columns that puts A into the block form

$$\begin{bmatrix} B & C \\ \hline O & D \end{bmatrix}$$

32. $A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$ **33.** $A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{bmatrix}$

34. $A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}$ **35.** $A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}$

36. (a) If A is the adjacency matrix of a graph G, show that A is irreducible if and only if G is connected. (A graph is **connected** if there is a path between every pair of vertices.)
    (b) Which of the graphs in Section 4.0 have an irreducible adjacency matrix? Which have a primitive adjacency matrix?

37. Let G be a bipartite graph with adjacency matrix A.
    (a) Show that A is not primitive.
    (b) Show that if λ is an eigenvalue of A, so is $-\lambda$. [*Hint:* Use Exercise 80 in Section 3.7 and partition an eigenvector for λ so that it is compatible with this partitioning of A. Use this partitioning to find an eigenvector for $-\lambda$.]

38. A graph is called **k-regular** if k edges meet at each vertex. Let G be a k-regular graph.
    (a) Show that the adjacency matrix A of G has $\lambda = k$ as an eigenvalue. [*Hint:* Adapt Theorem 4.30.]

(b) Show that if A is primitive, then the other eigenvalues are all less than k in absolute value. [*Hint:* Adapt Theorem 4.31.]

39. Explain the results of your exploration in Section 4.0 in light of Exercises 36–38 and Section 4.5.

In Exercise 40, the absolute value of a matrix $A = [a_{ij}]$ is defined to be the matrix $|A| = [|a_{ij}|]$.

40. Let A and B be $n \times n$ matrices, $\mathbf{x}$ a vector in $\mathbb{R}^n$, and c a scalar. Prove the following matrix inequalities:
    (a) $|cA| = |c| |A|$ **(b)** $|A + B| \le |A| + |B|$
    (c) $|A\mathbf{x}| \le |A| |\mathbf{x}|$ **(d)** $|AB| \le |A| |B|$

41. Prove that a 2×2 matrix $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ is reducible if and only if $a_{12} = 0$ or $a_{21} = 0$.

42. Let A be a nonnegative, irreducible matrix such that $I - A$ is invertible and $(I - A)^{-1} \ge O$. Let λ_1 and $\mathbf{v}_1$ be the Perron root and Perron eigenvector of A.
    (a) Prove that $0 < \lambda_1 < 1$. [*Hint:* Apply Exercise 22 in Section 4.3 and Theorem 4.18(b).]
    (b) Deduce from (a) that $\mathbf{v}_1 > A\mathbf{v}_1$.

Linear Recurrence Relations

In Exercises 43–46, write out the first six terms of the sequence defined by the recurrence relation with the given initial conditions.

43. $x_0 = 1, x_n = 2x_{n-1}$ for $n \ge 1$

44. $a_1 = 128, a_n = a_{n-1}/2$ for $n \ge 2$

45. $y_0 = 0, y_1 = 1, y_n = y_{n-1} - y_{n-2}$ for $n \ge 2$

46. $b_0 = 1, b_1 = 1, b_n = 2b_{n-1} + b_{n-2}$ for $n \ge 2$

In Exercises 47–52, solve the recurrence relation with the given initial conditions.

47. $x_0 = 0, x_1 = 5, x_n = 3x_{n-1} + 4x_{n-2}$ for $n \ge 2$

48. $x_0 = 0, x_1 = 1, x_n = 4x_{n-1} - 3x_{n-2}$ for $n \ge 2$

49. $y_1 = 1, y_2 = 6, y_n = 4y_{n-1} - 4y_{n-2}$ for $n \ge 3$

50. $a_0 = 4, a_1 = 1, a_n = a_{n-1} - a_{n-2}/4$ for $n \ge 2$

51. $b_0 = 0, b_1 = 1, b_n = 2b_{n-1} + 2b_{n-2}$ for $n \ge 2$

52. The recurrence relation in Exercise 45. Show that your solution agrees with the answer to Exercise 45.

53. Complete the proof of Theorem 4.38(a) by showing that if the recurrence relation $x_n = ax_{n-1} + bx_{n-2}$ has

distinct eigenvalues $\lambda_1 \neq \lambda_2$, then the solution will be of the form

$$x_n = c_1 \lambda_1^n + c_2 \lambda_2^n$$

[*Hint:* Show that the method of Example 4.40 works in general.]

54. **(a)** Show that for any choice of initial conditions $x_0 = r$ and $x_1 = s$, the scalars c_1 and c_2 can be found, as stated in Theorem 4.38(a) and (b).
   (b) If the eigenvalues λ_1 and λ_2 are distinct and the initial conditions are $\mathbf{x}_0 = 0$, $\mathbf{x}_1 = 1$, show that

$$\mathbf{x}_n = \left(\frac{1}{\lambda_1 - \lambda_2} \right)(\lambda_1^n - \lambda_2^n).$$

55. The Fibonacci recurrence $f_n = f_{n-1} + f_{n-2}$ has the associated matrix equation $\mathbf{x}_n = A\mathbf{x}_{n-1}$, where

$$\mathbf{x}_n = \begin{bmatrix} f_n \\ f_{n-1} \end{bmatrix} \text{ and } A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$$

   (a) With $f_0 = 0$ and $f_1 = 1$, use mathematical induction to prove that

$$A^n = \begin{bmatrix} f_{n+1} & f_n \\ f_n & f_{n-1} \end{bmatrix}$$

   for all $n \geq 1$.
   (b) Using part (a), prove that

$$f_{n+1}f_{n-1} - f_n^2 = (-1)^n$$

   for all $n \geq 1$. [This is called **Cassini's Identity**, after the astronomer Giovanni Domenico Cassini (1625–1712). Cassini was born in Italy but, on the invitation of Louis XIV, moved in 1669 to France, where he became director of the Paris Observatory. He became a French citizen and adopted the French version of his name: Jean-Dominique Cassini. Mathematics was one of his many interests other than astronomy. Cassini's Identity was published in 1680 in a paper submitted to the Royal Academy of Sciences in Paris.]
   (c) An 8×8 checkerboard can be dissected as shown in Figure 4.29(a) and the pieces reassembled to form the 5×13 rectangle in Figure 4.29(b).
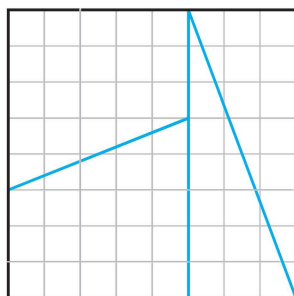
(a)
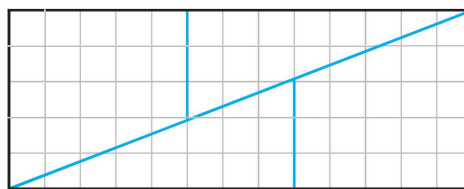
(b)

Figure 4.29

The area of the square is 64 square units, but the rectangle's area is 65 square units! Where did the extra square come from? [*Hint:* What does this have to do with the Fibonacci sequence?]

56. You have a supply of three kinds of tiles: two kinds of 1×2 tiles and one kind of 1×1 tile, as shown in Figure 4.30.

Figure 4.30

   Let t_n be the number of different ways to cover a $1 \times n$ rectangle with these tiles. For example, Figure 4.31 shows that $t_3 = 5$.

   (a) Find $t_1, \ldots, t_5$.
      (Does t_0 make any sense? If so, what is it?)
   (b) Set up a second order recurrence relation for t_n.
   (c) Using t_1 and t_2 as the initial conditions, solve the recurrence relation in part (b). Check your answer against the data in part (a).
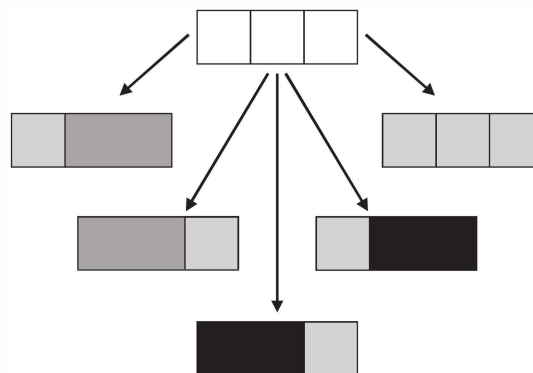
Figure 4.31
The five ways to tile a 1×3 rectangle

57. You have a supply of 1×2 dominoes with which to cover a $2 \times n$ rectangle. Let d_n be the number of different ways to cover the rectangle. For example, Figure 4.32 shows that $d_3 = 3$.

Figure 4.32

The three ways to cover a 2 × 3 rectangle with 1 × 2 dominoes

(a) Find $d_1, \ldots, d_5$.

▶ (Does d_0 make any sense? If so, what is it?)

(b) Set up a second order recurrence relation for d_n.

(c) Using d_1 and d_2 as the initial conditions, solve the recurrence relation in part (b). Check your answer against the data in part (a).
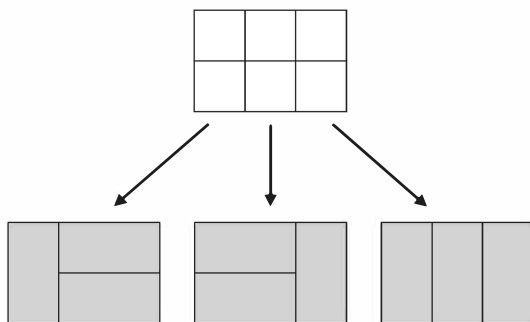
58. In Example 4.41, find eigenvectors $\mathbf{v}_1$ and $\mathbf{v}_2$ corresponding to $\lambda_1 = \dfrac{1 + \sqrt{5}}{2}$ and $\lambda_2 = \dfrac{1 - \sqrt{5}}{2}$. With $\mathbf{x}_k = \begin{bmatrix} f_k \\ f_{k-1} \end{bmatrix}$, verify formula (2) in Section 4.5. That is, show that, for some scalar c_1,

$$\lim_{k \to \infty} \frac{\mathbf{x}_k}{\lambda_1^k} = c_1 \mathbf{v}_1$$

Systems of Linear Differential Equations

$\boxed{a + bi}$ $\boxed{\frac{dy}{dx}}$

In Exercises 59–64, find the general solution to the given system of differential equations. Then find the specific solution that satisfies the initial conditions. (Consider all functions to be functions of t.)

59. $x' = x + 3y, \quad x(0) = 0$
    $y' = 2x + 2y, \quad y(0) = 5$

60. $x' = 2x - y, \quad x(0) = 1$
    $y' = -x + 2y, \quad y(0) = 1$

61. $x_1' = x_1 + x_2, \quad x_1(0) = 1$
    $x_2' = x_1 - x_2, \quad x_2(0) = 0$

62. $y_1' = y_1 - y_2, \quad y_1(0) = 1$
    $y_2' = y_1 + y_2, \quad y_2(0) = 1$

63. $x' = y - z, \quad x(0) = 1$
    $y' = x + z, \quad y(0) = 0$
    $z' = x + y, \quad z(0) = -1$

64. $x' = x + 3z, \quad x(0) = 2$
    $y' = x - 2y + z, \quad y(0) = 3$
    $z' = 3x + z, \quad z(0) = 4$

65. A scientist places two strains of bacteria, X and Y, in a petri dish. Initially, there are 400 of X and 500 of Y. The two bacteria compete for food and space but do not feed on each other. If $x = x(t)$ and $y = y(t)$ are the numbers of the strains at time t days, the growth rates of the two populations are given by the system

    $$x' = 1.2x - 0.2y$$
    $$y' = -0.2x + 1.5y$$

    (a) Determine what happens to these two populations by solving the system of differential equations.

    (b) Explore the effect of changing the initial populations by letting $x(0) = a$ and $y(0) = b$. Describe what happens to the populations in terms of a and b.

66. Two species, X and Y, live in a *symbiotic* relationship. That is, neither species can survive on its own and each depends on the other for its survival. Initially, there are 15 of X and 10 of Y. If $x = x(t)$ and $y = y(t)$ are the sizes of the populations at time t months, the growth rates of the two populations are given by the system

    $$x' = -0.8x + 0.4y$$
    $$y' = 0.4x - 0.2y$$

    Determine what happens to these two populations.

In Exercises 67 and 68, species X preys on species Y. The sizes of the populations are represented by $x = x(t)$ and $y = y(t)$. The growth rate of each population is governed by the system of differential equations $\mathbf{x}' = A\mathbf{x} + \mathbf{b}$, where $\mathbf{x} = \begin{bmatrix} x \\ y \end{bmatrix}$ and $\mathbf{b}$ is a constant vector. Determine what happens to the two populations for the given A and $\mathbf{b}$ and initial conditions $\mathbf{x}(0)$. (First show that there are constants a and b such that the substitutions $x = u + a$ and $y = v + b$ convert the system into an equivalent one with no constant terms.)

67. $A = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} -30 \\ -10 \end{bmatrix}, \mathbf{x}(0) = \begin{bmatrix} 20 \\ 30 \end{bmatrix}$

68. $A = \begin{bmatrix} -1 & 1 \\ -1 & -1 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} 0 \\ 40 \end{bmatrix}, \mathbf{x}(0) = \begin{bmatrix} 10 \\ 30 \end{bmatrix}$

69. Let $x = x(t)$ be a twice-differentiable function and consider the *second order differential equation*

    $$x'' + ax' + bx = 0 \qquad (11)$$

    (a) Show that the change of variables $y = x'$ and $z = x$ allows Equation (11) to be written as a system of two linear differential equations in y and z.

(b) Show that the characteristic equation of the system in part (a) is $\lambda^2 + a\lambda + b = 0$.

70. Show that there is a change of variables that converts the nth *order differential equation*

$$x^{(n)} + a_{n-1}x^{(n-1)} + \cdots + a_1x' + a_0 = 0$$

into a system of n linear differential equations whose coefficient matrix is the companion matrix $C(p)$ of the polynomial $p(\lambda) = \lambda^n + a_{n-1}\lambda^{n-1} + \cdots + a_1\lambda + a_0$. [The notation $x^{(k)}$ denotes the kth derivative of x. See Exercises 26–32 in Section 4.3 for the definition of a companion matrix.]

In Exercises 71 and 72, use Exercise 69 to find the general solution of the given equation.

71. $x'' - 5x' + 6x = 0$ 72. $x'' + 4x' + 3x = 0$

In Exercises 73–76, solve the system of differential equations in the given exercise using Theorem 4.41.

73. Exercise 59   74. Exercise 60

75. Exercise 63   76. Exercise 64

Discrete Linear Dynamical Systems

In Exercises 77–84, consider the dynamical system $\mathbf{x}_{k+1} = A\mathbf{x}_k$.

(a) *Compute and plot* $\mathbf{x}_0, \mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3$ *for* $\mathbf{x}_0 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

(b) *Compute and plot* $\mathbf{x}_0, \mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3$ *for* $\mathbf{x}_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

(c) *Using eigenvalues and eigenvectors, classify the origin as an attractor, repeller, saddle point, or none of these.*

(d) *Sketch several typical trajectories of the system.*

77. $A = \begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix}$ 78. $A = \begin{bmatrix} 0.5 & -0.5 \\ 0 & 0.5 \end{bmatrix}$

79. $A = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$ 80. $A = \begin{bmatrix} -4 & 2 \\ 1 & -3 \end{bmatrix}$

81. $A = \begin{bmatrix} 1.5 & -1 \\ -1 & 0 \end{bmatrix}$ 82. $A = \begin{bmatrix} 0.1 & 0.9 \\ 0.5 & 0.5 \end{bmatrix}$

83. $A = \begin{bmatrix} 0.2 & 0.4 \\ -0.2 & 0.8 \end{bmatrix}$ 84. $A = \begin{bmatrix} 0 & -1.5 \\ 1.2 & 3.6 \end{bmatrix}$

In Exercises 85–88, the given matrix is of the form $A = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$. *In each case, A can be factored as the product of a scaling matrix and a rotation matrix. Find the scaling factor r and the angle θ of rotation. Sketch the first four points of the trajectory for the dynamical system* $\mathbf{x}_{k+1} = A\mathbf{x}_k$ *with* $\mathbf{x}_0 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ *and classify the origin as a spiral attractor, spiral repeller, or orbital center.*

85. $A = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$ 86. $A = \begin{bmatrix} 0 & 0.5 \\ -0.5 & 0 \end{bmatrix}$

87. $A = \begin{bmatrix} 1 & \sqrt{3} \\ -\sqrt{3} & 1 \end{bmatrix}$ 88. $A = \begin{bmatrix} -\sqrt{3}/2 & -1/2 \\ 1/2 & -\sqrt{3}/2 \end{bmatrix}$

In Exercises 89–92, find an invertible matrix P and a matrix C of the form $C = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$ *such that $A = PCP^{-1}$. Sketch the first six points of the trajectory for the dynamical system $\mathbf{x}_{k+1} = A\mathbf{x}_k$ with $\mathbf{x}_0 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and classify the origin as a spiral attractor, spiral repeller, or orbital center.*

89. $A = \begin{bmatrix} 0.1 & -0.2 \\ 0.1 & 0.3 \end{bmatrix}$ 90. $A = \begin{bmatrix} 2 & 1 \\ -2 & 0 \end{bmatrix}$

91. $A = \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix}$ 92. $A = \begin{bmatrix} 0 & -1 \\ 1 & \sqrt{3} \end{bmatrix}$

Chapter Review

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Review Questions

1. Mark each of the following statements true or false:
   (a) For all square matrices A, $\det(-A) = -\det A$.
   (b) If A and B are $n \times n$ matrices, then $\det(AB) = \det(BA)$.
   (c) If A and B are $n \times n$ matrices whose columns are the same but in different orders, then $\det B = -\det A$.
   (d) If A is invertible, then $\det(A^{-1}) = \det A^T$.
   (e) If 0 is the only eigenvalue of a square matrix A, then A is the zero matrix.
   (f) Two eigenvectors corresponding to the same eigenvalue must be linearly dependent.
   (g) If an $n \times n$ matrix has n distinct eigenvalues, then it must be diagonalizable.
   (h) If an $n \times n$ matrix is diagonalizable, then it must have n distinct eigenvalues.
   (i) Similar matrices have the same eigenvectors.
   (j) If A and B are two $n \times n$ matrices with the same reduced row echelon form, then A is similar to B.

2. Let $A = \begin{bmatrix} 1 & 3 & 5 \\ 3 & 5 & 7 \\ 7 & 9 & 11 \end{bmatrix}$.

   (a) Compute $\det A$ by cofactor expansion along any row or column.
   (b) Compute $\det A$ by first reducing A to triangular form.

3. If $\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = 3$, find $\begin{vmatrix} 3d & 2e-4f & f \\ 3a & 2b-4c & c \\ 3g & 2h-4i & i \end{vmatrix}$.

4. Let A and B be 4×4 matrices with $\det A = 2$ and $\det B = -\frac{1}{4}$. Find $\det C$ for the indicated matrix C:
   (a) $C = (AB)^{-1}$ (b) $C = A^2B(3A^T)$

5. If A is a skew-symmetric $n \times n$ matrix and n is odd, prove that $\det A = 0$.

6. Find all values of k for which $\begin{vmatrix} 1 & -1 & 2 \\ 1 & 1 & k \\ 2 & 4 & k^2 \end{vmatrix} = 0$.

In Questions 7 and 8, show that $\mathbf{x}$ is an eigenvector of A and find the corresponding eigenvalue.

7. $\mathbf{x} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, $A = \begin{bmatrix} 3 & 1 \\ 4 & 3 \end{bmatrix}$

8. $\mathbf{x} = \begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix}$, $A = \begin{bmatrix} 13 & -60 & -45 \\ -5 & 18 & 15 \\ 10 & -40 & -32 \end{bmatrix}$

9. Let $A = \begin{bmatrix} -5 & -6 & 3 \\ 3 & 4 & -3 \\ 0 & 0 & -2 \end{bmatrix}$.

   (a) Find the characteristic polynomial of A.
   (b) Find all of the eigenvalues of A.
   (c) Find a basis for each of the eigenspaces of A.
   (d) Determine whether A is diagonalizable. If A is not diagonalizable, explain why not. If A is diagonalizable, find an invertible matrix P and a diagonal matrix D such that $P^{-1}AP = D$.

10. If A is a 3×3 diagonalizable matrix with eigenvalues -2, 3, and 4, find $\det A$.

11. If A is a 2×2 matrix with eigenvalues $\lambda_1 = \frac{1}{2}, \lambda_2 = -1$, and corresponding eigenvectors $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$, find $A^{-5}\begin{bmatrix} 3 \\ 7 \end{bmatrix}$.

12. If A is a diagonalizable matrix and all of its eigenvalues satisfy $|\lambda| < 1$, prove that A^n approaches the zero matrix as n gets large.

In Questions 13–15, determine, with reasons, whether A is similar to B. If $A \sim B$, give an invertible matrix P such that $P^{-1}AP = B$.

13. $A = \begin{bmatrix} 4 & 2 \\ 3 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 2 \\ 3 & 2 \end{bmatrix}$

14. $A = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$, $B = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}$

15. $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

16. Let $A = \begin{bmatrix} 2 & k \\ 1 & 0 \end{bmatrix}$. Find all values of k for which:
    (a) A has eigenvalues 3 and -1.
    (b) A has an eigenvalue with algebraic multiplicity 2.
    (c) A has no real eigenvalues.

17. If $A^3 = A$, what are the possible eigenvalues of A?

18. If a square matrix A has two equal rows, why must A have 0 as one of its eigenvalues?

19. If $\mathbf{x}$ is an eigenvector of A with eigenvalue $\lambda = 3$, show that $\mathbf{x}$ is also an eigenvector of $A^2 - 5A + 2I$. What is the corresponding eigenvalue?

20. If A is similar to B with $P^{-1}AP = B$ and $\mathbf{x}$ is an eigenvector of A, show that $P^{-1}\mathbf{x}$ is an eigenvector of B.