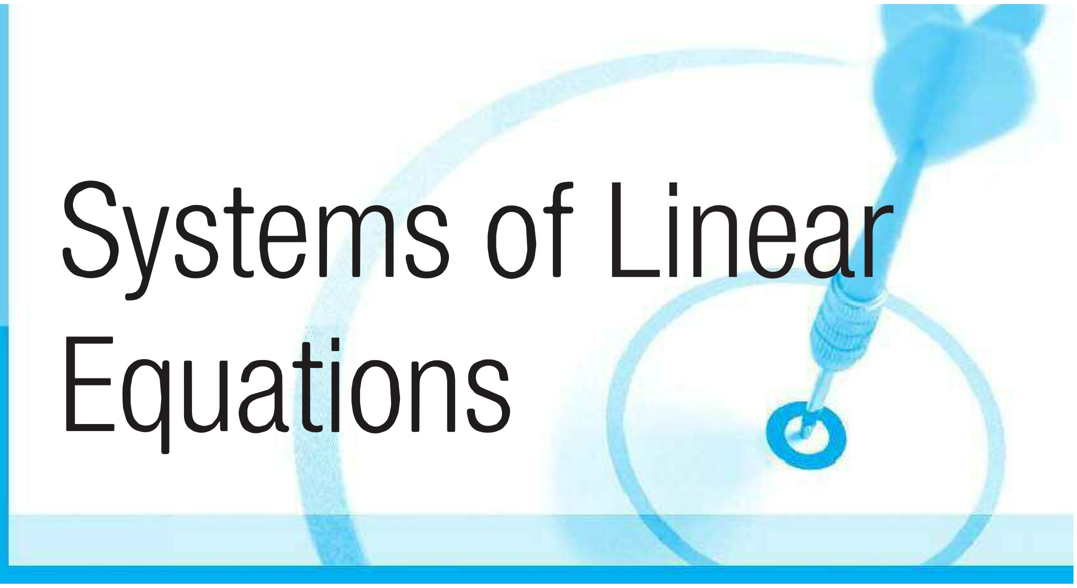


2 Systems of Linear Equations

The world was full of equations . . . .
There must be an answer for everything,
if only you knew how to set forth
the questions.

—Anne Tyler
The Accidental Tourist
Alfred A. Knopf, 1985, p. 235

2.0 Introduction: Triviality

The word *trivial* is derived from the Latin root *tri* ("three") and the Latin word *via* ("road"). Thus, speaking literally, a triviality is a place where three roads meet. This common meeting point gives rise to the other, more familiar meaning of *trivial*—commonplace, ordinary, or insignificant. In medieval universities, the *trivium* consisted of the three "common" subjects (grammar, rhetoric, and logic) that were taught before the *quadrivium* (arithmetic, geometry, music, and astronomy). The "three roads" that made up the trivium were the beginning of the liberal arts.

In this section, we begin to examine systems of linear equations. The same system of equations can be viewed in three different, yet equally important, ways—these will be our three roads, all leading to the same solution. You will need to get used to this threefold way of viewing systems of linear equations, so that it becomes commonplace (trivial!) for you.

The system of equations we are going to consider is

$$\begin{aligned} 2x + y &= 8 \\ x - 3y &= -3 \end{aligned}$$

Problem 1 Draw the two lines represented by these equations. What is their point of intersection?

Problem 2 Consider the vectors $\mathbf{u} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 1 \\ -3 \end{bmatrix}$. Draw the coordinate grid determined by $\mathbf{u}$ and $\mathbf{v}$. [*Hint:* Lightly draw the standard coordinate grid first and use it as an aid in drawing the new one.]

Problem 3 On the u-v grid, find the coordinates of $\mathbf{w} = \begin{bmatrix} 8 \\ -3 \end{bmatrix}$.

Problem 4 Another way to state Problem 3 is to ask for the coefficients x and y for which $x\mathbf{u} + y\mathbf{v} = \mathbf{w}$. Write out the two equations to which this vector equation is equivalent (one for each component). What do you observe?

Problem 5 Return now to the lines you drew for Problem 1. We will refer to the line whose equation is $2x + y = 8$ as line 1 and the line whose equation is $x - 3y = -3$ as line 2. Plot the point $(0, 0)$ on your graph from Problem 1 and label it P_0. Draw a

Table 2.1

Point	x	y
P_0	0	0
P_1		
P_2		
P_3		
P_4		
P_5		
P_6		

horizontal line segment from P_0 to line 1 and label this new point P_1. Next draw a *vertical* line segment from P_1 to line 2 and label this point P_2. Now draw a *horizontal* line segment from P_2 to line 1, obtaining point P_3. Continue in this fashion, drawing vertical segments to line 2 followed by horizontal segments to line 1. What appears to be happening?

Problem 6 Using a calculator with two-decimal-place accuracy, find the (approximate) coordinates of the points $P_1, P_2, P_3, \ldots, P_6$. (You will find it helpful to first solve the first equation for x in terms of y and the second equation for y in terms of x.) Record your results in Table 2.1, writing the x- and y-coordinates of each point separately.

The results of these problems show that the task of "solving" a system of linear equations may be viewed in several ways. Repeat the process described in the problems with the following systems of equations:

$$\text{(a)}\ \begin{aligned} 4x - 2y &= 0 \\ x + 2y &= 5 \end{aligned} \qquad \text{(b)}\ \begin{aligned} 3x + 2y &= 9 \\ x + 3y &= 10 \end{aligned} \qquad \text{(c)}\ \begin{aligned} x + y &= 5 \\ x - y &= 3 \end{aligned} \qquad \text{(d)}\ \begin{aligned} x + 2y &= 4 \\ 2x - y &= 3 \end{aligned}$$

Are all of your observations from Problems 1–6 still valid for these examples? Note any similarities or differences. In this chapter, we will explore these ideas in more detail.

2.1 Introduction to Systems of Linear Equations

Recall that the general equation of a line in $\mathbb{R}^2$ is of the form

$$ax + by = c$$

and that the general equation of a plane in $\mathbb{R}^3$ is of the form

$$ax + by + cz = d$$

Equations of this form are called ***linear equations.***

Definition A ***linear equation*** in the n variables $x_1, x_2, \ldots, x_n$ is an equation that can be written in the form

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = b$$

where the ***coefficients*** $a_1, a_2, \ldots, a_n$ and the ***constant term*** b are constants.

Example 2.1

The following equations are linear:

$$3x - 4y = -1 \qquad r - \tfrac{1}{2}s - \tfrac{15}{3}t = 9 \qquad x_1 + 5x_2 = 3 - x_3 + 2x_4$$

$$\sqrt{2}x + \frac{\pi}{4}y - \left(\sin\frac{\pi}{5}\right)z = 1 \qquad 3.2x_1 - 0.01x_2 = 4.6$$

Observe that the third equation is linear because it can be rewritten in the form $x_1 + 5x_2 + x_3 - 2x_4 = 3$. It is also important to note that, although in these examples (and in most applications) the coefficients and constant terms are real numbers, in some examples and applications they will be complex numbers or members of $\mathbb{Z}_p$ for some prime number p.

The following equations are not linear:

$$xy + 2z = 1 \qquad x_1^2 - x_2^3 = 3 \qquad \frac{x}{y} + z = 2$$

$$\sqrt{2x} + \frac{\pi}{4}y - \sin\left(\frac{\pi}{5}z\right) = 1 \qquad \sin x_1 - 3x_2 + 2^{x_3} = 0$$

Thus, linear equations do not contain products, reciprocals, or other functions of the variables; the variables occur only to the first power and are multiplied only by constants. Pay particular attention to the fourth example in each list: Why is it that the fourth equation in the first list is linear but the fourth equation in the second list is not?

A ***solution*** of a linear equation $a_1x_1 + a_2x_2 + \cdots + a_nx_n = b$ is a vector $[s_1, s_2, \ldots, s_n]$ whose components satisfy the equation when we substitute $x_1 = s_1$, $x_2 = s_2, \ldots, x_n = s_n$.

Example 2.2

(a) [5, 4] is a solution of $3x - 4y = -1$ because, when we substitute $x = 5$ and $y = 4$, the equation is satisfied: $3(5) - 4(4) = -1$. [1, 1] is another solution. In general, the solutions simply correspond to the points on the line determined by the given equation. Thus, setting $x = t$ and solving for y, we see that the complete set of solutions can be written in the parametric form $[t, \frac{1}{4} + \frac{3}{4}t]$. (We could also set y equal to some parameter—say, s—and solve for x instead; the two parametric solutions would look different but would be equivalent. Try this.)

(b) The linear equation $x_1 - x_2 + 2x_3 = 3$ has [3, 0, 0], [0, 1, 2], and [6, 1, −1] as specific solutions. The complete set of solutions corresponds to the set of points in the plane determined by the given equation. If we set $x_2 = s$ and $x_3 = t$, then a parametric solution is given by $[3 + s - 2t, s, t]$. (Which values of s and t produce the three specific solutions above?)

A ***system of linear equations*** is a finite set of linear equations, each with the same variables. A ***solution*** of a system of linear equations is a vector that is *simultaneously* a solution of each equation in the system. The ***solution set*** of a system of linear equations is the set of *all* solutions of the system. We will refer to the process of finding the solution set of a system of linear equations as "solving the system."

Example 2.3

The system

$$\begin{aligned} 2x - y &= 3 \\ x + 3y &= 5 \end{aligned}$$

has [2, 1] as a solution, since it is a solution of both equations. On the other hand, [1, −1] is not a solution of the system, since it satisfies only the first equation.

Example 2.4

Solve the following systems of linear equations:

(a) $\begin{aligned} x - y &= 1 \\ x + y &= 3 \end{aligned}$ (b) $\begin{aligned} x - y &= 2 \\ 2x - 2y &= 4 \end{aligned}$ (c) $\begin{aligned} x - y &= 1 \\ x - y &= 3 \end{aligned}$

Solution

(a) Adding the two equations together gives $2x = 4$, so $x = 2$, from which we find that $y = 1$. A quick check confirms that $[2, 1]$ is indeed a solution of both equations. That this is the *only* solution can be seen by observing that this solution corresponds to the (unique) point of intersection $(2, 1)$ of the lines with equations $x - y = 1$ and $x + y = 3$, as shown in Figure 2.1(a). Thus, $[2, 1]$ is a *unique solution.*

(b) The second equation in this system is just twice the first, so the solutions are the solutions of the first equation alone—namely, the points on the line $x - y = 2$. These can be represented parametrically as $[2 + t, t]$. Thus, this system has *infinitely many solutions* [Figure 2.1(b)].

(c) Two numbers x and y cannot simultaneously have a difference of 1 and 3. Hence, this system has *no solutions.* (A more algebraic approach might be to subtract the second equation from the first, yielding the absurd conclusion $0 = -2$.) As Figure 2.1(c) shows, the lines for the equations are parallel in this case.

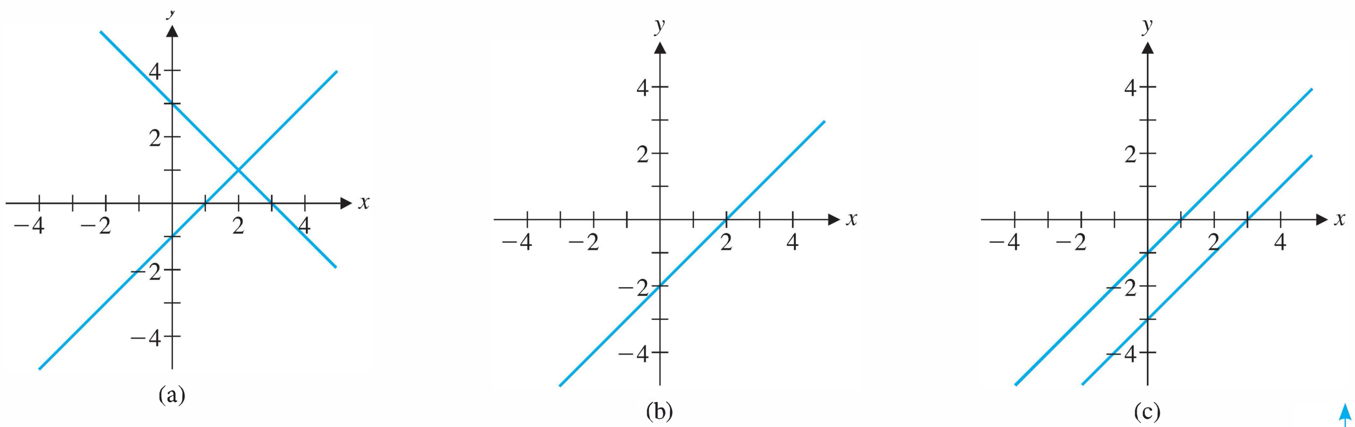


Figure 2.1

A system of linear equations is called ***consistent*** if it has at least one solution. A system with no solutions is called ***inconsistent.*** Even though they are small, the three systems in Example 2.4 illustrate the only three possibilities for the number of solutions of a system of linear equations with real coefficients. We will prove later that these same three possibilities hold for *any* system of linear equations over the real numbers.

A system of linear equations with real coefficients has either

(a) a unique solution (a consistent system) or
(b) infinitely many solutions (a consistent system) or
(c) no solutions (an inconsistent system).

Solving a System of Linear Equations

Two linear systems are called ***equivalent*** if they have the same solution sets. For example,

$$\begin{aligned} x - y &= 1 \\ x + y &= 3 \end{aligned} \qquad \text{and} \qquad \begin{aligned} x - y &= 1 \\ y &= 1 \end{aligned}$$

are equivalent, since they both have the unique solution $[2, 1]$. (Check this.)

Our approach to solving a system of linear equations is to transform the given system into an equivalent one that is easier to solve. The triangular pattern of the second example above (in which the second equation has one less variable than the first) is what we will aim for.

Example 2.5

Solve the system

$$\begin{aligned} x - y - z &= 2 \\ y + 3z &= 5 \\ 5z &= 10 \end{aligned}$$

Solution Starting from the last equation and working backward, we find successively that $z = 2$, $y = 5 - 3(2) = -1$, and $x = 2 + (-1) + 2 = 3$. So the unique solution is $[3, -1, 2]$.

The procedure used to solve Example 2.5 is called ***back substitution.***

We now turn to the general strategy for transforming a given system into an equivalent one that can be solved easily by back substitution. This process will be described in greater detail in the next section; for now, we will simply observe it in action in a single example.

Example 2.6

Solve the system

$$\begin{aligned} x - y - z &= 2 \\ 3x - 3y + 2z &= 16 \\ 2x - y + z &= 9 \end{aligned}$$

Solution To transform this system into one that exhibits the triangular structure of Example 2.5, we first need to eliminate the variable x from Equations 2 and 3. Observe that subtracting appropriate multiples of equation 1 from Equations 2 and 3 will do the trick. Next, observe that we are operating on the coefficients, not on the variables, so we can save ourselves some writing if we record the coefficients and constant terms in the *matrix*

$$\left[\begin{array}{rrr|r} 1 & -1 & -1 & 2 \\ 3 & -3 & 2 & 16 \\ 2 & -1 & 1 & 9 \end{array}\right]$$

The word *matrix* is derived from the Latin word *mater,* meaning "mother." When the suffix *-ix* is added, the meaning becomes "womb." Just as a womb surrounds a fetus, the brackets of a matrix surround its entries, and just as the womb gives rise to a baby, a matrix gives rise to certain types of functions called *linear transformations.* A matrix with m rows and n columns is called an $m \times n$ matrix (pronounced "m by n"). The plural of *matrix* is *matrices,* not "matrixes."

where the first three columns contain the coefficients of the variables in order, the final column contains the constant terms, and the vertical bar serves to remind us of the equal signs in the equations. This matrix is called the ***augmented matrix*** of the system.

There are various ways to convert the given system into one with the triangular pattern we are after. The steps we will use here are closest in spirit to the more general method described in the next section. We will perform the sequence of operations on the given system and simultaneously on the corresponding augmented matrix. We begin by eliminating x from Equations 2 and 3.

$$\begin{aligned} x - y - z &= 2 \\ 3x - 3y + 2z &= 16 \\ 2x - y + z &= 9 \end{aligned} \qquad \left[\begin{array}{rrr|r} 1 & -1 & -1 & 2 \\ 3 & -3 & 2 & 16 \\ 2 & -1 & 1 & 9 \end{array}\right]$$

Subtract 3 times the first equation from the second equation:

$$\begin{aligned} x - y - z &= 2 \\ 5z &= 10 \\ 2x - y + z &= 9 \end{aligned}$$

Subtract 3 times the first row from the second row:

$$\left[\begin{array}{rrr|r} 1 & -1 & -1 & 2 \\ 0 & 0 & 5 & 10 \\ 2 & -1 & 1 & 9 \end{array}\right]$$

Subtract 2 times the first equation from the third equation:

$$\begin{aligned} x - y - z &= 2 \\ 5z &= 10 \\ y + 3z &= 5 \end{aligned}$$

Subtract 2 times the first row from the third row:

$$\left[\begin{array}{rrr|r} 1 & -1 & -1 & 2 \\ 0 & 0 & 5 & 10 \\ 0 & 1 & 3 & 5 \end{array}\right]$$

Interchange Equations 2 and 3:

$$\begin{aligned} x - y - z &= 2 \\ y + 3z &= 5 \\ 5z &= 10 \end{aligned}$$

Interchange rows 2 and 3:

$$\left[\begin{array}{rrr|r} 1 & -1 & -1 & 2 \\ 0 & 1 & 3 & 5 \\ 0 & 0 & 5 & 10 \end{array}\right]$$

This is the same system that we solved using back substitution in Example 2.5, where we found that the solution was $[3, -1, 2]$. This is therefore also the solution to the system given in this example. Why? The calculations above show that *any solution of the given system is also a solution of the final one*. But since the steps we just performed are *reversible,* we could recover the original system, starting with the final system. (How?) So *any solution of the final system is also a solution of the given one*. Thus, the systems are equivalent (as are all of the ones obtained in the intermediate steps above). Moreover, we might just as well work with matrices instead of equations, since it is a simple matter to reinsert the variables before proceeding with the back substitution. (Working with matrices is the subject of the next section.)

Remark Calculators with matrix capabilities and computer algebra systems can facilitate solving systems of linear equations, particularly when the systems are large or have coefficients that are not "nice," as is often the case in real-life applications. As always, though, you should do as many examples as you can with pencil and paper until you are comfortable with the techniques. Even if a calculator or CAS is called for, think about *how* you would do the calculations manually before doing anything. After you have an answer, be sure to think about whether it is reasonable.

Do not be misled into thinking that technology will always give you the answer faster or more easily than calculating by hand. Sometimes it may not give you the answer at all! Roundoff errors associated with the floating-point arithmetic used by calculators and computers can cause serious problems and lead to wildly wrong answers to some problems. See Exploration: Lies My Computer Told Me for a glimpse of the problem. (You've been warned!)

Exercises 2.1

In Exercises 1–6, determine which equations are linear equations in the variables x, y, and z. If any equation is not linear, explain why not.

1. $x - \pi y + \sqrt[3]{5}z = 0$

2. $x^2 + y^2 + z^2 = 1$

3. $x^{-1} + 7y + z = \sin\left(\frac{\pi}{9}\right)$

4. $2x - xy - 5z = 0$

5. $3\cos x - 4y + z = \sqrt{3}$

6. $(\cos 3)x - 4y + z = \sqrt{3}$

In Exercises 7–10, find a linear equation that has the same solution set as the given equation (possibly with some restrictions on the variables).

7. $2x + y = 7 - 3y$

8. $\dfrac{x^2 - y^2}{x - y} = 1$

9. $\dfrac{1}{x} + \dfrac{1}{y} = \dfrac{4}{xy}$

10. $\log_{10} x - \log_{10} y = 2$

In Exercises 11–14, find the solution set of each equation.

11. $3x - 6y = 0$

12. $2x_1 + 3x_2 = 5$

13. $x + 2y + 3z = 4$

14. $4x_1 + 3x_2 + 2x_3 = 1$

In Exercises 15–18, draw graphs corresponding to the given linear systems. Determine geometrically whether each system has a unique solution, infinitely many solutions, or no solution. Then solve each system algebraically to confirm your answer.

15. $\begin{aligned} x + y &= 0 \\ 2x + y &= 3 \end{aligned}$

16. $\begin{aligned} x - 2y &= 7 \\ 3x + y &= 7 \end{aligned}$

17. $\begin{aligned} 3x - 6y &= 3 \\ -x + 2y &= 1 \end{aligned}$

18. $\begin{aligned} 0.10x - 0.05y &= 0.20 \\ -0.06x + 0.03y &= -0.12 \end{aligned}$

In Exercises 19–24, solve the given system by back substitution.

19. $\begin{aligned} x - 2y &= 1 \\ y &= 3 \end{aligned}$

20. $\begin{aligned} 2u - 3v &= 5 \\ 2v &= 6 \end{aligned}$

21. $\begin{aligned} x - y + z &= 0 \\ 2y - z &= 1 \\ 3z &= -1 \end{aligned}$

22. $\begin{aligned} x_1 + 2x_2 + 3x_3 &= 0 \\ -5x_2 + 2x_3 &= 0 \\ 4x_3 &= 0 \end{aligned}$

23. $\begin{aligned} x_1 + x_2 - x_3 - x_4 &= 1 \\ x_2 + x_3 + x_4 &= 0 \\ x_3 - x_4 &= 0 \\ x_4 &= 1 \end{aligned}$

24. $\begin{aligned} x - 3y + z &= 5 \\ y - 2z &= -1 \end{aligned}$

The systems in Exercises 25 and 26 exhibit a "lower triangular" pattern that makes them easy to solve by forward substitution. (We will encounter forward substitution again in Chapter 3.) Solve these systems.

25. $\begin{aligned} x &= 2 \\ 2x + y &= -3 \\ -3x - 4y + z &= -10 \end{aligned}$

26. $\begin{aligned} x_1 &= -1 \\ -\tfrac{1}{2}x_1 + x_2 &= 5 \\ \tfrac{3}{2}x_1 + 2x_2 + x_3 &= 7 \end{aligned}$

Find the augmented matrices of the linear systems in Exercises 27–30.

27. $\begin{aligned} x - y &= 0 \\ 2x + y &= 3 \end{aligned}$

28. $\begin{aligned} 2x_1 + 3x_2 - x_3 &= 1 \\ x_1 + x_3 &= 0 \\ -x_1 + 2x_2 - 2x_3 &= 0 \end{aligned}$

29. $\begin{aligned} x + 5y &= -1 \\ -x + y &= -5 \\ 2x + 4y &= 4 \end{aligned}$

30. $\begin{aligned} a - 2b + d &= 2 \\ -a + b - c - 3d &= 1 \end{aligned}$

In Exercises 31 and 32, find a system of linear equations that has the given matrix as its augmented matrix.

31. $\left[\begin{array}{ccc|c} 0 & 1 & 1 & 1 \\ 1 & -1 & 0 & 1 \\ 2 & -1 & 1 & 1 \end{array}\right]$

32. $\left[\begin{array}{ccccc|c} 1 & -1 & 0 & 3 & 1 & 2 \\ 1 & 1 & 2 & 1 & -1 & 4 \\ 0 & 1 & 0 & 2 & 3 & 0 \end{array}\right]$

For Exercises 33–38, solve the linear systems in the given exercises.

33. Exercise 27

34. Exercise 28

35. Exercise 29

36. Exercise 30

37. Exercise 31

38. Exercise 32

39. (a) Find a system of two linear equations in the variables x and y whose solution set is given by the parametric equations $x = t$ and $y = 3 - 2t$.

(b) Find another parametric solution to the system in part (a) in which the parameter is s and $y = s$.

40. (a) Find a system of two linear equations in the variables x_1, x_2, and x_3 whose solution set is given by the parametric equations $x_1 = t$, $x_2 = 1 + t$, and $x_3 = 2 - t$.

(b) Find another parametric solution to the system in part (a) in which the parameter is s and $x_3 = s$.

In Exercises 41–44, the systems of equations are nonlinear. Find substitutions (changes of variables) that convert each system into a linear system and use this linear system to help solve the given system.

41. $\dfrac{2}{x} + \dfrac{3}{y} = 0$
$\dfrac{3}{x} + \dfrac{4}{y} = 1$

42. $x^2 + 2y^2 = 6$
$x^2 - y^2 = 3$

43. $\tan x - 2 \sin y = 2$
$\tan x - \sin y + \cos z = 2$
$\sin y - \cos z = -1$

44. $-2^a + 2(3^b) = 1$
$3(2^a) - 4(3^b) = 1$

2.2 Direct Methods for Solving Linear Systems

In this section, we will look at a general, systematic procedure for solving a system of linear equations. This procedure is based on the idea of reducing the augmented matrix of the given system to a form that can then be solved by back substitution. The method is *direct* in the sense that it leads directly to the solution (if one exists) in a finite number of steps. In Section 2.5, we will consider some *indirect* methods that work in a completely different way.

Matrices and Echelon Form

There are two important matrices associated with a linear system. The ***coefficient matrix*** contains the coefficients of the variables, and the ***augmented matrix*** (which we have already encountered) is the coefficient matrix augmented by an extra column containing the constant terms.

For the system

$$\begin{aligned} 2x + y - z &= 3 \\ x + 5z &= 1 \\ -x + 3y - 2z &= 0 \end{aligned}$$

the coefficient matrix is

$$\begin{bmatrix} 2 & 1 & -1 \\ 1 & 0 & 5 \\ -1 & 3 & -2 \end{bmatrix}$$

and the augmented matrix is

$$\left[\begin{array}{ccc|c} 2 & 1 & -1 & 3 \\ 1 & 0 & 5 & 1 \\ -1 & 3 & -2 & 0 \end{array}\right]$$

Note that if a variable is missing (as y is in the second equation), its coefficient 0 is entered in the appropriate position in the matrix. If we denote the coefficient matrix of a linear system by A and the column vector of constant terms by $\mathbf{b}$, then the form of the augmented matrix is $[A \mid \mathbf{b}]$.

In solving a linear system, it will not always be possible to reduce the coefficient matrix to triangular form, as we did in Example 2.6. However, we can always achieve a staircase pattern in the nonzero entries of the final matrix.

The word echelon comes from the Latin word *scala*, meaning "ladder" or "stairs." The French word for "ladder," *échelle*, is also derived from this Latin base. A matrix in echelon form exhibits a staircase pattern.

Definition A matrix is in ***row echelon form*** if it satisfies the following properties:

1. Any rows consisting entirely of zeros are at the bottom.
2. In each nonzero row, the first nonzero entry (called the ***leading entry***) is in a column to the left of any leading entries below it.

Note that these properties guarantee that the leading entries form a staircase pattern. In particular, in any column containing a leading entry, all entries below the leading entry are zero, as the following examples illustrate.

Example 2.7

The following matrices are in row echelon form:

$$\begin{bmatrix} 2 & 4 & 1 \\ 0 & -1 & 2 \\ 0 & 0 & 0 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 5 \\ 0 & 0 & 4 \end{bmatrix} \quad \begin{bmatrix} 1 & 1 & 2 & 1 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \begin{bmatrix} 0 & 2 & 0 & 1 & -1 & 3 \\ 0 & 0 & -1 & 1 & 2 & 2 \\ 0 & 0 & 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 5 \end{bmatrix}$$

If a matrix in row echelon form is actually the augmented matrix of a linear system, the system is quite easy to solve by back substitution alone.

Example 2.8

Assuming that each of the matrices in Example 2.7 is an augmented matrix, write out the corresponding systems of linear equations and solve them.

Solution We first remind ourselves that the last column in an augmented matrix is the vector of constant terms. The first matrix then corresponds to the system

$$\begin{aligned} 2x_1 + 4x_2 &= 1 \\ -x_2 &= 2 \end{aligned}$$

(Notice that we have dropped the last equation $0 = 0$, or $0x_1 + 0x_2 = 0$, which is clearly satisfied for any values of x_1 and x_2.) Back substitution gives $x_2 = -2$ and then $2x_1 = 1 - 4(-2) = 9$, so $x_1 = \frac{9}{2}$. The solution is $[\frac{9}{2}, -2]$.

The second matrix has the corresponding system

$$\begin{aligned} x_1 \quad &= 1 \\ x_2 &= 5 \\ 0 &= 4 \end{aligned}$$

The last equation represents $0x_1 + 0x_2 = 4$, which clearly has no solutions. Therefore, the system has no solutions. Similarly, the system corresponding to the fourth matrix has no solutions. For the system corresponding to the third matrix, we have

$$\begin{aligned} x_1 + x_2 + 2x_3 &= 1 \\ x_3 &= 3 \end{aligned}$$

so $x_1 = 1 - 2(3) - x_2 = -5 - x_2$. There are infinitely many solutions, since we may assign x_2 any value t to get the parametric solution $[-5 - t, t, 3]$.

Elementary Row Operations

We now describe the procedure by which any matrix can be reduced to a matrix in row echelon form. The allowable operations, called ***elementary row operations,*** correspond to the operations that can be performed on a system of linear equations to transform it into an equivalent system.

Definition The following ***elementary row operations*** can be performed on a matrix:

1. Interchange two rows.
2. Multiply a row by a nonzero constant.
3. Add a multiple of a row to another row.

Observe that dividing a row by a nonzero constant is implied in the above definition, since, for example, dividing a row by 2 is the same as multiplying it by $\frac{1}{2}$. Similarly, subtracting a multiple of one row from another row is the same as adding a negative multiple of one row to another row.

We will use the following shorthand notation for the three elementary row operations:

1. $R_i \leftrightarrow R_j$ means interchange rows i and j.
2. kR_i means multiply row i by k.
3. $R_i + kR_j$ means add k times row j to row i (and replace row i with the result).

The process of applying elementary row operations to bring a matrix into row echelon form, called ***row reduction,*** is used to reduce a matrix to echelon form.

Example 2.9 Reduce the following matrix to echelon form:

$$\begin{bmatrix} 1 & 2 & -4 & -4 & 5 \\ 2 & 4 & 0 & 0 & 2 \\ 2 & 3 & 2 & 1 & 5 \\ -1 & 1 & 3 & 6 & 5 \end{bmatrix}$$

Solution We work column by column, from left to right and from top to bottom. The strategy is to create a leading entry in a column and then use it to create zeros below it. The entry chosen to become a leading entry is called a ***pivot,*** and this phase of the process is called ***pivoting.*** Although not strictly necessary, it is often convenient to use the second elementary row operation to make each leading entry a 1.

We begin by introducing zeros into the first column below the leading 1 in the first row:

$$\begin{bmatrix} 1 & 2 & -4 & -4 & 5 \\ 2 & 4 & 0 & 0 & 2 \\ 2 & 3 & 2 & 1 & 5 \\ -1 & 1 & 3 & 6 & 5 \end{bmatrix} \xrightarrow[R_4 + R_1]{\substack{R_2 - 2R_1 \\ R_3 - 2R_1}} \begin{bmatrix} 1 & 2 & -4 & -4 & 5 \\ 0 & 0 & 8 & 8 & -8 \\ 0 & -1 & 10 & 9 & -5 \\ 0 & 3 & -1 & 2 & 10 \end{bmatrix}$$

The first column is now as we want it, so the next thing to do is to create a leading entry in the second row, aiming for the staircase pattern of echelon form. In this case, we do this by interchanging rows. (We could also add row 3 or row 4 to row 2.)

$$\xrightarrow{R_2 \leftrightarrow R_3} \begin{bmatrix} 1 & 2 & -4 & -4 & 5 \\ 0 & -1 & 10 & 9 & -5 \\ 0 & 0 & 8 & 8 & -8 \\ 0 & 3 & -1 & 2 & 10 \end{bmatrix}$$

The pivot this time was -1. We now create a zero at the bottom of column 2, using the leading entry -1 in row 2:

$$\xrightarrow{R_4 + 3R_2} \begin{bmatrix} 1 & 2 & -4 & -4 & 5 \\ 0 & -1 & 10 & 9 & -5 \\ 0 & 0 & 8 & 8 & -8 \\ 0 & 0 & 29 & 29 & -5 \end{bmatrix}$$

Column 2 is now done. Noting that we already have a leading entry in column 3, we just pivot on the 8 to introduce a zero below it. This is easiest if we first divide row 3 by 8:

$$\xrightarrow{\frac{1}{8}R_3} \begin{bmatrix} 1 & 2 & -4 & -4 & 5 \\ 0 & -1 & 10 & 9 & -5 \\ 0 & 0 & 1 & 1 & -1 \\ 0 & 0 & 29 & 29 & -5 \end{bmatrix}$$

Now we use the leading 1 in row 3 to create a zero below it:

$$\xrightarrow{R_4 - 29R_3} \begin{bmatrix} 1 & 2 & -4 & -4 & 5 \\ 0 & -1 & 10 & 9 & -5 \\ 0 & 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 0 & 24 \end{bmatrix}$$

With this final step, we have reduced our matrix to echelon form.

Remarks

- The row echelon form of a matrix is not unique. (Find a different row echelon form for the matrix in Example 2.9.)

- The leading entry in each row is used to create the zeros below it.
- The pivots are not necessarily the entries that are originally in the positions eventually occupied by the leading entries. In Example 2.9, the pivots were 1, −1, 8, and 24. The original matrix had 1, 4, 2, and 5 in those positions on the "staircase."
- Once we have pivoted and introduced zeros below the leading entry in a column, that column does not change. In other words, the row echelon form emerges from left to right, top to bottom.

Elementary row operations are reversible—that is, they can be "undone." Thus, if some elementary row operation converts A into B, there is also an elementary row operation that converts B into A. (See Exercises 15 and 16.)

Definition Matrices A and B are ***row equivalent*** if there is a sequence of elementary row operations that converts A into B.

The matrices in Example 2.9,

$$\begin{bmatrix} 1 & 2 & -4 & -4 & 5 \\ 2 & 4 & 0 & 0 & 2 \\ 2 & 3 & 2 & 1 & 5 \\ -1 & 1 & 3 & 6 & 5 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1 & 2 & -4 & -4 & 5 \\ 0 & -1 & 10 & 9 & -5 \\ 0 & 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 0 & 24 \end{bmatrix}$$

are row equivalent. In general, though, how can we tell whether two matrices are row equivalent?

Theorem 2.1 Matrices A and B are row equivalent if and only if they can be reduced to the same row echelon form.

Proof If A and B are row equivalent, then further row operations will reduce B (and therefore A) to the (same) row echelon form.

Conversely, if A and B have the same row echelon form R, then, via elementary row operations, we can convert A into R and B into R. Reversing the latter sequence of operations, we can convert R into B, and therefore the sequence $A \to R \to B$ achieves the desired effect.

Remark In practice, Theorem 2.1 is easiest to use if R is the *reduced* row echelon form of A and B, as defined on page 73. See Exercises 17 and 18.

Gaussian Elimination

When row reduction is applied to the augmented matrix of a system of linear equations, we create an equivalent system that can be solved by back substitution. The entire process is known as ***Gaussian elimination.***

Gaussian Elimination

1. Write the augmented matrix of the system of linear equations.
2. Use elementary row operations to reduce the augmented matrix to row echelon form.
3. Using back substitution, solve the equivalent system that corresponds to the row-reduced matrix.

Remark When performed by hand, step 2 of Gaussian elimination allows quite a bit of choice. Here are some useful guidelines:

(a) Locate the leftmost column that is not all zeros.

(b) Create a leading entry at the top of this column. (It will usually be easiest if you make this a leading 1. See Exercise 22.)

(c) Use the leading entry to create zeros below it.

(d) Cover up the row containing the leading entry, and go back to step (a) to repeat the procedure on the remaining submatrix. Stop when the entire matrix is in row echelon form.

Example 2.10

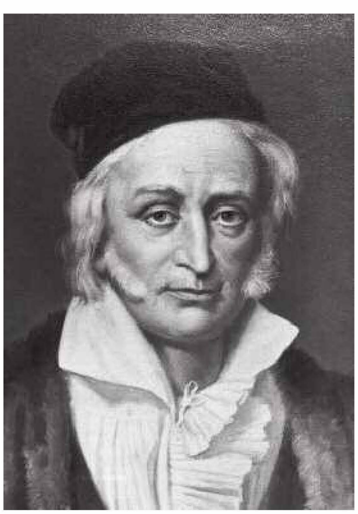
Solve the system

$$\begin{aligned} 2x_2 + 3x_3 &= 8 \\ 2x_1 + 3x_2 + x_3 &= 5 \\ x_1 - x_2 - 2x_3 &= -5 \end{aligned}$$

Solution The augmented matrix is

$$\left[\begin{array}{rrr|r} 0 & 2 & 3 & 8 \\ 2 & 3 & 1 & 5 \\ 1 & -1 & -2 & -5 \end{array}\right]$$

We proceed to reduce this matrix to row echelon form, following the guidelines given for step 2 of the process. The first nonzero column is column 1. We begin by creating



Bettmann/CORBIS

Carl Friedrich Gauss (1777–1855) is generally considered to be one of the three greatest mathematicians of all time, along with Archimedes and Newton. He is often called the "prince of mathematicians," a nickname that he richly deserves. A child prodigy, Gauss reportedly could do arithmetic before he could talk. At the age of 3, he corrected an error in his father's calculations for the company payroll, and as a young student, he found the formula $n(n + 1)/2$ for the sum of the first n natural numbers. When he was 19, he proved that a 17-sided polygon could be constructed using only a straightedge and a compass, and at the age of 21, he proved, in his doctoral dissertation, that every polynomial of degree n with real or complex coefficients has exactly n zeros, counting multiple zeros—the Fundamental Theorem of Algebra.

Gauss's 1801 publication *Disquisitiones Arithmeticae* is generally considered to be the foundation of modern number theory, but he made contributions to nearly every branch of mathematics as well as to statistics, physics, astronomy, and surveying. Gauss did not publish all of his findings, probably because he was too critical of his own work. He also did not like to teach and was often critical of other mathematicians, perhaps because he discovered—but did not publish—their results before they did.

The method called Gaussian elimination was known to the Chinese in the third century B.C. and was well known by Gauss's time. The method bears Gauss's name because of his use of it in a paper in which he solved a system of linear equations to describe the orbit of an asteroid.

a leading entry at the top of this column; interchanging rows 1 and 3 is the best way to achieve this.

$$\left[\begin{array}{rrr|r} 0 & 2 & 3 & 8 \\ 2 & 3 & 1 & 5 \\ 1 & -1 & -2 & -5 \end{array}\right] \xrightarrow{R_1 \leftrightarrow R_3} \left[\begin{array}{rrr|r} 1 & -1 & -2 & -5 \\ 2 & 3 & 1 & 5 \\ 0 & 2 & 3 & 8 \end{array}\right]$$

We now create a second zero in the first column, using the leading 1:

$$\xrightarrow{R_2 - 2R_1} \left[\begin{array}{rrr|r} 1 & -1 & -2 & -5 \\ 0 & 5 & 5 & 15 \\ 0 & 2 & 3 & 8 \end{array}\right]$$

We now cover up the first row and repeat the procedure. The second column is the first nonzero column of the submatrix. Multiplying row 2 by $\frac{1}{5}$ will create a leading 1.

$$\left[\begin{array}{rrr|r} 1 & -1 & -2 & -5 \\ 0 & 5 & 5 & 15 \\ 0 & 2 & 3 & 8 \end{array}\right] \xrightarrow{\frac{1}{5}R_2} \left[\begin{array}{rrr|r} 1 & -1 & -2 & -5 \\ 0 & 1 & 1 & 3 \\ 0 & 2 & 3 & 8 \end{array}\right]$$

We now need another zero at the bottom of column 2:

$$\xrightarrow{R_3 - 2R_2} \left[\begin{array}{rrr|r} 1 & -1 & -2 & -5 \\ 0 & 1 & 1 & 3 \\ 0 & 0 & 1 & 2 \end{array}\right]$$

The augmented matrix is now in row echelon form, and we move to step 3. The corresponding system is

$$\begin{aligned} x_1 - x_2 - 2x_3 &= -5 \\ x_2 + x_3 &= 3 \\ x_3 &= 2 \end{aligned}$$

and back substitution gives $x_3 = 2$, then $x_2 = 3 - x_3 = 3 - 2 = 1$, and finally $x_1 = -5 + x_2 + 2x_3 = -5 + 1 + 4 = 0$. We write the solution in vector form as

$$\begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$$

(We are going to write the vector solutions of linear systems as column vectors from now on. The reason for this will become clear in Chapter 3.)

Example 2.11

Solve the system

$$\begin{aligned} w - x - y + 2z &= 1 \\ 2w - 2x - y + 3z &= 3 \\ -w + x - y \qquad &= -3 \end{aligned}$$

Solution The augmented matrix is

$$\left[\begin{array}{rrrr|r} 1 & -1 & -1 & 2 & 1 \\ 2 & -2 & -1 & 3 & 3 \\ -1 & 1 & -1 & 0 & -3 \end{array}\right]$$

which can be row reduced as follows:

$$\left[\begin{array}{rrrr|r} 1 & -1 & -1 & 2 & 1 \\ 2 & -2 & -1 & 3 & 3 \\ -1 & 1 & -1 & 0 & -3 \end{array}\right] \xrightarrow[R_3+R_1]{R_2-2R_1} \left[\begin{array}{rrrr|r} 1 & -1 & -1 & 2 & 1 \\ 0 & 0 & 1 & -1 & 1 \\ 0 & 0 & -2 & 2 & -2 \end{array}\right]$$

$$\xrightarrow{R_3+2R_2} \left[\begin{array}{rrrr|r} 1 & -1 & -1 & 2 & 1 \\ 0 & 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

The associated system is now

$$\begin{aligned} w - x - y + 2z &= 1 \\ y - z &= 1 \end{aligned}$$

which has infinitely many solutions. There is more than one way to assign parameters, but we will proceed to use back substitution, writing the variables corresponding to the leading entries (the ***leading variables***) in terms of the other variables (the ***free variables***).

In this case, the leading variables are w and y, and the free variables are x and z. Thus, $y = 1 + z$, and from this we obtain

$$\begin{aligned} w &= 1 + x + y - 2z \\ &= 1 + x + (1 + z) - 2z \\ &= 2 + x - z \end{aligned}$$

If we assign parameters $x = s$ and $z = t$, the solution can be written in vector form as

$$\begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 + s - t \\ s \\ 1 + t \\ t \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 1 \\ 0 \end{bmatrix} + s\begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t\begin{bmatrix} -1 \\ 0 \\ 1 \\ 1 \end{bmatrix}$$

Example 2.11 highlights a very important property: In a consistent system, the free variables are just the variables that are not leading variables. Since the number of leading variables is the number of nonzero rows in the row echelon form of the coefficient matrix, we can predict the number of free variables (parameters) before we find the explicit solution using back substitution. In Chapter 3, we will prove that, although the row echelon form of a matrix is not unique, the number of nonzero rows is the same in *all* row echelon forms of a given matrix. Thus, it makes sense to give a name to this number.

Definition The ***rank*** of a matrix is the number of nonzero rows in its row echelon form.

We will denote the rank of a matrix A by rank(A). In Example 2.10, the rank of the coefficient matrix is 3, and in Example 2.11, the rank of the coefficient matrix is 2. The observations we have just made justify the following theorem, which we will prove in more generality in Chapters 3 and 6.

Theorem 2.2 **The Rank Theorem**

Let A be the coefficient matrix of a system of linear equations with n variables. If the system is consistent, then

$$\text{number of free variables} = n - \text{rank}(A)$$

Thus, in Example 2.10, we have $3 - 3 = 0$ free variables (in other words, a *unique* solution), and in Example 2.11, we have $4 - 2 = 2$ free variables, as we found.

Example 2.12 Solve the system

$$\begin{aligned} x_1 - x_2 + 2x_3 &= 3 \\ x_1 + 2x_2 - x_3 &= -3 \\ 2x_2 - 2x_3 &= 1 \end{aligned}$$

Solution When we row reduce the augmented matrix, we have

$$\left[\begin{array}{rrr|r} 1 & -1 & 2 & 3 \\ 1 & 2 & -1 & -3 \\ 0 & 2 & -2 & 1 \end{array}\right] \xrightarrow{R_2 - R_1} \left[\begin{array}{rrr|r} 1 & -1 & 2 & 3 \\ 0 & 3 & -3 & -6 \\ 0 & 2 & -2 & 1 \end{array}\right]$$

$$\xrightarrow{\frac{1}{3}R_2} \left[\begin{array}{rrr|r} 1 & -1 & 2 & 3 \\ 0 & 1 & -1 & -2 \\ 0 & 2 & -2 & 1 \end{array}\right]$$

$$\xrightarrow{R_3 - 2R_2} \left[\begin{array}{rrr|r} 1 & -1 & 2 & 3 \\ 0 & 1 & -1 & -2 \\ 0 & 0 & 0 & 5 \end{array}\right]$$

leading to the impossible equation $0 = 5$. (We could also have performed $R_3 - \frac{2}{3}R_2$ as the second elementary row operation, which would have given us the same contradiction but a different row echelon form.) Thus, the system has no solutions—it is inconsistent.

Wilhelm Jordan (1842–1899) was a German professor of geodesy whose contribution to solving linear systems was a systematic method of back substitution closely related to the method described here.

Gauss-Jordan Elimination

A modification of Gaussian elimination greatly simplifies the back substitution phase and is particularly helpful when calculations are being done by hand on a system with infinitely many solutions. This variant, known as ***Gauss-Jordan elimination,*** relies on reducing the augmented matrix even further.

Definition A matrix is in ***reduced row echelon form*** if it satisfies the following properties:

1. It is in row echelon form.
2. The leading entry in each nonzero row is a 1 (called a ***leading 1***).
3. Each column containing a leading 1 has zeros everywhere else.

The following matrix is in reduced row echelon form:

$$\begin{bmatrix} 1 & 2 & 0 & 0 & -3 & 1 & 0 \\ 0 & 0 & 1 & 0 & 4 & -1 & 0 \\ 0 & 0 & 0 & 1 & 3 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

For 2×2 matrices, the possible reduced row echelon forms are

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & * \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \text{ and } \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

where * can be any number.

For a short proof that the reduced row echelon form of a matrix is unique, see the article by Thomas Yuster, "The Reduced Row Echelon Form of a Matrix Is Unique: A Simple Proof," in the March 1984 issue of *Mathematics Magazine* (vol. 57, no. 2, pp. 93–94).

It is clear that after a matrix has been reduced to echelon form, further elementary row operations will bring it to reduced row echelon form. What is not clear (although intuition may suggest it) is that, unlike the row echelon form, the reduced row echelon form of a matrix is *unique*.

In Gauss-Jordan elimination, we proceed as in Gaussian elimination but reduce the augmented matrix to *reduced* row echelon form.

Gauss-Jordan Elimination

1. Write the augmented matrix of the system of linear equations.
2. Use elementary row operations to reduce the augmented matrix to reduced row echelon form.
3. If the resulting system is consistent, solve for the leading variables in terms of any remaining free variables.

Example 2.13

Solve the system in Example 2.11 by Gauss-Jordan elimination.

Solution The reduction proceeds as it did in Example 2.11 until we reach the echelon form:

$$\left[\begin{array}{cccc|c} 1 & -1 & -1 & 2 & 1 \\ 0 & 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

We now must create a zero above the leading 1 in the second row, third column. We do this by adding row 2 to row 1 to obtain

$$\left[\begin{array}{cccc|c} 1 & -1 & 0 & 1 & 2 \\ 0 & 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

The system has now been reduced to

$$\begin{aligned} w - x \quad\quad + z &= 2 \\ y - z &= 1 \end{aligned}$$

It is now much easier to solve for the leading variables:

$$w = 2 + x - z \qquad \text{and} \qquad y = 1 + z$$

If we assign parameters $x = s$ and $z = t$ as before, the solution can be written in vector form as

$$\begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 + s - t \\ s \\ 1 + t \\ t \end{bmatrix}$$

Remark From a computational point of view, it is more efficient (in the sense that it requires fewer calculations) to first reduce the matrix to row echelon form and then, working from *right to left,* make each leading entry a 1 and create zeros above these leading 1s. However, for manual calculation, you will find it easier to just work from left to right and create the leading 1s and the zeros in their columns as you go.

Let's return to the geometry that brought us to this point. Just as systems of linear equations in two variables correspond to lines in $\mathbb{R}^2$, so linear equations in three variables correspond to planes in $\mathbb{R}^3$. In fact, many questions about lines and planes can be answered by solving an appropriate linear system.

Example 2.14

Find the line of intersection of the planes $x + 2y - z = 3$ and $2x + 3y + z = 1$.

Solution First, observe that there *will* be a line of intersection, since the normal vectors of the two planes—$[1, 2, -1]$ and $[2, 3, 1]$—are not parallel. The points that lie in the intersection of the two planes correspond to the points in the solution set of the system

$$\begin{aligned} x + 2y - z &= 3 \\ 2x + 3y + z &= 1 \end{aligned}$$

Gauss-Jordan elimination applied to the augmented matrix yields

$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & 3 \\ 2 & 3 & 1 & 1 \end{array}\right] \xrightarrow{R_2 - 2R_1} \left[\begin{array}{ccc|c} 1 & 2 & -1 & 3 \\ 0 & -1 & 3 & -5 \end{array}\right]$$

$$\xrightarrow[-R_2]{R_1 + 2R_2} \left[\begin{array}{ccc|c} 1 & 0 & 5 & -7 \\ 0 & 1 & -3 & 5 \end{array}\right]$$

Replacing variables, we have

$$\begin{aligned} x \quad + 5z &= -7 \\ y - 3z &= 5 \end{aligned}$$

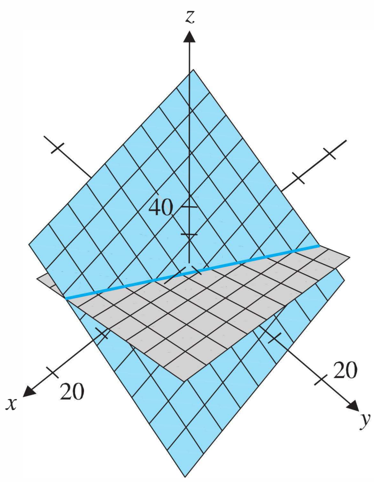


Figure 2.2
The intersection of two planes

We set the free variable z equal to a parameter t and thus obtain the parametric equations of the line of intersection of the two planes:

$$\begin{aligned} x &= -7 - 5t \\ y &= 5 + 3t \\ z &= t \end{aligned}$$

In vector form, the equation is

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -7 \\ 5 \\ 0 \end{bmatrix} + t\begin{bmatrix} -5 \\ 3 \\ 1 \end{bmatrix}$$

See Figure 2.2.

Example 2.15

Let $\mathbf{p} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$, $\mathbf{q} = \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}$, $\mathbf{u} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, and $\mathbf{v} = \begin{bmatrix} 3 \\ -1 \\ -1 \end{bmatrix}$. Determine whether the lines $\mathbf{x} = \mathbf{p} + t\mathbf{u}$ and $\mathbf{x} = \mathbf{q} + t\mathbf{v}$ intersect and, if so, find their point of intersection.

Solution We need to be careful here. Although t has been used as the parameter in the equations of both lines, the lines are independent and therefore so are their parameters. Let's use a different parameter—say, s—for the first line, so its equation becomes $\mathbf{x} = \mathbf{p} + s\mathbf{u}$. If the lines intersect, then we want to find an $\mathbf{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ that satisfies both equations simultaneously. That is, we want $\mathbf{x} = \mathbf{p} + s\mathbf{u} = \mathbf{q} + t\mathbf{v}$ or $s\mathbf{u} - t\mathbf{v} = \mathbf{q} - \mathbf{p}$.

Substituting the given $\mathbf{p}$, $\mathbf{q}$, $\mathbf{u}$, and $\mathbf{v}$, we obtain the equations

$$\begin{aligned} s - 3t &= -1 \\ s + t &= 2 \\ s + t &= 2 \end{aligned}$$

whose solution is easily found to be $s = \frac{5}{4}$, $t = \frac{3}{4}$. The point of intersection is therefore

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} + \frac{5}{4}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{9}{4} \\ \frac{5}{4} \\ \frac{1}{4} \end{bmatrix}$$

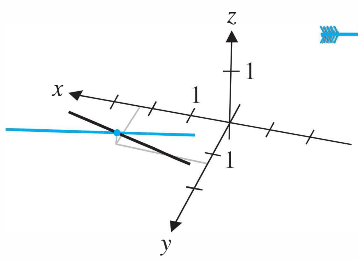


Figure 2.3
Two intersecting lines

See Figure 2.3. (Check that substituting $t = \frac{3}{4}$ in the other equation gives the same point.)

Remark In $\mathbb{R}^3$, it is possible for two lines to intersect in a point, to be parallel, or to do neither. Nonparallel lines that do not intersect are called *skew lines*.

Homogeneous Systems

We have seen that every system of linear equations has either no solution, a unique solution, or infinitely many solutions. However, there is one type of system that always has at least one solution.

Definition A system of linear equations is called ***homogeneous*** if the constant term in each equation is zero.

In other words, a homogeneous system has an augmented matrix of the form $[A \mid \mathbf{0}]$. The following system is homogeneous:

$$\begin{aligned} 2x + 3y - z &= 0 \\ -x + 5y + 2z &= 0 \end{aligned}$$

Since a homogeneous system cannot have no solution (forgive the double negative!), it will have either a unique solution (namely, the zero, or trivial, solution) or infinitely many solutions. The next theorem says that the latter case *must* occur if the number of variables is greater than the number of equations.

Theorem 2.3 If $[A \mid \mathbf{0}]$ is a homogeneous system of m linear equations with n variables, where $m < n$, then the system has infinitely many solutions.

Proof Since the system has at least the zero solution, it is consistent. Also, $\text{rank}(A) \le m$ (why?). By the Rank Theorem, we have

$$\text{number of free variables} = n - \text{rank}(A) \ge n - m > 0$$

So there is at least one free variable and, hence, there are infinitely many solutions.

Note Theorem 2.3 says nothing about the case where $m \ge n$. Exercise 44 asks you to give examples to show that, in this case, there can be either a unique solution or infinitely many solutions.

$\mathbb{R}$ and $\mathbb{Z}_p$ are examples of *fields*. The set of rational numbers $\mathbb{Q}$ and the set of complex numbers $\mathbb{C}$ are other examples. Fields are covered in detail in courses in abstract algebra.

Linear Systems over $\mathbb{Z}_p$

Thus far, all of the linear systems we have encountered have involved real numbers, and the solutions have accordingly been vectors in some $\mathbb{R}^n$. We have seen how other number systems arise—notably, $\mathbb{Z}_p$. When p is a prime number, $\mathbb{Z}_p$ behaves in many respects like $\mathbb{R}$; in particular, we can add, subtract, multiply, and divide (by nonzero numbers). Thus, we can also solve systems of linear equations when the variables and coefficients belong to some $\mathbb{Z}_p$. In such instances, we refer to solving a system *over* $\mathbb{Z}_p$.

For example, the linear equation $x_1 + x_2 + x_3 = 1$, when viewed as an equation over $\mathbb{Z}_2$, has exactly four solutions:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad \text{and} \quad \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

(where the last solution arises because $1 + 1 + 1 = 1$ in $\mathbb{Z}_2$).

When we view the equation $x_1 + x_2 + x_3 = 1$ over $\mathbb{Z}_3$, the solutions $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ are

$$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$$

(Check these.)

But we need not use trial-and-error methods; row reduction of augmented matrices works just as well over $\mathbb{Z}_p$ as over $\mathbb{R}$.

Example 2.16

Solve the following system of linear equations over $\mathbb{Z}_3$:

$$\begin{aligned} x_1 + 2x_2 + x_3 &= 0 \\ x_1 \quad\quad + x_3 &= 2 \\ x_2 + 2x_3 &= 1 \end{aligned}$$

Solution The first thing to note in examples like this one is that subtraction and division are not needed; we can accomplish the same effects using addition and multiplication. (This, however, requires that we be working over $\mathbb{Z}_p$, where p is a prime; see Exercise 60 at the end of this section and Exercise 57 in Section 1.1.)

We row reduce the augmented matrix of the system, using calculations modulo 3.

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 1 & 0 & 1 & 2 \\ 0 & 1 & 2 & 1 \end{array}\right] \xrightarrow{R_2 + 2R_1} \left[\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 0 & 1 & 0 & 2 \\ 0 & 1 & 2 & 1 \end{array}\right]$$

$$\xrightarrow[R_3 + 2R_2]{R_1 + R_2} \left[\begin{array}{ccc|c} 1 & 0 & 1 & 2 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 2 & 2 \end{array}\right]$$

$$\xrightarrow[2R_3]{R_1+R_3} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

Thus, the solution is $x_1 = 1$, $x_2 = 2$, $x_3 = 1$.

Example 2.17

Solve the following system of linear equations over $\mathbb{Z}_2$:

$$\begin{aligned} x_1 + x_2 + x_3 + x_4 &= 1 \\ x_1 + x_2 \qquad\qquad &= 1 \\ x_2 + x_3 \qquad &= 0 \\ x_3 + x_4 &= 0 \\ x_1 \qquad\qquad + x_4 &= 1 \end{aligned}$$

Solution The row reduction proceeds as follows:

$$\left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 & 1 \end{array}\right] \xrightarrow[R_5+R_1]{R_2+R_1} \left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 \end{array}\right]$$

$$\xrightarrow[\substack{R_1+R_2 \\ R_5+R_2}]{R_2 \leftrightarrow R_3} \left[\begin{array}{cccc|c} 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$\xrightarrow[R_4+R_3]{R_2+R_3} \left[\begin{array}{cccc|c} 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore, we have

$$\begin{aligned} x_1 \qquad\qquad + x_4 &= 1 \\ x_2 \qquad + x_4 &= 0 \\ x_3 + x_4 &= 0 \end{aligned}$$

Setting the free variable $x_4 = t$ yields

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1+t \\ t \\ t \\ t \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + t\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

Since t can take on the two values 0 and 1, there are exactly two solutions:

$$\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \text{ and } \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

Remark For linear systems over $\mathbb{Z}_p$, there can never be infinitely many solutions. (Why not?) Rather, when there is more than one solution, the number of solutions is finite and is a function of the number of free variables and p. (See Exercise 59.)

Exercises 2.2

In Exercises 1–8, determine whether the given matrix is in row echelon form. If it is, state whether it is also in reduced row echelon form.

1. $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 3 \\ 0 & 1 & 0 \end{bmatrix}$

2. $\begin{bmatrix} 7 & 0 & 1 & 0 \\ 0 & 1 & -1 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

3. $\begin{bmatrix} 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

4. $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

5. $\begin{bmatrix} 1 & 0 & 3 & -4 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 5 & 0 & 1 \end{bmatrix}$

6. $\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$

7. $\begin{bmatrix} 1 & 2 & 3 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$

8. $\begin{bmatrix} 2 & 1 & 3 & 5 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

In Exercises 9–14, use elementary row operations to reduce the given matrix to (a) row echelon form and (b) reduced row echelon form.

9. $\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$

10. $\begin{bmatrix} 4 & 3 \\ 2 & 1 \end{bmatrix}$

11. $\begin{bmatrix} 3 & 5 \\ 5 & -2 \\ 2 & 4 \end{bmatrix}$

12. $\begin{bmatrix} 2 & -4 & -2 & 6 \\ 3 & 1 & 6 & 6 \end{bmatrix}$

13. $\begin{bmatrix} 3 & -2 & -1 \\ 2 & -1 & -1 \\ 4 & -3 & -1 \end{bmatrix}$

14. $\begin{bmatrix} -2 & -4 & 7 \\ -3 & -6 & 10 \\ 1 & 2 & -3 \end{bmatrix}$

15. Reverse the elementary row operations used in Example 2.9 to show that we can convert

$$\begin{bmatrix} 1 & 2 & -4 & -4 & 5 \\ 0 & -1 & 10 & 9 & -5 \\ 0 & 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 0 & 24 \end{bmatrix} \text{ into}$$

$$\begin{bmatrix} 1 & 2 & -4 & -4 & 5 \\ 2 & 4 & 0 & 0 & 2 \\ 2 & 3 & 2 & 1 & 5 \\ -1 & 1 & 3 & 6 & 5 \end{bmatrix}$$

16. In general, what is the elementary row operation that "undoes" each of the three elementary row operations $R_i \leftrightarrow R_j$, kR_i, and $R_i + kR_j$?

In Exercises 17 and 18, show that the given matrices are row equivalent and find a sequence of elementary row operations that will convert A into B.

17. $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, B = \begin{bmatrix} 3 & -1 \\ 1 & 0 \end{bmatrix}$

18. $A = \begin{bmatrix} 2 & 0 & -1 \\ 1 & 1 & 0 \\ -1 & 1 & 1 \end{bmatrix}, B = \begin{bmatrix} 3 & 1 & -1 \\ 3 & 5 & 1 \\ 2 & 2 & 0 \end{bmatrix}$

19. What is wrong with the following "proof" that every matrix with at least two rows is row equivalent to a matrix with a zero row?

Perform $R_2 + R_1$ and $R_1 + R_2$. Now rows 1 and 2 are identical. Now perform $R_2 - R_1$ to obtain a row of zeros in the second row.

20. What is the net effect of performing the following sequence of elementary row operations on a matrix (with at least two rows)?

$$R_2 + R_1, R_1 - R_2, R_2 + R_1, -R_1$$

21. Students frequently perform the following type of calculation to introduce a zero into a matrix:

$$\begin{bmatrix} 3 & 1 \\ 2 & 4 \end{bmatrix} \xrightarrow{3R_2 - 2R_1} \begin{bmatrix} 3 & 1 \\ 0 & 10 \end{bmatrix}$$

However, $3R_2 - 2R_1$ is *not* an elementary row operation. Why not? Show how to achieve the same result using elementary row operations.

22. Consider the matrix $A = \begin{bmatrix} 3 & 2 \\ 1 & 4 \end{bmatrix}$. Show that any of the three types of elementary row operations can be used to create a leading 1 at the top of the first column. Which do you prefer and why?

23. What is the rank of each of the matrices in Exercises 1–8?

24. What are the possible reduced row echelon forms of 3×3 matrices?

In Exercises 25–34, solve the given system of equations using either Gaussian or Gauss-Jordan elimination.

25. $\begin{aligned} x_1 + 2x_2 - 3x_3 &= 9 \\ 2x_1 - x_2 + x_3 &= 0 \\ 4x_1 - x_2 + x_3 &= 4 \end{aligned}$

26. $\begin{aligned} x - y + z &= 0 \\ -x + 3y + z &= 5 \\ 3x + y + 7z &= 2 \end{aligned}$

27. $\begin{aligned} x_1 - 3x_2 - 2x_3 &= 0 \\ -x_1 + 2x_2 + x_3 &= 0 \\ 2x_1 + 4x_2 + 6x_3 &= 0 \end{aligned}$

28. $\begin{aligned} 2w + 3x - y + 4z &= 1 \\ 3w - x + z &= 1 \\ 3w - 4x + y - z &= 2 \end{aligned}$

29. $\begin{aligned} 2r + s &= 3 \\ 4r + s &= 7 \\ 2r + 5s &= -1 \end{aligned}$

30. $\begin{aligned} -x_1 + 3x_2 - 2x_3 + 4x_4 &= 0 \\ 2x_1 - 6x_2 + x_3 - 2x_4 &= -3 \\ x_1 - 3x_2 + 4x_3 - 8x_4 &= 2 \end{aligned}$

31. $\begin{aligned} \tfrac{1}{2}x_1 + x_2 - x_3 - 6x_4 &= 2 \\ \tfrac{1}{6}x_1 + \tfrac{1}{2}x_2 - 3x_4 + x_5 &= -1 \\ \tfrac{1}{3}x_1 - 2x_3 - 4x_5 &= 8 \end{aligned}$

32. $\begin{aligned} \sqrt{2}x + y + 2z &= 1 \\ \sqrt{2}y - 3z &= -\sqrt{2} \\ -y + \sqrt{2}z &= 1 \end{aligned}$

33. $\begin{aligned} w + x + 2y + z &= 1 \\ w - x - y + z &= 0 \\ x + y &= -1 \\ w + x + z &= 2 \end{aligned}$

34. $\begin{aligned} a + b + c + d &= 4 \\ a + 2b + 3c + 4d &= 10 \\ a + 3b + 6c + 10d &= 20 \\ a + 4b + 10c + 20d &= 35 \end{aligned}$

In Exercises 35–38, determine by inspection (i.e., without performing any calculations) whether a linear system with the given augmented matrix has a unique solution, infinitely many solutions, or no solution. Justify your answers.

35. $\left[\begin{array}{ccc|c} 0 & 0 & 1 & 2 \\ 0 & 1 & 3 & 1 \\ 1 & 0 & 1 & 1 \end{array}\right]$

36. $\left[\begin{array}{cccc|c} 3 & -2 & 0 & 1 & 1 \\ 1 & 2 & -3 & 1 & -1 \\ 2 & 4 & -6 & 2 & 0 \end{array}\right]$

37. $\left[\begin{array}{cccc|c} 1 & 2 & 3 & 4 & 0 \\ 5 & 6 & 7 & 8 & 0 \\ 9 & 10 & 11 & 12 & 0 \end{array}\right]$

38. $\left[\begin{array}{ccccc|c} 1 & 2 & 3 & 4 & 5 & 6 \\ 6 & 5 & 4 & 3 & 2 & 1 \\ 7 & 7 & 7 & 7 & 7 & 7 \end{array}\right]$

39. Show that if $ad - bc \neq 0$, then the system

$$\begin{aligned} ax + by &= r \\ cx + dy &= s \end{aligned}$$

has a unique solution.

In Exercises 40–43, for what value(s) of k, if any, will the systems have (a) no solution, (b) a unique solution, and (c) infinitely many solutions?

40. $\begin{aligned} kx + 2y &= 3 \\ 2x - 4y &= -6 \end{aligned}$

41. $\begin{aligned} x + ky &= 1 \\ kx + y &= 1 \end{aligned}$

42. $\begin{aligned} x - 2y + 3z &= 2 \\ x + y + z &= k \\ 2x - y + 4z &= k^2 \end{aligned}$

43. $\begin{aligned} x + y + kz &= 1 \\ x + ky + z &= 1 \\ kx + y + z &= -2 \end{aligned}$

44. Give examples of homogeneous systems of m linear equations in n variables with $m = n$ and with $m > n$ that have (a) infinitely many solutions and (b) a unique solution.

In Exercises 45 and 46, find the line of intersection of the given planes.

45. $3x + 2y + z = -1$ and $2x - y + 4z = 5$

46. $4x + y + z = 0$ and $2x - y + 3z = 2$

47. **(a)** Give an example of three planes that have a common line of intersection (Figure 2.4).

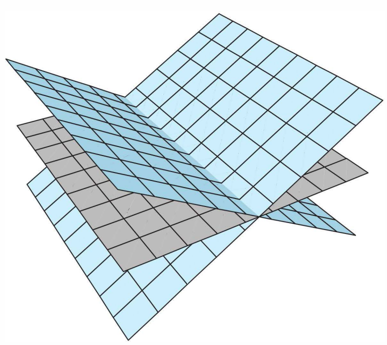


Figure 2.4

(b) Give an example of three planes that intersect in pairs but have no common point of intersection (Figure 2.5).

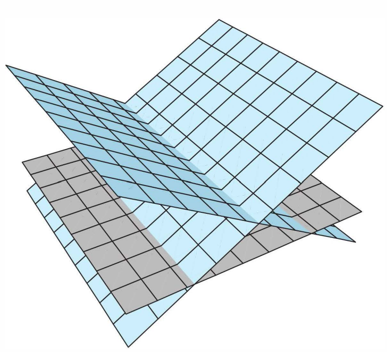


Figure 2.5

(c) Give an example of three planes, exactly two of which are parallel (Figure 2.6).

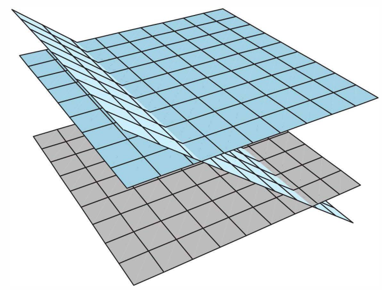


Figure 2.6

(d) Give an example of three planes that intersect in a single point (Figure 2.7).

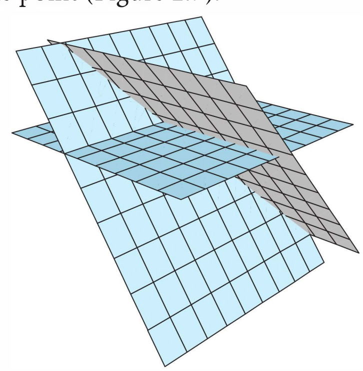


Figure 2.7

In Exercises 48 and 49, determine whether the lines $\mathbf{x} = \mathbf{p} + s\mathbf{u}$ *and* $\mathbf{x} = \mathbf{q} + t\mathbf{v}$ *intersect and, if they do, find their point of intersection.*

48. $\mathbf{p} = \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}, \mathbf{q} = \begin{bmatrix} 2 \\ 2 \\ 0 \end{bmatrix}, \mathbf{u} = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}, \mathbf{v} = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$

49. $\mathbf{p} = \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix}, \mathbf{q} = \begin{bmatrix} -1 \\ 1 \\ -1 \end{bmatrix}, \mathbf{u} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \mathbf{v} = \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}$

50. Let $\mathbf{p} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \mathbf{u} = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$, and $\mathbf{v} = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$. Describe all points $Q = (a, b, c)$ such that the line through Q with direction vector $\mathbf{v}$ intersects the line with equation $\mathbf{x} = \mathbf{p} + s\mathbf{u}$.

51. Recall that the cross product of vectors $\mathbf{u}$ and $\mathbf{v}$ is a vector $\mathbf{u} \times \mathbf{v}$ that is orthogonal to both $\mathbf{u}$ and $\mathbf{v}$. (See Exploration: The Cross Product in Chapter 1.) If

$$\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} \quad \text{and} \quad \mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$$

show that there are infinitely many vectors

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

that simultaneously satisfy $\mathbf{u} \cdot \mathbf{x} = 0$ and $\mathbf{v} \cdot \mathbf{x} = 0$ and that all are multiples of

$$\mathbf{u} \times \mathbf{v} = \begin{bmatrix} u_2v_3 - u_3v_2 \\ u_3v_1 - u_1v_3 \\ u_1v_2 - u_2v_1 \end{bmatrix}$$

52. Let $\mathbf{p} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$, $\mathbf{q} = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$, $\mathbf{u} = \begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix}$, and $\mathbf{v} = \begin{bmatrix} 0 \\ 6 \\ -1 \end{bmatrix}$.

Show that the lines $\mathbf{x} = \mathbf{p} + s\mathbf{u}$ and $\mathbf{x} = \mathbf{q} + t\mathbf{v}$ are skew lines. Find vector equations of a pair of parallel planes, one containing each line.

In Exercises 53–58, solve the systems of linear equations over the indicated $\mathbb{Z}_p$.

53. $\begin{aligned} x + 2y &= 1 \\ x + y &= 2 \end{aligned}$ over $\mathbb{Z}_3$

54. $\begin{aligned} x + y &= 1 \\ y + z &= 0 \\ x + z &= 1 \end{aligned}$ over $\mathbb{Z}_2$

55. $\begin{aligned} x + y &= 1 \\ y + z &= 0 \\ x + z &= 1 \end{aligned}$ over $\mathbb{Z}_3$

56. $\begin{aligned} 3x + 2y &= 1 \\ x + 4y &= 1 \end{aligned}$ over $\mathbb{Z}_5$

57. $\begin{aligned} 3x + 2y &= 1 \\ x + 4y &= 1 \end{aligned}$ over $\mathbb{Z}_7$

58. $\begin{aligned} x_1 + 4x_4 &= 1 \\ x_1 + 2x_2 + 4x_3 &= 3 \\ 2x_1 + 2x_2 + x_4 &= 1 \\ x_1 + 3x_3 &= 2 \end{aligned}$ over $\mathbb{Z}_5$

59. Prove the following corollary to the Rank Theorem: Let A be an $m \times n$ matrix with entries in $\mathbb{Z}_p$. Any consistent system of linear equations with coefficient matrix A has exactly $p^{n-\text{rank}(A)}$ solutions over $\mathbb{Z}_p$.

60. When p is not prime, extra care is needed in solving a linear system (or, indeed, any equation) over $\mathbb{Z}_p$. Using Gaussian elimination, solve the following system over $\mathbb{Z}_6$. What complications arise?

$$\begin{aligned} 2x + 3y &= 4 \\ 4x + 3y &= 2 \end{aligned}$$

Writing Project

A History of Gaussian Elimination

As noted in the biographical sketch of Gauss in this section, Gauss did not actually "invent" the method known as Gaussian elimination. It was known in some form as early as the third century B.C. and appears in the mathematical writings of cultures throughout Europe and Asia.

Write a report on the history of elimination methods for solving systems of linear equations. What role did Gauss actually play in this history, and why is his name attached to the method?

1. S. Athloen and R. McLaughlin, Gauss-Jordan reduction: A brief history, *American Mathematical Monthly* 94 (1987), pp. 130–142.
2. Joseph F. Grcar, Mathematicians of Gaussian Elimination, *Notices of the AMS*, Vol. 58, No. 6 (2011), pp. 782–792. (Available online at http://www.ams.org/notices/201106/index.html)
3. Roger Hart, *The Chinese Roots of Linear Algebra* (Baltimore: Johns Hopkins University Press, 2011).
4. Victor J. Katz, *A History of Mathematics: An Introduction* (Third Edition) (Reading, MA: Addison Wesley Longman, 2008).

Explorations

Lies My Computer Told Me

Computers and calculators store real numbers in *floating-point form*. For example, 2001 is stored as 0.2001×10^4, and -0.00063 is stored as -0.63×10^{-3}. In general, the floating-point form of a number is $\pm M \times 10^k$, where k is an integer and the *mantissa* M is a (decimal) real number that satisfies $0.1 \le M < 1$.

The maximum number of decimal places that can be stored in the mantissa depends on the computer, calculator, or computer algebra system. If the maximum number of decimal places that can be stored is d, we say that there are d *significant digits*. Many calculators store 8 or 12 significant digits; computers can store more but still are subject to a limit. Any digits that are not stored are either omitted (in which case we say that the number has been *truncated*) or used to *round* the number to d significant digits.

For example, $\pi \approx 3.141592654$, and its floating-point form is 0.3141592654×10^1. In a computer that truncates to five significant digits, π would be stored as 0.31415×10^1 (and displayed as 3.1415); a computer that rounds to five significant digits would store π as 0.31416×10^1 (and display 3.1416). When the dropped digit is a solitary 5, the last remaining digit is rounded so that it becomes even. Thus, rounded to two significant digits, 0.735 becomes 0.74 while 0.725 becomes 0.72.

Whenever truncation or rounding occurs, a *roundoff error* is introduced, which can have a dramatic effect on the calculations. The more operations that are performed, the more the error accumulates. Sometimes, unfortunately, there is nothing we can do about this. This exploration illustrates this phenomenon with very simple systems of linear equations.

1. Solve the following system of linear equations *exactly* (that is, work with rational numbers throughout the calculations).

$$\begin{aligned} x + \phantom{\tfrac{801}{800}}y &= 0 \\ x + \tfrac{801}{800}y &= 1 \end{aligned}$$

2. As a decimal, $\frac{801}{800} = 1.00125$, so, rounded to five significant digits, the system becomes

$$\begin{aligned} x + y &= 0 \\ x + 1.0012y &= 1 \end{aligned}$$

Using your calculator or CAS, solve this system, rounding the result of every calculation to five significant digits.

3. Solve the system two more times, rounding first to four significant digits and then to three significant digits. What happens?

4. Clearly, a very small roundoff error (less than or equal to 0.00125) can result in very large errors in the solution. Explain why geometrically. (Think about the graphs of the various linear systems you solved in Problems 1–3.)

Systems such as the one you just worked with are called ***ill-conditioned.*** They are extremely sensitive to roundoff errors, and there is not much we can do about it. We will encounter ill-conditioned systems again in Chapters 3 and 7. Here is another example to experiment with:

$$\begin{aligned} 4.552x + 7.083y &= 1.931 \\ 1.731x + 2.693y &= 2.001 \end{aligned}$$

Play around with various numbers of significant digits to see what happens, starting with eight significant digits (if you can).

Partial Pivoting

In Exploration: Lies My Computer Told Me, we saw that ill-conditioned linear systems can cause trouble when roundoff error occurs. In this exploration, you will discover another way in which linear systems are sensitive to roundoff error and see that very small changes in the coefficients can lead to huge inaccuracies in the solution. Fortunately, there is something that can be done to minimize or even eliminate this problem (unlike the problem with ill-conditioned systems).

1. (a) Solve the single linear equation $0.00021x = 1$ for x.

   (b) Suppose your calculator can carry only four decimal places. The equation will be rounded to $0.0002x = 1$. Solve this equation.

The difference between the answers in parts (a) and (b) can be thought of as the effect of an error of 0.00001 on the solution of the given equation.

2. Now extend this idea to a system of linear equations.

   (a) With Gaussian elimination, solve the linear system

$$\begin{aligned} 0.400x + 99.6y &= 100 \\ 75.3x - 45.3y &= 30.0 \end{aligned}$$

   using three significant digits. Begin by pivoting on 0.400 and take each calculation to three significant digits. You should obtain the "solution" $x = -1.00$, $y = 1.01$. Check that the actual solution is $x = 1.00$, $y = 1.00$. This is a huge error—200% in the x value! Can you discover what caused it?

   (b) Solve the system in part (a) again, this time interchanging the two equations (or, equivalently, the two rows of its augmented matrix) and pivoting on 75.3. Again, take each calculation to three significant digits. What is the solution this time?

The moral of the story is that, when using Gaussian or Gauss-Jordan elimination to obtain a numerical solution to a system of linear equations (i.e., a decimal approximation), you should choose the pivots with care. Specifically, at each pivoting step, choose from among all possible pivots in a column the entry with the largest absolute value. Use row interchanges to bring this element into the correct position and use it to create zeros where needed in the column. This strategy is known as ***partial pivoting.***

3. Solve the following systems by Gaussian elimination, first without and then with partial pivoting. Take each calculation to three significant digits. (The exact solutions are given.)

(a)
$$\begin{aligned} 0.001x + 0.995y &= 1.00 \\ -10.2x + 1.00y &= -50.0 \end{aligned}$$

Exact solution: $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 5.00 \\ 1.00 \end{bmatrix}$

(b)
$$\begin{aligned} 10x - 7y &= 7 \\ -3x + 2.09y + 6z &= 3.91 \\ 5x - y + 5z &= 6 \end{aligned}$$

Exact solution: $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0.00 \\ -1.00 \\ 1.00 \end{bmatrix}$

Counting Operations: An Introduction to the Analysis of Algorithms

Gaussian and Gauss-Jordan elimination are examples of ***algorithms:*** systematic procedures designed to implement a particular task—in this case, the row reduction of the augmented matrix of a system of linear equations. Algorithms are particularly well suited to computer implementation, but not all algorithms are created equal. Apart from the speed, memory, and other attributes of the computer system on which they are running, some algorithms are faster than others. One measure of the so-called *complexity* of an algorithm (a measure of its efficiency, or ability to perform its task in a reasonable number of steps) is the number of basic operations it performs as a function of the number of variables that are input.

Let's examine this proposition in the case of the two algorithms we have for solving a linear system: Gaussian and Gauss-Jordan elimination. For our purposes, the basic operations are multiplication and division; we will assume that all other operations are performed much more rapidly and can be ignored. (This is a reasonable assumption, but we will not attempt to justify it.) We will consider only systems of equations with *square* coefficient matrices, so, if the coefficient matrix is $n \times n$, the number of input variables is n. Thus, our task is to find the number of operations performed by Gaussian and Gauss-Jordan elimination as a function of n. Furthermore, we will not worry about special cases that may arise, but rather establish the *worst case* that can arise—when the algorithm takes as long as possible. Since this will give us an estimate of the time it will take a computer to perform the algorithm (if we know how long it takes a computer to perform a single operation), we will denote the number of operations performed by an algorithm by $T(n)$. We will typically be interested in $T(n)$ for large values of n, so comparing this function for different algorithms will allow us to determine which will take less time to execute.

Abu Ja'far Muhammad ibn Musa al-Khwarizmi (c. 780–850) was a Persian mathematician whose book *Hisab al-jabr w'al muqabalah* (c. 825) described the use of Hindu-Arabic numerals and the rules of basic arithmetic. The second word of the book's title gives rise to the English word *algebra*, and the word *algorithm* is derived from al-Khwarizmi's name.

1. Consider the augmented matrix

$$[A \mid \mathbf{b}] = \left[\begin{array}{rrr|r} 2 & 4 & 6 & 8 \\ 3 & 9 & 6 & 12 \\ -1 & 1 & -1 & 1 \end{array}\right]$$

Count the number of operations required to bring $[A \mid \mathbf{b}]$ to the row echelon form

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 4 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

(By "operation" we mean a multiplication or a division.) Now count the number of operations needed to complete the back substitution phase of Gaussian elimination. Record the total number of operations.

2. Count the number of operations needed to perform Gauss-Jordan elimination—that is, to reduce $[A \mid \mathbf{b}]$ to its reduced row echelon form

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

(where the zeros are introduced into each column immediately after the leading 1 is created in that column). What do your answers suggest about the relative efficiency of the two algorithms?

We will now attempt to analyze the algorithms in a general, systematic way. Suppose the augmented matrix $[A \mid \mathbf{b}]$ arises from a linear system with n equations and n variables; thus, $[A \mid \mathbf{b}]$ is $n \times (n+1)$:

$$[A \mid \mathbf{b}] = \left[\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} & b_n \end{array}\right]$$

We will assume that row interchanges are never needed—that we can always create a leading 1 from a pivot by dividing by the pivot.

3. (a) Show that n operations are needed to create the first leading 1:

$$\left[\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} & b_n \end{array}\right] \longrightarrow \left[\begin{array}{cccc|c} 1 & * & \cdots & * & * \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} & b_n \end{array}\right]$$

(Why don't we need to count an operation for the creation of the leading 1?) Now show that n operations are needed to obtain the first zero in column 1:

$$\left[\begin{array}{cccc|c} 1 & * & \cdots & * & * \\ 0 & * & \cdots & * & * \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} & b_n \end{array}\right]$$

(Why don't we need to count an operation for the creation of the zero itself?) When the first column has been "swept out," we have the matrix

$$\left[\begin{array}{cccc|c} 1 & * & \cdots & * & * \\ 0 & * & \cdots & * & * \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & * & \cdots & * & * \end{array}\right]$$

Show that the total number of operations needed up to this point is $n + (n-1)n$.

(b) Show that the total number of operations needed to reach the row echelon form

$$\left[\begin{array}{cccc|c} 1 & * & \cdots & * & * \\ 0 & 1 & \cdots & * & * \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & * \end{array}\right]$$

is

$$[n + (n-1)n] + [(n-1) + (n-2)(n-1)] + [(n-2) + (n-3)(n-2)] + \cdots + [2 + 1 \cdot 2] + 1$$

which simplifies to

$$n^2 + (n-1)^2 + \cdots + 2^2 + 1^2$$

(c) Show that the number of operations needed to complete the back substitution phase is

$$1 + 2 + \cdots + (n-1)$$

(d) Using summation formulas for the sums in parts (b) and (c) (see Exercises 51 and 52 in Section 2.4 and Appendix B), show that the total number of operations, $T(n)$, performed by Gaussian elimination is

$$T(n) = \tfrac{1}{3}n^3 + n^2 - \tfrac{1}{3}n$$

Since every polynomial function is dominated by its leading term for large values of the variable, we see that $T(n) \approx \frac{1}{3}n^3$ for large values of n.

4. Show that Gauss-Jordan elimination has $T(n) \approx \frac{1}{2}n^3$ total operations if we create zeros above and below the leading 1s as we go. (This shows that, for large systems of linear equations, Gaussian elimination is faster than this version of Gauss-Jordan elimination.)

2.3 Spanning Sets and Linear Independence

The second of the three roads in our "trivium" is concerned with linear combinations of vectors. We have seen that we can view solving a system of linear equations as asking whether a certain vector is a linear combination of certain other vectors. We explore this idea in more detail in this section. It leads to some very important concepts, which we will encounter repeatedly in later chapters.

Spanning Sets of Vectors

We can now easily answer the question raised in Section 1.1: When is a given vector a linear combination of other given vectors?

Example 2.18

(a) Is the vector $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ a linear combination of the vectors $\begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}$ and $\begin{bmatrix} -1 \\ 1 \\ -3 \end{bmatrix}$?

(b) Is $\begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}$ a linear combination of the vectors $\begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}$ and $\begin{bmatrix} -1 \\ 1 \\ -3 \end{bmatrix}$?

Solution

(a) We want to find scalars x and y such that

$$x\begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix} + y\begin{bmatrix} -1 \\ 1 \\ -3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

Expanding, we obtain the system

$$\begin{aligned} x - y &= 1 \\ y &= 2 \\ 3x - 3y &= 3 \end{aligned}$$

whose augmented matrix is

$$\left[\begin{array}{rr|r} 1 & -1 & 1 \\ 0 & 1 & 2 \\ 3 & -3 & 3 \end{array}\right]$$

(Observe that the columns of the augmented matrix are just the given vectors; notice the order of the vectors—in particular, which vector is the constant vector.)

The reduced echelon form of this matrix is

$$\left[\begin{array}{rr|r} 1 & 0 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{array}\right]$$

(Verify this.) So the solution is $x = 3$, $y = 2$, and the corresponding linear combination is

$$3\begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix} + 2\begin{bmatrix} -1 \\ 1 \\ -3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

(b) Utilizing our observation in part (a), we obtain a linear system whose augmented matrix is

$$\left[\begin{array}{cc|c} 1 & -1 & 2 \\ 0 & 1 & 3 \\ 3 & -3 & 4 \end{array}\right]$$

which reduces to

$$\left[\begin{array}{cc|c} 1 & 0 & 5 \\ 0 & 1 & 3 \\ 0 & 0 & -2 \end{array}\right]$$

revealing that the system has no solution. Thus, in this case, $\begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}$ is not a linear combination of $\begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}$ and $\begin{bmatrix} -1 \\ 1 \\ -3 \end{bmatrix}$.

The notion of a spanning set is intimately connected with the solution of linear systems. Look back at Example 2.18. There we saw that a system with augmented matrix $[A \mid \mathbf{b}]$ has a solution precisely when $\mathbf{b}$ is a linear combination of the columns of A. This is a general fact, summarized in the next theorem.

Theorem 2.4 A system of linear equations with augmented matrix $[A \mid \mathbf{b}]$ is consistent if and only if $\mathbf{b}$ is a linear combination of the columns of A.

Let's revisit Example 2.4, interpreting it in light of Theorem 2.4.

(a) The system

$$\begin{aligned} x - y &= 1 \\ x + y &= 3 \end{aligned}$$

has the unique solution $x = 2$, $y = 1$. Thus,

$$2\begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

See Figure 2.8(a).

(b) The system

$$\begin{aligned} x - y &= 2 \\ 2x - 2y &= 4 \end{aligned}$$

has infinitely many solutions of the form $x = 2 + t$, $y = t$. This implies that

$$(2 + t)\begin{bmatrix} 1 \\ 2 \end{bmatrix} + t\begin{bmatrix} -1 \\ -2 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$$

for all values of t. Geometrically, the vectors $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$, $\begin{bmatrix} -1 \\ -2 \end{bmatrix}$, and $\begin{bmatrix} 2 \\ 4 \end{bmatrix}$ are all parallel and so all lie along the same line through the origin [see Figure 2.8(b)].

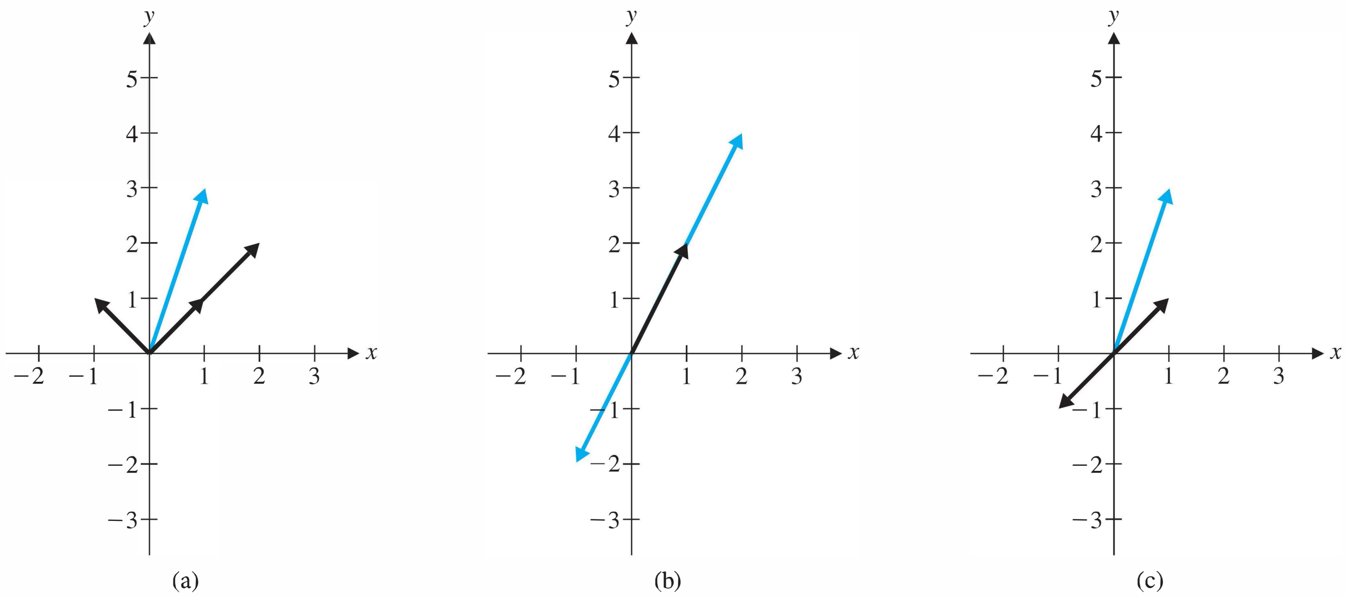


Figure 2.8

(c) The system

$$\begin{aligned} x - y &= 1 \\ x - y &= 3 \end{aligned}$$

has no solutions, so there are no values of x and y that satisfy

$$x\begin{bmatrix} 1 \\ 1 \end{bmatrix} + y\begin{bmatrix} -1 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

In this case, $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} -1 \\ -1 \end{bmatrix}$ are parallel, but $\begin{bmatrix} 1 \\ 3 \end{bmatrix}$ does not lie along the same line through the origin [see Figure 2.8(c)].

We will often be interested in the collection of *all* linear combinations of a given set of vectors.

Definition If $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ is a set of vectors in $\mathbb{R}^n$, then the set of all linear combinations of $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ is called the ***span*** of $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ and is denoted by $\text{span}(\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k)$ or $\text{span}(S)$. If $\text{span}(S) = \mathbb{R}^n$, then S is called a ***spanning set*** for $\mathbb{R}^n$.

Example 2.19

Show that $\mathbb{R}^2 = \text{span}\left(\begin{bmatrix} 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \end{bmatrix}\right)$.

Solution We need to show that an arbitrary vector $\begin{bmatrix} a \\ b \end{bmatrix}$ can be written as a linear combination of $\begin{bmatrix} 2 \\ -1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 3 \end{bmatrix}$; that is, we must show that the equation $x\begin{bmatrix} 2 \\ -1 \end{bmatrix} + y\begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} a \\ b \end{bmatrix}$ can always be solved for x and y (in terms of a and b), regardless of the values of a and b.

The augmented matrix is $\left[\begin{array}{rr|r} 2 & 1 & a \\ -1 & 3 & b \end{array}\right]$, and row reduction produces

$$\left[\begin{array}{rr|r} 2 & 1 & a \\ -1 & 3 & b \end{array}\right] \xrightarrow{R_1 \leftrightarrow R_2} \left[\begin{array}{rr|r} -1 & 3 & b \\ 2 & 1 & a \end{array}\right] \xrightarrow{R_2 + 2R_1} \left[\begin{array}{rr|c} -1 & 3 & b \\ 0 & 7 & a + 2b \end{array}\right]$$

at which point it is clear that the system has a (unique) solution. (Why?) If we continue, we obtain

$$\xrightarrow{\frac{1}{7}R_2} \left[\begin{array}{rr|c} -1 & 3 & b \\ 0 & 1 & (a + 2b)/7 \end{array}\right] \xrightarrow{R_1 - 3R_2} \left[\begin{array}{rr|c} -1 & 0 & (b - 3a)/7 \\ 0 & 1 & (a + 2b)/7 \end{array}\right]$$

from which we see that $x = (3a - b)/7$ and $y = (a + 2b)/7$. Thus, for any choice of a and b, we have

$$\left(\frac{3a - b}{7}\right)\begin{bmatrix} 2 \\ -1 \end{bmatrix} + \left(\frac{a + 2b}{7}\right)\begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} a \\ b \end{bmatrix}$$

(Check this.)

Remark It is also true that $\mathbb{R}^2 = \text{span}\left(\begin{bmatrix} 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 5 \\ 7 \end{bmatrix}\right)$: If, given $\begin{bmatrix} a \\ b \end{bmatrix}$, we can find x and y such that $x\begin{bmatrix} 2 \\ -1 \end{bmatrix} + y\begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} a \\ b \end{bmatrix}$, then we also have $x\begin{bmatrix} 2 \\ -1 \end{bmatrix} + y\begin{bmatrix} 1 \\ 3 \end{bmatrix} + 0\begin{bmatrix} 5 \\ 7 \end{bmatrix} = \begin{bmatrix} a \\ b \end{bmatrix}$. In fact, any set of vectors that *contains* a spanning set for $\mathbb{R}^2$ will also be a spanning set for $\mathbb{R}^2$ (see Exercise 20).

The next example is an important (easy) case of a spanning set. We will encounter versions of this example many times.

Example 2.20

Let $\mathbf{e}_1$, $\mathbf{e}_2$, and $\mathbf{e}_3$ be the standard unit vectors in $\mathbb{R}^3$. Then for any vector $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$, we have

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = x\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + y\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + z\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = x\mathbf{e}_1 + y\mathbf{e}_2 + z\mathbf{e}_3$$

Thus, $\mathbb{R}^3 = \text{span}(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$.

You should have no difficulty seeing that, in general, $\mathbb{R}^n = \text{span}(\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n)$.

When the span of a set of vectors in $\mathbb{R}^n$ is not all of $\mathbb{R}^n$, it is reasonable to ask for a description of the vectors' span.

Example 2.21

Find the span of $\begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}$ and $\begin{bmatrix} -1 \\ 1 \\ -3 \end{bmatrix}$. (See Example 2.18.)

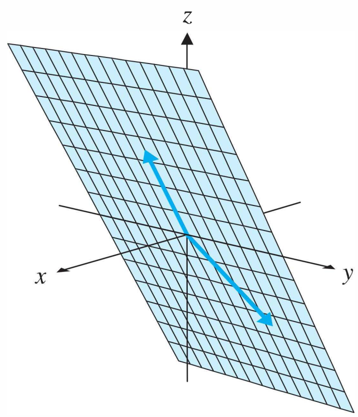


Figure 2.9
Two nonparallel vectors span a plane

Solution Thinking geometrically, we can see that the set of all linear combinations of $\begin{bmatrix}1\\0\\3\end{bmatrix}$ and $\begin{bmatrix}-1\\1\\-3\end{bmatrix}$ is just the plane through the origin with $\begin{bmatrix}1\\0\\3\end{bmatrix}$ and $\begin{bmatrix}-1\\1\\-3\end{bmatrix}$ as direction vectors (Figure 2.9). The vector equation of this plane is $\begin{bmatrix}x\\y\\z\end{bmatrix} = s\begin{bmatrix}1\\0\\3\end{bmatrix} + t\begin{bmatrix}-1\\1\\-3\end{bmatrix}$, which is just another way of saying that $\begin{bmatrix}x\\y\\z\end{bmatrix}$ is in the span of $\begin{bmatrix}1\\0\\3\end{bmatrix}$ and $\begin{bmatrix}-1\\1\\-3\end{bmatrix}$.

Suppose we want to obtain the general equation of this plane. There are several ways to proceed. One is to use the fact that the equation $ax + by + cz = 0$ must be satisfied by the points $(1, 0, 3)$ and $(-1, 1, -3)$ determined by the direction vectors. Substitution then leads to a system of equations in a, b, and c. (See Exercise 17.)

Another method is to use the system of equations arising from the vector equation:

$$\begin{aligned} s - t &= x \\ t &= y \\ 3s - 3t &= z \end{aligned}$$

If we row reduce the augmented matrix, we obtain

$$\left[\begin{array}{rr|c} 1 & -1 & x \\ 0 & 1 & y \\ 3 & -3 & z \end{array}\right] \xrightarrow{R_3 - 3R_1} \left[\begin{array}{rr|c} 1 & -1 & x \\ 0 & 1 & y \\ 0 & 0 & z - 3x \end{array}\right]$$

Now we know that this system is consistent, since $\begin{bmatrix}x\\y\\z\end{bmatrix}$ *is* in the span of $\begin{bmatrix}1\\0\\3\end{bmatrix}$ and $\begin{bmatrix}-1\\1\\-3\end{bmatrix}$ by assumption. So we *must* have $z - 3x = 0$ (or $3x - z = 0$, in more standard form), giving us the general equation we seek.

Remark A normal vector to the plane in this example is also given by the cross product

$$\begin{bmatrix}1\\0\\3\end{bmatrix} \times \begin{bmatrix}-1\\1\\-3\end{bmatrix} = \begin{bmatrix}-3\\0\\1\end{bmatrix}$$

Linear Independence

In Example 2.18, we found that $3\begin{bmatrix}1\\0\\3\end{bmatrix} + 2\begin{bmatrix}-1\\1\\-3\end{bmatrix} = \begin{bmatrix}1\\2\\3\end{bmatrix}$. Let's abbreviate this equation as $3\mathbf{u} + 2\mathbf{v} = \mathbf{w}$. The vector $\mathbf{w}$ "depends" on $\mathbf{u}$ and $\mathbf{v}$ in the sense that it is a linear combination of them. We say that a set of vectors is ***linearly dependent*** if one of them can be written as a linear combination of the others. Note that we also have $\mathbf{u} = -\frac{2}{3}\mathbf{v} + \frac{1}{3}\mathbf{w}$ and $\mathbf{v} = -\frac{3}{2}\mathbf{u} + \frac{1}{2}\mathbf{w}$. To get around the question of which vector to express in terms of the rest, the formal definition is stated as follows:

Definition A set of vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ is ***linearly dependent*** if there are scalars $c_1, c_2, \ldots, c_k$, *at least one of which is not zero*, such that

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_k\mathbf{v}_k = \mathbf{0}$$

A set of vectors that is not linearly dependent is called ***linearly independent.***

Remarks

- In the definition of linear dependence, the requirement that at least one of the scalars $c_1, c_2, \ldots, c_k$ must be nonzero allows for the possibility that some may be zero. In the example above, $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ are linearly dependent, since $3\mathbf{u} + 2\mathbf{v} - \mathbf{w} = \mathbf{0}$ and, in fact, *all* of the scalars are nonzero. On the other hand,

$$\begin{bmatrix} 2 \\ 6 \end{bmatrix} - 2\begin{bmatrix} 1 \\ 3 \end{bmatrix} + 0\begin{bmatrix} 4 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

so $\begin{bmatrix} 2 \\ 6 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \end{bmatrix}$, and $\begin{bmatrix} 4 \\ 1 \end{bmatrix}$ are linearly dependent, since at least one (in fact, two) of the three scalars 1, -2, and 0 is nonzero. (Note that the actual dependence arises simply from the fact that the first two vectors are multiples.) (See Exercise 44.)

- Since $0\mathbf{v}_1 + 0\mathbf{v}_2 + \cdots + 0\mathbf{v}_k = \mathbf{0}$ for *any* vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$, linear dependence essentially says that the zero vector can be expressed as a *nontrivial* linear combination of $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$. Thus, linear independence means that the zero vector can be expressed as a linear combination of $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ *only* in the trivial way: $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_k\mathbf{v}_k = \mathbf{0}$ only if $c_1 = 0, c_2 = 0, \ldots, c_k = 0$.

The relationship between the intuitive notion of dependence and the formal definition is given in the next theorem. Happily, the two notions are equivalent!

Theorem 2.5 Vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m$ in $\mathbb{R}^n$ are linearly dependent if and only if at least one of the vectors can be expressed as a linear combination of the others.

Proof If one of the vectors—say, $\mathbf{v}_1$—is a linear combination of the others, then there are scalars $c_2, \ldots, c_m$ such that $\mathbf{v}_1 = c_2\mathbf{v}_2 + \cdots + c_m\mathbf{v}_m$. Rearranging, we obtain $\mathbf{v}_1 - c_2\mathbf{v}_2 - \cdots - c_m\mathbf{v}_m = \mathbf{0}$, which implies that $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m$ are linearly dependent, since at least one of the scalars (namely, the coefficient 1 of $\mathbf{v}_1$) is nonzero.

Conversely, suppose that $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m$ are linearly dependent. Then there are scalars $c_1, c_2, \ldots, c_m$, not all zero, such that $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_m\mathbf{v}_m = \mathbf{0}$. Suppose $c_1 \neq 0$. Then

$$c_1\mathbf{v}_1 = -c_2\mathbf{v}_2 - \cdots - c_m\mathbf{v}_m$$

and we may multiply both sides by $1/c_1$ to obtain $\mathbf{v}_1$ as a linear combination of the other vectors:

$$\mathbf{v}_1 = -\left(\frac{c_2}{c_1}\right)\mathbf{v}_2 - \cdots - \left(\frac{c_m}{c_1}\right)\mathbf{v}_m$$

Note It may appear as if we are cheating a bit in this proof. After all, we cannot be sure that $\mathbf{v}_1$ is a linear combination of the other vectors, nor that c_1 is nonzero. However, the argument is analogous for some other vector $\mathbf{v}_i$ or for a different scalar c_j. Alternatively, we can just relabel things so that they work out as in the above proof. In a situation like this, a mathematician might begin by saying, "without loss of generality, we may assume that $\mathbf{v}_1$ is a linear combination of the other vectors" and then proceed as above.

Example 2.22

Any set of vectors containing the zero vector is linearly dependent. For if $\mathbf{0}, \mathbf{v}_2, \ldots, \mathbf{v}_m$ are in $\mathbb{R}^n$, then we can find a nontrivial combination of the form $c_1\mathbf{0} + c_2\mathbf{v}_2 + \cdots + c_m\mathbf{v}_m = \mathbf{0}$ by setting $c_1 = 1$ and $c_2 = c_3 = \cdots = c_m = 0$.

Example 2.23

Determine whether the following sets of vectors are linearly independent:

(a) $\begin{bmatrix} 1 \\ 4 \end{bmatrix}$ and $\begin{bmatrix} -1 \\ 2 \end{bmatrix}$ (b) $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$, and $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$

(c) $\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$, and $\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$ (d) $\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$, and $\begin{bmatrix} 1 \\ 4 \\ 2 \end{bmatrix}$

Solution In answering any question of this type, it is a good idea to see if you can determine by inspection whether one vector is a linear combination of the others. A little thought may save a lot of computation!

(a) The only way two vectors can be linearly dependent is if one is a multiple of the other. (Why?) These two vectors are clearly not multiples, so they are linearly independent.

(b) There is no obvious dependence relation here, so we try to find scalars c_1, c_2, c_3 such that

$$c_1\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + c_2\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} + c_3\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The corresponding linear system is

$$\begin{aligned} c_1 + c_3 &= 0 \\ c_1 + c_2 &= 0 \\ c_2 + c_3 &= 0 \end{aligned}$$

and the augmented matrix is

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{array}\right]$$

Once again, we make the fundamental observation that the columns of the coefficient matrix are just the vectors in question!

The reduced row echelon form is

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

(check this), so $c_1 = 0$, $c_2 = 0$, $c_3 = 0$. Thus, the given vectors are linearly independent.

(c) A little reflection reveals that

$$\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} + \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

so the three vectors are linearly dependent. [Set up a linear system as in part (b) to check this algebraically.]

(d) Once again, we observe no obvious dependence, so we proceed directly to reduce a homogeneous linear system whose augmented matrix has as its columns the given vectors:

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 2 & 1 & 4 & 0 \\ 0 & -1 & 2 & 0 \end{array}\right] \xrightarrow{R_2 - 2R_1} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & -1 & 2 & 0 \\ 0 & -1 & 2 & 0 \end{array}\right] \xrightarrow[-R_2]{\substack{R_1 + R_2 \\ R_3 - R_2}} \left[\begin{array}{ccc|c} 1 & 0 & 3 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

If we let the scalars be c_1, c_2, and c_3, we have

$$\begin{aligned} c_1 + \quad 3c_3 &= 0 \\ c_2 - 2c_3 &= 0 \end{aligned}$$

from which we see that the system has infinitely many solutions. In particular, there must be a nonzero solution, so the given vectors are linearly dependent.

If we continue, we can describe these solutions exactly: $c_1 = -3c_3$ and $c_2 = 2c_3$. Thus, for any nonzero value of c_3, we have the linear dependence relation

$$-3c_3\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} + 2c_3\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} + c_3\begin{bmatrix} 1 \\ 4 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

(Once again, check that this is correct.)

We summarize this procedure for testing for linear independence as a theorem.

Theorem 2.6 Let $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m$ be (column) vectors in $\mathbb{R}^n$ and let A be the $n \times m$ matrix $[\mathbf{v}_1\ \mathbf{v}_2 \cdots \mathbf{v}_m]$ with these vectors as its columns. Then $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m$ are linearly dependent if and only if the homogeneous linear system with augmented matrix $[A \mid \mathbf{0}]$ has a nontrivial solution.

Proof $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m$ are linearly dependent if and only if there are scalars $c_1, c_2, \ldots, c_m$, not all zero, such that $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_m\mathbf{v}_m = \mathbf{0}$. By Theorem 2.4, this is equivalent to saying that the nonzero vector $\begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_m \end{bmatrix}$ is a solution of the system whose augmented matrix is $[\mathbf{v}_1\ \mathbf{v}_2 \cdots \mathbf{v}_m \mid \mathbf{0}]$.

Example 2.24

The standard unit vectors $\mathbf{e}_1$, $\mathbf{e}_2$, and $\mathbf{e}_3$ are linearly independent in $\mathbb{R}^3$, since the system with augmented matrix $[\mathbf{e}_1\ \mathbf{e}_2\ \mathbf{e}_3 \mid \mathbf{0}]$ is already in the reduced row echelon form

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

and so clearly has only the trivial solution. In general, we see that $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n$ will be linearly independent in $\mathbb{R}^n$.

Performing elementary row operations on a matrix constructs linear combinations of the rows. We can use this fact to come up with another way to test vectors for linear independence.

Example 2.25

Consider the three vectors of Example 2.23(d) as *row* vectors:

$$[1, 2, 0], \quad [1, 1, -1], \quad \text{and} \quad [1, 4, 2]$$

We construct a matrix with these vectors as its rows and proceed to reduce it to echelon form. Each time a row changes, we denote the new row by adding a prime symbol:

$$\begin{bmatrix} 1 & 2 & 0 \\ 1 & 1 & -1 \\ 1 & 4 & 2 \end{bmatrix} \xrightarrow[R_3' = R_3 - R_1]{R_2' = R_2 - R_1} \begin{bmatrix} 1 & 2 & 0 \\ 0 & -1 & -1 \\ 0 & 2 & 2 \end{bmatrix} \xrightarrow{R_3'' = R_3' + 2R_2'} \begin{bmatrix} 1 & 2 & 0 \\ 0 & -1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

From this we see that

$$\mathbf{0} = R_3'' = R_3' + 2R_2' = (R_3 - R_1) + 2(R_2 - R_1) = -3R_1 + 2R_2 + R_3$$

or, in terms of the original vectors,

$$-3[1, 2, 0] + 2[1, 1, -1] + [1, 4, 2] = [0, 0, 0]$$

[Notice that this approach corresponds to taking $c_3 = 1$ in the solution to Example 2.23(d).]

Thus, the rows of a matrix will be linearly dependent if elementary row operations can be used to create a zero row. We summarize this finding as follows:

Theorem 2.7

Let $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m$ be (row) vectors in $\mathbb{R}^n$ and let A be the $m \times n$ matrix $\begin{bmatrix} \mathbf{v}_1 \\ \mathbf{v}_2 \\ \vdots \\ \mathbf{v}_m \end{bmatrix}$ with these vectors as its rows. Then $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m$ are linearly dependent if and only if $\operatorname{rank}(A) < m$.

Proof Assume that $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m$ are linearly dependent. Then, by Theorem 2.2, at least one of the vectors can be written as a linear combination of the others.

We relabel the vectors, if necessary, so that we can write $\mathbf{v}_m = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_{m-1}\mathbf{v}_{m-1}$. Then the elementary row operations $R_m - c_1R_1, R_m - c_2R_2, \ldots, R_m - c_{m-1}R_{m-1}$ applied to A will create a zero row in row m. Thus, $\operatorname{rank}(A) < m$.

Conversely, assume that $\operatorname{rank}(A) < m$. Then there is some sequence of row operations that will create a zero row. A successive substitution argument analogous to that used in Example 2.25 can be used to show that $\mathbf{0}$ is a nontrivial linear combination of $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m$. Thus, $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m$ are linearly dependent.

In some situations, we can deduce that a set of vectors is linearly dependent without doing any work. One such situation is when the zero vector is in the set (as in Example 2.22). Another is when there are "too many" vectors to be independent. The following theorem summarizes this case. (We will see a sharper version of this result in Chapter 6.)

Theorem 2.8 Any set of m vectors in $\mathbb{R}^n$ is linearly dependent if $m > n$.

Proof Let $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m$ be (column) vectors in $\mathbb{R}^n$ and let A be the $n \times m$ matrix $[\mathbf{v}_1\ \mathbf{v}_2 \cdots \mathbf{v}_m]$ with these vectors as its columns. By Theorem 2.6, $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m$ are linearly dependent if and only if the homogeneous linear system with augmented matrix $[A \mid \mathbf{0}]$ has a nontrivial solution. But, according to Theorem 2.6, this will always be the case if A has more columns than rows; it is the case here, since number of columns m is greater than number of rows n.

Example 2.26

The vectors $\begin{bmatrix}1\\3\end{bmatrix}, \begin{bmatrix}2\\4\end{bmatrix}$, and $\begin{bmatrix}3\\1\end{bmatrix}$ are linearly dependent, since there cannot be more than two linearly independent vectors in $\mathbb{R}^2$. (Note that if we want to find the actual dependence relation among these three vectors, we must solve the homogeneous system whose coefficient matrix has the given vectors as columns. Do this!)

Exercises 2.3

In Exercises 1–6, determine if the vector $\mathbf{v}$ *is a linear combination of the remaining vectors.*

1. $\mathbf{v} = \begin{bmatrix}1\\2\end{bmatrix}, \mathbf{u}_1 = \begin{bmatrix}1\\-1\end{bmatrix}, \mathbf{u}_2 = \begin{bmatrix}2\\-1\end{bmatrix}$

2. $\mathbf{v} = \begin{bmatrix}2\\1\end{bmatrix}, \mathbf{u}_1 = \begin{bmatrix}4\\-2\end{bmatrix}, \mathbf{u}_2 = \begin{bmatrix}-2\\1\end{bmatrix}$

3. $\mathbf{v} = \begin{bmatrix}1\\2\\3\end{bmatrix}, \mathbf{u}_1 = \begin{bmatrix}1\\1\\0\end{bmatrix}, \mathbf{u}_2 = \begin{bmatrix}0\\1\\1\end{bmatrix}$

4. $\mathbf{v} = \begin{bmatrix}3\\2\\-1\end{bmatrix}, \mathbf{u}_1 = \begin{bmatrix}1\\1\\0\end{bmatrix}, \mathbf{u}_2 = \begin{bmatrix}0\\1\\1\end{bmatrix}$

5. $\mathbf{v} = \begin{bmatrix}1\\2\\3\end{bmatrix}, \mathbf{u}_1 = \begin{bmatrix}1\\1\\0\end{bmatrix}, \mathbf{u}_2 = \begin{bmatrix}0\\1\\1\end{bmatrix},$

$\mathbf{u}_3 = \begin{bmatrix}1\\0\\1\end{bmatrix}$

CAS **6.** $\mathbf{v} = \begin{bmatrix}3.2\\2.0\\-2.6\end{bmatrix}, \mathbf{u}_1 = \begin{bmatrix}1.0\\0.4\\4.8\end{bmatrix}, \mathbf{u}_2 = \begin{bmatrix}3.4\\1.4\\-6.4\end{bmatrix},$

$\mathbf{u}_3 = \begin{bmatrix}-1.2\\0.2\\-1.0\end{bmatrix}$

In Exercises 7 and 8, determine if the vector **b** *is in the span of the columns of the matrix A.*

7. $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} 5 \\ 6 \end{bmatrix}$

8. $A = \begin{bmatrix} 1 & 2 & 3 \\ 5 & 6 & 7 \\ 9 & 10 & 11 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} 4 \\ 8 \\ 12 \end{bmatrix}$

9. Show that $\mathbb{R}^2 = \text{span}\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix}\right)$.

10. Show that $\mathbb{R}^2 = \text{span}\left(\begin{bmatrix} 3 \\ -2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right)$.

11. Show that $\mathbb{R}^3 = \text{span}\left(\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}\right)$.

12. Show that $\mathbb{R}^3 = \text{span}\left(\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}\right)$.

In Exercises 13–16, describe the span of the given vectors (a) geometrically and (b) algebraically.

13. $\begin{bmatrix} 2 \\ -4 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \end{bmatrix}$

14. $\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 4 \end{bmatrix}$

15. $\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \\ -1 \end{bmatrix}$

16. $\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$

17. The general equation of the plane that contains the points $(1, 0, 3)$, $(-1, 1, -3)$, and the origin is of the form $ax + by + cz = 0$. Solve for a, b, and c.

18. Prove that $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ are all in $\text{span}(\mathbf{u}, \mathbf{v}, \mathbf{w})$.

19. Prove that $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ are all in $\text{span}(\mathbf{u}, \mathbf{u} + \mathbf{v}, \mathbf{u} + \mathbf{v} + \mathbf{w})$.

20. (a) Prove that if $\mathbf{u}_1, \ldots, \mathbf{u}_m$ are vectors in $\mathbb{R}^n$, $S = \{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_k\}$, and $T = \{\mathbf{u}_1, \ldots, \mathbf{u}_k, \mathbf{u}_{k+1}, \ldots, \mathbf{u}_m\}$, then $\text{span}(S) \subseteq \text{span}(T)$. [*Hint:* Rephrase this question in terms of linear combinations.]

(b) Deduce that if $\mathbb{R}^n = \text{span}(S)$, then $\mathbb{R}^n = \text{span}(T)$ also.

21. (a) Suppose that vector $\mathbf{w}$ is a linear combination of vectors $\mathbf{u}_1, \ldots, \mathbf{u}_k$ and that each $\mathbf{u}_i$ is a linear combination of vectors $\mathbf{v}_1, \ldots, \mathbf{v}_m$. Prove that $\mathbf{w}$ is a linear combination of $\mathbf{v}_1, \ldots, \mathbf{v}_m$ and therefore $\text{span}(\mathbf{u}_1, \ldots, \mathbf{u}_k) \subseteq \text{span}(\mathbf{v}_1, \ldots, \mathbf{v}_m)$.

(b) In part (a), suppose in addition that each $\mathbf{v}_j$ is also a linear combination of $\mathbf{u}_1, \ldots, \mathbf{u}_k$. Prove that $\text{span}(\mathbf{u}_1, \ldots, \mathbf{u}_k) = \text{span}(\mathbf{v}_1, \ldots, \mathbf{v}_m)$.

(c) Use the result of part (b) to prove that

$$\mathbb{R}^3 = \text{span}\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right)$$

[*Hint:* We know that $\mathbb{R}^3 = \text{span}(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$.]

Use the method of Example 2.23 and Theorem 2.6 to determine if the sets of vectors in Exercises 22–31 are linearly independent. If, for any of these, the answer can be determined by inspection (i.e., without calculation), state why. For any sets that are linearly dependent, find a dependence relationship among the vectors.

22. $\begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \\ 3 \end{bmatrix}$

23. $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$

24. $\begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ -5 \\ 2 \end{bmatrix}$

25. $\begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$

26. $\begin{bmatrix} -2 \\ 3 \\ 7 \end{bmatrix}, \begin{bmatrix} 4 \\ -1 \\ 5 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 5 \\ 0 \\ 2 \end{bmatrix}$

27. $\begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix}, \begin{bmatrix} 6 \\ 7 \\ 8 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

28. $\begin{bmatrix} -1 \\ 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \\ 2 \\ 4 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 1 \\ -1 \end{bmatrix}$

29. $\begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \\ 1 \end{bmatrix}$

30. $\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 3 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 4 \\ 3 \\ 2 \\ 1 \end{bmatrix}$

31. $\begin{bmatrix} 3 \\ -1 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} -1 \\ 3 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 3 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ -1 \\ 1 \\ 3 \end{bmatrix}$

In Exercises 32–41, determine if the sets of vectors in the given exercise are linearly independent by converting the *vectors to row vectors and using the method of Example 2.25 and Theorem 2.7. For any sets that are linearly dependent, find a dependence relationship among the vectors.*

32. Exercise 22

33. Exercise 23

34. Exercise 24

35. Exercise 25

36. Exercise 26

37. Exercise 27

38. Exercise 28

39. Exercise 29

40. Exercise 30

41. Exercise 31

42. **(a)** If the columns of an $n \times n$ matrix A are linearly independent as vectors in $\mathbb{R}^n$, what is the rank of A? Explain.

(b) If the rows of an $n \times n$ matrix A are linearly independent as vectors in $\mathbb{R}^n$, what is the rank of A? Explain.

43. **(a)** If vectors $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ are linearly independent, will $\mathbf{u} + \mathbf{v}$, $\mathbf{v} + \mathbf{w}$, and $\mathbf{u} + \mathbf{w}$ also be linearly independent? Justify your answer.

(b) If vectors $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ are linearly independent, will $\mathbf{u} - \mathbf{v}$, $\mathbf{v} - \mathbf{w}$, and $\mathbf{u} - \mathbf{w}$ also be linearly independent? Justify your answer.

44. Prove that two vectors are linearly dependent if and only if one is a scalar multiple of the other. [*Hint:* Separately consider the case where one of the vectors is $\mathbf{0}$.]

45. Give a "row vector proof" of Theorem 2.8.

46. Prove that every subset of a linearly independent set is linearly independent.

47. Suppose that $S = \{\mathbf{v}_1, \ldots, \mathbf{v}_k, \mathbf{v}\}$ is a set of vectors in some $\mathbb{R}^n$ and that $\mathbf{v}$ is a linear combination of $\mathbf{v}_1, \ldots, \mathbf{v}_k$. If $S' = \{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$, prove that $\text{span}(S) = \text{span}(S')$. [*Hint:* Exercise 21(b) is helpful here.]

48. Let $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ be a linearly independent set of vectors in $\mathbb{R}^n$, and let $\mathbf{v}$ be a vector in $\mathbb{R}^n$. Suppose that $\mathbf{v} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_k\mathbf{v}_k$ with $c_1 \neq 0$. Prove that $\{\mathbf{v}, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ is linearly independent.

2.4 Applications

There are too many applications of systems of linear equations to do them justice in a single section. This section will introduce a few applications, to illustrate the diverse settings in which they arise.

Allocation of Resources

A great many applications of systems of linear equations involve allocating limited resources subject to a set of constraints.

Example 2.27

A biologist has placed three strains of bacteria (denoted I, II, and III) in a test tube, where they will feed on three different food sources (A, B, and C). Each day 2300 units of A, 800 units of B, and 1500 units of C are placed in the test tube, and each bacterium consumes a certain number of units of each food per day, as shown in Table 2.2. How many bacteria of each strain can coexist in the test tube and consume all of the food?

Table 2.2

	Bacteria Strain I	Bacteria Strain II	Bacteria Strain III
Food A	2	2	4
Food B	1	2	0
Food C	1	3	1

Solution Let x_1, x_2, and x_3 be the numbers of bacteria of strains I, II, and III, respectively. Since each of the x_1 bacteria of strain I consumes 2 units of A per day, strain I consumes a total of $2x_1$ units per day. Similarly, strains II and III consume a total of $2x_2$ and $4x_3$ units of food A daily. Since we want to use up all of the 2300 units of A, we have the equation

$$2x_1 + 2x_2 + 4x_3 = 2300$$

Likewise, we obtain equations corresponding to the consumption of B and C:

$$\begin{aligned} x_1 + 2x_2 \phantom{{}+x_3} &= 800 \\ x_1 + 3x_2 + x_3 &= 1500 \end{aligned}$$

Thus, we have a system of three linear equations in three variables. Row reduction of the corresponding augmented matrix gives

$$\left[\begin{array}{ccc|c} 2 & 2 & 4 & 2300 \\ 1 & 2 & 0 & 800 \\ 1 & 3 & 1 & 1500 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & 100 \\ 0 & 1 & 0 & 350 \\ 0 & 0 & 1 & 350 \end{array}\right]$$

Therefore, $x_1 = 100$, $x_2 = 350$, and $x_3 = 350$. The biologist should place 100 bacteria of strain I and 350 of each of strains II and III in the test tube if she wants all the food to be consumed.

Example 2.28

Repeat Example 2.27, using the data on daily consumption of food (units per day) shown in Table 2.3. Assume this time that 1500 units of A, 3000 units of B, and 4500 units of C are placed in the test tube each day.

Table 2.3

	Bacteria Strain I	Bacteria Strain II	Bacteria Strain III
Food A	1	1	1
Food B	1	2	3
Food C	1	3	5

Solution Let x_1, x_2, and x_3 again be the numbers of bacteria of each type. The augmented matrix for the resulting linear system and the corresponding reduced echelon form are

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 1500 \\ 1 & 2 & 3 & 3000 \\ 1 & 3 & 5 & 4500 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 2 & 1500 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

We see that in this case we have more than one solution, given by

$$\begin{aligned} x_1 \phantom{{}+2x_2} - x_3 &= 0 \\ x_2 + 2x_3 &= 1500 \end{aligned}$$

Letting $x_3 = t$, we obtain $x_1 = t$, $x_2 = 1500 - 2t$, and $x_3 = t$. In any applied problem, we must be careful to interpret solutions properly. Certainly the number of bacteria cannot be negative. Therefore, $t \geq 0$ and $1500 - 2t \geq 0$. The latter inequality implies that $t \leq 750$, so we have $0 \leq t \leq 750$. Presumably the number of bacteria must be a whole number, so there are exactly 751 values of t that satisfy the inequality. Thus, our 751 solutions are of the form

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} t \\ 1500 - 2t \\ t \end{bmatrix} = \begin{bmatrix} 0 \\ 1500 \\ 0 \end{bmatrix} + t\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$$

one for each integer value of t such that $0 \leq t \leq 750$. (So, although mathematically this system has infinitely many solutions, *physically* there are only finitely many.)

Balancing Chemical Equations

When a chemical reaction occurs, certain molecules (the *reactants*) combine to form new molecules (the *products*). A ***balanced chemical equation*** is an algebraic equation that gives the relative numbers of reactants and products in the reaction and has the same number of atoms of each type on the left- and right-hand sides. The equation is usually written with the reactants on the left, the products on the right, and an arrow in between to show the direction of the reaction.

For example, for the reaction in which hydrogen gas (H_2) and oxygen (O_2) combine to form water (H_2O), a balanced chemical equation is

$$2H_2 + O_2 \longrightarrow 2H_2O$$

indicating that two molecules of hydrogen combine with one molecule of oxygen to form two molecules of water. Observe that the equation is balanced, since there are four hydrogen atoms and two oxygen atoms on each side. Note that there will never be a unique balanced equation for a reaction, since any positive integer multiple of a balanced equation will also be balanced. For example, $6H_2 + 3O_2 \longrightarrow 6H_2O$ is also balanced. Therefore, we usually look for the *simplest* balanced equation for a given reaction.

While trial and error will often work in simple examples, the process of balancing chemical equations really involves solving a homogeneous system of linear equations, so we can use the techniques we have developed to remove the guesswork.

Example 2.29

The combustion of ammonia (NH_3) in oxygen produces nitrogen (N_2) and water. Find a balanced chemical equation for this reaction.

Solution If we denote the numbers of molecules of ammonia, oxygen, nitrogen, and water by w, x, y, and z, respectively, then we are seeking an equation of the form

$$wNH_3 + xO_2 \longrightarrow yN_2 + zH_2O$$

Comparing the numbers of nitrogen, hydrogen, and oxygen atoms in the reactants and products, we obtain three linear equations:

$$\begin{aligned} \text{Nitrogen:}\quad & w = 2y \\ \text{Hydrogen:}\quad & 3w = 2z \\ \text{Oxygen:}\quad & 2x = z \end{aligned}$$

Rewriting these equations in standard form gives us a homogeneous system of three linear equations in four variables. [Notice that Theorem 2.3 guarantees that such a system will have (infinitely many) nontrivial solutions.] We reduce the corresponding augmented matrix by Gauss-Jordan elimination.

$$\begin{aligned} w \quad - 2y \quad &= 0 \\ 3w \quad - 2z &= 0 \\ 2x \quad - z &= 0 \end{aligned} \longrightarrow \left[\begin{array}{cccc|c} 1 & 0 & -2 & 0 & 0 \\ 3 & 0 & 0 & -2 & 0 \\ 0 & 2 & 0 & -1 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{cccc|c} 1 & 0 & 0 & -\frac{2}{3} & 0 \\ 0 & 1 & 0 & -\frac{1}{2} & 0 \\ 0 & 0 & 1 & -\frac{1}{3} & 0 \end{array}\right]$$

Thus, $w = \frac{2}{3}z$, $x = \frac{1}{2}z$, and $y = \frac{1}{3}z$. The smallest positive value of z that will produce *integer* values for all four variables is the least common denominator of the fractions $\frac{2}{3}$, $\frac{1}{2}$, and $\frac{1}{3}$—namely, 6—which gives $w = 4$, $x = 3$, $y = 2$, and $z = 6$. Therefore, the balanced chemical equation is

$$4NH_3 + 3O_2 \longrightarrow 2N_2 + 6H_2O$$

Network Analysis

Many practical situations give rise to networks: transportation networks, communications networks, and economic networks, to name a few. Of particular interest are the possible *flows* through networks. For example, vehicles flow through a network of roads, information flows through a data network, and goods and services flow through an economic network.

For us, a ***network*** will consist of a finite number of ***nodes*** (also called ***junctions*** or ***vertices***) connected by a series of directed edges known as ***branches*** or ***arcs.*** Each branch will be labeled with a ***flow*** that represents the amount of some commodity that can flow along or through that branch in the indicated direction. (Think of cars traveling along a network of one-way streets.) The fundamental rule governing flow through a network is ***conservation of flow:***

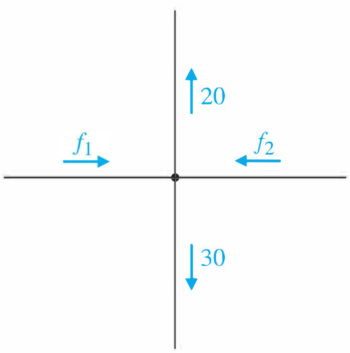


Figure 2.10
Flow at a node: $f_1 + f_2 = 50$

At each node, the flow in equals the flow out.

Figure 2.10 shows a portion of a network, with two branches entering a node and two leaving. The conservation of flow rule implies that the total incoming flow, $f_1 + f_2$ units, must match the total outgoing flow, 20 + 30 units. Thus, we have the linear equation $f_1 + f_2 = 50$ corresponding to this node.

We can analyze the flow through an entire network by constructing such equations and solving the resulting system of linear equations.

Example 2.30

Describe the possible flows through the network of water pipes shown in Figure 2.11, where flow is measured in liters per minute.

Solution At each node, we write out the equation that represents the conservation of flow there. We then rewrite each equation with the variables on the left and the constant on the right, to get a linear system in standard form.

$$\begin{aligned} &\text{Node } A\text{:} \quad 15 = f_1 + f_4 \\ &\text{Node } B\text{:} \quad f_1 = f_2 + 10 \\ &\text{Node } C\text{:} \quad f_2 + f_3 + 5 = 30 \\ &\text{Node } D\text{:} \quad f_4 + 20 = f_3 \end{aligned} \longrightarrow \begin{aligned} f_1 \qquad\qquad + f_4 &= 15 \\ f_1 - f_2 \qquad\qquad &= 10 \\ f_2 + f_3 \qquad &= 25 \\ f_3 - f_4 &= 20 \end{aligned}$$

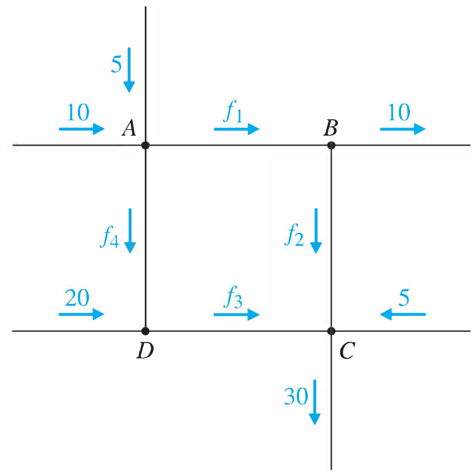


Figure 2.11

Using Gauss-Jordan elimination, we reduce the augmented matrix:

$$\left[\begin{array}{rrrr|r} 1 & 0 & 0 & 1 & 15 \\ 1 & -1 & 0 & 0 & 10 \\ 0 & 1 & 1 & 0 & 25 \\ 0 & 0 & 1 & -1 & 20 \end{array}\right] \longrightarrow \left[\begin{array}{rrrr|r} 1 & 0 & 0 & 1 & 15 \\ 0 & 1 & 0 & 1 & 5 \\ 0 & 0 & 1 & -1 & 20 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

(Check this.) We see that there is one free variable, f_4, so we have infinitely many solutions. Setting $f_4 = t$ and expressing the leading variables in terms of f_4, we obtain

$$\begin{aligned} f_1 &= 15 - t \\ f_2 &= 5 - t \\ f_3 &= 20 + t \\ f_4 &= t \end{aligned}$$

These equations describe all possible flows and allow us to analyze the network. For example, we see that if we control the flow on branch AD so that $t = 5$ L/min, then the other flows are $f_1 = 10, f_2 = 0$, and $f_3 = 25$.

We can do even better: We can find the minimum and maximum possible flows on each branch. Each of the flows must be nonnegative. Examining the first and second equations in turn, we see that $t \leq 15$ (otherwise f_1 would be negative) and $t \leq 5$ (otherwise f_2 would be negative). The second of these inequalities is more restrictive than the first, so we must use it. The third equation contributes no further restrictions on our parameter t, so we have deduced that $0 \leq t \leq 5$. Combining this result with the four equations, we see that

$$\begin{aligned} 10 \leq f_1 \leq 15 \\ 0 \leq f_2 \leq 5 \\ 20 \leq f_3 \leq 25 \\ 0 \leq f_4 \leq 5 \end{aligned}$$

We now have a complete description of the possible flows through this network.

Electrical Networks

Electrical networks are a specialized type of network providing information about power sources, such as batteries, and devices powered by these sources, such as light bulbs or motors. A power source "forces" a current of electrons to flow through the network, where it encounters various *resistors,* each of which requires that a certain amount of force be applied in order for the current to flow through it.

The fundamental law of electricity is Ohm's law, which states exactly how much force E is needed to drive a current I through a resistor with resistance R.

Ohm's Law

$$\text{force} = \text{resistance} \times \text{current}$$

or

$$E = RI$$

Force is measured in *volts,* resistance in *ohms,* and current in *amperes* (or *amps,* for short). Thus, in terms of these units, Ohm's law becomes "volts $=$ ohms $\times$ amps," and it tells us what the "voltage drop" is when a current passes through a resistor—that is, how much voltage is used up.

Current flows out of the positive terminal of a battery and flows back into the negative terminal, traveling around one or more closed circuits in the process. In a diagram of an electrical network, batteries are represented by —|⊢ (where the positive terminal is the longer vertical bar) and resistors are represented by —/\/\/\—. The following two laws, whose discovery we owe to Kirchhoff, govern electrical networks. The first is a "conservation of flow" law at each node; the second is a "balancing of voltage" law around each circuit.

Kirchhoff's Laws

Current Law (nodes)
The sum of the currents flowing into any node is equal to the sum of the currents flowing out of that node.

Voltage Law (circuits)
The sum of the voltage drops around any circuit is equal to the total voltage around the circuit (provided by the batteries).

Figure 2.12 illustrates Kirchhoff's laws. In part (a), the current law gives $I_1 = I_2 + I_3$ (or $I_1 - I_2 - I_3 = 0$, as we will write it); part (b) gives $4I = 10$, where we have used Ohm's law to compute the voltage drop $4I$ at the resistor. Using Kirchhoff's laws, we can set up a system of linear equations that will allow us to determine the currents in an electrical network.

Example 2.31

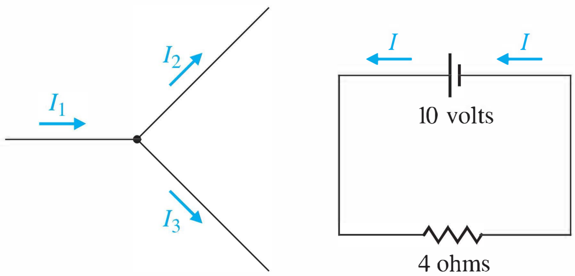
Determine the currents I_1, I_2, and I_3 in the electrical network shown in Figure 2.13.

Solution This network has two batteries and four resistors. Current I_1 flows through the top branch BCA, current I_2 flows across the middle branch AB, and current I_3 flows through the bottom branch BDA.

At node A, the current law gives $I_1 + I_3 = I_2$, or

$$I_1 - I_2 + I_3 = 0$$

(Observe that we get the same equation at node B.)



(a) $I_1 = I_2 + I_3$ (b) $4I = 10$

Figure 2.12

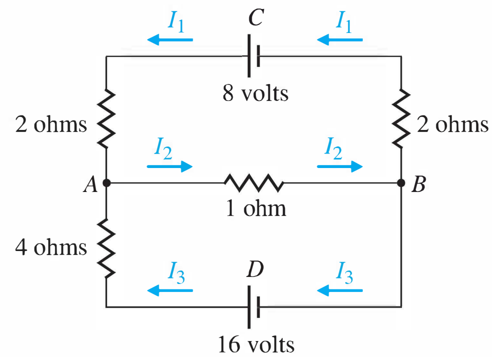


Figure 2.13

Next we apply the voltage law for each circuit. For the circuit *CABC*, the voltage drops at the resistors are $2I_1$, I_2, and $2I_1$. Thus, we have the equation

$$4I_1 + I_2 = 8$$

Similarly, for the circuit *DABD*, we obtain

$$I_2 + 4I_3 = 16$$

(Notice that there is actually a third circuit, *CADBC*, if we "go against the flow." In this case, we must treat the voltages and resistances on the "reversed" paths as negative. Doing so gives $2I_1 + 2I_1 - 4I_3 = 8 - 16 = -8$ or $4I_1 - 4I_3 = -8$, which we observe is just the difference of the voltage equations for the other two circuits. Thus, we can omit this equation, as it contributes no new information. On the other hand, including it does no harm.)

We now have a system of three linear equations in three variables:

$$\begin{aligned} I_1 - I_2 + I_3 &= 0 \\ 4I_1 + I_2 &= 8 \\ I_2 + 4I_3 &= 16 \end{aligned}$$

Gauss-Jordan elimination produces

$$\left[\begin{array}{rrr|r} 1 & -1 & 1 & 0 \\ 4 & 1 & 0 & 8 \\ 0 & 1 & 4 & 16 \end{array}\right] \longrightarrow \left[\begin{array}{rrr|r} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

Hence, the currents are $I_1 = 1$ amp, $I_2 = 4$ amps, and $I_3 = 3$ amps.

Remark In some electrical networks, the currents may have fractional values or may even be negative. A negative value simply means that the current in the corresponding branch flows in the direction opposite that shown on the network diagram.

CAS

Example 2.32

The network shown in Figure 2.14 has a single power source *A* and five resistors. Find the currents $I, I_1, \ldots, I_5$. This is an example of what is known in electrical engineering as a *Wheatstone bridge circuit*.

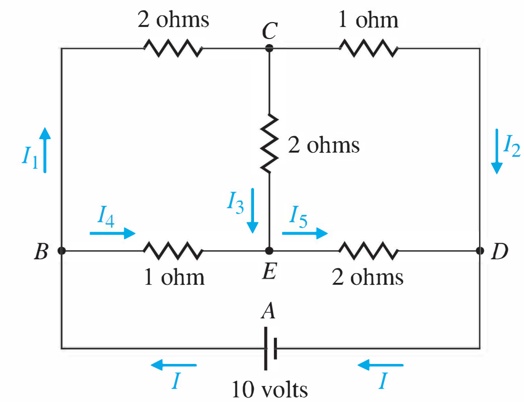


Figure 2.14
A bridge circuit

Solution Kirchhoff's current law gives the following equations at the four nodes:

$$\begin{aligned} &\text{Node } B\text{:} \quad I - I_1 - I_4 = 0 \\ &\text{Node } C\text{:} \quad I_1 - I_2 - I_3 = 0 \\ &\text{Node } D\text{:} \quad I - I_2 - I_5 = 0 \\ &\text{Node } E\text{:} \quad I_3 + I_4 - I_5 = 0 \end{aligned}$$

For the three basic circuits, the voltage law gives

$$\begin{aligned} &\text{Circuit } ABEDA\text{:} \quad I_4 + 2I_5 = 10 \\ &\text{Circuit } BCEB\text{:} \quad 2I_1 + 2I_3 - I_4 = 0 \\ &\text{Circuit } CDEC\text{:} \quad I_2 - 2I_5 - 2I_3 = 0 \end{aligned}$$

(Observe that branch *DAB* has no resistor and therefore no voltage drop; thus, there is no *I* term in the equation for circuit *ABEDA*. Note also that we had to change signs three times because we went "against the current." This poses no problem, since we will let the sign of the answer determine the direction of current flow.)

We now have a system of seven equations in six variables. Row reduction gives

$$\left[\begin{array}{rrrrrr|r} 1 & -1 & 0 & 0 & -1 & 0 & 0 \\ 0 & 1 & -1 & -1 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 2 & 10 \\ 0 & 2 & 0 & 2 & -1 & 0 & 0 \\ 0 & 0 & 1 & -2 & 0 & -2 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{rrrrrr|r} 1 & 0 & 0 & 0 & 0 & 0 & 7 \\ 0 & 1 & 0 & 0 & 0 & 0 & 3 \\ 0 & 0 & 1 & 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 4 \\ 0 & 0 & 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

(Use your calculator or CAS to check this.) Thus, the solution (in amps) is $I = 7$, $I_1 = I_5 = 3$, $I_2 = I_4 = 4$, and $I_3 = -1$. The significance of the negative value here is that the current through branch *CE* is flowing in the direction opposite that marked on the diagram.

Remark There is only one power source in this example, so the single 10-volt battery sends a current of 7 amps through the network. If we substitute these values into

Ohm's law, $E = RI$, we get $10 = 7R$ or $R = \frac{10}{7}$. Thus, the entire network behaves as if there were a single $\frac{10}{7}$-ohm resistor. This value is called the *effective resistance* (R_{eff}) of the network.

Linear Economic Models

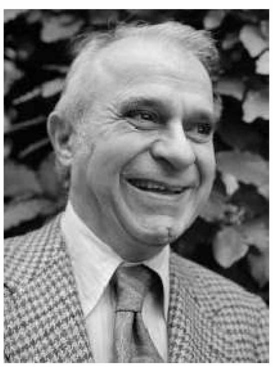
An economy is a very complex system with many interrelationships among the various sectors of the economy and the goods and services they produce and consume. Determining optimal prices and levels of production subject to desired economic goals requires sophisticated mathematical models. Linear algebra has proven to be a powerful tool in developing and analyzing such economic models.

In this section, we introduce two models based on the work of Harvard economist Wassily Leontief in the 1930s. His methods, often referred to as ***input-output analysis***, are now standard tools in mathematical economics and are used by cities, corporations, and entire countries for economic planning and forecasting.

We begin with a simple example.

Example 2.33

The economy of a region consists of three industries, or sectors: service, electricity, and oil production. For simplicity, we assume that each industry produces a single commodity (goods or services) in a given year and that ***income (output)*** is generated from the sale of this commodity. Each industry purchases commodities from the other industries, including itself, in order to generate its output. No commodities are purchased from outside the region and no output is sold outside the region. Furthermore, for each industry, we assume that production exactly equals consumption (output equals input, income equals expenditure). In this sense, this is a ***closed economy*** that is in ***equilibrium***. Table 2.4 summarizes how much of each industry's output is consumed by each industry.



Bettmann/CORBIS

Wassily Leontief (1906–1999) was born in St. Petersburg, Russia. He studied at the University of Leningrad and received his Ph.D. from the University of Berlin. He emigrated to the United States in 1931, teaching at Harvard University and later at New York University. In 1932, Leontief began compiling data for the monumental task of conducting an input-output analysis of the United States economy, the results of which were published in 1941. He was also an early user of computers, which he needed to solve the large-scale linear systems in his models. For his pioneering work, Leontief was awarded the Nobel Prize in Economics in 1973.

Table 2.4

		Produced by (output)		
		Service	**Electricity**	**Oil**
Consumed by (input)	**Service**	1/4	1/3	1/2
	Electricity	1/4	1/3	1/4
	Oil	1/2	1/3	1/4

From the first column of the table, we see that the service industry consumes 1/4 of its own output, electricity consumes another 1/4, and the oil industry uses 1/2 of the service industry's output. The other two columns have similar interpretations. Notice that the sum of each column is 1, indicating that all of the output of each industry is consumed.

Let x_1, x_2, and x_3 denote the annual output (income) of the service, electricity, and oil industries, respectively, in millions of dollars. Since consumption corresponds to expenditure, the service industry spends $\frac{1}{4}x_1$ on its own commodity, $\frac{1}{3}x_2$ on electricity, and $\frac{1}{2}x_3$ on oil. This means that the service industry's total annual expenditure is $\frac{1}{4}x_1 + \frac{1}{3}x_2 + \frac{1}{2}x_3$. Since the economy is in equilibrium, the service industry's expenditure must equal its annual income x_1. This gives the first of the following equations; the other two equations are obtained by analyzing the expenditures of the electricity and oil industries.

$$\begin{aligned} &\text{Service:} && \tfrac{1}{4}x_1 + \tfrac{1}{3}x_2 + \tfrac{1}{2}x_3 = x_1 \\ &\text{Electricity:} && \tfrac{1}{4}x_1 + \tfrac{1}{3}x_2 + \tfrac{1}{4}x_3 = x_2 \\ &\text{Oil:} && \tfrac{1}{2}x_1 + \tfrac{1}{3}x_2 + \tfrac{1}{4}x_3 = x_3 \end{aligned}$$

Rearranging each equation, we obtain a homogeneous system of linear equations, which we then solve. (Check this!)

$$\begin{aligned} -\tfrac{3}{4}x_1 + \tfrac{1}{3}x_2 + \tfrac{1}{2}x_3 &= 0 \\ \tfrac{1}{4}x_1 - \tfrac{2}{3}x_2 - \tfrac{1}{4}x_3 &= 0 \\ \tfrac{1}{2}x_1 + \tfrac{1}{3}x_2 - \tfrac{3}{4}x_3 &= 0 \end{aligned} \longrightarrow \left[\begin{array}{ccc|c} -\frac{3}{4} & \frac{1}{3} & \frac{1}{2} & 0 \\ \frac{1}{4} & -\frac{2}{3} & \frac{1}{4} & 0 \\ \frac{1}{2} & \frac{1}{3} & -\frac{3}{4} & 0 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & -\frac{3}{4} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Setting $x_3 = t$, we find that $x_1 = t$ and $x_2 = \frac{3}{4}t$. Thus, we see that the *relative* outputs of the service, electricity, and oil industries need to be in the ratios $x_1 : x_2 : x_3 = 4 : 3 : 4$ for the economy to be in equilibrium.

Remarks

- The last example illustrates what is commonly called the ***Leontief closed model***.
- Since output corresponds to income, we can also think of x_1, x_2, and x_3 as the *prices* of the three commodities.

We now modify the model in Example 2.33 to accommodate an ***open economy***, one in which there is an *external* as well as an internal demand for the commodities that are produced. Not surprisingly, this version is called the ***Leontief open model***.

Example 2.34

Consider the three industries of Example 2.33 but with consumption given by Table 2.5. We see that, of the commodities produced by the service industry, 20% are consumed by the service industry, 40% by the electricity industry, and 10% by the oil industry. Thus, only 70% of the service industry's output is consumed by this economy. The implication of this calculation is that there is an excess of output (income) over input (expenditure) for the service industry. We say that the service industry is ***productive***. Likewise, the oil industry is productive but the electricity industry is ***non-productive***. (This is reflected in the fact that the sums of the first and third columns are less than 1 but the sum of the second column is equal to 1). The excess output may be applied to satisfy an external demand.

Table 2.5

		Produced by (output)		
		Service	**Electricity**	**Oil**
Consumed by (input)	**Service**	0.20	0.50	0.10
	Electricity	0.40	0.20	0.20
	Oil	0.10	0.30	0.30

For example, suppose there is an annual external demand (in millions of dollars) for 10, 10, and 30 from the service, electricity, and oil industries, respectively. Then, equating expenditures (internal demand and external demand) with income (output), we obtain the following equations:

	output	**internal demand**	**external demand**
Service	x_1	$= 0.2x_1 + 0.5x_2 + 0.1x_3$	$+ 10$
Electricity	x_2	$= 0.4x_1 + 0.2x_2 + 0.2x_3$	$+ 10$
Oil	x_3	$= 0.1x_1 + 0.3x_2 + 0.3x_3$	$+ 30$

Rearranging, we obtain the following linear system and augmented matrix:

$$\begin{aligned} 0.8x_1 - 0.5x_2 - 0.1x_3 &= 10 \\ -0.4x_1 + 0.8x_2 - 0.2x_3 &= 10 \\ -0.1x_1 - 0.3x_2 + 0.7x_3 &= 30 \end{aligned} \longrightarrow \left[\begin{array}{rrr|r} 0.8 & -0.5 & -0.1 & 10 \\ -0.4 & 0.8 & -0.2 & 10 \\ -0.1 & -0.3 & 0.7 & 30 \end{array}\right]$$

CAS

Row reduction yields

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 61.74 \\ 0 & 1 & 0 & 63.04 \\ 0 & 0 & 1 & 78.70 \end{array}\right]$$

from which we see that the service, electricity, and oil industries must have an annual production of \$61.74, \$63.04, and \$78.70 (million), respectively, in order to meet both the internal and external demand for their commodities.

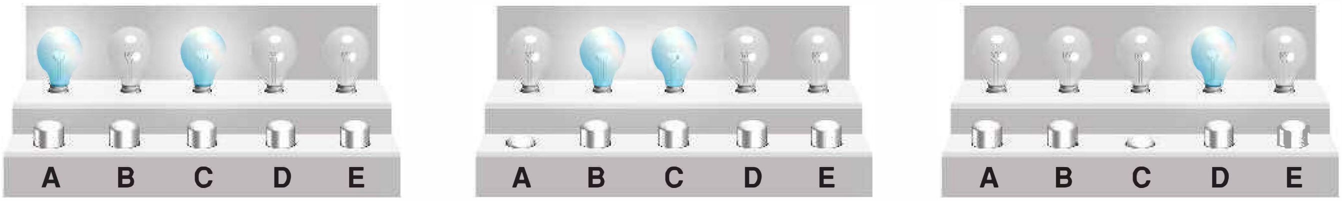
We will revisit these models in Section 3.7.

Finite Linear Games

There are many situations in which we must consider a physical system that has only a finite number of *states*. Sometimes these states can be altered by applying certain processes, each of which produces finitely many outcomes. For example, a light bulb can be on or off and a switch can change the state of the light bulb from on to off and vice versa. Digital systems that arise in computer science are often of this type. More frivolously, many computer games feature puzzles in which a certain device must be manipulated by various switches to produce a desired outcome. The finiteness of such situations is perfectly suited to analysis using modular arithmetic, and often linear systems over some $\mathbb{Z}_p$ play a role. Problems involving this type of situation are often called ***finite linear games.***

Example 2.35

A row of five lights is controlled by five switches. Each switch changes the state (on or off) of the light directly above it and the states of the lights immediately adjacent to the left and right. For example, if the first and third lights are on, as in Figure 2.15(a), then pushing switch A changes the state of the system to that shown in Figure 2.15(b). If we next push switch C, then the result is the state shown in Figure 2.15(c).



(a) (b) (c)

Figure 2.15

Suppose that initially all the lights are off. Can we push the switches in some order so that only the first, third, and fifth lights will be on? Can we push the switches in some order so that only the first light will be on?

Solution The on/off nature of this problem suggests that binary notation will be helpful and that we should work with $\mathbb{Z}_2$. Accordingly, we represent the states of the five lights by a vector in $\mathbb{Z}_2^5$, where 0 represents off and 1 represents on. Thus, for example, the vector

$$\begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

corresponds to Figure 2.15(b).

We may also use vectors in $\mathbb{Z}_2^5$ to represent the action of each switch. If a switch changes the state of a light, the corresponding component is a 1; otherwise, it is 0. With this convention, the actions of the five switches are given by

$$\mathbf{a} = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \ \mathbf{b} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \ \mathbf{c} = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \ \mathbf{d} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \ \mathbf{e} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}$$

The situation depicted in Figure 2.15(a) corresponds to the initial state

$$\mathbf{s} = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

followed by

$$\mathbf{a} = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

It is the vector sum (in $\mathbb{Z}_2^5$)

$$\mathbf{s} + \mathbf{a} = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

Observe that this result agrees with Figure 2.15(b).

Starting with any initial configuration $\mathbf{s}$, suppose we push the switches in the order A, C, D, A, C, B. This corresponds to the vector sum $\mathbf{s} + \mathbf{a} + \mathbf{c} + \mathbf{d} + \mathbf{a} + \mathbf{c} + \mathbf{b}$. But in $\mathbb{Z}_2^5$, addition is commutative, so we have

$$\begin{aligned} \mathbf{s} + \mathbf{a} + \mathbf{c} + \mathbf{d} + \mathbf{a} + \mathbf{c} + \mathbf{b} &= \mathbf{s} + 2\mathbf{a} + \mathbf{b} + 2\mathbf{c} + \mathbf{d} \\ &= \mathbf{s} + \mathbf{b} + \mathbf{d} \end{aligned}$$

where we have used the fact that $2 = 0$ in $\mathbb{Z}_2$. Thus, we would achieve the same result by pushing only B and D—and the order does not matter. (Check that this is correct.) Hence, in this example, we do not need to push any switch more than once.

So, to see if we can achieve a target configuration $\mathbf{t}$ starting from an initial configuration $\mathbf{s}$, we need to determine whether there are scalars $x_1, \ldots, x_5$ in $\mathbb{Z}_2$ such that

$$\mathbf{s} + x_1\mathbf{a} + x_2\mathbf{b} + \cdots + x_5\mathbf{e} = \mathbf{t}$$

In other words, we need to solve (if possible) the linear system over $\mathbb{Z}_2$ that corresponds to the vector equation

$$x_1\mathbf{a} + x_2\mathbf{b} + \cdots + x_5\mathbf{e} = \mathbf{t} - \mathbf{s} = \mathbf{t} + \mathbf{s}$$

In this case, $\mathbf{s} = \mathbf{0}$ and our first target configuration is

$$\mathbf{t} = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}$$

The augmented matrix of this system has the given vectors as columns:

$$\left[\begin{array}{ccccc|c} 1 & 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{array}\right]$$

We reduce it over $\mathbb{Z}_2$ to obtain

$$\left[\begin{array}{ccccc|c} 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Thus, x_5 is a free variable. Hence, there are exactly two solutions (corresponding to $x_5 = 0$ and $x_5 = 1$). Solving for the other variables in terms of x_5, we obtain

$$\begin{aligned} x_1 &= \phantom{1 + {}} x_5 \\ x_2 &= 1 + x_5 \\ x_3 &= 1 \\ x_4 &= 1 + x_5 \end{aligned}$$

So, when $x_5 = 0$ and $x_5 = 1$, we have the solutions

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \\ 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}$$

respectively. (Check that these both work.)

Similarly, in the second case, we have

$$\mathbf{t} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

The augmented matrix reduces as follows:

$$\left[\begin{array}{ccccc|c} 1 & 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{ccccc|c} 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right]$$

showing that there is no solution in this case; that is, it is impossible to start with all of the lights off and turn only the first light on.

Example 2.35 shows the power of linear algebra. Even though we might have found out by trial and error that there was no solution, checking all possible ways to push the switches would have been extremely tedious. We might also have missed the fact that no switch need ever be pushed more than once.

Example 2.36

Consider a row with only three lights, each of which can be off, light blue, or dark blue. Below the lights are three switches, A, B, and C, each of which changes the states of particular lights to the *next* state, in the order shown in Figure 2.16. Switch A changes the states of the first two lights, switch B all three lights, and switch C the last two

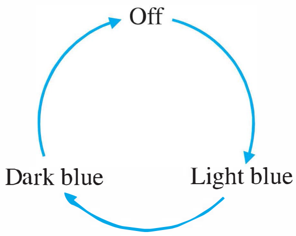


Figure 2.16

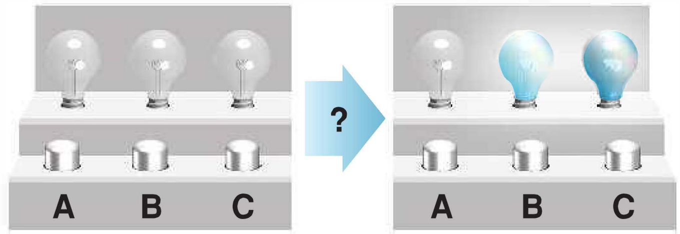


Figure 2.17

lights. If all three lights are initially off, is it possible to push the switches in some order so that the lights are off, light blue, and dark blue, in that order (as in Figure 2.17)?

Solution Whereas Example 2.35 involved $\mathbb{Z}_2$, this one clearly (is it clear?) involves $\mathbb{Z}_3$. Accordingly, the switches correspond to the vectors

$$\mathbf{a} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \ \mathbf{b} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \ \mathbf{c} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

in $\mathbb{Z}_3^3$, and the final configuration we are aiming for is $\mathbf{t} = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$. (Off is 0, light blue is 1, and dark blue is 2.) We wish to find scalars x_1, x_2, x_3 in $\mathbb{Z}_3$ such that

$$x_1\mathbf{a} + x_2\mathbf{b} + x_3\mathbf{c} = \mathbf{t}$$

(where x_i represents the number of times the ith switch is pushed). This equation gives rise to the augmented matrix $[\mathbf{a}\ \mathbf{b}\ \mathbf{c} \mid \mathbf{t}]$, which reduces over $\mathbb{Z}_3$ as follows:

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 2 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

Hence, there is a unique solution: $x_1 = 2, x_2 = 1, x_3 = 1$. In other words, we must push switch A twice and the other two switches once each. (Check this.)

Exercises 2.4

Allocation of Resources

1. Suppose that, in Example 2.27, 400 units of food A, 600 units of B, and 600 units of C are placed in the test tube each day and the data on daily food consumption by the bacteria (in units per day) are as shown in Table 2.6. How many bacteria of each strain can coexist in the test tube and consume all of the food?

Table 2.6

	Bacteria Strain I	Bacteria Strain II	Bacteria Strain III
Food A	1	2	0
Food B	2	1	1
Food C	1	1	2

2. Suppose that in Example 2.27, 400 units of food A, 500 units of B, and 600 units of C are placed in the test tube each day and the data on daily food consumption by the bacteria (in units per day) are as shown in Table 2.7. How many bacteria of each

Table 2.7

	Bacteria Strain I	Bacteria Strain II	Bacteria Strain III
Food A	1	2	0
Food B	2	1	3
Food C	1	1	1

strain can coexist in the test tube and consume all of the food?

3. A florist offers three sizes of flower arrangements containing roses, daisies, and chrysanthemums. Each small arrangement contains one rose, three daisies, and three chrysanthemums. Each medium arrangement contains two roses, four daisies, and six chrysanthemums. Each large arrangement contains four roses, eight daisies, and six chrysanthemums. One day, the florist noted that she used a total of 24 roses, 50 daisies, and 48 chrysanthemums in filling orders for these three types of arrangements. How many arrangements of each type did she make?

4. **(a)** In your pocket you have some nickels, dimes, and quarters. There are 20 coins altogether and exactly twice as many dimes as nickels. The total value of the coins is \$3.00. Find the number of coins of each type.

(b) Find *all* possible combinations of 20 coins (nickels, dimes, and quarters) that will make exactly \$3.00.

5. A coffee merchant sells three blends of coffee. A bag of the house blend contains 300 grams of Colombian beans and 200 grams of French roast beans. A bag of the special blend contains 200 grams of Colombian beans, 200 grams of Kenyan beans, and 100 grams of French roast beans. A bag of the gourmet blend contains 100 grams of Colombian beans, 200 grams of Kenyan beans, and 200 grams of French roast beans. The merchant has on hand 30 kilograms of Colombian beans, 15 kilograms of Kenyan beans, and 25 kilograms of French roast beans. If he wishes to use up all of the beans, how many bags of each type of blend can be made?

6. Redo Exercise 5, assuming that the house blend contains 300 grams of Colombian beans, 50 grams of Kenyan beans, and 150 grams of French roast beans and the gourmet blend contains 100 grams of Colombian beans, 350 grams of Kenyan beans, and 50 grams of French roast beans. This time the merchant has on hand 30 kilograms of Colombian beans, 15 kilograms of Kenyan beans, and 15 kilograms of French roast beans. Suppose one bag of the house blend produces a profit of \$0.50, one bag of the special blend produces a profit of \$1.50, and one bag of the gourmet blend produces a profit of \$2.00. How many bags of each type should the merchant prepare if he wants to use up all of the beans *and* maximize his profit? What is the maximum profit?

Balancing Chemical Equations

In Exercises 7–14, balance the chemical equation for each reaction.

7. $FeS_2 + O_2 \longrightarrow Fe_2O_3 + SO_2$

8. $CO_2 + H_2O \longrightarrow C_6H_{12}O_6 + O_2$ (This reaction takes place when a green plant converts carbon dioxide and water to glucose and oxygen during photosynthesis.)

9. $C_4H_{10} + O_2 \longrightarrow CO_2 + H_2O$ (This reaction occurs when butane, C_4H_{10}, burns in the presence of oxygen to form carbon dioxide and water.)

10. $C_7H_6O_2 + O_2 \longrightarrow H_2O + CO_2$

11. $C_5H_{11}OH + O_2 \longrightarrow H_2O + CO_2$ (This equation represents the combustion of amyl alcohol.)

12. $HClO_4 + P_4O_{10} \longrightarrow H_3PO_4 + Cl_2O_7$

13. $Na_2CO_3 + C + N_2 \longrightarrow NaCN + CO$

CAS **14.** $C_2H_2Cl_4 + Ca(OH)_2 \longrightarrow C_2HCl_3 + CaCl_2 + H_2O$

Network Analysis

15. Figure 2.18 shows a network of water pipes with flows measured in liters per minute.

(a) Set up and solve a system of linear equations to find the possible flows.

(b) If the flow through AB is restricted to 5 L/min, what will the flows through the other two branches be?

(c) What are the minimum and maximum possible flows through each branch?

(d) We have been assuming that flow is always *positive*. What would *negative* flow mean, assuming we allowed it? Give an illustration for this example.

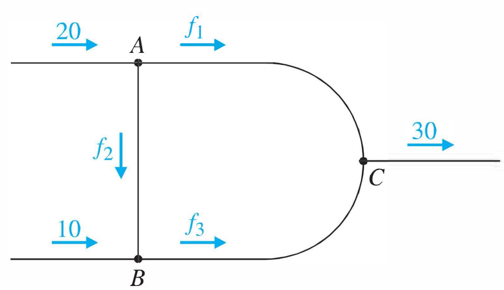


Figure 2.18

16. The downtown core of Gotham City consists of one-way streets, and the traffic flow has been measured at each intersection. For the city block shown in Figure 2.19, the numbers represent the average numbers of vehicles per minute entering and leaving intersections A, B, C, and D during business hours.

 (a) Set up and solve a system of linear equations to find the possible flows $f_1, \ldots, f_4$.
 (b) If traffic is regulated on CD so that $f_4 = 10$ vehicles per minute, what will the average flows on the other streets be?
 (c) What are the minimum and maximum possible flows on each street?
 (d) How would the solution change if *all* of the directions were reversed?

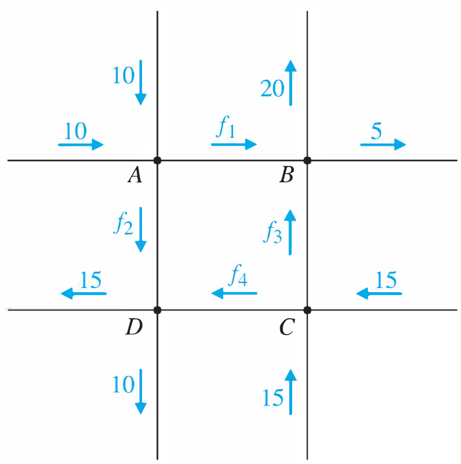


Figure 2.19

17. A network of irrigation ditches is shown in Figure 2.20, with flows measured in thousands of liters per day.

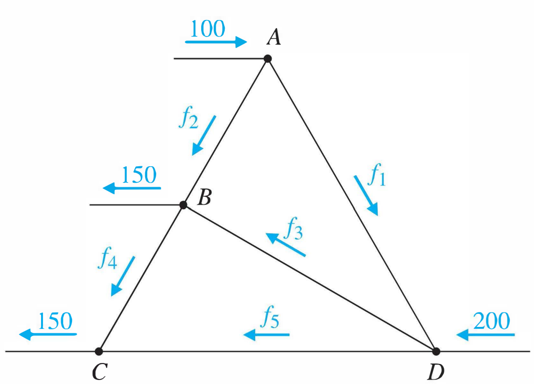


Figure 2.20

 (a) Set up and solve a system of linear equations to find the possible flows $f_1, \ldots, f_5$.
 (b) Suppose DC is closed. What range of flow will need to be maintained through DB?
 (c) From Figure 2.20 it is clear that DB cannot be closed. (Why not?) How does your solution in part (a) show this?
 (d) From your solution in part (a), determine the minimum and maximum flows through DB.

18. **(a)** Set up and solve a system of linear equations to find the possible flows in the network shown in Figure 2.21.
 (b) Is it possible for $f_1 = 100$ and $f_6 = 150$? [Answer this question first with reference to your solution in part (a) and then directly from Figure 2.21.]
 (c) If $f_4 = 0$, what will the range of flow be on each of the other branches?

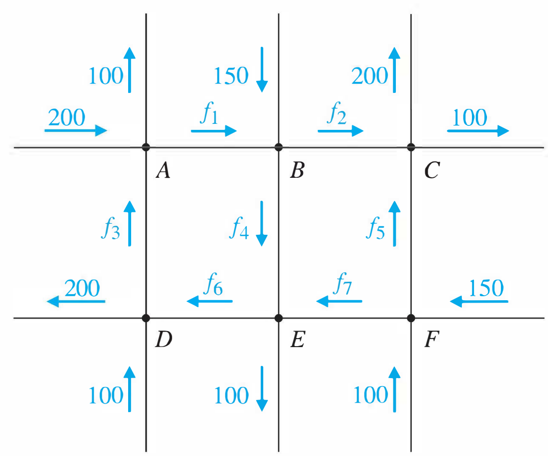
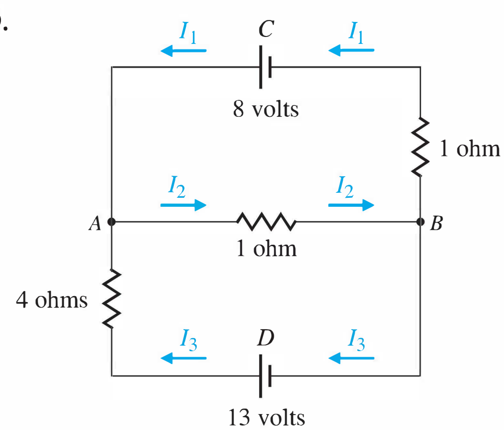


Figure 2.21

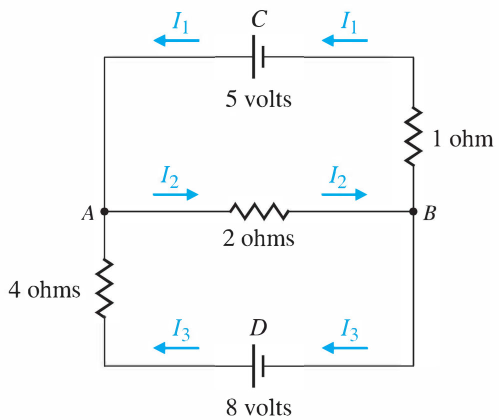
Electrical Networks

For Exercises 19 and 20, determine the currents for the given electrical networks.

19.



20.



21. **(a)** Find the currents $I, I_1, \ldots, I_5$ in the bridge circuit in Figure 2.22.

(b) Find the effective resistance of this network.

(c) Can you change the resistance in branch BC (but leave everything else unchanged) so that the current through branch CE becomes 0?

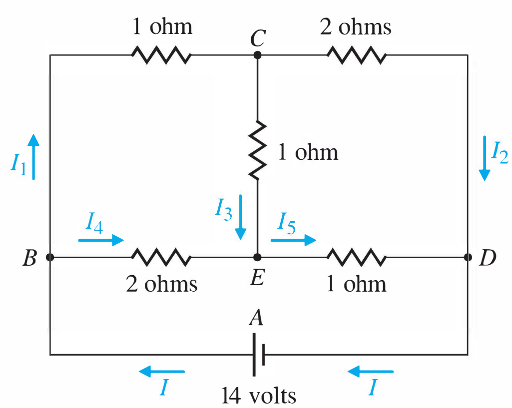


Figure 2.22

22. The networks in parts (a) and (b) of Figure 2.23 show two resistors coupled in *series* and in *parallel*, respectively. We wish to find a general formula for the effective resistance of each network—that is, find R_{eff} such that $E = R_{\text{eff}}I$.

(a) Show that the effective resistance R_{eff} of a network with two resistors coupled in series [Figure 2.23(a)] is given by

$$R_{\text{eff}} = R_1 + R_2$$

(b) Show that the effective resistance R_{eff} of a network with two resistors coupled in parallel [Figure 2.23(b)] is given by

$$R_{\text{eff}} = \frac{1}{\dfrac{1}{R_1} + \dfrac{1}{R_2}}$$

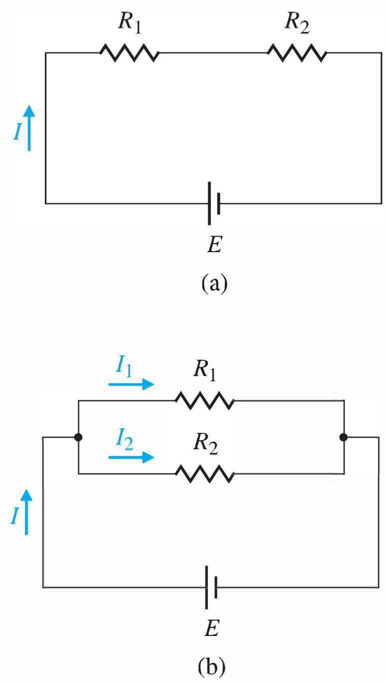


Figure 2.23

Resistors in series and in parallel

Linear Economic Models

23. Consider a simple economy with just two industries: farming and manufacturing. Farming consumes 1/2 of the food and 1/3 of the manufactured goods. Manufacturing consumes 1/2 of the food and 2/3 of the manufactured goods. Assuming the economy is closed and in equilibrium, find the relative outputs of the farming and manufacturing industries.

24. Suppose the coal and steel industries form a closed economy. Every \$1 produced by the coal industry requires \$0.30 of coal and \$0.70 of steel. Every \$1 produced by steel requires \$0.80 of coal and \$0.20 of steel. Find the annual production (output) of coal and steel if the total annual production is \$20 million.

25. A painter, a plumber, and an electrician enter into a cooperative arrangement in which each of them agrees to work for himself/herself and the other two for a total of 10 hours per week according to the schedule shown in Table 2.8. For tax purposes, each person must establish a value for his/her services. They agree to do this so that they each come out even—that is, so that the total amount paid out by each person equals the amount he/she receives. What hourly rate should each person charge if the rates are all whole numbers between \$30 and \$60 per hour?

Table 2.8

		Supplier		
		Painter	Plumber	Electrician
Consumer	Painter	2	1	5
	Plumber	4	5	1
	Electrician	4	4	4

26. Four neighbors, each with a vegetable garden, agree to share their produce. One will grow beans (**B**), one will grow lettuce (**L**), one will grow tomatoes (**T**), and one will grow zucchini (**Z**). Table 2.9 shows what fraction of each crop each neighbor will receive. What prices should the neighbors charge for their crops if each person is to break even and the lowest-priced crop has a value of \$50?

Table 2.9

		Producer			
		B	L	T	Z
Consumer	B	0	1/4	1/8	1/6
	L	1/2	1/4	1/4	1/6
	T	1/4	1/4	1/2	1/3
	Z	1/4	1/4	1/8	1/3

27. Suppose the coal and steel industries form an open economy. Every \$1 produced by the coal industry requires \$0.15 of coal and \$0.20 of steel. Every \$1 produced by steel requires \$0.25 of coal and \$0.10 of steel. Suppose that there is an annual outside demand for \$45 million of coal and \$124 million of steel.

(a) How much should each industry produce to satisfy the demands?

(b) If the demand for coal decreases by \$5 million per year while the demand for steel increases by \$6 million per year, how should the coal and steel industries adjust their production?

28. In Gotham City, the departments of Administration (A), Health (H), and Transportation (T) are interdependent. For every dollar's worth of services they produce, each department uses a certain amount of the services produced by the other departments and itself, as shown in Table 2.10. Suppose that, during the year, other city departments require \$1 million in Administrative services, \$1.2 million in Health services, and \$0.8 million in Transportation services. What does the annual dollar value of the services produced by each department need to be in order to meet the demands?

Table 2.10

		Department		
		A	H	T
Buy	A	\$0.20	0.10	0.20
	H	0.10	0.10	0.20
	T	0.20	0.40	0.30

Finite Linear Games

29. (a) In Example 2.35, suppose all the lights are initially off. Can we push the switches in some order so that only the second and fourth lights will be on?

(b) Can we push the switches in some order so that only the second light will be on?

30. (a) In Example 2.35, suppose the fourth light is initially on and the other four lights are off. Can we push the switches in some order so that only the second and fourth lights will be on?

(b) Can we push the switches in some order so that only the second light will be on?

31. In Example 2.35, describe all possible configurations of lights that can be obtained if we start with all the lights off.

32. (a) In Example 2.36, suppose that all of the lights are initially off. Show that it is possible to push the switches in some order so that the lights are off, dark blue, and light blue, in that order.

(b) Show that it is possible to push the switches in some order so that the lights are light blue, off, and light blue, in that order.

(c) Prove that *any* configuration of the three lights can be achieved.

33. Suppose the lights in Example 2.35 can be off, light blue, or dark blue and the switches work as described in Example 2.36. (That is, the switches control the same lights as in Example 2.35 but cycle through the colors as in Example 2.36.) Show that it is possible to start with all of the lights off and push the switches in some order so that the lights are dark blue, light blue, dark blue, light blue, and dark blue, in that order.

34. For Exercise 33, describe all possible configurations of lights that can be obtained, starting with all the lights off.

CAS **35.** Nine squares, each one either black or white, are arranged in a 3×3 grid. Figure 2.24 shows one possible

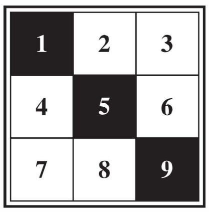


Figure 2.24
The nine squares puzzle

arrangement. When touched, each square changes its own state and the states of some of its neighbors (black → white and white → black). Figure 2.25 shows

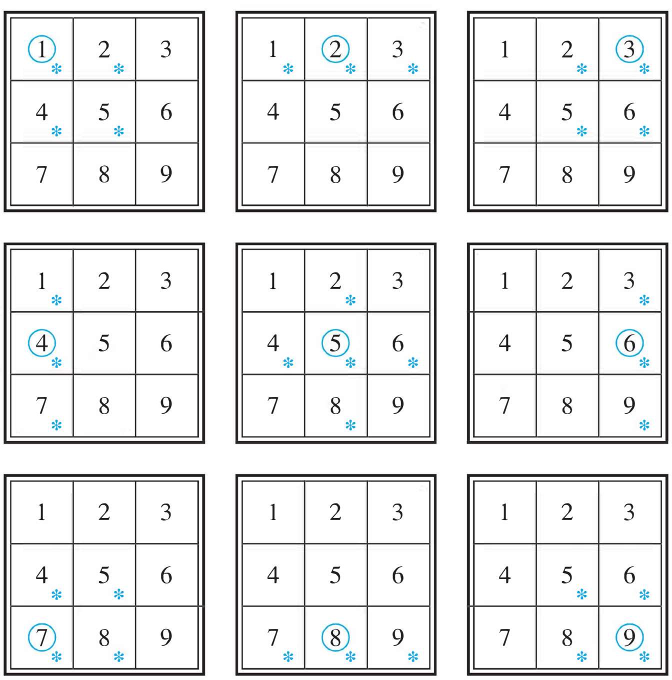


Figure 2.25
State changes for the nine squares puzzle

how the state changes work. (Touching the square whose number is circled causes the states of the squares marked * to change.) The object of the game is to turn all nine squares black. [Exercises 35 and 36 are adapted from puzzles that can be found in the interactive CD-ROM game *The Seventh Guest* (Trilobyte Software/Virgin Games, 1992).]

(a) If the initial configuration is the one shown in Figure 2.24, show that the game can be won and describe a winning sequence of moves.

(b) Prove that the game can always be won, no matter what the initial configuration.

CAS **36.** Consider a variation on the nine squares puzzle. The game is the same as that described in Exercise 35 except that there are three possible states for each square: white, gray, or black. The squares change as shown in Figure 2.25, but now the state changes follow the cycle white → gray → black → white. Show how the winning all-black configuration can be achieved from the initial configuration shown in Figure 2.26.

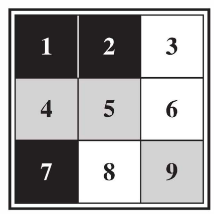


Figure 2.26
The nine squares puzzle with more states

Miscellaneous Problems

In Exercises 37–53, set up and solve an appropriate system of linear equations to answer the questions.

37. Grace is three times as old as Hans, but in 5 years she will be twice as old as Hans is then. How old are they now?

38. The sum of Annie's, Bert's, and Chris's ages is 60. Annie is older than Bert by the same number of years that Bert is older than Chris. When Bert is as old as Annie is now, Annie will be three times as old as Chris is now. What are their ages?

The preceding two problems are typical of those found in popular books of mathematical puzzles. However, they have their origins in antiquity. A Babylonian clay tablet that survives from about 300 B.C. contains the following problem.

39. There are two fields whose total area is 1800 square yards. One field produces grain at the rate of $\frac{2}{3}$ bushel per square yard; the other field produces grain at the rate of $\frac{1}{2}$ bushel per square yard. If the total yield is 1100 bushels, what is the size of each field?

Over 2000 years ago, the Chinese developed methods for solving systems of linear equations, including a version of Gaussian elimination that did not become well known in Europe until the 19th century. (There is no evidence that Gauss was aware of the Chinese methods when he developed what we now call Gaussian elimination. However, it is clear that the Chinese knew the essence of the method, even though they did not justify its use.) The following problem is taken from the Chinese text Jiuzhang suanshu (Nine Chapters in the Mathematical Art), *written during the early Han Dynasty, about 200* B.C.

40. There are three types of corn. Three bundles of the first type, two of the second, and one of the third make 39 measures. Two bundles of the first type, three of the second, and one of the third make 34 measures. And one bundle of the first type, two of the second, and three of the third make 26 measures. How many measures of corn are contained in one bundle of each type?

41. Describe all possible values of a, b, c, and d that will make each of the following a valid addition table. [Problems 41–44 are based on the article "An Application of Matrix Theory" by Paul Glaister in *The Mathematics Teacher,* 85 (1992), pp. 220–223.]

(a)

+	a	b
c	2	3
d	4	5

(b)

+	a	b
c	3	6
d	4	5

42. What conditions on w, x, y, and z will guarantee that we can find a, b, c, and d so that the following is a valid addition table?

+	a	b
c	w	x
d	y	z

43. Describe all possible values of a, b, c, d, e, and f that will make each of the following a valid addition table.

(a)

+	a	b	c
d	3	2	1
e	5	4	3
f	4	3	1

(b)

+	a	b	c
d	1	2	3
e	3	4	5
f	4	5	6

44. Generalizing Exercise 42, find conditions on the entries of a 3×3 addition table that will guarantee that we can solve for a, b, c, d, e, and f as previously.

45. From elementary geometry we know that there is a unique straight line through any two points in a plane. Less well known is the fact that there is a unique parabola through any *three* noncollinear points in a plane. For each set of points below, find a parabola with an equation of the form $y = ax^2 + bx + c$ that passes through the given points. (Sketch the resulting parabola to check the validity of your answer.)

(a) $(0, 1)$, $(-1, 4)$, and $(2, 1)$
(b) $(-3, 1)$, $(-2, 2)$, and $(-1, 5)$

46. Through any three noncollinear points there also passes a unique circle. Find the circles (whose general equations are of the form $x^2 + y^2 + ax + by + c = 0$) that pass through the sets of points in Exercise 45. (To check the validity of your answer, find the center and radius of each circle and draw a sketch.)

The process of adding rational functions (ratios of polynomials) by placing them over a common denominator is the analogue of adding rational numbers. The reverse process of taking a rational function apart by writing it as a sum of simpler rational functions is useful in several areas of mathematics; for example, it arises in calculus when we need to integrate a rational function and in discrete mathematics when we use generating functions to solve recurrence relations. The decomposition of a rational function as a sum of partial fractions leads to a system of linear equations. In Exercises 47–50, find the partial fraction decomposition of the given form. (The capital letters denote constants.)

47. $\dfrac{3x+1}{x^2+2x-3} = \dfrac{A}{x-1} + \dfrac{B}{x+3}$

48. $\dfrac{x^2-3x+3}{x^3+2x^2+x} = \dfrac{A}{x} + \dfrac{B}{x+1} + \dfrac{C}{(x+1)^2}$

CAS **49.** $\dfrac{x-1}{(x+1)(x^2+1)(x^2+4)}$
$= \dfrac{A}{x+1} + \dfrac{Bx+C}{x^2+1} + \dfrac{Dx+E}{x^2+4}$

CAS **50.** $\dfrac{x^3+x+1}{x(x-1)(x^2+x+1)(x^2+1)^3} = \dfrac{A}{x} + \dfrac{B}{x-1}$
$+ \dfrac{Cx+D}{x^2+x+1} + \dfrac{Ex+F}{x^2+1} + \dfrac{Gx+H}{(x^2+1)^2} + \dfrac{Ix+J}{(x^2+1)^3}$

Following are two useful formulas for the sums of powers of consecutive natural numbers:

$$1 + 2 + \cdots + n = \frac{n(n+1)}{2}$$

and

$$1^2 + 2^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$$

The validity of these formulas for all values of $n \geq 1$ *(or even* $n \geq 0$*) can be established using mathematical induction (see Appendix B). One way to make an educated guess as to what the formulas are, though, is to observe that we can rewrite the two formulas above as*

$$\tfrac{1}{2}n^2 + \tfrac{1}{2} \quad \textit{and} \quad \tfrac{1}{3}n^3 + \tfrac{1}{2}n^2 + \tfrac{1}{6}n$$

respectively. This leads to the conjecture that the sum of pth powers of the first n natural numbers is a polynomial of degree $p + 1$ *in the variable n.*

51. Assuming that $1 + 2 + \cdots + n = an^2 + bn + c$, find a, b, and c by substituting three values for n and thereby obtaining a system of linear equations in a, b, and c.

52. Assume that $1^2 + 2^2 + \cdots + n^2 = an^3 + bn^2 + cn + d$. Find a, b, c, and d. [*Hint:* It is legitimate to use $n = 0$. What is the left-hand side in that case?]

53. Show that $1^3 + 2^3 + \cdots + n^3 = (n(n+1)/2)^2$.

Vignette

The Global Positioning System

The Global Positioning System (GPS) is used in a variety of situations for determining geographical locations. The military, surveyors, airlines, shipping companies, and hikers all make use of it. GPS technology is becoming so commonplace that some automobiles, cellular phones, and various handheld devices are now equipped with it.

The basic idea of GPS is a variant on three-dimensional triangulation: A point on Earth's surface is uniquely determined by knowing its distances from three other points. Here the point we wish to determine is the location of the GPS receiver, the other points are satellites, and the distances are computed using the travel times of radio signals from the satellites to the receiver.

We will assume that Earth is a sphere on which we impose an *xyz*-coordinate system with Earth centered at the origin and with the positive z-axis running through the north pole and fixed relative to Earth.

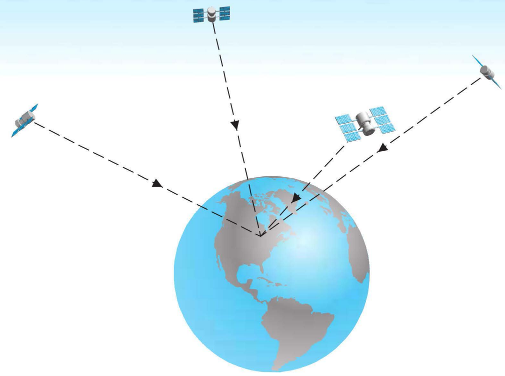
For simplicity, let's take one unit to be equal to the radius of Earth. Thus Earth's surface becomes the unit sphere with equation $x^2 + y^2 + z^2 = 1$. Time will be measured in hundredths of a second. GPS finds distances by knowing how long it takes a radio signal to get from one point to another. For this we need to know the speed of light, which is approximately equal to 0.47 (Earth radii per hundredths of a second).

Let's imagine that you are a hiker lost in the woods at point (x, y, z) at some time t. You don't know where you are, and furthermore, you have no watch, so you don't know what time it is. However, you have your GPS device, and it receives simultaneous signals from four satellites, giving their positions and times as shown in Table 2.11. (Distances are measured in Earth radii and time in hundredths of a second past midnight.)

This application is based on the article "An Underdetermined Linear System for GPS" by Dan Kalman in *The College Mathematics Journal,* 33 (2002), pp. 384–390. For a more in-depth treatment of the ideas introduced here, see G. Strang and K. Borre, *Linear Algebra, Geodesy, and GPS* (Wellesley-Cambridge Press, MA, 1997).

Table 2.11 Satellite Data

Satellite	Position	Time
1	(1.11, 2.55, 2.14)	1.29
2	(2.87, 0.00, 1.43)	1.31
3	(0.00, 1.08, 2.29)	2.75
4	(1.54, 1.01, 1.23)	4.06



Let (x, y, z) be your position, and let t be the time when the signals arrive. The goal is to solve for x, y, z, and t. Your distance from Satellite 1 can be computed as follows. The signal, traveling at a speed of 0.47 Earth radii/10^{-2} sec, was sent at time 1.29 and arrived at time t, so it took $t - 1.29$ hundredths of a second to reach you. Distance equals velocity multiplied by (elapsed) time, so

$$d = 0.47(t - 1.29)$$

We can also express d in terms of (x, y, z) and the satellite's position (1.11, 2.55, 2.14) using the distance formula:

$$d = \sqrt{(x - 1.11)^2 + (y - 2.55)^2 + (z - 2.14)^2}$$

Combining these results leads to the equation

$$(x - 1.11)^2 + (y - 2.55)^2 + (z - 2.14)^2 = 0.47^2(t - 1.29)^2 \qquad (1)$$

Expanding, simplifying, and rearranging, we find that Equation (1) becomes

$$2.22x + 5.10y + 4.28z - 0.57t = x^2 + y^2 + z^2 - 0.22t^2 + 11.95$$

Similarly, we can derive a corresponding equation for each of the other three satellites. We end up with a system of four equations in x, y, z, and t:

$$\begin{aligned}
2.22x + 5.10y + 4.28z - 0.57t &= x^2 + y^2 + z^2 - 0.22t^2 + 11.95 \\
5.74x \qquad\qquad + 2.86z - 0.58t &= x^2 + y^2 + z^2 - 0.22t^2 + 9.90 \\
2.16y + 4.58z - 1.21t &= x^2 + y^2 + z^2 - 0.22t^2 + 4.74 \\
3.08x + 2.02y + 2.46z - 1.79t &= x^2 + y^2 + z^2 - 0.22t^2 + 1.26
\end{aligned}$$

These are not linear equations, but the nonlinear terms are the same in each equation. If we subtract the first equation from each of the other three equations, we obtain a linear system:

$$\begin{aligned}
3.52x - 5.10y - 1.42z - 0.01t &= 2.05 \\
-2.22x - 2.94y + 0.30z - 0.64t &= 7.21 \\
0.86x - 3.08y - 1.82z - 1.22t &= -10.69
\end{aligned}$$

The augmented matrix row reduces as

$$\left[\begin{array}{cccc|c} 3.52 & -5.10 & -1.42 & -0.01 & -2.05 \\ -2.22 & -2.94 & 0.30 & -0.64 & -7.21 \\ 0.86 & -3.08 & -1.82 & -1.22 & -10.69 \end{array}\right] \longrightarrow \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0.36 & 2.97 \\ 0 & 1 & 0 & 0.03 & 0.81 \\ 0 & 0 & 1 & 0.79 & 5.91 \end{array}\right]$$

from which we see that

$$\begin{aligned} x &= 2.97 - 0.36t \\ y &= 0.81 - 0.03t \\ z &= 5.91 - 0.79t \end{aligned} \tag{2}$$

with t free. Substituting these equations into (1), we obtain

$$(2.97 - 0.36t - 1.11)^2 + (0.81 - 0.03t - 2.55)^2 + (5.91 - 0.79t - 2.14)^2 = 0.47^2(t - 1.29)^2$$

which simplifies to the quadratic equation

$$0.54t^2 - 6.65t + 20.32 = 0$$

There are two solutions:

$$t = 6.74 \quad \text{and} \quad t = 5.60$$

Substituting into (2), we find that the first solution corresponds to $(x, y, z) = (0.55, 0.61, 0.56)$ and the second solution to $(x, y, z) = (0.96, 0.65, 1.46)$. The second solution is clearly not on the unit sphere (Earth), so we reject it. The first solution produces $x^2 + y^2 + z^2 = 0.99$, so we are satisfied that, within acceptable roundoff error, we have located your coordinates as (0.55, 0.61, 0.56).

In practice, GPS takes significantly more factors into account, such as the fact that Earth's surface is not exactly spherical, so additional refinements are needed involving such techniques as least squares approximation (see Chapter 7). In addition, the results of the GPS calculation are converted from rectangular (Cartesian) coordinates into latitude and longitude, an interesting exercise in itself and one involving yet other branches of mathematics.

CAS

2.5 Iterative Methods for Solving Linear Systems

The direct methods for solving linear systems, using elementary row operations, lead to exact solutions in many cases but are subject to errors due to roundoff and other factors, as we have seen. The third road in our "trivium" takes us down quite a different path indeed. In this section, we explore methods that proceed *iteratively* by successively generating sequences of vectors that approach a solution to a linear system. In many instances (such as when the coefficient matrix is *sparse*—that is, contains many zero entries), iterative methods can be faster and more accurate than direct methods. Also, iterative methods can be stopped whenever the approximate solution they generate is sufficiently accurate. In addition, iterative methods often *benefit* from inaccuracy: Roundoff error can actually accelerate their convergence toward a solution.

We will explore two iterative methods for solving linear systems: ***Jacobi's method*** and a refinement of it, the ***Gauss-Seidel method.*** In all examples, we will be considering linear systems with the same number of variables as equations, and we will assume that there is a unique solution. Our interest is in finding this solution using iterative methods.

Example 2.37

Consider the system

$$\begin{aligned} 7x_1 - x_2 &= 5 \\ 3x_1 - 5x_2 &= -7 \end{aligned}$$

Jacobi's method begins with solving the first equation for x_1 and the second equation for x_2, to obtain

$$\begin{aligned} x_1 &= \frac{5 + x_2}{7} \\ x_2 &= \frac{7 + 3x_1}{5} \end{aligned} \qquad (1)$$

We now need an ***initial approximation*** to the solution. It turns out that it does not matter what this initial approximation is, so we might as well take $x_1 = 0$, $x_2 = 0$. We use these values in Equations (1) to get new values of x_1 and x_2:

$$\begin{aligned} x_1 &= \frac{5 + 0}{7} = \frac{5}{7} \approx 0.714 \\ x_2 &= \frac{7 + 3 \cdot 0}{5} = \frac{7}{5} = 1.400 \end{aligned}$$

Now we substitute these values into (1) to get

$$\begin{aligned} x_1 &= \frac{5 + 1.4}{7} \approx 0.914 \\ x_2 &= \frac{7 + 3 \cdot \frac{5}{7}}{5} \approx 1.829 \end{aligned}$$

(written to three decimal places). We repeat this process (using the old values of x_2 and x_1 to get the new values of x_1 and x_2), producing the sequence of approximations given in Table 2.12.

Courtesy of the Smithsonian Institution Libraries, Washington, D.C.

Carl Gustav Jacobi (1804–1851) was a German mathematician who made important contributions to many fields of mathematics and physics, including geometry, number theory, analysis, mechanics, and fluid dynamics. Although much of his work was in applied mathematics, Jacobi believed in the importance of doing mathematics for its own sake. A fine teacher, he held positions at the Universities of Berlin and Königsberg and was one of the most famous mathematicians in Europe.

Table 2.12

n	0	1	2	3	4	5	6
x_1	0	0.714	0.914	0.976	0.993	0.998	0.999
x_2	0	1.400	1.829	1.949	1.985	1.996	1.999

The successive vectors $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ are called ***iterates,*** so, for example, when $n = 4$, the fourth iterate is $\begin{bmatrix} 0.993 \\ 1.985 \end{bmatrix}$. We can see that the iterates in this example are approaching $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$, which is the exact solution of the given system. (Check this.) We say in this case that Jacobi's method ***converges.***

Jacobi's method calculates the successive iterates in a two-variable system according to the crisscross pattern shown in Table 2.13.

Table 2.13

n	0	1	2	3
x_1				
x_2				

The Gauss-Seidel method is named after C. F. Gauss and Philipp Ludwig von Seidel (1821–1896). Seidel worked in analysis, probability theory, astronomy, and optics. Unfortunately, he suffered from eye problems and retired at a young age. The paper in which he described the method now known as Gauss-Seidel was published in 1874. Gauss, it seems, was unaware of the method!

Before we consider Jacobi's method in the general case, we will look at a modification of it that often converges faster to the solution. The *Gauss-Seidel method* is the same as the Jacobi method except that we use each new value *as soon as we can.* So in our example, we begin by calculating $x_1 = (5 + 0)/7 = \frac{5}{7} \approx 0.714$ as before, but we now use this value of x_1 to get the next value of x_2:

$$x_2 = \frac{7 + 3 \cdot \frac{5}{7}}{5} \approx 1.829$$

We then use this value of x_2 to recalculate x_1, and so on. The iterates this time are shown in Table 2.14.

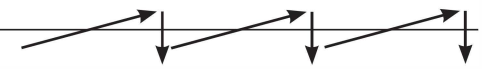
We observe that the Gauss-Seidel method has converged faster to the solution. The iterates this time are calculated according to the zigzag pattern shown in Table 2.15.

Table 2.14

n	0	1	2	3	4	5
x_1	0	0.714	0.976	0.998	1.000	1.000
x_2	0	1.829	1.985	1.999	2.000	2.000

Table 2.15

n	0	1	2	3
x_1				
x_2				



The Gauss-Seidel method also has a nice geometric interpretation in the case of two variables. We can think of x_1 and x_2 as the coordinates of points in the plane. Our starting point is the point corresponding to our initial approximation, (0, 0). Our first calculation gives $x_1 = \frac{5}{7}$, so we move to the point $(\frac{5}{7}, 0) \approx (0.714, 0)$. Then we compute $x_2 = \frac{64}{35} \approx 1.829$, which moves us to the point $(\frac{5}{7}, \frac{64}{35}) \approx (0.714, 1.829)$. Continuing in this fashion, our calculations from the Gauss-Seidel method give rise to a sequence of points, each one differing from the preceding point in exactly one coordinate. If we plot the lines $7x_1 - x_2 = 5$ and $3x_1 - 5x_2 = -7$ corresponding to the two given equations, we find that the points calculated above fall alternately on the two lines, as shown in Figure 2.27. Moreover, they approach the point of intersection of the lines, which corresponds to the solution of the system of equations. This is what *convergence* means!

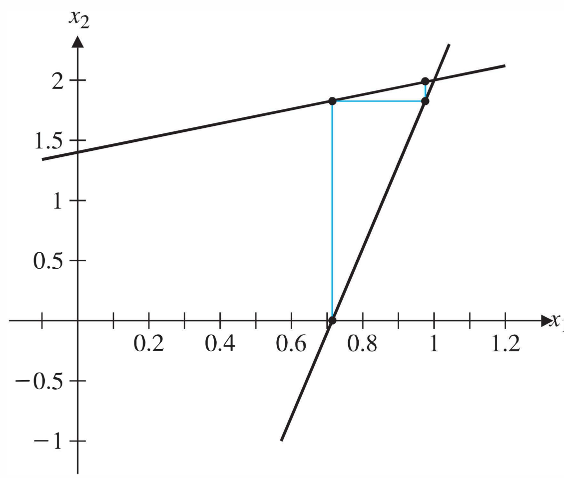


Figure 2.27
Converging iterates

The general cases of the two methods are analogous. Given a system of n linear equations in n variables,

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ &\vdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n &= b_n \end{aligned} \tag{2}$$

we solve the first equation for x_1, the second for x_2, and so on. Then, beginning with an initial approximation, we use these new equations to iteratively update each variable. Jacobi's method uses *all* of the values at the kth iteration to compute the $(k + 1)$st iterate, whereas the Gauss-Seidel method always uses the *most recent* value of each variable in every calculation. Example 2.39 later illustrates the Gauss-Seidel method in a three-variable problem.

At this point, you should have some questions and concerns about these iterative methods. (Do you?) Several come to mind: Must these methods converge? If not, when *do* they converge? *If* they converge, must they converge to the solution? The answer to the first question is no, as Example 2.38 illustrates.

Example 2.38

Apply the Gauss-Seidel method to the system

$$\begin{aligned} x_1 - x_2 &= 1 \\ 2x_1 + x_2 &= 5 \end{aligned}$$

with initial approximation $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

Solution We rearrange the equations to get

$$\begin{aligned} x_1 &= 1 + x_2 \\ x_2 &= 5 - 2x_1 \end{aligned}$$

The first few iterates are given in Table 2.16. (Check these.)

The actual solution to the given system is $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$. Clearly, the iterates in Table 2.16 are not approaching this point, as Figure 2.28 makes graphically clear in an example of ***divergence.***

Table 2.16

n	0	1	2	3	4	5
x_1	0	1	4	−2	10	−14
x_2	0	3	−3	9	−15	33

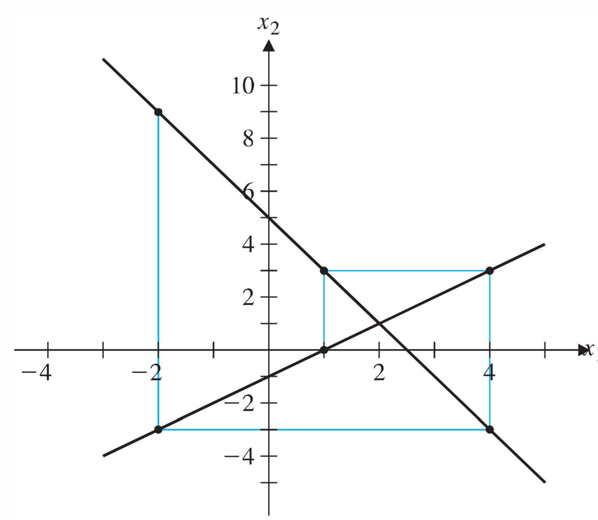


Figure 2.28
Diverging iterates

So when do these iterative methods converge? Unfortunately, the answer to this question is rather tricky. We will answer it completely in Chapter 7, but for now we will give a partial answer, without proof.

Let A be the $n \times n$ matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$$

We say that A is ***strictly diagonally dominant*** if

$$\begin{aligned} |a_{11}| &> |a_{12}| + |a_{13}| + \cdots + |a_{1n}| \\ |a_{22}| &> |a_{21}| + |a_{23}| + \cdots + |a_{2n}| \\ &\vdots \\ |a_{nn}| &> |a_{n1}| + |a_{n2}| + \cdots + |a_{n,n-1}| \end{aligned}$$

That is, the absolute value of each diagonal entry $a_{11}, a_{22}, \ldots, a_{nn}$ is greater than the sum of the absolute values of the remaining entries in that row.

Theorem 2.9 If a system of n linear equations in n variables has a strictly diagonally dominant coefficient matrix, then it has a unique solution and both the Jacobi and the Gauss-Seidel method converge to it.

Remark Be warned! This theorem is a one-way implication. The fact that a system is *not* strictly diagonally dominant does *not* mean that the iterative methods diverge. They may or may not converge. (See Exercises 15–19.) Indeed, there are examples in which one of the methods converges and the other diverges. However, *if* either of these methods converges, then it must converge to the solution—it cannot converge to some other point.

Theorem 2.10 If the Jacobi or the Gauss-Seidel method converges for a system of n linear equations in n variables, then it must converge to the solution of the system.

Proof We will illustrate the idea behind the proof by sketching it out for the case of Jacobi's method, using the system of equations in Example 2.37. The general proof is similar.

Convergence means that "as iterations increase, the values of the iterates get closer and closer to a limiting value." This means that x_1 and x_2 converge to r and s, respectively, as shown in Table 2.17.

We must prove that $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} r \\ s \end{bmatrix}$ is the solution of the system of equations. In other words, at the $(k + 1)$st iteration, the values of x_1 and x_2 must stay the same as at

Table 2.17

n	$\cdots$	k	$k+1$	$k+2$	$\cdots$
x_1	$\cdots$	r	r	r	$\cdots$
x_2	$\cdots$	s	s	s	$\cdots$

the kth iteration. But the calculations give $x_1 = (5 + x_2)/7 = (5 + s)/7$ and $x_2 = (7 + 3x_1)/5 = (7 + 3r)/5$. Therefore,

$$\frac{5+s}{7} = r \quad \text{and} \quad \frac{7+3r}{5} = s$$

Rearranging, we see that

$$\begin{aligned} 7r - s &= 5 \\ 3r - 5s &= -7 \end{aligned}$$

Thus, $x_1 = r$, $x_2 = s$ satisfy the original equations, as required.

By now you may be wondering: If iterative methods don't always converge to the solution, what good are they? Why don't we just use Gaussian elimination? First, we have seen that Gaussian elimination is sensitive to roundoff errors, and this sensitivity can lead to inaccurate or even wildly wrong answers. Also, even if Gaussian elimination does not go astray, we cannot improve on a solution once we have found it. For example, if we use Gaussian elimination to calculate a solution to two decimal places, there is no way to obtain the solution to four decimal places except to start over again and work with increased accuracy.

In contrast, we can achieve additional accuracy with iterative methods simply by doing more iterations. For large systems, particularly those with sparse coefficient matrices, iterative methods are much faster than direct methods when implemented on a computer. In many applications, the systems that arise are strictly diagonally dominant, and thus iterative methods are guaranteed to converge. The next example illustrates one such application.

Example 2.39

Suppose we heat each edge of a metal plate to a constant temperature, as shown in Figure 2.29.

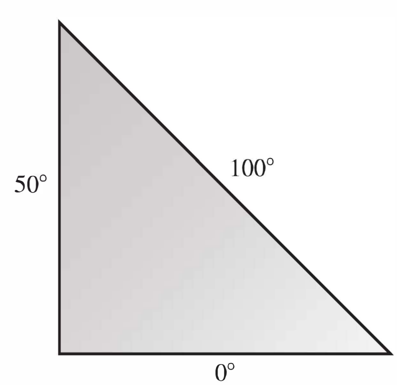


Figure 2.29
A heated metal plate

Eventually the temperature at the interior points will reach *equilibrium*, where the following property can be shown to hold:

The temperature at each interior point P on a plate is the average of the temperatures on the circumference of any circle centered at P inside the plate (Figure 2.30).

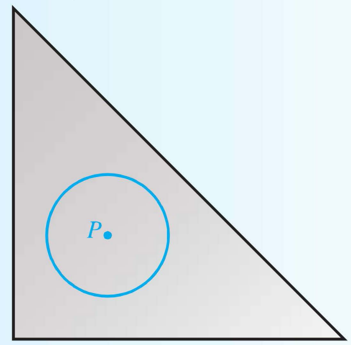


Figure 2.30

To apply this property in an actual example requires techniques from calculus. As an alternative, we can approximate the situation by overlaying the plate with a grid, or mesh, that has a finite number of interior points, as shown in Figure 2.31.

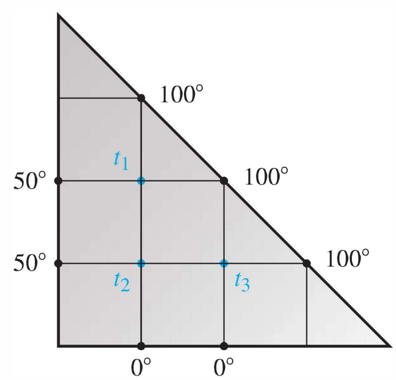


Figure 2.31
The discrete version of the heated plate problem

The discrete analogue of the averaging property governing equilibrium temperatures is stated as follows:

The temperature at each interior point P is the average of the temperatures at the points adjacent to P.

For the example shown in Figure 2.31, there are three interior points, and each is adjacent to four other points. Let the equilibrium temperatures of the interior points be t_1, t_2, and t_3, as shown. Then, by the temperature-averaging property, we have

$$\begin{aligned} t_1 &= \frac{100 + 100 + t_2 + 50}{4} \\ t_2 &= \frac{t_1 + t_3 + 0 + 50}{4} \\ t_3 &= \frac{100 + 100 + 0 + t_2}{4} \end{aligned} \tag{3}$$

or

$$\begin{aligned} 4t_1 - t_2 \qquad &= 250 \\ -t_1 + 4t_2 - t_3 &= 50 \\ - t_2 + 4t_3 &= 200 \end{aligned}$$

Notice that this system is strictly diagonally dominant. Notice also that Equations (3) are in the form required for Jacobi or Gauss-Seidel iteration. With an initial approximation of $t_1 = 0$, $t_2 = 0$, $t_3 = 0$, the Gauss-Seidel method gives the following iterates.

Iteration 1:

$$\begin{aligned} t_1 &= \frac{100 + 100 + 0 + 50}{4} = 62.5 \\ t_2 &= \frac{62.5 + 0 + 0 + 50}{4} = 28.125 \\ t_3 &= \frac{100 + 100 + 0 + 28.125}{4} = 57.031 \end{aligned}$$

Iteration 2:

$$\begin{aligned} t_1 &= \frac{100 + 100 + 28.125 + 50}{4} = 69.531 \\ t_2 &= \frac{69.531 + 57.031 + 0 + 50}{4} = 44.141 \\ t_3 &= \frac{100 + 100 + 0 + 44.141}{4} = 61.035 \end{aligned}$$

Continuing, we find the iterates listed in Table 2.18. We work with five-significant-digit accuracy and stop when two successive iterates agree within 0.001 in all variables.

Thus, the equilibrium temperatures at the interior points are (to an accuracy of 0.001) $t_1 = 74.108$, $t_2 = 46.430$, and $t_3 = 61.607$. (Check these calculations.)

By using a finer grid (with more interior points), we can get as precise information as we like about the equilibrium temperatures at various points on the plate.

Table 2.18

n	0	1	2	3	. . .	7	8
t_1	0	62.500	69.531	73.535	. . .	74.107	74.107
t_2	0	28.125	44.141	46.143	. . .	46.429	46.429
t_3	0	57.031	61.035	61.536	. . .	61.607	61.607

Exercises 2.5

CAS

In Exercises 1–6, apply Jacobi's method to the given system. Take the zero vector as the initial approximation and work with four-significant-digit accuracy until two successive iterates agree within 0.001 in each variable. In each case, compare your answer with the exact solution found using any direct method you like.

1. $$\begin{aligned} 7x_1 - x_2 &= 6 \\ x_1 - 5x_2 &= -4 \end{aligned}$$

2. $$\begin{aligned} 2x_1 + x_2 &= 5 \\ x_1 - x_2 &= 1 \end{aligned}$$

3. $$\begin{aligned} 4.5x_1 - 0.5x_2 &= 1 \\ x_1 - 3.5x_2 &= -1 \end{aligned}$$

4. $$\begin{aligned} 20x_1 + x_2 - x_3 &= 17 \\ x_1 - 10x_2 + x_3 &= 13 \\ -x_1 + x_2 + 10x_3 &= 18 \end{aligned}$$

5. $$\begin{aligned} 3x_1 + x_2 &= 1 \\ x_1 + 4x_2 + x_3 &= 1 \\ x_2 + 3x_3 &= 1 \end{aligned}$$

6. $$\begin{aligned} 3x_1 - x_2 &= 1 \\ -x_1 + 3x_2 - x_3 &= 0 \\ -x_2 + 3x_3 - x_4 &= 1 \\ -x_3 + 3x_4 &= 1 \end{aligned}$$

In Exercises 7–12, repeat the given exercise using the Gauss-Seidel method. Take the zero vector as the initial approximation and work with four-significant-digit accuracy until two successive iterates agree within 0.001 in each variable. Compare the number of iterations required by the Jacobi and Gauss-Seidel methods to reach such an approximate solution.

7. Exercise 1 **8.** Exercise 2

9. Exercise 3 **10.** Exercise 4

11. Exercise 5 **12.** Exercise 6

In Exercises 13 and 14, draw diagrams to illustrate the convergence of the Gauss-Seidel method with the given system.

13. The system in Exercise 1

14. The system in Exercise 2

In Exercises 15 and 16, compute the first four iterates, using the zero vector as the initial approximation, to show that the Gauss-Seidel method diverges. Then show that the equations can be rearranged to give a strictly diagonally dominant coefficient matrix, and apply the Gauss-Seidel method to obtain an approximate solution that is accurate to within 0.001.

15. $$\begin{aligned} x_1 - 2x_2 &= 3 \\ 3x_1 + 2x_2 &= 1 \end{aligned}$$

16. $$\begin{aligned} x_1 - 4x_2 + 2x_3 &= 2 \\ 2x_2 + 4x_3 &= 1 \\ 6x_1 - x_2 - 2x_3 &= 1 \end{aligned}$$

17. Draw a diagram to illustrate the divergence of the Gauss-Seidel method in Exercise 15.

In Exercises 18 and 19, the coefficient matrix is not strictly diagonally dominant, nor can the equations be rearranged to make it so. However, both the Jacobi and the Gauss-Seidel method converge anyway. Demonstrate that this is true of the Gauss-Seidel method, starting with the zero vector as the initial approximation and obtaining a solution that is accurate to within 0.01.

18. $$\begin{aligned} -4x_1 + 5x_2 &= 14 \\ x_1 - 3x_2 &= -7 \end{aligned}$$

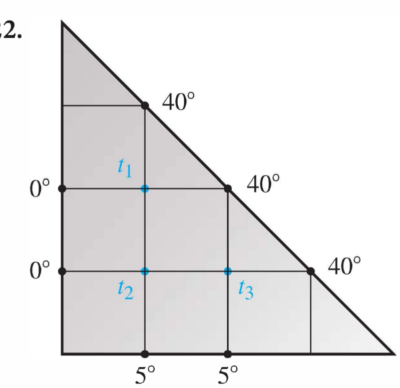
19. $$\begin{aligned} 5x_1 - 2x_2 + 3x_3 &= -8 \\ x_1 + 4x_2 - 4x_3 &= 102 \\ -2x_1 - 2x_2 + 4x_3 &= -90 \end{aligned}$$

20. Continue performing iterations in Exercise 18 to obtain a solution that is accurate to within 0.001.

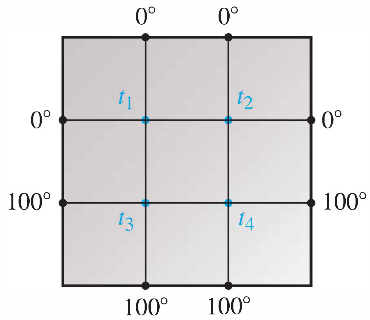
21. Continue performing iterations in Exercise 19 to obtain a solution that is accurate to within 0.001.

In Exercises 22–24, the metal plate has the constant temperatures shown on its boundaries. Find the equilibrium temperature at each of the indicated interior points by setting up a system of linear equations and applying either the Jacobi or the Gauss-Seidel method. Obtain a solution that is accurate to within 0.001.

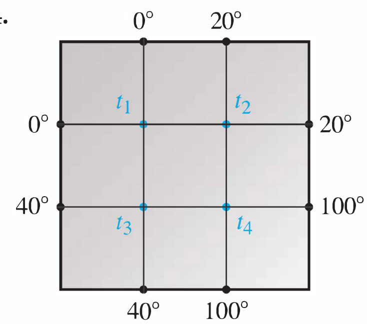
22.



23.

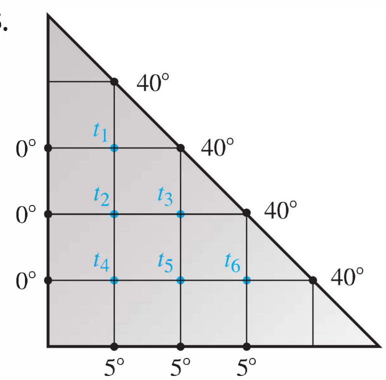


24.

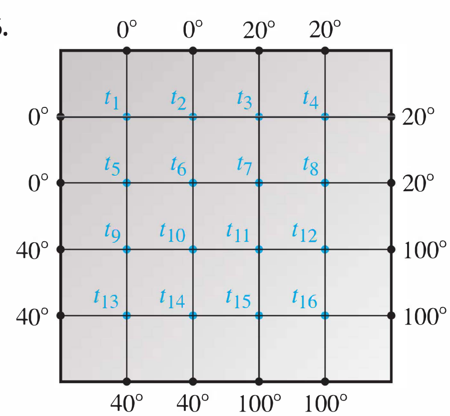


In Exercises 25 and 26, we refine the grids used in Exercises 22 and 24 to obtain more accurate information about the equilibrium temperatures at interior points of the plates. Obtain solutions that are accurate to within 0.001, using either the Jacobi or the Gauss-Seidel method.

25.



26.



Exercises 27 and 28 demonstrate that sometimes, if we are lucky, the form of an iterative problem may allow us to use a little insight to obtain an exact solution.

27. A narrow strip of paper 1 unit long is placed along a number line so that its ends are at 0 and 1. The paper is folded in half, right end over left, so that its ends are now at 0 and $\frac{1}{2}$. Next, it is folded in half again, this time left end over right, so that its ends are at $\frac{1}{4}$ and $\frac{1}{2}$. Figure 2.32 shows this process. We continue folding the paper in half, alternating right-over-left and left-over-right. If we could continue indefinitely, it is clear that the ends of the paper would converge to a point. It is this point that we want to find.

(a) Let x_1 correspond to the left-hand end of the paper and x_2 to the right-hand end. Make a table with the first six values of $[x_1, x_2]$ and plot the corresponding points on x_1, x_2 coordinate axes.

(b) Find two linear equations of the form $x_2 = ax_1 + b$ and $x_1 = cx_2 + d$ that determine the new values of the endpoints at each iteration. Draw the corresponding lines on your coordinate axes and show that this diagram would result from applying the Gauss-Seidel method to the system of linear equations you have found. (Your diagram should resemble Figure 2.27 on page 126.)

(c) Switching to decimal representation, continue applying the Gauss-Seidel method to approximate the point to which the ends of the paper are converging to within 0.001 accuracy.

(d) Solve the system of equations exactly and compare your answers.

28. An ant is standing on a number line at point A. It walks halfway to point B and turns around. Then it walks halfway back to point A, turns around again, and walks halfway to point B. It continues to do this indefinitely. Let point A be at 0 and point B be at 1. The ant's walk is made up of a sequence of overlapping line segments. Let x_1 record the positions of the left-hand endpoints of these segments and x_2 their right-hand endpoints. (Thus, we begin with $x_1 = 0$ and $x_2 = \frac{1}{2}$. Then we have $x_1 = \frac{1}{4}$ and $x_2 = \frac{1}{2}$, and so on.) Figure 2.33 shows the start of the ant's walk.

(a) Make a table with the first six values of $[x_1, x_2]$ and plot the corresponding points on x_1, x_2 coordinate axes.

(b) Find two linear equations of the form $x_2 = ax_1 + b$ and $x_1 = cx_2 + d$ that determine the new values of the endpoints at each iteration. Draw the corresponding

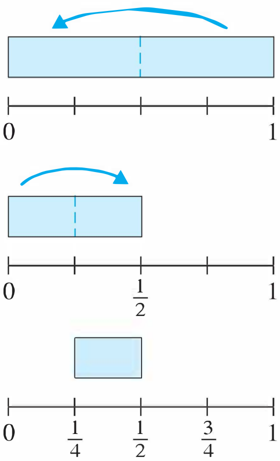


Figure 2.32
Folding a strip of paper

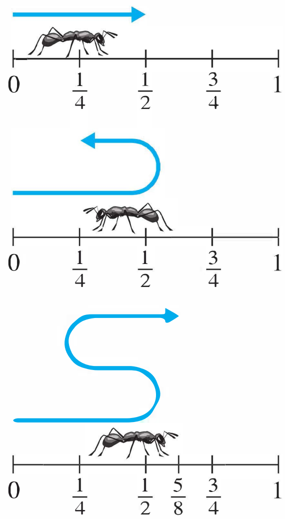


Figure 2.33
The ant's walk

lines on your coordinate axes and show that this diagram would result from applying the Gauss-Seidel method to the system of linear equations you have found. (Your diagram should resemble Figure 2.27 on page 126.)

(c) Switching to decimal representation, continue applying the Gauss-Seidel method to approximate the values to which x_1 and x_2 are converging to within 0.001 accuracy.

(d) Solve the system of equations exactly and compare your answers. Interpret your results.

Chapter Review

Key Definitions and Concepts

Review Questions

1. Mark each of the following statements true or false:

(a) Every system of linear equations has a solution.

(b) Every homogeneous system of linear equations has a solution.

(c) If a system of linear equations has more variables than equations, then it has infinitely many solutions.

(d) If a system of linear equations has more equations than variables, then it has no solution.

(e) Determining whether $\mathbf{b}$ is in span$(\mathbf{a}_1, \ldots, \mathbf{a}_n)$ is equivalent to determining whether the system $[A \mid \mathbf{b}]$ is consistent, where $A = [\mathbf{a}_1 \cdots \mathbf{a}_n]$.

(f) In $\mathbb{R}^3$, span$(\mathbf{u}, \mathbf{v})$ is always a plane through the origin.

(g) In $\mathbb{R}^3$, if nonzero vectors $\mathbf{u}$ and $\mathbf{v}$ are not parallel, then they are linearly independent.

(h) In $\mathbb{R}^3$, if a set of vectors can be drawn head to tail, one after the other so that a closed path (polygon) is formed, then the vectors are linearly dependent.

(i) If a set of vectors has the property that no two vectors in the set are scalar multiples of one another, then the set of vectors is linearly independent.

(j) If there are more vectors in a set of vectors than the number of entries in each vector, then the set of vectors is linearly dependent.

2. Find the rank of the matrix $\begin{bmatrix} 1 & -2 & 0 & 3 & 2 \\ 3 & -1 & 1 & 3 & 4 \\ 3 & 4 & 2 & -3 & 2 \\ 0 & -5 & -1 & 6 & 2 \end{bmatrix}$.

3. Solve the linear system

$$\begin{aligned} x + y - 2z &= 4 \\ x + 3y - z &= 7 \\ 2x + y - 5z &= 7 \end{aligned}$$

4. Solve the linear system

$$\begin{aligned} 3w + 8x - 18y + z &= 35 \\ w + 2x - 4y \phantom{{}+z} &= 11 \\ w + 3x - 7y + z &= 10 \end{aligned}$$

5. Solve the linear system

$$\begin{aligned} 2x + 3y &= 4 \\ x + 2y &= 3 \end{aligned}$$

over $\mathbb{Z}_7$.

6. Solve the linear system

$$\begin{aligned} 3x + 2y &= 1 \\ x + 4y &= 2 \end{aligned}$$

over $\mathbb{Z}_5$.

7. For what value(s) of k is the linear system with augmented matrix $\left[\begin{array}{cc|c} k & 2 & 1 \\ 1 & 2k & 1 \end{array}\right]$ inconsistent?

8. Find parametric equations for the line of intersection of the planes $x + 2y + 3z = 4$ and $5x + 6y + 7z = 8$.

9. Find the point of intersection of the following lines, if it exists.

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + s\begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 5 \\ -2 \\ -4 \end{bmatrix} + t\begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$$

10. Determine whether $\begin{bmatrix} 3 \\ 5 \\ -1 \end{bmatrix}$ is in the span of $\begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix}$.

11. Find the general equation of the plane spanned by $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$.

12. Determine whether $\begin{bmatrix} 2 \\ 1 \\ -3 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ -2 \end{bmatrix}, \begin{bmatrix} 3 \\ 9 \\ -2 \end{bmatrix}$ are linearly independent.

13. Determine whether $\mathbb{R}^3 = \text{span}(\mathbf{u}, \mathbf{v}, \mathbf{w})$ if:

(a) $\mathbf{u} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \mathbf{v} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \mathbf{w} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$

(b) $\mathbf{u} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \mathbf{v} = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \mathbf{w} = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$

14. Let $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3$ be linearly independent vectors in $\mathbb{R}^3$, and let $A = [\mathbf{a}_1\ \mathbf{a}_2\ \mathbf{a}_3]$. Which of the following statements are true?

(a) The reduced row echelon form of A is I_3.

(b) The rank of A is 3.

(c) The system $[A \mid \mathbf{b}]$ has a unique solution for any vector $\mathbf{b}$ in $\mathbb{R}^3$.

(d) (a), (b), and (c) are all true.

(e) (a) and (b) are both true, but not (c).

15. Let $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3$ be linearly dependent vectors in $\mathbb{R}^3$, not all zero, and let $A = [\mathbf{a}_1\ \mathbf{a}_2\ \mathbf{a}_3]$. What are the possible values of the rank of A?

16. What is the maximum rank of a 5×3 matrix? What is the minimum rank of a 5×3 matrix?

17. Show that if $\mathbf{u}$ and $\mathbf{v}$ are linearly independent vectors, then so are $\mathbf{u} + \mathbf{v}$ and $\mathbf{u} - \mathbf{v}$.

18. Show that $\text{span}(\mathbf{u}, \mathbf{v}) = \text{span}(\mathbf{u}, \mathbf{u} + \mathbf{v})$ for any vectors $\mathbf{u}$ and $\mathbf{v}$.

19. In order for a linear system with augmented matrix $[A \mid \mathbf{b}]$ to be consistent, what must be true about the ranks of A and $[A \mid \mathbf{b}]$?

20. Are the matrices $\begin{bmatrix} 1 & 1 & 1 \\ 2 & 3 & -1 \\ -1 & 4 & 1 \end{bmatrix}$ and $\begin{bmatrix} 1 & 0 & -1 \\ 1 & 1 & 1 \\ 0 & 1 & 3 \end{bmatrix}$ row equivalent? Why or why not?