

Answers to Selected Odd-Numbered Exercises

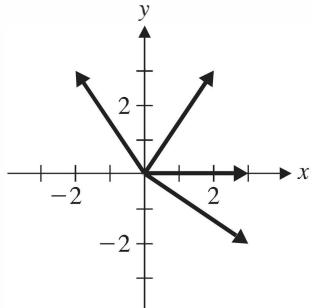
*Answers are easy. It's asking
the right questions [that's] hard.*

—Doctor Who
“The Face of Evil,”
By Chris Boucher
BBC, 1977

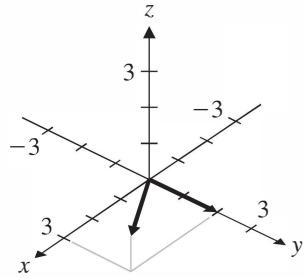
Chapter 1

Exercises 1.1

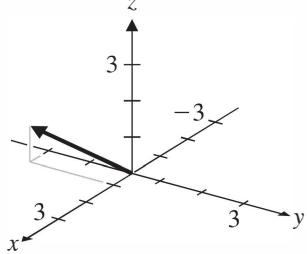
1.



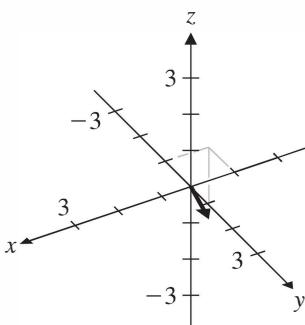
3. (a), (b)



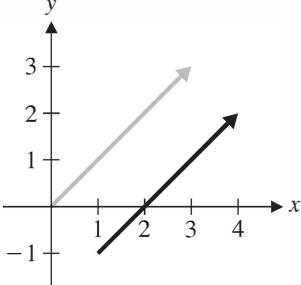
(c)



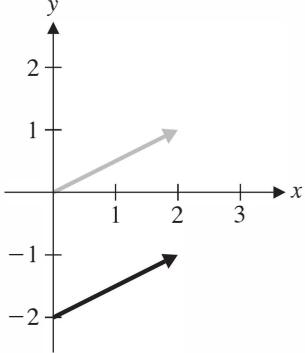
(d)



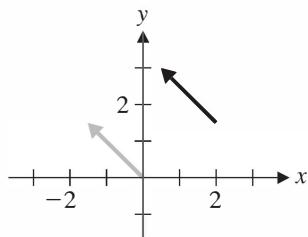
5. (a)



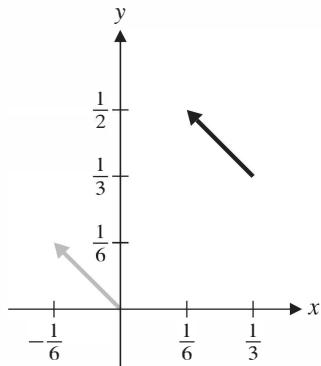
(b)



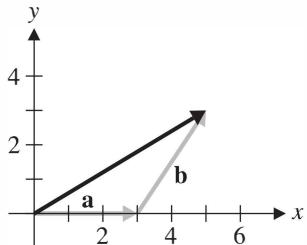
(c)



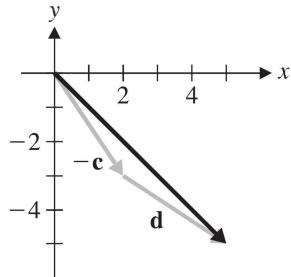
(d)



7. $\mathbf{a} + \mathbf{b} = [5, 3]$



9. $\mathbf{d} - \mathbf{c} = [5, -5]$



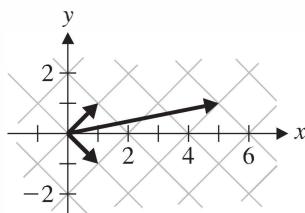
11. $[3, -2, 3]$

13. $\mathbf{u} = \begin{bmatrix} 1/2 \\ \sqrt{3}/2 \end{bmatrix}, \mathbf{v} = \begin{bmatrix} -\sqrt{3}/2 \\ -1/2 \end{bmatrix}, \mathbf{u} + \mathbf{v} = \begin{bmatrix} (1 - \sqrt{3})/2 \\ (\sqrt{3} - 1)/2 \end{bmatrix}, \mathbf{u} - \mathbf{v} = \begin{bmatrix} (1 + \sqrt{3})/2 \\ (1 + \sqrt{3})/2 \end{bmatrix}$

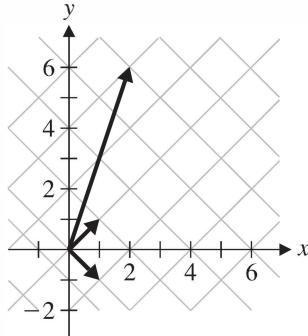
15. 5a

17. $x = 3a$

19.



21. $\mathbf{w} = -2\mathbf{u} + 4\mathbf{v}$



25. $\mathbf{u} + \mathbf{v} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

27. $\mathbf{u} + \mathbf{v} = [0, 1, 0, 0]$

$$\begin{array}{r|rrrr} 29. + & 0 & 1 & 2 & 3 \\ \hline 0 & 0 & 1 & 2 & 3 \\ 1 & 1 & 2 & 3 & 0 \\ 2 & 2 & 3 & 0 & 1 \\ 3 & 3 & 0 & 1 & 2 \end{array} \quad \begin{array}{r|rrrr} \cdot & 0 & 1 & 2 & 3 \\ \hline 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 2 & 3 \\ 2 & 0 & 2 & 0 & 2 \\ 3 & 0 & 3 & 2 & 1 \end{array}$$

31. 0

33. 1

35. 0

37. 2, 0, 3

39. 5

41. $[1, 1, 0]$

43. $[0, 0, 2, 2], [2, 3, 1, 1]$

45. $x = 2$

47. No solution

49. $x = 3$

51. No solution

53. $x = 2$

55. $x = 1$, or $x = 5$

57. (a) All $a \neq 0$ (b) $a = 1, 5$

(c) a and m can have no common factors other than 1 [i.e., the greatest common divisor (gcd) of a and m is 1].

Exercises 1.2

1. -1

3. 11

5. 2

7. $\sqrt{5}, \begin{bmatrix} -1/\sqrt{5} \\ 2/\sqrt{5} \end{bmatrix}$

9. $\sqrt{14}, \begin{bmatrix} 1/\sqrt{14} \\ 2/\sqrt{14} \\ 3/\sqrt{14} \end{bmatrix}$

11. $\sqrt{6}$, $[1/\sqrt{6}, 1/\sqrt{3}, 1/\sqrt{2}, 0]$

13. $\sqrt{17}$ 15. $\sqrt{6}$

17. (a) $\mathbf{u} \cdot \mathbf{v}$ is a scalar, not a vector.(c) $\mathbf{v} \cdot \mathbf{w}$ is a scalar and \mathbf{u} is a vector.

19. Acute

25. 60°

21. Acute

27. $\approx 88.10^\circ$

23. Acute

29. $\approx 14.34^\circ$

31. Since $\overrightarrow{AB} \cdot \overrightarrow{AC} = \begin{bmatrix} -4 \\ 1 \\ -1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ -3 \end{bmatrix} = 0$, $\angle BAC$ is a right angle.

33. If we take the cube to be a unit cube (as in Figure 1.34), the four diagonals are given by the vectors

$$\mathbf{d}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \mathbf{d}_2 = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}, \mathbf{d}_3 = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}, \mathbf{d}_4 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

Since $\mathbf{d}_i \cdot \mathbf{d}_j \neq 0$ for all $i \neq j$ (six possibilities), no two diagonals are perpendicular.

35. $D = (-2, 1, 1)$

37. 5 mi/h at an angle of $\approx 53.13^\circ$ to the bank

39. 60°

41. $\begin{bmatrix} -\frac{3}{5} \\ \frac{4}{5} \end{bmatrix}$

43. $\begin{bmatrix} \frac{3}{2} \\ -\frac{3}{2} \\ \frac{3}{2} \\ -\frac{3}{2} \end{bmatrix}$

45. $\begin{bmatrix} -0.301 \\ 0.033 \\ -0.252 \end{bmatrix}$

47. $\mathcal{A} = \sqrt{45}/2$

49. $k = -2, 3$

51. \mathbf{v} is of the form $k \begin{bmatrix} b \\ -a \end{bmatrix}$, where k is a scalar.

53. The Cauchy-Schwarz Inequality would be violated.

Exercises 1.3

1. (a) $\begin{bmatrix} 3 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = 0$ (b) $3x + 2y = 0$

3. (a) $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -1 \\ 3 \end{bmatrix}$

(b) $x = 1 - t$
 $y = 3t$

5. (a) $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = t \begin{bmatrix} 1 \\ -1 \\ 4 \end{bmatrix}$ (b) $x = t$
 $y = -t$
 $z = 4t$

7. (a) $\begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 2$ (b) $3x + 2y + z = 2$

9. (a) $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = s \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} + t \begin{bmatrix} -3 \\ 2 \\ 1 \end{bmatrix}$

(b) $x = 2s - 3t$
 $y = s + 2t$
 $z = 2s + t$

11. $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \end{bmatrix} + t \begin{bmatrix} 2 \\ 2 \end{bmatrix}$

13. $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + s \begin{bmatrix} 3 \\ -1 \\ 1 \end{bmatrix} + t \begin{bmatrix} -1 \\ 0 \\ -2 \end{bmatrix}$

15. (a) $x = t$ (b) $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \end{bmatrix} + t \begin{bmatrix} 1 \\ 3 \end{bmatrix}$
 $y = -1 + 3t$

17. Direction vectors for the two lines are given by

 $\mathbf{d}_1 = \begin{bmatrix} 1 \\ m_1 \end{bmatrix}$ and $\mathbf{d}_2 = \begin{bmatrix} 1 \\ m_2 \end{bmatrix}$. The lines are perpendicular if and only if \mathbf{d}_1 and \mathbf{d}_2 are orthogonal. But $\mathbf{d}_1 \cdot \mathbf{d}_2 = 0$ if and only if $1 + m_1 m_2 = 0$ or, equivalently,

$m_1 m_2 = -1$.

19. (a) Perpendicular

(c) Perpendicular

(b) Parallel

(d) Perpendicular

21. $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \end{bmatrix} + t \begin{bmatrix} 3 \\ 2 \end{bmatrix}$

23. $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 3 \end{bmatrix} + t \begin{bmatrix} -1 \\ 3 \\ -1 \end{bmatrix}$

25. (a) $x = 0, x = 1, y = 0, y = 1, z = 0, z = 1$
(b) $x - y = 0$ (c) $x + y - z = 0$

27. $3\sqrt{2}/2$ 29. $2\sqrt{3}/3$ 31. $(\frac{1}{2}, \frac{1}{2})$

33. $(\frac{4}{3}, \frac{4}{3}, \frac{8}{3})$ 35. $18\sqrt{13}/13$ 37. $\frac{5}{3}$

43. $\approx 78.9^\circ$ 45. $\approx 80.4^\circ$

Exercises 1.4

1. 13 N at approx N 67.38 E

3. $8\sqrt{3}$ N at an angle of 30° to \mathbf{f}_1 5. 4 N at an angle of 60° to \mathbf{f}_2 7. 5 N at an angle of 60° to the given force, $5\sqrt{3}$ N perpendicular to the 5 N force9. $750\sqrt{2}$ N

11. 980 N

13. ≈ 117.6 N in the 15 cm wire, ≈ 88.2 N in the 20 cm wire

Review Questions

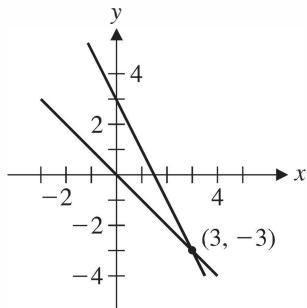
1. (a) T (c) F (e) T (g) F (i) T
 3. $x = \begin{bmatrix} 8 \\ 11 \end{bmatrix}$ 5. 120° 7. $\begin{bmatrix} -2/\sqrt{5} \\ 1/\sqrt{5} \\ 0 \end{bmatrix}$
 9. $2x + 3y - z = 7$ 11. $\sqrt{6}/2$

13. The Cauchy-Schwarz Inequality would be violated.
 15. $2\sqrt{6}/3$ 17. $x = 2$ 19. 3

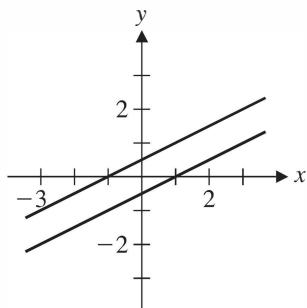
Chapter 2

Exercises 2.1

1. Linear 3. Not linear because of the x^{-1} term
 5. Not linear 7. $2x + 4y = 7$
 9. $x + y = 4(x, y \neq 0)$
 11. $\left\{ \begin{bmatrix} 2t \\ t \end{bmatrix} \right\}$ 13. $\left\{ \begin{bmatrix} 4 - 2s - 3t \\ s \\ t \end{bmatrix} \right\}$
 15. Unique solution, $x = 3, y = -3$



17. No solution



19. $[7, 3]$ 21. $[\frac{2}{3}, \frac{1}{3}, -\frac{1}{3}]$
 23. $[5, -2, 1, 1]$ 25. $[2, -7, -32]$
 27. $\left[\begin{array}{cc|c} 1 & -1 & 0 \\ 2 & 1 & 3 \end{array} \right]$ 29. $\left[\begin{array}{cc|c} 1 & 5 & -1 \\ -1 & 1 & -5 \\ 2 & 4 & 4 \end{array} \right]$

31. $y + z = 1$ 33. $[1, 1]$
 $x - y = 1$
 $2x - y + z = 1$
 35. $[4, -1]$ 37. No solution
 39. (a) $2x + y = 3$ (b) $x = \frac{3}{2} - \frac{1}{2}s$
 $4x + 2y = 6$ $y = s$
 41. Let $u = \frac{1}{x}$ and $v = \frac{1}{y}$. The solution is $x = \frac{1}{3}, y = -\frac{1}{2}$.
 43. Let $u = \tan x, v = \sin y, w = \cos z$. One solution is $x = \pi/4, y = -\pi/6, z = \pi/3$. (There are infinitely many solutions.)

Exercises 2.2

1. No 3. Reduced row echelon form
 5. No 7. No
 9. (a) $\left[\begin{array}{ccc} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{array} \right]$ 11. (b) $\left[\begin{array}{cc} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{array} \right]$
 13. (b) $\left[\begin{array}{ccc} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{array} \right]$

15. Perform elementary row operations in the order $R_4 + 29R_3, 8R_3, R_4 - 3R_2, R_2 \leftrightarrow R_3, R_4 - R_1, R_3 + 2R_1$, and, finally, $R_2 + 2R_1$.
 17. One possibility is to perform elementary row operations on A in the order $R_2 - 3R_1, \frac{1}{2}R_2, R_1 + 2R_2, R_2 + 3R_1, R_1 \leftrightarrow R_2$.
 19. Hint: Pick a random 2×2 matrix and try this—carefully!
 21. This is really two elementary row operations combined: $3R_2$ and $R_2 - 2R_1$.
 23. Exercise 1: 3; Exercise 3: 2; Exercise 5: 2; Exercise 7: 3
 25. $\left[\begin{array}{c} 2 \\ 5 \\ 1 \end{array} \right]$ 27. $t \left[\begin{array}{c} 1 \\ 1 \\ -1 \end{array} \right]$ 29. $\left[\begin{array}{c} 2 \\ -1 \end{array} \right]$
 31. $\left[\begin{array}{c} 24 \\ -10 \\ 0 \\ 0 \\ 0 \end{array} \right] + r \left[\begin{array}{c} 6 \\ -2 \\ 1 \\ 0 \\ 0 \end{array} \right] + s \left[\begin{array}{c} 0 \\ 6 \\ 0 \\ 1 \\ 0 \end{array} \right] + t \left[\begin{array}{c} 12 \\ -6 \\ 0 \\ 0 \\ 1 \end{array} \right]$
 33. No solution
 35. Unique solution

37. Infinitely many solutions

39. Hint: Show that if $ad - bc \neq 0$, the rank of $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$

is 2. (There are two cases: $a = 0$ and $a \neq 0$.) Use the Rank Theorem to deduce that the given system must have a unique solution.

- 41.** (a) No solution if $k = -1$
 (b) A unique solution if $k \neq \pm 1$
 (c) Infinitely many solutions if $k = 1$
- 43.** (a) No solution if $k = 1$
 (b) A unique solution if $k \neq -2, 1$
 (c) Infinitely many solutions if $k = -2$

45.
$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} + t \begin{bmatrix} 9 \\ -10 \\ -7 \end{bmatrix}$$

49. No intersection

51. The required vectors $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ are the solutions of the homogeneous system with augmented matrix

$$\left[\begin{array}{ccc|c} u_1 & u_2 & u_3 & 0 \\ v_1 & v_2 & v_3 & 0 \end{array} \right]$$

By Theorem 3, there are infinitely many solutions. If $u_1 \neq 0$ and $u_1v_2 - u_2v_1 \neq 0$, the solutions are given by

$$t \begin{bmatrix} u_2v_3 - u_3v_2 \\ u_3v_1 - u_1v_3 \\ u_1v_2 - u_2v_1 \end{bmatrix}$$

But a direct check shows that these are still solutions even if $u_1 = 0$ and/or $u_1v_2 - u_2v_1 = 0$.

53. $\begin{bmatrix} 0 \\ 2 \end{bmatrix}$ **55.** $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ **57.** $\begin{bmatrix} 3 \\ 3 \end{bmatrix}$

Exercises 2.3

- 1.** Yes **3.** No **5.** Yes **7.** Yes

9. We need to show that the vector equation $x \begin{bmatrix} 1 \\ 1 \end{bmatrix} + y \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} a \\ b \end{bmatrix}$ has a solution for all values of a and b .

This vector equation is equivalent to the linear system whose augmented matrix is $\left[\begin{array}{cc|c} 1 & 1 & a \\ 1 & -1 & b \end{array} \right]$. Row

reduction yields $\left[\begin{array}{cc|c} 1 & 1 & a \\ 0 & -2 & b - a \end{array} \right]$, from which we can see that there is a (unique) solution.

[Further row operations yield $x = (a + b)/2$, $y = (a - b)/2$.] Hence, $\mathbb{R}^2 = \text{span}\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix}\right)$.

11. We need to show that the vector equation $x \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + y \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + z \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ has a solution for all values

of a , b , and c . This vector equation is equivalent to the linear system whose augmented matrix is

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & a \\ 0 & 1 & 1 & b \\ 1 & 0 & 1 & c \end{array} \right]. \text{ Row reduction yields}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & a \\ 0 & 1 & 1 & b \\ 0 & 0 & 2 & b + c - a \end{array} \right], \text{ from which we can see}$$

that there is a (unique) solution. [Further row operations yield $x = (a - b + c)/2$, $y = (a + b - c)/2$, $z = (-a + b + c)/2$.]

Hence, $\mathbb{R}^3 = \text{span}\left(\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}\right)$.

13. (a) The line through the origin with direction vector $\begin{bmatrix} -1 \\ 2 \end{bmatrix}$

(b) The line with general equation $2x + y = 0$

15. (a) The plane through the origin with direction

$$\text{vectors } \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \\ -1 \end{bmatrix}$$

(b) The plane with general equation $2x - y + 4z = 0$

17. Substitution yields the linear system

$$\begin{aligned} a &+ 3c = 0 \\ -a &+ b - 3c = 0 \end{aligned}$$

whose solution is $t \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}$. It follows that there are

infinitely many solutions, the simplest perhaps being $a = -3$, $b = 0$, $c = 1$.

19. $\mathbf{u} = \mathbf{u} + 0(\mathbf{u} + \mathbf{v}) + 0(\mathbf{u} + \mathbf{v} + \mathbf{w})$

$$\mathbf{v} = (-1)\mathbf{u} + (\mathbf{u} + \mathbf{v}) + 0(\mathbf{u} + \mathbf{v} + \mathbf{w})$$

$$\mathbf{w} = 0\mathbf{u} + (-1)(\mathbf{u} + \mathbf{v}) + (\mathbf{u} + \mathbf{v} + \mathbf{w})$$

21. (c) We must show that $\text{span}(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3) = \text{span}(\mathbf{e}_1, \mathbf{e}_1 + \mathbf{e}_2, \mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3)$. We know that $\text{span}(\mathbf{e}_1, \mathbf{e}_1 + \mathbf{e}_2, \mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3) \subseteq \mathbb{R}^3 = \text{span}(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$. From Exercise 19, $\mathbf{e}_1, \mathbf{e}_2$, and \mathbf{e}_3 all belong to $\text{span}(\mathbf{e}_1, \mathbf{e}_1 + \mathbf{e}_2, \mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3)$. Therefore, by Exercise 21(b), $\text{span}(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3) = \text{span}(\mathbf{e}_1, \mathbf{e}_1 + \mathbf{e}_2, \mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3)$.

23. Linearly independent

$$\begin{matrix} \text{25. Linearly dependent, } - & \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} + \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} \end{matrix}$$

27. Linearly dependent, since the set contains the zero vector

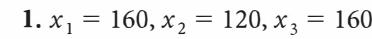
29. Linearly independent

$$\begin{matrix} \text{31. Linearly dependent, } & \begin{bmatrix} 3 \\ -1 \\ 1 \\ -1 \end{bmatrix} + \begin{bmatrix} -1 \\ 3 \\ 1 \\ -1 \end{bmatrix} + \begin{bmatrix} -1 \\ -1 \\ 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \end{matrix}$$

43. (a) Yes

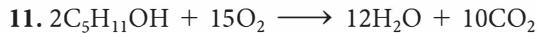
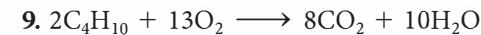
(b) No

Exercises 2.4



3. two small, three medium, four large

5. 65 bags of house blend, 30 bags of special blend, 45 bags of gourmet blend



15. (a) $f_1 = 30 - t$ **(b)** $f_1 = 15, f_3 = 15$

$$f_2 = -10 + t$$

$$f_3 = t$$

(c) $0 \leq f_1 \leq 20$

$$0 \leq f_2 \leq 20$$

$$10 \leq f_3 \leq 30$$

(d) Negative flow would mean that water was flowing backward, against the direction of the arrow.

17. (a) $f_1 = -200 + s + t$ **(b)** $200 \leq f_3 \leq 300$

$$f_2 = 300 - s - t$$

$$f_3 = s$$

$$f_4 = 150 - t$$

$$f_5 = t$$

(c) If $f_3 = s = 0$, then $f_5 = t \geq 200$ (from the f_1 equation), but $f_5 = t \leq 150$ (from the f_4 equation). This is a contradiction.

(d) $50 \leq f_3 \leq 300$

19. $I_1 = 3$ amps, $I_2 = 5$ amps, $I_3 = 2$ amps

21. (a) $I = 10$ amps, $I_1 = I_5 = 6$ amps, $I_2 = I_4 = 4$ amps, $I_3 = 2$ amps

$$\text{(b)} \quad R_{\text{eff}} = \frac{7}{5} \text{ ohms}$$

(c) Yes; change it to 4 ohms.

23. Farming : Manufacturing = 2 : 3

25. The painter charges \$39/hr, the plumber \$42/hr, the electrician \$54/hr.

27. (a) Coal should produce \$100 million and steel \$160 million.

(b) Coal should reduce production by $\approx \$4.2$ million and steel should increase production by $\approx \$5.7$ million.

29. (a) Yes; push switches 1, 2, and 3 or switches 3, 4, and 5.

(b) No

31. The states that can be obtained are represented by those vectors

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix}$$

in \mathbb{Z}_2^5 for which $x_1 + x_2 + x_4 + x_5 = 0$.

(There are 16 such possibilities.)

33. If 0 = off, 1 = light blue, and 2 = dark blue, then the linear system that arises has augmented matrix

$$\left[\begin{array}{ccccc|c} 1 & 1 & 0 & 0 & 0 & 2 \\ 1 & 1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 2 \end{array} \right]$$

which reduces over \mathbb{Z}_3 to

$$\left[\begin{array}{ccccc|c} 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 2 & 1 \\ 0 & 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array} \right]$$

This yields the solutions

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 2 \\ 2 \\ 0 \end{bmatrix} + t \begin{bmatrix} 2 \\ 1 \\ 0 \\ 2 \\ 1 \end{bmatrix}$$

where t is in \mathbb{Z}_3 . Hence, there are exactly three solutions:

$$\begin{bmatrix} 1 \\ 1 \\ 2 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 2 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 2 \\ 0 \\ 2 \end{bmatrix}$$

where each entry indicates the number of times the corresponding switch should be pushed.

35. (a) Push squares 3 and 7.
 (b) The 9×9 coefficient matrix A is row equivalent to \mathbb{Z}_2 , so for any \mathbf{b} in \mathbb{Z}_2^9 , $A\mathbf{x} = \mathbf{b}$ has a unique solution.

37. Grace is 15, and Hans is 5.

39. 1200 and 600 square yards

41. (a) $a = 4 - d$, $b = 5 - d$, $c = -2 + d$, d is arbitrary

(b) No solution

43. (a) No solution

(b) $[a, b, c, d, e, f] = [4, 5, 6, -3, -1, 0] + f[-1, -1, -1, 1, 1, 1]$

45. (a) $y = x^2 - 2x + 1$ (b) $y = x^2 + 6x + 10$

47. $A = 1$, $B = 2$

49. $A = -\frac{1}{5}$, $B = \frac{1}{3}$, $C = 0$, $D = -\frac{2}{15}$, $E = -\frac{1}{5}$

51. $a = \frac{1}{2}$, $b = \frac{1}{2}$, $c = 0$

Exercises 2.5

n	0	1	2	3	4	5
x_1	0	0.8571	0.9714	0.9959	0.9991	0.9998
x_2	0	0.8000	0.9714	0.9943	0.9992	0.9998

Exact solution: $x_1 = 1$, $x_2 = 1$

n	0	1	2	3	4	5	6
x_1	0	0.2222	0.2539	0.2610	0.2620	0.2622	0.2623
x_2	0	0.2857	0.3492	0.3582	0.3603	0.3606	0.3606

Exact solution (to four decimal places): $x_1 = 0.2623$, $x_2 = 0.3606$

n	0	1	2	3	4	5	6	7	8
x_1	0	0.3333	0.2500	0.3055	0.2916	0.3009	0.2986	0.3001	0.2997
x_2	0	0.2500	0.0834	0.1250	0.0972	0.1042	0.0996	0.1008	0.1000
x_3	0	0.3333	0.2500	0.3055	0.2916	0.3009	0.2986	0.3001	0.2997

Exact solution: $x_1 = 0.3$, $x_2 = 0.1$, $x_3 = 0.3$

n	0	1	2	3	4
x_1	0	0.8571	0.9959	0.9998	1.0000
x_2	0	0.9714	0.9992	1.0000	1.0000

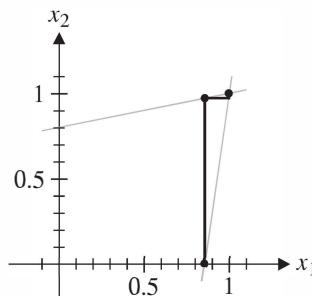
After three iterations, the Gauss-Seidel method is within 0.001 of the exact solution. Jacobi's method took four iterations to reach the same accuracy.

n	0	1	2	3	4
x_1	0	0.2222	0.2610	0.2622	0.2623
x_2	0	0.3492	0.3603	0.3606	0.3606

After three iterations, the Gauss-Seidel method is within 0.001 of the exact solution. Jacobi's method took four iterations to reach the same accuracy.

n	0	1	2	3	4	5	6
x_1	0	0.3333	0.2777	0.2962	0.2993	0.2998	0.3000
x_2	0	0.1667	0.1112	0.1020	0.1004	0.1000	0.1000
x_3	0	0.2777	0.2962	0.2993	0.2998	0.3000	0.3000

After four iterations, the Gauss-Seidel method is within 0.001 of the exact solution. Jacobi's method took seven iterations to reach the same accuracy.

13.

n	0	1	2	3	4
x_1	0	3	-5	19	-53
x_2	0	-4	8	-28	80

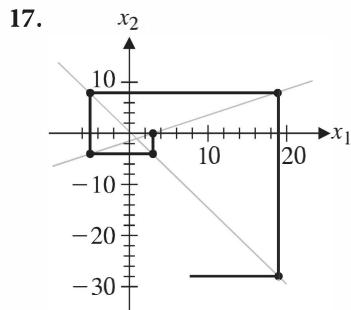
If the equations are interchanged and the Gauss-Seidel method is applied to the equivalent system

$$\begin{aligned}3x_1 + 2x_2 &= 1 \\x_1 - 2x_2 &= 3\end{aligned}$$

we obtain

n	0	1	2	3	4	5	6	7	8
x_1	0	0.3333	1.2222	0.9260	1.0247	0.9918	1.0027	0.9991	1.0003
x_2	0	-1.3333	-0.8889	-1.0370	-0.9876	-1.0041	-0.9986	-1.0004	-0.9998

After seven iterations, the process has converged to within 0.001 of the exact solution $x_1 = 1$, $x_2 = -1$.



n	0	1	2	3	4	5	6
x_1	0	-1.6	14.97	8.550	10.740	9.839	10.120
x_2	0	25.9	11.408	14.051	11.615	11.718	11.249
x_3	0	-10.35	-9.311	-11.200	-11.322	-11.721	-11.816

n	7	8	9	10	11	12
x_1	9.989	10.022	10.002	10.005	10.001	10.001
x_2	11.187	11.082	11.052	11.026	11.015	11.008
x_3	-11.912	-11.948	-11.973	-11.985	-11.992	-11.996

After 12 iterations, the Gauss-Seidel method has converged to within 0.01 of the exact solution $x_1 = 10$, $x_2 = 11$, $x_3 = -12$.

21.

n	13	14	15	16
x_1	10.0004	10.0003	10.0001	10.0001
x_2	11.0043	11.0023	11.0014	11.0007
x_3	-11.9976	-11.9986	-11.9993	-11.9996

23. The Gauss-Seidel method produces

n	0	1	2	3	4	5	6	7	8	9
x_1	0	0	12.5	21.875	24.219	24.805	24.951	24.988	24.997	24.999
x_2	0	0	18.75	21.438	24.609	24.902	24.976	24.994	24.998	24.999
x_3	0	50	68.75	73.438	74.609	74.902	74.976	74.994	74.998	74.999
x_4	0	62.5	71.875	74.219	74.805	74.951	74.988	74.997	74.999	75.000

The exact solution is $x_1 = 25$, $x_2 = 25$, $x_3 = 75$, $x_4 = 75$.

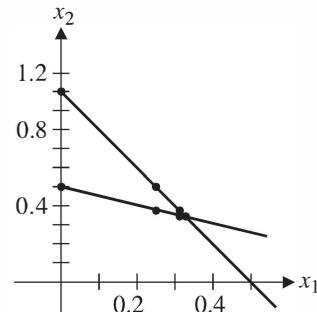
25. The Gauss-Seidel method produces the following iterates:

n	0	1	2	3	4	5	6
t_1	0	20	21.25	22.8125	23.3301	23.6596	23.7732
t_2	0	5	11.25	13.3203	14.6386	15.0926	15.2732
t_3	0	21.25	24.6094	26.9873	27.7303	27.9626	28.0352
t_4	0	2.5	5.8594	8.2373	8.9804	9.2126	9.2852
t_5	0	7.1875	14.6289	16.2829	16.7578	16.9036	16.9491
t_6	0	23.0469	24.9072	25.3207	25.4394	25.4759	25.4873

n	7	8	9	10	11	12
t_1	23.8093	23.8206	23.8242	23.8252	23.8256	23.8257
t_2	15.2824	15.2966	15.3010	15.3024	15.3029	15.3029
t_3	28.0579	28.0650	28.0671	28.0678	28.0681	28.0681
t_4	9.3079	9.3150	9.3172	9.3178	9.3181	9.3181
t_5	16.9633	16.9677	16.9690	16.9695	16.9696	16.9696
t_6	25.4908	25.4919	25.4922	25.4924	25.4924	25.4924

27. (a)

n	0	1	2	3	4	5	6
x_1	0	0	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{5}{16}$	$\frac{5}{16}$	$\frac{21}{64}$
x_2	1	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{11}{32}$	$\frac{11}{32}$



(b) $2x_1 + x_2 = 1$
 $x_1 + 2x_2 = 1$

(c)	n	0	1	2	3	4	5	6	7
	x_1	0	0	0.25	0.3125	0.3281	0.3320	0.3330	0.3332
	x_2	1	0.5	0.375	0.3438	0.3360	0.3340	0.3335	0.3334

[Columns 1, 2, and 3 of this table are the odd-numbered columns 1, 3, and 5 from the table in part (a).] The iterates are converging to $x_1 = x_2 = 0.3333$.

(d) $x_1 = x_2 = \frac{1}{3}$

Review Questions

1. (a) F (c) F (e) T (g) T (i) F
 3. $\begin{bmatrix} 0 \\ 2 \\ -1 \end{bmatrix}$ 5. $\begin{bmatrix} 6 \\ 2 \end{bmatrix}$ 7. $k = -1$ 9. $(0, 3, 1)$
 11. $x - 2y + z = 0$ 13. (a) Yes 15. 1 or 2
 17. If $c_1(\mathbf{u} + \mathbf{v}) + c_2(\mathbf{u} - \mathbf{v}) = \mathbf{0}$, then $(c_1 + c_2)\mathbf{u} + (c_1 - c_2)\mathbf{v} = \mathbf{0}$. Linear independence of \mathbf{u} and \mathbf{v} implies $c_1 + c_2 = 0$ and $c_1 - c_2 = 0$. Solving this system, we get $c_1 = c_2 = 0$. Hence $\mathbf{u} + \mathbf{v}$ and $\mathbf{u} - \mathbf{v}$ are linearly independent.
 19. Their ranks must be equal.

Chapter 3

Exercises 3.1

1. $\begin{bmatrix} 3 & -6 \\ -5 & 7 \end{bmatrix}$ 3. Not possible
 5. $\begin{bmatrix} 12 & -6 & 3 \\ -4 & 12 & 14 \end{bmatrix}$ 7. $\begin{bmatrix} 3 & 3 \\ 19 & 27 \end{bmatrix}$
 9. [10] 11. $\begin{bmatrix} -4 & -2 \\ 8 & 4 \end{bmatrix}$
 13. $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ 15. $\begin{bmatrix} 27 & 0 \\ -49 & 125 \end{bmatrix}$
 17. $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$

19. $B = \begin{bmatrix} 1.50 & 1.00 & 2.00 \\ 1.75 & 1.50 & 1.00 \end{bmatrix}$, $BA = \begin{bmatrix} 650.00 & 462.50 \\ 675.00 & 406.25 \end{bmatrix}$

Column i corresponds to warehouse i , row 1 contains the costs of shipping by truck, and row 2 contains the costs of shipping by train.

21. $\begin{bmatrix} 1 & -2 & 3 \\ 2 & 1 & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 4 \end{bmatrix}$

23. $AB = [2\mathbf{a}_1 + \mathbf{a}_2 - \mathbf{a}_3 \quad 3\mathbf{a}_1 - \mathbf{a}_2 + 6\mathbf{a}_3 \quad \mathbf{a}_2 + 4\mathbf{a}_3]$
 (where \mathbf{a}_i is the i th column of A)

25. $\begin{bmatrix} 2 & 3 & 0 \\ -6 & -9 & 0 \\ 4 & 6 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 1 & -1 & 1 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 2 & -12 & -8 \\ -1 & 6 & 4 \\ 1 & -6 & -4 \end{bmatrix}$

27. $BA = \begin{bmatrix} 2\mathbf{A}_1 + 3\mathbf{A}_2 \\ \mathbf{A}_1 - \mathbf{A}_2 + \mathbf{A}_3 \\ -\mathbf{A}_1 + 6\mathbf{A}_2 + 4\mathbf{A}_3 \end{bmatrix}$ (where \mathbf{A}_i is the i th row of A)

29. If \mathbf{b}_i is the i th column of B , then $A\mathbf{b}_i$ is the i th column of AB . If the columns of B are linearly dependent, then there are scalars c_1, \dots, c_n (not all zero) such that $c_1\mathbf{b}_1 + \dots + c_n\mathbf{b}_n = \mathbf{0}$. But then $c_1(A\mathbf{b}_1) + \dots + c_n(A\mathbf{b}_n) = A(c_1\mathbf{b}_1 + \dots + c_n\mathbf{b}_n) = A\mathbf{0} = \mathbf{0}$, so the columns of AB are linearly dependent.

31. $\begin{array}{c|c} 3 & 2 \\ \hline -1 & 1 \\ \hline 0 & 0 \end{array} \begin{array}{c|c} 1 & 2 \\ \hline 3 & 4 \\ \hline 1 & 0 \\ \hline 0 & 1 \end{array} \begin{array}{c|c} 2 & 0 \\ \hline 5 & 3 \\ \hline 1 & 2 \\ \hline 0 & -1 \end{array}$

35. (a) $A^2 = \begin{bmatrix} -1 & 1 \\ -1 & 0 \end{bmatrix}$, $A^3 = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$,
 $A^4 = \begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix}$, $A^5 = \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix}$, $A^6 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$,
 $A^7 = \begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix}$

(b) $A^{2001} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$

37. $A^n = \begin{bmatrix} 1 & n \\ 0 & 1 \end{bmatrix}$

39. (a) $\begin{bmatrix} 1 & -1 & 1 & -1 \\ -1 & 1 & -1 & 1 \\ 1 & -1 & 1 & -1 \\ -1 & 1 & -1 & 1 \end{bmatrix}$ (c) $\begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 2 & 4 & 8 & 16 \\ 3 & 9 & 27 & 81 \end{bmatrix}$

Exercises 3.2

1. $X = \begin{bmatrix} 5 & 4 \\ 3 & 5 \end{bmatrix}$

5. $B = 2A_1 + A_2$

9. $\text{span}(A_1, A_2) = \left\{ \begin{bmatrix} c_1 & 2c_1 + c_2 \\ -c_1 + 2c_2 & c_1 + c_2 \end{bmatrix} \right\} = \left\{ \begin{bmatrix} w & x \\ 2x - 5w & x - w \end{bmatrix} \right\}$

11. $\text{span}(A_1, A_2, A_3) =$

$$\left\{ \begin{bmatrix} c_1 - c_2 + c_3 & 2c_2 + c_3 & -c_1 + c_3 \\ 0 & c_1 + c_2 & 0 \end{bmatrix} \right\} = \left\{ \begin{bmatrix} -3b + 4c + 5e & b & c \\ 0 & e & 0 \end{bmatrix} \right\}$$

13. Linearly independent

23. $a = d, c = 0$

27. $a = d, b = c = 0$

29. Let $A = [a_{ij}]$ and $B = [b_{ij}]$ be upper triangular $n \times n$ matrices and let $i > j$. Then, by the definition of an upper triangular matrix,

$a_{ii} = a_{i2} = \dots = a_{i,i-1} = 0 \quad \text{and}$

$b_{ij} = b_{i+1,j} = \dots = b_{nj} = 0$

Now let $C = AB$. Then

$$\begin{aligned} c_{ij} &= a_{i1}b_{1j} + a_{i2}b_{2j} + \dots + a_{i,i-1}b_{i-1,j} + a_{ii}b_{ij} \\ &\quad + a_{i,i+1}b_{i+1,j} + \dots + a_{in}b_{nj} \\ &= 0 \cdot b_{1j} + 0 \cdot b_{2j} + \dots + 0 \cdot b_{i-1,j} + a_{ii} \cdot 0 \\ &\quad + a_{i,i+1} \cdot 0 + \dots + a_{in} \cdot 0 = 0 \end{aligned}$$

from which it follows that C is upper triangular.35. (a) A, B symmetric $\Rightarrow (A + B)^T = A^T + B^T = A + B \Rightarrow A + B$ is symmetric

37. Matrices (b) and (c) are skew-symmetric.

41. Either A or B (or both) must be the zero matrix.

43. (b) $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} = \begin{bmatrix} 1 & 3 & 5 \\ 3 & 5 & 7 \\ 5 & 7 & 9 \end{bmatrix} + \begin{bmatrix} 0 & -1 & -2 \\ 1 & 0 & -1 \\ 2 & 1 & 0 \end{bmatrix}$

47. Hint: Use the trace.

Exercises 3.3

1. $\begin{bmatrix} 2 & -7 \\ -1 & 4 \end{bmatrix}$

5. Not invertible

9. $\begin{bmatrix} a/(a^2 + b^2) & b/(a^2 + b^2) \\ -b/(a^2 + b^2) & a/(a^2 + b^2) \end{bmatrix}$

3. Not invertible

7. $\begin{bmatrix} -1.6 & -2.8 \\ 0.3 & 1 \end{bmatrix}$

11. $\begin{bmatrix} -5 \\ 9 \end{bmatrix}$

13. (a) $\mathbf{x}_1 = \begin{bmatrix} 4 \\ -\frac{1}{2} \end{bmatrix}, \mathbf{x}_2 = \begin{bmatrix} -5 \\ 2 \end{bmatrix}, \mathbf{x}_3 = \begin{bmatrix} 6 \\ -2 \end{bmatrix}$

(c) The method in part (b) uses fewer multiplications.

17. (b) $(AB)^{-1} = A^{-1}B^{-1}$ if and only if $AB = BA$

21. $X = A^{-1}(BA)^2B^{-1}$

23. $X = (AB)^{-1}BA + A$

25. $E = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$

27. $E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$

31. $\begin{bmatrix} \frac{1}{3} & 0 \\ 0 & 1 \end{bmatrix}$

33. $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

35. $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$

37. $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/c & 0 \\ 0 & 0 & 1 \end{bmatrix}$

39. $A = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix}, A^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & -\frac{1}{2} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$

43. (a) If A is invertible, then $BA = CA \Rightarrow (BA)A^{-1} = (CA)A^{-1} \Rightarrow B(AA^{-1}) = C(AA^{-1}) \Rightarrow BI = CI \Rightarrow B = C$.45. Hint: Rewrite $A^2 - 2A + I = O$ as $A(2I - A) = I$.47. If AB is invertible, then there exists a matrix X such that $(AB)X = I$. But then $A(BX) = I$ too, so A is invertible (with inverse BX).

49. $\begin{bmatrix} \frac{1}{10} & \frac{2}{5} \\ \frac{3}{10} & \frac{1}{5} \end{bmatrix}$

51. $\begin{bmatrix} 1/(a^2 + 1) & -a/(a^2 + 1) \\ a/(a^2 + 1) & 1/(a^2 + 1) \end{bmatrix}$

53. Not invertible

55. $\begin{bmatrix} 1/a & 0 & 0 \\ -1/a^2 & 1/a & 0 \\ 1/a^3 & -1/a^2 & 1/a \end{bmatrix}, a \neq 0$

57. $\begin{bmatrix} -11 & -2 & 5 & -4 \\ 4 & 1 & -2 & 2 \\ 5 & 1 & -2 & 2 \\ 9 & 2 & -4 & 3 \end{bmatrix}$

59. $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -a/d & -b/d & -c/d & 1/d \end{bmatrix}, d \neq 0$

61. Not invertible

63. $\begin{bmatrix} 4 & 6 & 4 \\ 5 & 3 & 2 \\ 0 & 6 & 5 \end{bmatrix}$

43. If $a = -1$, then $\text{rank}(A) = 1$; if $a = 2$, then $\text{rank}(A) = 2$; for $a \neq -1, 2$, $\text{rank}(A) = 3$.

45. Yes 47. Yes 49. No

51. \mathbf{w} is in $\text{span}(\mathcal{B})$ if and only if the linear system with augmented matrix $[\mathcal{B} | \mathbf{w}]$ is consistent, which is true in this case, since

$$[\mathcal{B} | \mathbf{w}] = \left[\begin{array}{cc|c} 1 & 1 & 1 \\ 2 & 0 & 6 \\ 0 & -1 & 2 \end{array} \right] \longrightarrow \left[\begin{array}{cc|c} 1 & 0 & 3 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{array} \right]$$

From this reduced row echelon form, it is also clear that $[\mathbf{w}]_{\mathcal{B}} = \begin{bmatrix} 3 \\ -2 \end{bmatrix}$.

53. $\text{rank}(A) = 2$, $\text{nullity}(A) = 1$

55. $\text{rank}(A) = 3$, $\text{nullity}(A) = 1$

57. Let $\mathbf{A}_1, \dots, \mathbf{A}_m$ be the row vectors of A so that $\text{row}(A) = \text{span}(\mathbf{A}_1, \dots, \mathbf{A}_m)$. If \mathbf{x} is in $\text{null}(A)$, then, since $A\mathbf{x} = \mathbf{0}$, we also have $\mathbf{A}_i \cdot \mathbf{x} = 0$ for $i = 1, \dots, m$, by the row-column definition of matrix multiplication. If \mathbf{r} is in $\text{row}(A)$, then \mathbf{r} is of the form $\mathbf{r} = c_1\mathbf{A}_1 + \dots + c_m\mathbf{A}_m$. Therefore,

$$\begin{aligned} \mathbf{r} \cdot \mathbf{x} &= (c_1\mathbf{A}_1 + \dots + c_m\mathbf{A}_m) \cdot \mathbf{x} \\ &= c_1(\mathbf{A}_1 \cdot \mathbf{x}) + \dots + c_m(\mathbf{A}_m \cdot \mathbf{x}) = 0 \end{aligned}$$

59. (a) If a set of columns of AB is linearly independent, then the corresponding columns of B are linearly independent (by an argument similar to that needed to prove Exercise 29 in Section 3.1). It follows that the maximum number k of linearly independent columns of AB [i.e., $k = \text{rank}(AB)$] is not more than the maximum number r of linearly independent columns of B [i.e., $r = \text{rank}(B)$]. In other words, $\text{rank}(AB) \leq \text{rank}(B)$.

61. (a) From Exercise 59(a), $\text{rank}(UA) \leq \text{rank}(A)$ and $\text{rank}(A) = \text{rank}((U^{-1}U)A) = \text{rank}(U^{-1}(UA)) \leq \text{rank}(UA)$. Hence, $\text{rank}(UA) = \text{rank}(A)$.

Exercises 3.6

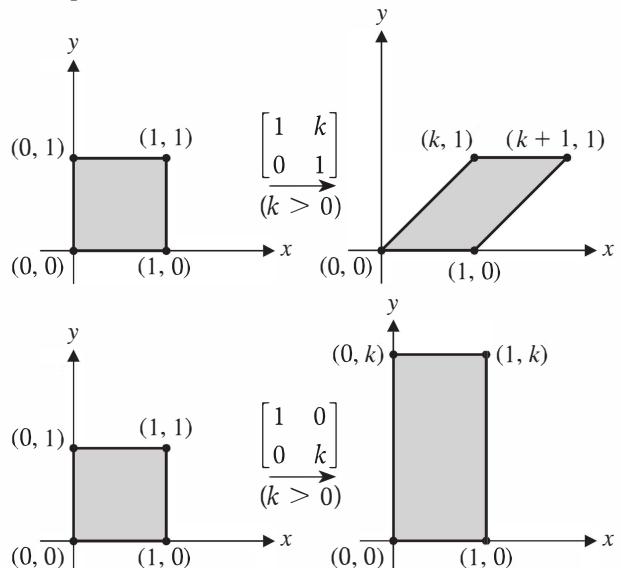
1. $T(\mathbf{u}) = \begin{bmatrix} 0 \\ 11 \end{bmatrix}, T(\mathbf{v}) = \begin{bmatrix} 8 \\ 1 \end{bmatrix}$

11. $\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$ 13. $\begin{bmatrix} 1 & -1 & 1 \\ 2 & 1 & -3 \end{bmatrix}$

15. $[F] = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$ 17. $[D] = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$

19. $\begin{bmatrix} k & 0 \\ 0 & 1 \end{bmatrix}$ stretches or contracts in the x -direction (combined with a reflection in the y -axis if $k < 0$); $\begin{bmatrix} 1 & 0 \\ 0 & k \end{bmatrix}$ stretches or contracts in the y -direction (combined

with a reflection in the x -axis if $k < 0$); $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ is a reflection in the line $y = x$; $\begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}$ is a shear in the x -direction; $\begin{bmatrix} 1 & 0 \\ k & 1 \end{bmatrix}$ is a shear in the y -direction. For example,



21. $\begin{bmatrix} \sqrt{3}/2 & 1/2 \\ -1/2 & \sqrt{3}/2 \end{bmatrix}$

25. $\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$

23. $\begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix}$

27. $\begin{bmatrix} -\frac{3}{5} & \frac{4}{5} \\ \frac{4}{5} & \frac{3}{5} \end{bmatrix}$

31. $[S \circ T] = \begin{bmatrix} -8 & 5 \\ 4 & 1 \end{bmatrix}$

33. $[S \circ T] = \begin{bmatrix} 0 & 6 & -6 \\ 1 & -2 & 2 \end{bmatrix}$

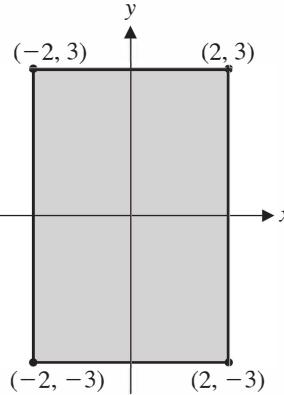
35. $[S \circ T] = \begin{bmatrix} 1 & 0 & -1 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$

37. $\begin{bmatrix} -\sqrt{3}/2 & 1/2 \\ 1/2 & \sqrt{3}/2 \end{bmatrix}$ 39. $\begin{bmatrix} -\sqrt{3}/2 & -1/2 \\ 1/2 & -\sqrt{3}/2 \end{bmatrix}$

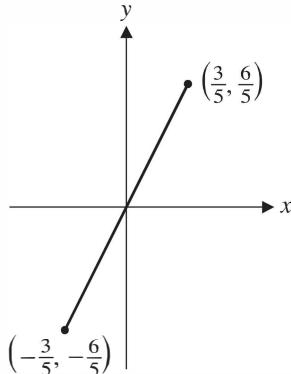
45. In vector form, let the parallel lines be given by $\mathbf{x} = \mathbf{p} + t\mathbf{d}$ and $\mathbf{x}' = \mathbf{p}' + t\mathbf{d}$. Their images are $T(\mathbf{x}) = T(\mathbf{p} + t\mathbf{d}) = T(\mathbf{p}) + tT(\mathbf{d})$ and $T(\mathbf{x}') = T(\mathbf{p}' + t\mathbf{d}) = T(\mathbf{p}') + tT(\mathbf{d})$. Suppose $T(\mathbf{d}) \neq \mathbf{0}$. If $T(\mathbf{p}') - T(\mathbf{p})$ is parallel to $T(\mathbf{d})$, then the images represent the same line; otherwise the images represent distinct parallel lines. On the other hand, if $T(\mathbf{d}) = \mathbf{0}$,

then the images represent two distinct points if $T(\mathbf{p}') \neq T(\mathbf{p})$ and single point otherwise.

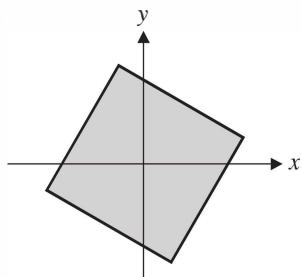
47.



49.



51.



Exercises 3.7

$$1. \mathbf{x}_1 = \begin{bmatrix} 0.4 \\ 0.6 \end{bmatrix}, \mathbf{x}_2 = \begin{bmatrix} 0.38 \\ 0.62 \end{bmatrix}$$

3. 64%

$$5. \mathbf{x}_1 = \begin{bmatrix} 150 \\ 120 \\ 120 \end{bmatrix}, \mathbf{x}_2 = \begin{bmatrix} 155 \\ 120 \\ 115 \end{bmatrix}$$

7. $\frac{5}{18}$

$$9. (a) P = \begin{bmatrix} 0.662 & 0.250 \\ 0.338 & 0.750 \end{bmatrix}$$

(c) 42.5% wet, 57.5% dry

$$11. (a) P = \begin{bmatrix} 0.08 & 0.09 & 0.11 \\ 0.07 & 0.11 & 0.05 \\ 0.85 & 0.80 & 0.84 \end{bmatrix}$$

(b) 0.08, 0.1062, 0.1057, 0.1057, 0.1057

(c) 10.6% good, 5.5% fair, 83.9% poor

13. The entries of the vector $\mathbf{j}P$ are just the column sums of the matrix P . So P is stochastic if and only if $\mathbf{j}P = \mathbf{j}$.

15. 4

17. 9.375

$$19. \text{Yes, } \mathbf{x} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

21. No

23. No

$$25. \text{Yes, } \mathbf{x} = \begin{bmatrix} 10 \\ 27 \\ 35 \end{bmatrix}$$

27. Productive

29. Not productive

$$31. \mathbf{x} = \begin{bmatrix} 10 \\ 16 \end{bmatrix}$$

$$33. \text{Yes, } \mathbf{x} = \begin{bmatrix} 10 \\ 6 \\ 8 \end{bmatrix}$$

$$37. \mathbf{x}_1 = \begin{bmatrix} 45 \\ 6 \end{bmatrix}, \mathbf{x}_2 = \begin{bmatrix} 120 \\ 27 \end{bmatrix}, \mathbf{x}_3 = \begin{bmatrix} 375 \\ 72 \end{bmatrix}$$

$$39. \mathbf{x}_1 = \begin{bmatrix} 500 \\ 70 \\ 50 \end{bmatrix}, \mathbf{x}_2 = \begin{bmatrix} 720 \\ 350 \\ 35 \end{bmatrix}, \mathbf{x}_3 = \begin{bmatrix} 1175 \\ 504 \\ 175 \end{bmatrix}$$

$$41. (a) \text{For } L_1, \text{ we have } \mathbf{x}_1 = \begin{bmatrix} 50 \\ 8 \end{bmatrix}, \mathbf{x}_2 = \begin{bmatrix} 40 \\ 40 \end{bmatrix},$$

$$\mathbf{x}_3 = \begin{bmatrix} 200 \\ 32 \end{bmatrix}, \mathbf{x}_4 = \begin{bmatrix} 160 \\ 160 \end{bmatrix}, \mathbf{x}_5 = \begin{bmatrix} 800 \\ 128 \end{bmatrix}, \mathbf{x}_6 = \begin{bmatrix} 640 \\ 640 \end{bmatrix},$$

$$\mathbf{x}_7 = \begin{bmatrix} 3200 \\ 512 \end{bmatrix}, \mathbf{x}_8 = \begin{bmatrix} 2560 \\ 2560 \end{bmatrix}, \mathbf{x}_9 = \begin{bmatrix} 12800 \\ 2048 \end{bmatrix},$$

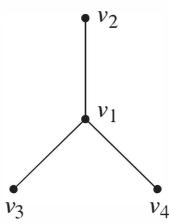
$$\mathbf{x}_{10} = \begin{bmatrix} 10240 \\ 10240 \end{bmatrix}.$$

(b) The first population oscillates between two states, while the second approaches a steady state.

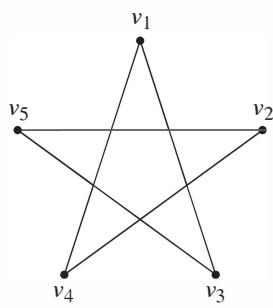
43. The population oscillates through a cycle of three states (for the relative population): If $0.1 < s \leq 1$, the actual population is growing; if $s = 0.1$, the actual population goes through a cycle of length 3; and if $0 \leq s < 0.1$, the actual population is declining (and will eventually die out).

$$45. A = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix} \quad 47. A = \begin{bmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 \end{bmatrix}$$

49.



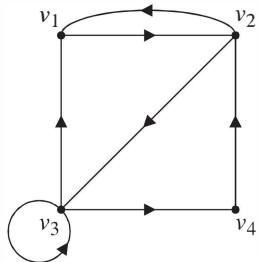
51.



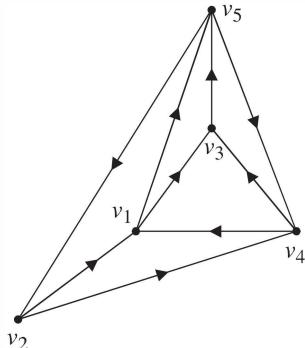
$$53. A = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

$$55. A = \begin{bmatrix} 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

57.



59.



61. 2

63. 3

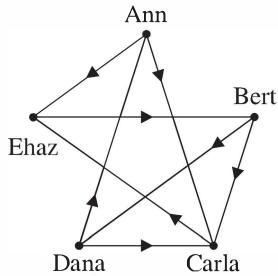
65. 0

67. 3

69. (a) Vertex i is not adjacent to any other vertices.

71. If we use direct wins only, P_2 is in first place; P_3, P_4 , and P_6 tie for second place; and P_1 and P_5 tie for third place. If we combine direct and indirect wins, the players rank as follows: P_2 in first place, followed by P_6, P_4, P_3, P_5 , and P_1 .

73. (a)



$$A = \begin{bmatrix} 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{bmatrix}$$

- (b) two steps; all of the off-diagonal entries of the second row of $A + A^2$ are nonzero.
 (d) If the graph has n vertices, check the (i, j) entry of the powers A^k for $k = 1, \dots, n - 1$. Vertex i is

connected to vertex j by a path of length k if and only if $(A^k)_{ij} \neq 0$.

75. $(AA^T)_{ij}$ counts the number of vertices adjacent to both vertex i and vertex j .

77. Bipartite

79. Bipartite

Review Questions

1. (a) T (c) F (e) T (g) T (i) T

3. Impossible 5. $\begin{bmatrix} \frac{17}{83} & -\frac{1}{83} \\ -\frac{1}{83} & \frac{5}{166} \end{bmatrix}$

7. $\begin{bmatrix} 1 & 3 \\ 3 & 9 \end{bmatrix} + \begin{bmatrix} 4 & 10 \\ 10 & 25 \end{bmatrix}$ 9. $\begin{bmatrix} 0 & -9 \\ 2 & 4 \\ 1 & -6 \end{bmatrix}$

11. Because $(I - A)(I + A + A^2) = I - A^3 = I - O = I$, $(I - A)^{-1} = I + A + A^2$.

13. A basis for $\text{row}(A)$ is $\{[1, -2, 0, -1, 0], [0, 0, 1, 2, 0], [0, 0, 0, 0, 1]\}$; a basis for $\text{col}(A)$ is $\left\{ \begin{bmatrix} 2 \\ 1 \\ 0 \\ 4 \\ 0 \end{bmatrix}, \begin{bmatrix} 5 \\ 2 \\ 3 \\ 6 \\ 0 \end{bmatrix}, \begin{bmatrix} 5 \\ 1 \\ 1 \\ 6 \\ 0 \end{bmatrix} \right\}$ (or the standard basis for \mathbb{R}^3); and a basis for $\text{null}(A)$ is $\left\{ \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -2 \\ 1 \\ 0 \end{bmatrix} \right\}$.

15. An invertible matrix has a trivial (zero) null space. If A is invertible, then so is A^T , and so both A and A^T have trivial null spaces. If A is not invertible, then A and A^T need not have the same null space. For example, take $A = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$.

17. Because A has n linearly independent columns, $\text{rank}(A) = n$. Hence $\text{rank}(A^TA) = n$ by Theorem 3.28. Because A^TA is $n \times n$, this implies that A^TA is invertible, by the Fundamental Theorem of Invertible Matrices. AA^T need not be invertible. For example, take $A = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

19. $\begin{bmatrix} -1/5\sqrt{2} & -3/5\sqrt{2} \\ 2/5\sqrt{2} & 6/5\sqrt{2} \end{bmatrix}$

Chapter 4

Exercises 4.1

1. $A\mathbf{v} = \begin{bmatrix} 3 \\ 3 \end{bmatrix} = 3\mathbf{v}, \lambda = 3$

3. $A\mathbf{v} = \begin{bmatrix} -3 \\ 6 \end{bmatrix} = -3\mathbf{v}$, $\lambda = -3$

5. $A\mathbf{v} = \begin{bmatrix} 6 \\ -3 \\ 3 \end{bmatrix} = 3\mathbf{v}$, $\lambda = 3$

7. $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$

9. $\begin{bmatrix} 4 \\ 1 \end{bmatrix}$

11. $\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$

13. $\lambda = 1$, $E_1 = \text{span}\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right)$; $\lambda = -1$, $E_{-1} = \text{span}\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right)$

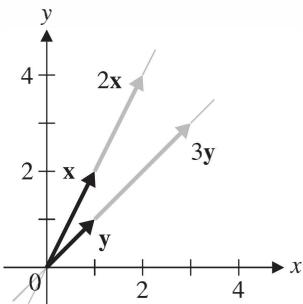
15. $\lambda = 0$, $E_0 = \text{span}\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right)$; $\lambda = 1$, $E_1 = \text{span}\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right)$

17. $\lambda = 2$, $E_2 = \text{span}\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right)$; $\lambda = 3$, $E_3 = \text{span}\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right)$

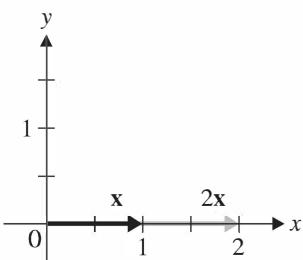
19. $\mathbf{v} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $\lambda = 1$; $\mathbf{v} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $\lambda = 2$

21. $\mathbf{v} = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$, $\lambda = 2$; $\mathbf{v} = \begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$, $\lambda = 0$

23. $\lambda = 2$, $E_2 = \text{span}\left(\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right)$; $\lambda = 3$, $E_3 = \text{span}\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right)$



25. $\lambda = 2$, $E_2 = \text{span}\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right)$



27. $\lambda = 1 + i$, $E_{1+i} = \text{span}\left(\begin{bmatrix} 1 \\ i \end{bmatrix}\right)$; $\lambda = 1 - i$, $E_{1-i} =$

$\text{span}\left(\begin{bmatrix} 1 \\ -i \end{bmatrix}\right)$

29. $\lambda = 1 + i$, $E_{1+i} = \text{span}\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right)$; $\lambda = 1 - i$, $E_{1-i} = \text{span}\left(\begin{bmatrix} 1 \\ -1 \end{bmatrix}\right)$

31. $\lambda = 1, 2$

33. $\lambda = 4$

Exercises 4.2

1. 16

9. -12

17. 0

31. 0

39. -8

51. $(-2)^{3^n}$

53. $\det(AB) = (\det A)(\det B) = (\det B)(\det A) = \det(BA)$

55. 0, 1

57. $x = \frac{3}{2}, y = -\frac{1}{2}$

59. $x = -1, y = 0, z = 1$

61. $\begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{bmatrix}$

63. $\begin{bmatrix} \frac{1}{2} & -\frac{1}{2} & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$

Exercises 4.3

1. (a) $\lambda^2 - 7\lambda + 12$ (b) $\lambda = 3, 4$

(c) $E_3 = \text{span}\left(\begin{bmatrix} 3 \\ 2 \end{bmatrix}\right)$; $E_4 = \text{span}\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right)$

(d) The algebraic and geometric multiplicities are all 1.

3. (a) $-\lambda^3 + 2\lambda^2 + 5\lambda - 6$

(b) $\lambda = -2, 1, 3$

(c) $E_{-2} = \text{span}\left(\begin{bmatrix} 1 \\ -3 \\ 0 \end{bmatrix}\right)$; $E_1 = \text{span}\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right)$

$E_3 = \text{span}\left(\begin{bmatrix} 1 \\ 2 \\ 10 \end{bmatrix}\right)$

(d) The algebraic and geometric multiplicities are all 1.

5. (a) $-\lambda^3 + \lambda^2$ (b) $\lambda = 0, 1$

(c) $E_0 = \text{span}\left(\begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}\right)$; $E_1 = \text{span}\left(\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}\right)$

(d) $\lambda = 0$ has algebraic multiplicity 2 and geometric multiplicity 1; $\lambda = 1$ has algebraic and geometric multiplicity 1.

7. (a) $-\lambda^3 + 9\lambda^2 - 27\lambda + 27$

(b) $\lambda = 3$

(c) $E_3 = \text{span}\left(\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right)$

(d) $\lambda = 3$ has algebraic multiplicity 3 and geometric multiplicity 2.

9. (a) $\lambda^4 - 6\lambda^3 + 9\lambda^2 + 4\lambda - 12$

(b) $\lambda = -1, 2, 3$

(c) $E_{-1} = \text{span}\left(\begin{bmatrix} 0 \\ 0 \\ -2 \\ 1 \end{bmatrix}\right); E_2 = \text{span}\left(\begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix}\right);$

$$E_3 = \text{span}\left(\begin{bmatrix} 0 \\ 0 \\ 2 \\ 1 \end{bmatrix}\right)$$

(d) $\lambda = -1$ and $\lambda = 3$ have algebraic and geometric multiplicity 1; $\lambda = 2$ has algebraic multiplicity 2 and geometric multiplicity 1.

11. (a) $\lambda^4 - 4\lambda^3 + 2\lambda^2 + 4\lambda - 3$

(b) $\lambda = -1, 1, 3$

(c) $E_{-1} = \text{span}\left(\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}\right);$

$$E_1 = \text{span}\left(\begin{bmatrix} -2 \\ 0 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} -2 \\ 2 \\ 0 \\ 3 \end{bmatrix}\right);$$

$$E_3 = \text{span}\left(\begin{bmatrix} 0 \\ 0 \\ 2 \\ 1 \end{bmatrix}\right)$$

(d) $\lambda = -1$ and $\lambda = 3$ have algebraic and geometric multiplicity 1; $\lambda = 1$ has algebraic and geometric multiplicity 2.

15. $\begin{bmatrix} 2^{-9} + 3 \cdot 2^{10} \\ -2^{-9} + 3 \cdot 2^{10} \end{bmatrix}$ 17. $\begin{bmatrix} 2 \\ (2 \cdot 3^{20} - 1)/3^{20} \\ 2 \end{bmatrix}$

23. (a) $\lambda = -2, E_{-2} = \text{span}\left(\begin{bmatrix} 2 \\ -5 \end{bmatrix}\right); \lambda = 5, E_5 =$

$$\text{span}\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right)$$

(b) (i) $\lambda = -\frac{1}{2}, E_{-1/2} = \text{span}\left(\begin{bmatrix} 2 \\ -5 \end{bmatrix}\right); \lambda = \frac{1}{5}, E_{1/5} = \text{span}\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right)$

(iii) $\lambda = 0, E_0 = \text{span}\left(\begin{bmatrix} 2 \\ -5 \end{bmatrix}\right); \lambda = 7, E_7 = \text{span}\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right)$

27. $\begin{bmatrix} -3 & 4 & -12 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, -\lambda^3 - 3\lambda^2 + 4\lambda - 12$

35. $A^2 = 4A - 5I, A^3 = 11A - 20I$

$A^4 = 24A - 55I$

37. $A^{-1} = -\frac{1}{5}A + \frac{4}{5}I, A^{-2} = -\frac{4}{25}A + \frac{11}{25}I$

Exercises 4.4

1. The characteristic polynomial of A is $\lambda^2 - 5\lambda + 1$, but that of B is $\lambda^2 - 2\lambda + 1$.

3. The eigenvalues of A are $\lambda = 2$ and $\lambda = 4$, but those of B are $\lambda = 1$ and $\lambda = 4$.

5. $\lambda_1 = 4, E_4 = \text{span}\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right); \lambda_2 = 3, E_3 = \text{span}\left(\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right)$

7. $\lambda_1 = 6, E_6 = \text{span}\left(\begin{bmatrix} 3 \\ 2 \\ 3 \end{bmatrix}\right); \lambda_2 = -2, E_{-2} = \text{span}\left(\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}\right)$

9. Not diagonalizable

11. $P = \begin{bmatrix} 1 & 1 & -1 \\ 1 & 1 & 1 \\ 1 & -2 & 0 \end{bmatrix}, D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

13. Not diagonalizable

15. $P = \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, D = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & -2 \end{bmatrix}$

17. $\begin{bmatrix} 35839 & -69630 \\ -11605 & 24234 \end{bmatrix}$

19. $\begin{bmatrix} (3^k + 3(-1)^k)/4 & (3^{k+1} - 3(-1)^k)/4 \\ (3^k - (-1)^k)/4 & (3^{k+1} + (-1)^k)/4 \end{bmatrix}$

21. $\begin{bmatrix} 1 & 1 & 1 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$

23.

$$\begin{bmatrix} (5 + 2^{k+2} + (-3)^k)/10 & (2^k - (-3)^k)/5 & (-5 + 2^{k+2} + (-3)^k)/10 \\ (2^{k+1} - 2(-3)^k)/5 & (2^k + 4(-3)^k)/5 & (2^{k+1} - 2(-3)^k)/5 \\ (-5 + 2^{k+2} + (-3)^k)/10 & (2^k - (-3)^k)/5 & (5 + 2^{k+2} + (-3)^k)/10 \end{bmatrix}$$

25. $k = 0$ 27. $k = 0$

29. All real values of k

37. If $A \sim B$, then there is an invertible matrix P such that

$B = P^{-1}AP$. Therefore, we have

$$\begin{aligned} \text{tr}(B) &= \text{tr}(P^{-1}AP) = \text{tr}(P^{-1}(AP)) = \text{tr}((AP)P^{-1}) \\ &= \text{tr}(APP^{-1}) = \text{tr}(AI) = \text{tr}(A) \end{aligned}$$

using Exercise 45 in Section 3.2.

39. $P = \begin{bmatrix} 7 & -2 \\ 10 & -3 \end{bmatrix}$

41. $P = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} & 0 \\ -\frac{3}{2} & -\frac{3}{2} & 1 \\ -\frac{5}{2} & -\frac{3}{2} & 0 \end{bmatrix}$

51. (b) $\dim E_{-1} = 1$, $\dim E_1 = 2$,
 $\dim E_2 = 3$

Exercises 4.5

1. (a) $\begin{bmatrix} 1 \\ 2.5 \end{bmatrix}$, 6.000
(b) $\lambda_1 = 6$

3. (a) $\begin{bmatrix} 1 \\ 0.618 \end{bmatrix}$, 2.618
(b) $\lambda_1 = (3 + \sqrt{5})/2 \approx 2.618$

5. (a) $m_5 = 11.001$, $y_5 = \begin{bmatrix} -0.333 \\ 1.000 \end{bmatrix}$

7. (a) $m_8 = 10.000$, $y_8 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$

9.	k	0	1	2	3	4	5
x_k		$\begin{bmatrix} 1 \\ 1 \end{bmatrix}$	$\begin{bmatrix} 26 \\ 8 \end{bmatrix}$	$\begin{bmatrix} 17.692 \\ 5.923 \end{bmatrix}$	$\begin{bmatrix} 18.018 \\ 6.004 \end{bmatrix}$	$\begin{bmatrix} 17.999 \\ 6.000 \end{bmatrix}$	$\begin{bmatrix} 18.000 \\ 6.000 \end{bmatrix}$
y_k		$\begin{bmatrix} 1 \\ 1 \end{bmatrix}$	$\begin{bmatrix} 1 \\ 0.308 \end{bmatrix}$	$\begin{bmatrix} 1 \\ 0.335 \end{bmatrix}$	$\begin{bmatrix} 1 \\ 0.333 \end{bmatrix}$	$\begin{bmatrix} 1 \\ 0.333 \end{bmatrix}$	$\begin{bmatrix} 1 \\ 0.333 \end{bmatrix}$
m_k		1	26	17.692	18.018	17.999	18.000

Therefore, $\lambda_1 \approx 18$, $v_1 \approx \begin{bmatrix} 1 \\ 0.333 \end{bmatrix}$.

11.	k	0	1	2	3	4	5	6
x_k		$\begin{bmatrix} 1 \\ 0 \end{bmatrix}$	$\begin{bmatrix} 7 \\ 2 \end{bmatrix}$	$\begin{bmatrix} 7.571 \\ 2.857 \end{bmatrix}$	$\begin{bmatrix} 7.755 \\ 3.132 \end{bmatrix}$	$\begin{bmatrix} 7.808 \\ 3.212 \end{bmatrix}$	$\begin{bmatrix} 7.823 \\ 3.234 \end{bmatrix}$	$\begin{bmatrix} 7.827 \\ 3.240 \end{bmatrix}$
y_k		$\begin{bmatrix} 1 \\ 0 \end{bmatrix}$	$\begin{bmatrix} 1 \\ 0.286 \end{bmatrix}$	$\begin{bmatrix} 1 \\ 0.377 \end{bmatrix}$	$\begin{bmatrix} 1 \\ 0.404 \end{bmatrix}$	$\begin{bmatrix} 1 \\ 0.411 \end{bmatrix}$	$\begin{bmatrix} 1 \\ 0.413 \end{bmatrix}$	$\begin{bmatrix} 1 \\ 0.414 \end{bmatrix}$
m_k		1	7	7.571	7.755	7.808	7.823	7.827

Therefore, $\lambda_1 \approx 7.827$, $v_1 \approx \begin{bmatrix} 1 \\ 0.414 \end{bmatrix}$.

k	0	1	2	3	4	5
\mathbf{x}_k	$\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$	$\begin{bmatrix} 21 \\ 15 \\ 13 \end{bmatrix}$	$\begin{bmatrix} 16.809 \\ 12.238 \\ 10.714 \end{bmatrix}$	$\begin{bmatrix} 17.011 \\ 12.371 \\ 10.824 \end{bmatrix}$	$\begin{bmatrix} 16.999 \\ 12.363 \\ 10.818 \end{bmatrix}$	$\begin{bmatrix} 17.000 \\ 12.363 \\ 10.818 \end{bmatrix}$
\mathbf{y}_k	$\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$	$\begin{bmatrix} 1 \\ 0.714 \\ 0.619 \end{bmatrix}$	$\begin{bmatrix} 1 \\ 0.728 \\ 0.637 \end{bmatrix}$	$\begin{bmatrix} 1 \\ 0.727 \\ 0.636 \end{bmatrix}$	$\begin{bmatrix} 1 \\ 0.727 \\ 0.636 \end{bmatrix}$	$\begin{bmatrix} 1 \\ 0.727 \\ 0.636 \end{bmatrix}$
m_k	1	21	16.809	17.011	16.999	17.000

Therefore, $\lambda_1 \approx 17$, $\mathbf{v}_1 \approx \begin{bmatrix} 1 \\ 0.727 \\ 0.636 \end{bmatrix}$.

15. $\lambda_1 \approx 5$, $\mathbf{v}_1 \approx \begin{bmatrix} 1 \\ 0 \\ 0.333 \end{bmatrix}$

k	0	1	2	3	4	5	6
\mathbf{x}_k	$\begin{bmatrix} 1 \\ 0 \end{bmatrix}$	$\begin{bmatrix} 7 \\ 2 \end{bmatrix}$	$\begin{bmatrix} 7.571 \\ 2.857 \end{bmatrix}$	$\begin{bmatrix} 7.755 \\ 3.132 \end{bmatrix}$	$\begin{bmatrix} 7.808 \\ 3.212 \end{bmatrix}$	$\begin{bmatrix} 7.823 \\ 3.234 \end{bmatrix}$	$\begin{bmatrix} 7.827 \\ 3.240 \end{bmatrix}$
$R(\mathbf{x}_k)$	7	7.755	7.823	7.828	7.828	7.828	7.828
\mathbf{y}_k	$\begin{bmatrix} 1 \\ 0 \end{bmatrix}$	$\begin{bmatrix} 1 \\ 0.286 \end{bmatrix}$	$\begin{bmatrix} 1 \\ 0.377 \end{bmatrix}$	$\begin{bmatrix} 1 \\ 0.404 \end{bmatrix}$	$\begin{bmatrix} 1 \\ 0.411 \end{bmatrix}$	$\begin{bmatrix} 1 \\ 0.413 \end{bmatrix}$	$\begin{bmatrix} 1 \\ 0.414 \end{bmatrix}$

k	0	1	2	3	4	5
\mathbf{x}_k	$\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$	$\begin{bmatrix} 21 \\ 15 \\ 13 \end{bmatrix}$	$\begin{bmatrix} 16.809 \\ 12.238 \\ 10.714 \end{bmatrix}$	$\begin{bmatrix} 17.011 \\ 12.371 \\ 10.824 \end{bmatrix}$	$\begin{bmatrix} 16.999 \\ 12.363 \\ 10.818 \end{bmatrix}$	$\begin{bmatrix} 17.000 \\ 12.363 \\ 10.818 \end{bmatrix}$
$R(\mathbf{x}_k)$	16.333	16.998	17.000	17.000	17.000	17.000
\mathbf{y}_k	$\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$	$\begin{bmatrix} 1 \\ 0.714 \\ 0.619 \end{bmatrix}$	$\begin{bmatrix} 1 \\ 0.728 \\ 0.637 \end{bmatrix}$	$\begin{bmatrix} 1 \\ 0.727 \\ 0.636 \end{bmatrix}$	$\begin{bmatrix} 1 \\ 0.727 \\ 0.636 \end{bmatrix}$	$\begin{bmatrix} 1 \\ 0.727 \\ 0.636 \end{bmatrix}$

k	0	1	2	3	4	5	6	7	8
\mathbf{x}_k	$\begin{bmatrix} 1 \\ 1 \end{bmatrix}$	$\begin{bmatrix} 5 \\ 4 \end{bmatrix}$	$\begin{bmatrix} 4.8 \\ 3.2 \end{bmatrix}$	$\begin{bmatrix} 4.667 \\ 2.667 \end{bmatrix}$	$\begin{bmatrix} 4.571 \\ 2.286 \end{bmatrix}$	$\begin{bmatrix} 4.500 \\ 2.000 \end{bmatrix}$	$\begin{bmatrix} 4.444 \\ 1.778 \end{bmatrix}$	$\begin{bmatrix} 4.400 \\ 1.600 \end{bmatrix}$	$\begin{bmatrix} 4.364 \\ 1.455 \end{bmatrix}$
\mathbf{y}_k	$\begin{bmatrix} 1 \\ 1 \end{bmatrix}$	$\begin{bmatrix} 1 \\ 0.8 \end{bmatrix}$	$\begin{bmatrix} 1 \\ 0.667 \end{bmatrix}$	$\begin{bmatrix} 1 \\ 0.571 \end{bmatrix}$	$\begin{bmatrix} 1 \\ 0.500 \end{bmatrix}$	$\begin{bmatrix} 1 \\ 0.444 \end{bmatrix}$	$\begin{bmatrix} 1 \\ 0.400 \end{bmatrix}$	$\begin{bmatrix} 1 \\ 0.364 \end{bmatrix}$	$\begin{bmatrix} 1 \\ 0.333 \end{bmatrix}$
m_k	1	5	4.8	4.667	4.571	4.500	4.444	4.400	4.364

Since $\lambda_1 = \lambda_2 = 4$, $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, m_k is converging slowly to the exact answer.

k	0	1	2	3	4	5	6	7	8
\mathbf{x}_k	$\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$	$\begin{bmatrix} 5 \\ 4 \\ 1 \end{bmatrix}$	$\begin{bmatrix} 4.2 \\ 3.2 \\ 0.2 \end{bmatrix}$	$\begin{bmatrix} 4.048 \\ 3.048 \\ 0.048 \end{bmatrix}$	$\begin{bmatrix} 4.012 \\ 3.012 \\ 0.012 \end{bmatrix}$	$\begin{bmatrix} 4.003 \\ 3.003 \\ 0.003 \end{bmatrix}$	$\begin{bmatrix} 4.001 \\ 3.001 \\ 0.001 \end{bmatrix}$	$\begin{bmatrix} 4.000 \\ 3.000 \\ 0.000 \end{bmatrix}$	$\begin{bmatrix} 4.000 \\ 3.000 \\ 0.000 \end{bmatrix}$
\mathbf{y}_k	$\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$	$\begin{bmatrix} 1 \\ 0.8 \\ 0.2 \end{bmatrix}$	$\begin{bmatrix} 1 \\ 0.762 \\ 0.048 \end{bmatrix}$	$\begin{bmatrix} 1 \\ 0.753 \\ 0.012 \end{bmatrix}$	$\begin{bmatrix} 1 \\ 0.751 \\ 0.003 \end{bmatrix}$	$\begin{bmatrix} 1 \\ 0.750 \\ 0.001 \end{bmatrix}$	$\begin{bmatrix} 1 \\ 0.750 \\ 0 \end{bmatrix}$	$\begin{bmatrix} 1 \\ 0.750 \\ 0 \end{bmatrix}$	$\begin{bmatrix} 1 \\ 0.750 \\ 0 \end{bmatrix}$
m_k	1	5	4.2	4.048	4.012	4.003	4.001	4.000	4.000

In this case, $\lambda_1 = \lambda_2 = 4$ and $E_4 = \text{span}\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right)$.

Clearly, m_k is converging to 4 and \mathbf{y}_k is converging to a

vector in the eigenspace E_4 —namely, $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + 0.75\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$.

k	0	1	2	3	4	5
\mathbf{x}_k	$\begin{bmatrix} 1 \\ 1 \end{bmatrix}$	$\begin{bmatrix} 1 \\ 0 \end{bmatrix}$	$\begin{bmatrix} -1 \\ -1 \end{bmatrix}$	$\begin{bmatrix} 1 \\ 0 \end{bmatrix}$	$\begin{bmatrix} -1 \\ -1 \end{bmatrix}$	$\begin{bmatrix} 1 \\ 0 \end{bmatrix}$
\mathbf{y}_k	$\begin{bmatrix} 1 \\ 1 \end{bmatrix}$	$\begin{bmatrix} 1 \\ 0 \end{bmatrix}$	$\begin{bmatrix} 1 \\ 1 \end{bmatrix}$	$\begin{bmatrix} 1 \\ 0 \end{bmatrix}$	$\begin{bmatrix} 1 \\ 1 \end{bmatrix}$	$\begin{bmatrix} 1 \\ 0 \end{bmatrix}$
m_k	1	1	-1	1	-1	1

The exact eigenvalues are complex (i and $-i$), so the power method cannot possibly converge to either the dominant eigenvalue or the dominant eigenvector if we start with a *real* initial iterate. Instead, the power method oscillates between two sets of real vectors.

k	0	1	2	3	4	5
\mathbf{x}_k	$\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$	$\begin{bmatrix} 3 \\ 4 \\ 3 \end{bmatrix}$	$\begin{bmatrix} 2.500 \\ 4.000 \\ 2.500 \end{bmatrix}$	$\begin{bmatrix} 2.250 \\ 4.000 \\ 2.250 \end{bmatrix}$	$\begin{bmatrix} 2.125 \\ 4.000 \\ 2.125 \end{bmatrix}$	$\begin{bmatrix} 2.063 \\ 4.000 \\ 2.063 \end{bmatrix}$
\mathbf{y}_k	$\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$	$\begin{bmatrix} 0.750 \\ 1 \\ 0.750 \end{bmatrix}$	$\begin{bmatrix} 0.625 \\ 1 \\ 0.625 \end{bmatrix}$	$\begin{bmatrix} 0.562 \\ 1 \\ 0.562 \end{bmatrix}$	$\begin{bmatrix} 0.531 \\ 1 \\ 0.531 \end{bmatrix}$	$\begin{bmatrix} 0.516 \\ 1 \\ 0.516 \end{bmatrix}$
m_k	1	4	4	4	4	4

The eigenvalues are $\lambda_1 = -12$, $\lambda_2 = 4$, $\lambda_3 = 2$, with corresponding eigenvectors

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}.$$

Since $\mathbf{x}_0 = \frac{1}{2}\mathbf{v}_2 + \frac{1}{2}\mathbf{v}_3$, the initial vector \mathbf{x}_0 has a zero component in the direction of the dominant eigenvector, so the power method cannot converge to the dominant eigenvalue/eigenvector. Instead, it converges to a *second* eigenvalue/eigenvector pair, as the calculations show.

29. Apply the power method to $A - 18I = \begin{bmatrix} -4 & 12 \\ 5 & -15 \end{bmatrix}$.

k	0	1	2	3
\mathbf{x}_k	$\begin{bmatrix} 1 \\ 1 \end{bmatrix}$	$\begin{bmatrix} 8 \\ -10 \end{bmatrix}$	$\begin{bmatrix} 15.2 \\ -19 \end{bmatrix}$	$\begin{bmatrix} 15.2 \\ -19 \end{bmatrix}$
\mathbf{y}_k	$\begin{bmatrix} 1 \\ 1 \end{bmatrix}$	$\begin{bmatrix} -0.8 \\ 1 \end{bmatrix}$	$\begin{bmatrix} -0.8 \\ 1 \end{bmatrix}$	$\begin{bmatrix} -0.8 \\ 1 \end{bmatrix}$
m_k	1	-10	-19	-19
$R(\mathbf{x}_k)$	-0.667	-18	-18	-18

Thus, -19 is the dominant eigenvalue of $A - 18I$, and $\lambda_2 = -19 + 18 = -1$ is the second eigenvalue of A .

31. Apply the power method to $A - 17I = \begin{bmatrix} -8 & 4 & 8 \\ 4 & -2 & -4 \\ 8 & -4 & -8 \end{bmatrix}$.

k	0	1	2	3
\mathbf{x}_k	$\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$	$\begin{bmatrix} 4 \\ -2 \\ -4 \end{bmatrix}$	$\begin{bmatrix} -18 \\ 9 \\ 18 \end{bmatrix}$	$\begin{bmatrix} -18 \\ 9 \\ 18 \end{bmatrix}$
\mathbf{y}_k	$\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$	$\begin{bmatrix} 1 \\ -0.5 \\ -1 \end{bmatrix}$	$\begin{bmatrix} 1 \\ -0.5 \\ -1 \end{bmatrix}$	$\begin{bmatrix} 1 \\ -0.5 \\ -1 \end{bmatrix}$
m_k	1	4	-18	-18
$R(\mathbf{x}_k)$	-0.667	-18	-18	-18

In this case, there is no dominant eigenvalue. (We could choose either 18 or -18 for m_k , $k \geq 2$.) However, the Rayleigh quotient method (Exercises 17–20) converges to -18. Thus, -18 is the dominant eigenvalue of $A - 17I$, and $\lambda_2 = -18 + 17 = -1$ is the second eigenvalue of A .

k	0	1	2	3	4	5
\mathbf{x}_k	$\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$	$\begin{bmatrix} 0.5 \\ -0.5 \end{bmatrix}$	$\begin{bmatrix} -0.833 \\ 1.056 \end{bmatrix}$	$\begin{bmatrix} 0.798 \\ -0.997 \end{bmatrix}$	$\begin{bmatrix} 0.800 \\ -1.000 \end{bmatrix}$	$\begin{bmatrix} 0.800 \\ -1.000 \end{bmatrix}$
\mathbf{y}_k	$\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$	$\begin{bmatrix} 1 \\ -1 \end{bmatrix}$	$\begin{bmatrix} -0.789 \\ 1 \end{bmatrix}$	$\begin{bmatrix} -0.801 \\ 1 \end{bmatrix}$	$\begin{bmatrix} -0.800 \\ 1 \end{bmatrix}$	$\begin{bmatrix} -0.800 \\ 1 \end{bmatrix}$
m_k	1	0.5	1.056	-0.997	-1.000	-1.000

Thus, the eigenvalue of A that is smallest in magnitude is $1/(-1) = -1$.

35.	k	0	1	2	3	4	5
\mathbf{x}_k		$\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$	$\begin{bmatrix} -0.500 \\ 0.000 \\ 0.500 \end{bmatrix}$	$\begin{bmatrix} 0.500 \\ 0.333 \\ -0.500 \end{bmatrix}$	$\begin{bmatrix} 0.500 \\ 0.111 \\ -0.500 \end{bmatrix}$	$\begin{bmatrix} 0.500 \\ 0.259 \\ -0.500 \end{bmatrix}$	$\begin{bmatrix} 0.500 \\ 0.160 \\ -0.500 \end{bmatrix}$
\mathbf{y}_k		$\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$	$\begin{bmatrix} -1.000 \\ 0.000 \\ 1.000 \end{bmatrix}$	$\begin{bmatrix} -1.000 \\ -0.667 \\ 1.000 \end{bmatrix}$	$\begin{bmatrix} -1.000 \\ -0.222 \\ 1.000 \end{bmatrix}$	$\begin{bmatrix} -1.000 \\ -0.518 \\ 1.000 \end{bmatrix}$	$\begin{bmatrix} -1.000 \\ -0.321 \\ 1.000 \end{bmatrix}$
m_k		1	-0.500	-0.500	-0.500	-0.500	-0.500

Clearly, m_k converges to -0.5 , so the smallest eigenvalue of A is $1/(-0.5) = -2$.

37. The calculations are the same as for Exercise 33.

39. We apply the inverse power method to $A - 5I =$

$$\begin{bmatrix} -1 & 0 & 6 \\ -1 & -2 & 1 \\ 6 & 0 & -1 \end{bmatrix}. \text{ Taking } \mathbf{x}_0 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \text{ we have}$$

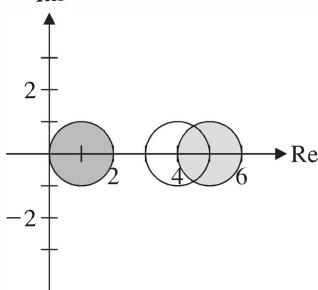
k	0	1	2	3
\mathbf{x}_k	$\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$	$\begin{bmatrix} 0.200 \\ -0.500 \\ 0.200 \end{bmatrix}$	$\begin{bmatrix} -0.080 \\ -0.500 \\ -0.080 \end{bmatrix}$	$\begin{bmatrix} 0.032 \\ -0.500 \\ 0.032 \end{bmatrix}$
\mathbf{y}_k	$\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$	$\begin{bmatrix} -0.400 \\ 1 \\ -0.400 \end{bmatrix}$	$\begin{bmatrix} 0.160 \\ 1 \\ 0.160 \end{bmatrix}$	$\begin{bmatrix} -0.064 \\ 1 \\ -0.064 \end{bmatrix}$
m_k	1	-0.500	-0.500	-0.500

Clearly, m_k converges to -0.5 , so the eigenvalue of A closest to 5 is $5 + 1/(-0.5) = 5 - 2 = 3$.

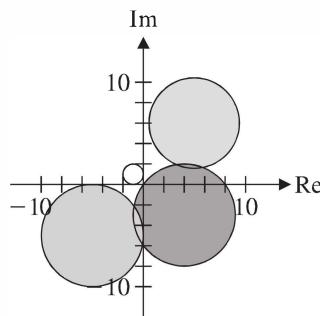
41. 0.732

43. -0.619

47.



49.



51. Hint: Show that 0 is not contained in any Gershgorin disk and then apply Theorem 4.16.

53. Exercise 52 implies that $|\lambda|$ is less than or equal to all of the column sums of A for every eigenvalue λ . But for a stochastic matrix, all column sums are 1. Hence $|\lambda| \leq 1$.

Exercises 4.6

1. Not regular

3. Regular

5. Not regular

$$7. L = \begin{bmatrix} \frac{1}{5} & \frac{1}{5} \\ \frac{4}{5} & \frac{4}{5} \end{bmatrix}$$

9. $L = \begin{bmatrix} 0.304 & 0.304 & 0.304 \\ 0.354 & 0.354 & 0.354 \\ 0.342 & 0.342 & 0.342 \end{bmatrix}$

11. 1, $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$

$$13. 2, \begin{bmatrix} 16 \\ 4 \\ 1 \end{bmatrix}$$

15. The population is increasing, decreasing, and constant, respectively.

17.

$$P^{-1}LP = \begin{bmatrix} b_1 & b_2s_1 & b_3s_1s_2 & \cdots & b_{n-1}s_1s_2 \cdots s_{n-2} & b_ns_1s_2 \cdots s_{n-1} \\ 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 1 & 0 \end{bmatrix}.$$

The characteristic polynomial of L is $(\lambda^n - b_1\lambda^{n-1} - b_2s_1\lambda^{n-2} - b_3s_1s_2\lambda^{n-3} - \cdots - b_ns_1s_2 \cdots s_{n-1})(-1)^n$.

$$19. \lambda \approx 1.746, \mathbf{p} \approx \begin{bmatrix} 0.660 \\ 0.264 \\ 0.076 \end{bmatrix}$$

$$21. \lambda \approx 1.092, \mathbf{p} \approx \begin{bmatrix} 0.535 \\ 0.147 \\ 0.094 \\ 0.078 \\ 0.064 \\ 0.053 \\ 0.029 \end{bmatrix}$$

$$25. (a) h \approx 0.082 \quad 29. 3, \begin{bmatrix} \frac{3}{5} \\ \frac{2}{5} \end{bmatrix}$$

$$31. 3, \begin{bmatrix} \frac{1}{2} \\ \frac{1}{4} \\ \frac{1}{4} \end{bmatrix}$$

33. Reducible

35. Irreducible

43. 1, 2, 4, 8, 16

$$45. 0, 1, 1, 0, -1$$

$$47. x_n = 4^n - (-1)^n$$

$$49. y_n = (n - \frac{1}{2})2^n$$

$$51. b_n = \frac{1}{2\sqrt{3}}[(1 + \sqrt{3})^n - (1 - \sqrt{3})^n]$$

$$57. (a) d_1 = 1, d_2 = 2, d_3 = 3, d_4 = 5, d_5 = 8$$

$$(b) d_n = d_{n-1} + d_{n-2}$$

$$(c) d_n = \frac{1}{\sqrt{5}} \left[\left(\frac{1 + \sqrt{5}}{2} \right)^{n+1} - \left(\frac{1 - \sqrt{5}}{2} \right)^{n+1} \right]$$

59. The general solution is $x(t) = -3C_1e^{-t} + C_2e^{4t}$, $y(t) = 2C_1e^{-t} + C_2e^{4t}$. The specific solution is $x(t) = -3e^{-t} + 3e^{4t}$, $y(t) = 2e^{-t} + 3e^{4t}$.

61. The general solution is $x_1(t) = (1 + \sqrt{2})C_1e^{\sqrt{2}t} + (1 - \sqrt{2})C_2e^{-\sqrt{2}t}$, $x_2(t) = C_1e^{\sqrt{2}t} + C_2e^{-\sqrt{2}t}$. The specific solution is $x_1(t) = (2 + \sqrt{2})e^{\sqrt{2}t}/4 + (2 - \sqrt{2})e^{-\sqrt{2}t}/4$, $x_2(t) = \sqrt{2}e^{\sqrt{2}t}/4 - \sqrt{2}e^{-\sqrt{2}t}/4$.

63. The general solution is $x(t) = -C_1 + C_3e^{-t}$, $y(t) = C_1 + C_2e^t - C_3e^{-t}$, $z(t) = C_1 + C_2e^t$. The specific solution is $x(t) = 2 - e^{-t}$, $y(t) = -2 + e^t + e^{-t}$, $z(t) = -2 + e^t$.

65. (a) $x(t) = -120e^{8t/5} + 520e^{11t/10}$, $y(t) = 240e^{8t/5} + 260e^{11t/10}$. Strain X dies out after approximately 2.93 days; strain Y continues to grow.

67. $a = 10$, $b = 20$; $x(t) = 10e^t(\cos t + \sin t) + 10$, $y(t) = 10e^t(\cos t - \sin t) + 20$. Species Y dies out when $t \approx 1.22$.

$$71. x(t) = C_1e^{2t} + C_2e^{3t}$$

$$77. (a) \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 3 \end{bmatrix}, \begin{bmatrix} 9 \\ 9 \end{bmatrix}, \begin{bmatrix} 27 \\ 27 \end{bmatrix} \quad (c) \text{Repeller}$$

$$79. (a) \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad (c) \text{Neither}$$

$$81. (a) \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0.5 \\ -1 \end{bmatrix}, \begin{bmatrix} 1.75 \\ -0.5 \end{bmatrix}, \begin{bmatrix} 3.125 \\ -1.75 \end{bmatrix} \quad (c) \text{Saddle point}$$

$$83. (a) \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0.6 \\ 0.6 \end{bmatrix}, \begin{bmatrix} 0.36 \\ 0.36 \end{bmatrix}, \begin{bmatrix} 0.216 \\ 0.216 \end{bmatrix} \quad (c) \text{Attractor}$$

85. $r = \sqrt{2}$, $\theta = 45^\circ$, spiral repeller87. $r = 2$, $\theta = -60^\circ$, spiral repeller

$$89. P = \begin{bmatrix} -1 & -1 \\ 1 & 0 \end{bmatrix}, C = \begin{bmatrix} 0.2 & -0.1 \\ 0.1 & 0.2 \end{bmatrix}, \text{spiral attractor}$$

$$91. P = \begin{bmatrix} 1/2 & -\sqrt{3}/2 \\ 1 & 0 \end{bmatrix}, C = \begin{bmatrix} 1/2 & -\sqrt{3}/2 \\ \sqrt{3}/2 & 1/2 \end{bmatrix}, \text{orbital center}$$

Review Questions

$$1. (a) F \quad (c) F \quad (e) F \quad (g) T \quad (i) F$$

$$3. -18$$

5. Since $A^T = -A$, we have $\det A = \det(A^T) = \det(-A) = (-1)^n \det A = -\det A$ by Theorem 4.7 and the fact that n is odd. It follows that $\det A = 0$.

$$7. Ax = \begin{bmatrix} 5 \\ 10 \end{bmatrix} = 5x, \lambda = 5$$

$$9. (a) 4 - 3\lambda^2 - \lambda^3$$

$$(c) E_1 = \text{span} \left(\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \right), E_{-2} = \text{span} \left(\begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right)$$

$$11. \begin{bmatrix} 162 \\ 158 \end{bmatrix}$$

13. Not similar

15. Not similar

$$17. 0, 1, \text{ or } -1$$

$$19. \text{If } Ax = \lambda x, \text{ then } (A^2 - 5A + 2I)x = A^2x - 5Ax + 2x = 3^2x - 5(3x) + 2x = -4x.$$

Chapter 5*Exercises 5.1*

1. Orthogonal 3. Not orthogonal 5. Orthogonal

7. $[\mathbf{w}]_{\mathcal{B}} = \begin{bmatrix} \frac{1}{2} \\ -1 \end{bmatrix}$ 9. $[\mathbf{w}]_{\mathcal{B}} = \begin{bmatrix} 0 \\ \frac{2}{3} \\ \frac{1}{3} \end{bmatrix}$ 11. Orthonormal

13. $\begin{bmatrix} 1/3 \\ 2/3 \\ 2/3 \end{bmatrix}, \begin{bmatrix} 2/\sqrt{5} \\ -1/\sqrt{5} \\ 0 \end{bmatrix}, \begin{bmatrix} 2/3\sqrt{5} \\ 4/3\sqrt{5} \\ -5/3\sqrt{5} \end{bmatrix}$

15. Orthonormal

17. Orthogonal, $\begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$

19. Orthogonal, $\begin{bmatrix} \cos \theta & \sin \theta & \cos^2 \theta & \sin \theta \\ -\cos \theta & \sin \theta & 0 \\ -\sin^2 \theta & -\cos \theta & \sin \theta & \cos \theta \end{bmatrix}$

21. Not orthogonal

27. $\cos(\angle(Q\mathbf{x}, Q\mathbf{y})) = \frac{Q\mathbf{x} \cdot Q\mathbf{y}}{\|Q\mathbf{x}\| \|Q\mathbf{y}\|} = \frac{\mathbf{x} \cdot \mathbf{y}}{\|\mathbf{x}\| \|\mathbf{y}\|}$
= $\cos(\angle(\mathbf{x}, \mathbf{y}))$ by Theorem 5.6

29. Rotation, $\theta = 45^\circ$ 31. Reflection, $y = \sqrt{3}x$

33. (a) $A(A^T + B^T)B = AA^T B + AB^T B = IB + AI = B + A = A + B$

(b) From part (a),

$$\begin{aligned} \det(A + B) &= \det(A(A^T + B^T)B) \\ &= \det A \det(A^T + B^T) \det B \\ &= \det A \det((A + B)^T) \det B \\ &= \det A \det(A + B) \det B \end{aligned}$$

Assume that $\det A + \det B = 0$ (so that $\det B = -\det A$) but that $A + B$ is invertible.Then $\det(A + B) \neq 0$, so $1 = \det A \det B = \det A(-\det A) = -(\det A)^2$. This is impossible, so we conclude that $A + B$ cannot be invertible.*Exercises 5.2*

1. $W^\perp = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} : x + 2y = 0 \right\}, \mathcal{B}^\perp = \left\{ \begin{bmatrix} -2 \\ 1 \end{bmatrix} \right\}$

3. $W^\perp = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} : x = t, y = t, z = -t \right\},$

$$\mathcal{B}^\perp = \left\{ \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} \right\}$$

5. $W^\perp = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} : x - y + 3z = 0 \right\}, \mathcal{B}^\perp = \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 3 \\ 1 \end{bmatrix} \right\}$

7. $\text{row}(A): \{[1 \ 0 \ 1], [0 \ 1 \ -2]\}, \text{null}(A): \left\{ \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \right\}$

9. $\text{col}(A): \left\{ \begin{bmatrix} 1 \\ 5 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \\ 1 \\ -1 \end{bmatrix} \right\}, \text{null}(A^T):$

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} -5 \\ 1 \\ -7 \\ 0 \end{bmatrix} \right\}$$

11. $\left\{ \begin{bmatrix} 1 \\ -10 \\ -4 \end{bmatrix} \right\}$ 13. $\left\{ \begin{bmatrix} -4 \\ 1 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 1 \\ 0 \end{bmatrix} \right\}$

15. $\begin{bmatrix} \frac{3}{2} \\ \frac{3}{2} \\ \frac{3}{2} \end{bmatrix}$ 17. $\begin{bmatrix} -\frac{1}{2} \\ \frac{1}{2} \\ 3 \end{bmatrix}$

19. $\mathbf{v} = \begin{bmatrix} -\frac{2}{5} \\ -\frac{6}{5} \end{bmatrix} + \begin{bmatrix} \frac{12}{5} \\ -\frac{4}{5} \end{bmatrix}$ 21. $\mathbf{v} = \begin{bmatrix} \frac{7}{2} \\ -2 \\ \frac{7}{2} \end{bmatrix} + \begin{bmatrix} \frac{1}{2} \\ 0 \\ -\frac{1}{2} \end{bmatrix}$

25. No

Exercises 5.3

1. $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} -1/2 \\ 1/2 \end{bmatrix}; \mathbf{q}_1 = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}, \mathbf{q}_2 = \begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$

3. $\mathbf{v}_1 = \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}; \mathbf{q}_1 = \begin{bmatrix} 1/\sqrt{3} \\ -1/\sqrt{3} \\ -1/\sqrt{3} \end{bmatrix},$

$$\mathbf{q}_2 = \begin{bmatrix} 2/\sqrt{6} \\ 1/\sqrt{6} \\ 1/\sqrt{6} \end{bmatrix}, \mathbf{q}_3 = \begin{bmatrix} 0 \\ -1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$$

5. $\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -\frac{1}{2} \\ \frac{1}{2} \\ 2 \end{bmatrix} \right\}$ 7. $\mathbf{v} = \begin{bmatrix} -\frac{2}{9} \\ \frac{2}{9} \\ \frac{8}{9} \end{bmatrix} + \begin{bmatrix} \frac{38}{9} \\ -\frac{38}{9} \\ \frac{19}{9} \end{bmatrix}$

9. $\left\{ \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -\frac{1}{2} \\ \frac{1}{2} \end{bmatrix}, \begin{bmatrix} \frac{2}{3} \\ \frac{2}{3} \\ -\frac{2}{3} \end{bmatrix} \right\}$

11. $\left\{ \begin{bmatrix} 3 \\ 1 \\ 5 \end{bmatrix}, \begin{bmatrix} -\frac{3}{35} \\ \frac{34}{35} \\ -\frac{1}{7} \end{bmatrix}, \begin{bmatrix} -\frac{15}{34} \\ 0 \\ \frac{9}{34} \end{bmatrix} \right\}$

13. $Q = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{3} & 1/\sqrt{6} \\ 0 & 1/\sqrt{3} & -2/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{3} & 1/\sqrt{6} \end{bmatrix}$

15. $\begin{bmatrix} 0 & 2/\sqrt{6} & 1/\sqrt{3} \\ 1/\sqrt{2} & -1/\sqrt{6} & 1/\sqrt{3} \\ 1/\sqrt{2} & 1/\sqrt{6} & -1/\sqrt{3} \end{bmatrix} \begin{bmatrix} \sqrt{2} & 1/\sqrt{2} & 1/\sqrt{2} \\ 0 & 3/\sqrt{6} & 1/\sqrt{6} \\ 0 & 0 & 2/\sqrt{3} \end{bmatrix}$

17. $R = \begin{bmatrix} 3 & 9 & \frac{1}{3} \\ 0 & 6 & \frac{2}{3} \\ 0 & 0 & \frac{7}{3} \end{bmatrix}$

19. $A = AI$

21. $A^{-1} = (QR)^{-1} = R^{-1}Q^{-1} = R^{-1}Q^T =$

$$\begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{6} & -1/2\sqrt{3} \\ 0 & 2/\sqrt{6} & -1/2\sqrt{3} \\ 0 & 0 & 3/2\sqrt{3} \end{bmatrix} \cdot \begin{bmatrix} 0 & 1/\sqrt{2} & 1/\sqrt{2} \\ 2/\sqrt{6} & -1/\sqrt{6} & 1/\sqrt{6} \\ 1/\sqrt{3} & 1/\sqrt{3} & -1/\sqrt{3} \end{bmatrix}$$

23. Let $Rx = \mathbf{0}$. Then $Ax = QRx = Q\mathbf{0} = \mathbf{0}$. Since Ax represents a linear combination of the columns of A (which are linearly independent), we must have $x = \mathbf{0}$. Hence, R is invertible, by the Fundamental Theorem.

Exercises 5.4

1. $Q = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix}, D = \begin{bmatrix} 5 & 0 \\ 0 & 3 \end{bmatrix}$

3. $Q = \begin{bmatrix} 2/\sqrt{6} & 1/\sqrt{3} \\ 1/\sqrt{3} & -2/\sqrt{6} \end{bmatrix}, D = \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix}$

5. $Q = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/\sqrt{2} & -1/\sqrt{2} \\ 0 & 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}, D = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & -2 \end{bmatrix}$

7. $Q = \begin{bmatrix} -1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 0 & 1 & 0 \\ 1/\sqrt{2} & 0 & 1/\sqrt{2} \end{bmatrix}, D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

9. $Q = \begin{bmatrix} 1/\sqrt{2} & 0 & 1/\sqrt{2} & 0 \\ 1/\sqrt{2} & 0 & -1/\sqrt{2} & 0 \\ 0 & 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 0 & 1/\sqrt{2} & 0 & -1/\sqrt{2} \end{bmatrix}$

$$D = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

11. $Q^T A Q = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix} \begin{bmatrix} a & b \\ b & a \end{bmatrix}.$

$$\begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix} = \begin{bmatrix} a+b & 0 \\ 0 & a-b \end{bmatrix} = D$$

13. (a) If A and B are orthogonally diagonalizable, then each is symmetric, by the Spectral Theorem. Therefore, $A + B$ is symmetric, by Exercise 35 in Section 3.2, and so is orthogonally diagonalizable, by the Spectral Theorem.

15. If A and B are orthogonally diagonalizable, then each is symmetric, by the Spectral Theorem. Since $AB = BA$, AB is also symmetric, by Exercise 36 in Section 3.2. Hence, AB is orthogonally diagonalizable, by the Spectral Theorem.

17. $A = \begin{bmatrix} \frac{5}{2} & \frac{5}{2} \\ \frac{5}{2} & \frac{5}{2} \end{bmatrix} + \begin{bmatrix} \frac{3}{2} & -\frac{3}{2} \\ -\frac{3}{2} & \frac{3}{2} \end{bmatrix}$

19. $A = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 2 & 2 \\ 0 & 2 & 2 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & -1 & 1 \\ 0 & 1 & -1 \end{bmatrix}$

21. $\begin{bmatrix} \frac{1}{2} & -\frac{3}{2} \\ -\frac{3}{2} & \frac{1}{2} \end{bmatrix}$ 23. $\begin{bmatrix} \frac{5}{3} & -\frac{2}{3} & -\frac{1}{3} \\ -\frac{2}{3} & \frac{5}{3} & \frac{1}{3} \\ -\frac{1}{3} & \frac{1}{3} & \frac{8}{3} \end{bmatrix}$

Exercises 5.5

1. $2x^2 + 6xy + 4y^2$ 3. 123

5. -5

7. $\begin{bmatrix} 1 & 3 \\ 3 & 2 \end{bmatrix}$ 9. $\begin{bmatrix} 3 & -\frac{3}{2} \\ -\frac{3}{2} & -1 \end{bmatrix}$ 11. $\begin{bmatrix} 5 & 1 & -2 \\ 1 & -1 & 2 \\ -2 & 2 & 2 \end{bmatrix}$

13. $Q = \begin{bmatrix} 2/\sqrt{5} & 1/\sqrt{5} \\ 1/\sqrt{5} & -2/\sqrt{5} \end{bmatrix}, y_1^2 + 6y_2^2$

15. $Q = \begin{bmatrix} 2/\sqrt{5} & 2/3\sqrt{5} & -1/3 \\ 0 & 5/3\sqrt{5} & 2/3 \\ 1/\sqrt{5} & -4/3\sqrt{5} & 2/3 \end{bmatrix}, 9y_1^2 + 9y_2^2 - 9y_3^2$

17. $Q = \begin{bmatrix} 1/\sqrt{3} & 1/\sqrt{2} & 1/\sqrt{6} \\ -1/\sqrt{3} & 0 & 2/\sqrt{6} \\ -1/\sqrt{3} & 1/\sqrt{2} & -1/\sqrt{6} \end{bmatrix}, 2(x')^2 + (y')^2 - (z')^2$

19. Positive definite

21. Negative definite

23. Positive definite

25. Indefinite

27. For any vector \mathbf{x} , we have $\mathbf{x}^T A \mathbf{x} = \mathbf{x}^T B^T B \mathbf{x} = (B\mathbf{x})^T (B\mathbf{x}) = \|B\mathbf{x}\|^2 \geq 0$. If $\mathbf{x}^T A \mathbf{x} = 0$, then $\|B\mathbf{x}\|^2 = 0$, so $B\mathbf{x} = \mathbf{0}$. Since B is invertible, this implies that $\mathbf{x} = \mathbf{0}$. Therefore, $\mathbf{x}^T A \mathbf{x} > 0$ for all $\mathbf{x} \neq \mathbf{0}$, and hence $A = B^T B$ is positive definite.

29. (a) Every eigenvalue of cA is of the form $c\lambda$ for some eigenvalue λ of A . By Theorem 5.24, $\lambda > 0$, so $c\lambda > 0$, since c is positive. Hence, cA is positive definite, by Theorem 5.24.

(c) Let $\mathbf{x} \neq \mathbf{0}$. Then $\mathbf{x}^T A \mathbf{x} > 0$ and $\mathbf{x}^T B \mathbf{x} > 0$, since A and B are positive definite. But then $\mathbf{x}^T (A + B) \mathbf{x} = \mathbf{x}^T A \mathbf{x} + \mathbf{x}^T B \mathbf{x} > 0$, so $A + B$ is positive definite.

31. The maximum value of $f(\mathbf{x})$ is 2 when $\mathbf{x} = \pm \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix}$;

the minimum value of $f(\mathbf{x})$ is 0 when $\mathbf{x} = \pm \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$.

33. The maximum value of $f(\mathbf{x})$ is 4 when $\mathbf{x} = \pm \begin{bmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix}$;

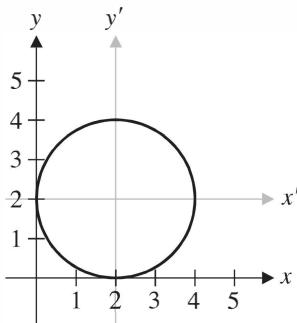
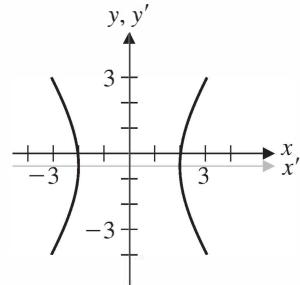
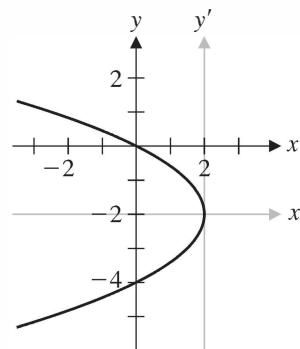
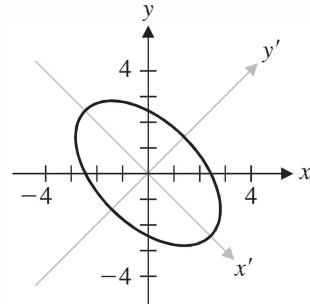
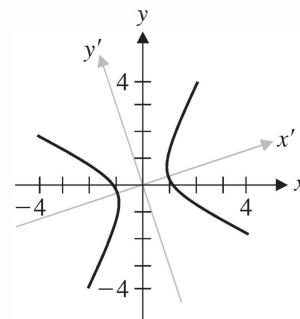
the minimum value of $f(\mathbf{x})$ is 1 when $\mathbf{x} =$

$$\pm \begin{bmatrix} 1/\sqrt{2} \\ 0 \\ -1/\sqrt{2} \end{bmatrix} \text{ or } \pm \begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{bmatrix}.$$

35. Ellipse

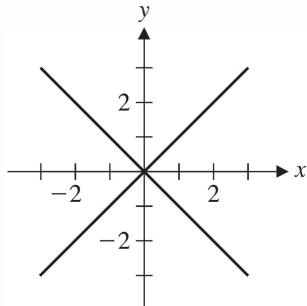
37. Parabola

39. Hyperbola

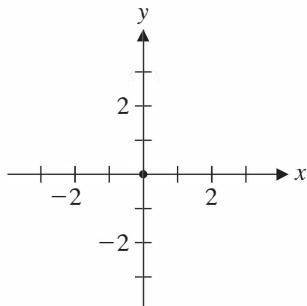
41. Circle, $x' = x - 2$, $y' = y - 2$, $(x')^2 + (y')^2 = 4$ 43. Hyperbola, $x' = x$, $y' = y + \frac{1}{2}$, $(x')^2/4 - (y')^2/9 = 1$ 45. Parabola, $x' = x - 2$, $y' = y + 2$, $x' = -\frac{1}{2}(y')^2$ 47. Ellipse, $(x')^2/4 + (y')^2/12 = 1$ 49. Hyperbola, $(x')^2 - (y')^2 = 1$ 

51. Ellipse, $(x'')^2/50 + (y'')^2/10 = 1$ 53. Hyperbola, $(x'')^2 - (y'')^2 = 1$

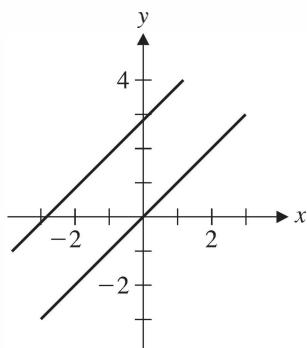
55. Degenerate (two lines)



57. Degenerate (a point)



59. Degenerate (two lines)

61. Hyperboloid of one sheet, $(x')^2 - (y')^2 + 3(z')^2 = 1$ 63. Hyperbolic paraboloid, $z = -(x')^2 + (y')^2$ 65. Hyperbolic paraboloid, $x' = -\sqrt{3}(y')^2 + \sqrt{3}(z')^2$ 67. Ellipsoid, $3(x'')^2 + (y'')^2 + 2(z'')^2 = 4$ **Review Questions**

1. (a) T (c) T (e) F (g) F (i) F

3. $\begin{bmatrix} 9/2 \\ 2/3 \\ -11/6 \end{bmatrix}$

5. Verify that $Q^T Q = I$.

7. Theorem 5.6(c) shows that if $\mathbf{v}_i \cdot \mathbf{v}_j = 0$, then $Q\mathbf{v}_i \cdot Q\mathbf{v}_j = 0$. Theorem 5.6(b) shows that $\{Q\mathbf{v}_1, \dots, Q\mathbf{v}_k\}$ consists of unit vectors, because $\{\mathbf{v}_1, \dots, \mathbf{v}_k\}$ does. Hence, $\{Q\mathbf{v}_1, \dots, Q\mathbf{v}_k\}$ is an orthonormal set.

9. $\left\{ \begin{bmatrix} 2 \\ -5 \end{bmatrix} \right\}$

11. $\left\{ \begin{bmatrix} -1 \\ 3 \\ 1 \end{bmatrix} \right\}$

13. $\text{row}(A): \{[1 \ 0 \ 2 \ 3 \ 4], [0 \ 1 \ 0 \ 2 \ 1]\}$

$\text{col}(A): \left\{ \begin{bmatrix} 1 \\ -1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \\ 1 \\ -5 \end{bmatrix} \right\}$

$\text{null}(A): \left\{ \begin{bmatrix} -2 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ -2 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -4 \\ -1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$

$\text{null}(A^T): \left\{ \begin{bmatrix} -5 \\ -3 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \\ 0 \\ 1 \end{bmatrix} \right\}$

15. (a) $\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} \frac{1}{4} \\ \frac{1}{4} \\ \frac{1}{4} \\ -\frac{3}{4} \end{bmatrix}, \begin{bmatrix} -\frac{2}{3} \\ \frac{1}{3} \\ \frac{1}{3} \\ 0 \end{bmatrix} \right\}$

17. $\left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} \frac{1}{3} \\ \frac{1}{3} \\ \frac{1}{3} \\ -1 \end{bmatrix} \right\}$

19. $\begin{bmatrix} -\frac{1}{2} & \frac{3}{2} & 0 \\ \frac{3}{2} & -\frac{1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Chapter 6**Exercises 6.1**

1. Vector space
3. Not a vector space; axiom 1 fails.
5. Not a vector space; axiom 8 fails.
7. Vector space 9. Vector space
11. Vector space 15. Complex vector space

17. Not a complex vector space; axiom 6 fails.
 19. Not a vector space; axioms 1, 4, and 6 fail.
 21. Not a vector space; the operations of addition and multiplication are not even the same.
 25. Subspace 27. Not a subspace
 29. Not a subspace 31. Subspace
 33. Subspace 35. Subspace
 37. Not a subspace 39. Subspace
 41. Subspace 43. Not a subspace
 45. Not a subspace
 47. Take U to be the x -axis and W the y -axis, for example.
 Then $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ are in $U \cup W$, but $\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ is not.
51. No
 53. Yes; $s(x) = (3 + 2t)p(x) + (1 + t)q(x) + tr(x)$ for any scalar t .
 55. Yes; $h(x) = f(x) + g(x)$
 57. No
 59. No
 61. Yes

Exercises 6.2

1. Linearly independent
 3. Linearly dependent; $\begin{bmatrix} -1 & 0 \\ -1 & 7 \end{bmatrix} = 4\begin{bmatrix} -1 & 1 \\ -2 & 2 \end{bmatrix} + \begin{bmatrix} 3 & 0 \\ 1 & 1 \end{bmatrix} - 2\begin{bmatrix} 0 & 2 \\ -3 & 1 \end{bmatrix}$
 5. Linearly independent
 7. Linearly dependent; $3x + 2x^2 = 7x - 2(2x - x^2)$
 9. Linearly independent

11. Linearly dependent; $1 = \sin^2 x + \cos^2 x$
 13. Linearly dependent; $\ln(x^2) = -2 \ln 2 \cdot 1 + 2 \cdot \ln(2x)$

17. (a) Linearly independent
 (b) Linearly dependent

19. Basis 21. Not a basis
 23. Not a basis 25. Not a basis

27. $[A]_B = \begin{bmatrix} -1 \\ -1 \\ -1 \\ 4 \end{bmatrix}$ 29. $[p(x)]_B = \begin{bmatrix} 6 \\ -1 \\ 3 \end{bmatrix}$

35. $\dim V = 2, \mathcal{B} = \{1 - x, 1 - x^2\}$
 37. $\dim V = 3, \mathcal{B} = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$
 39. $\dim V = 2, \mathcal{B} = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \right\}$
 41. $(n^2 - n)/2$
 43. (a) $\dim(U \times V) = \dim U + \dim V$
 (b) Show that if $\{\mathbf{w}_1, \dots, \mathbf{w}_n\}$ is a basis for W , then $\{(\mathbf{w}_1, \mathbf{w}_1), \dots, (\mathbf{w}_n, \mathbf{w}_n)\}$ is a basis for Δ .
 45. $\{1 + x, 1 + x + x^2, 1\}$
 47. $\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \right\}$
 49. $\{1, 1 + x\}$
 51. $\{1 - x, x - x^2\}$
 53. $\{\sin^2 x, \cos^2 x\}$
 59. (a) $p_0(x) = \frac{1}{2}x^2 - \frac{5}{2}x + 3, p_1(x) = -x^2 + 4x - 3,$
 $p_2(x) = \frac{1}{2}x^2 - \frac{3}{2}x + 1$
 61. (c) (i) $3x^2 - 16x + 19$ (ii) $x^2 - 4x + 5$
 63. $(p^n - 1)(p^n - p)(p^n - p^2) \cdots (p^n - p^{n-1})$

Exercises 6.3

1. $[\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}, [\mathbf{x}]_{\mathcal{C}} = \begin{bmatrix} \frac{5}{2} \\ -\frac{1}{2} \end{bmatrix}, P_{\mathcal{C} \leftarrow \mathcal{B}} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{bmatrix}, P_{\mathcal{B} \leftarrow \mathcal{C}} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$
 3. $[\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, [\mathbf{x}]_{\mathcal{C}} = \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}, P_{\mathcal{C} \leftarrow \mathcal{B}} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}, P_{\mathcal{B} \leftarrow \mathcal{C}} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$
 5. $[p(x)]_{\mathcal{B}} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}, [p(x)]_{\mathcal{C}} = \begin{bmatrix} -3 \\ 2 \end{bmatrix}, P_{\mathcal{C} \leftarrow \mathcal{B}} = \begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix}, P_{\mathcal{B} \leftarrow \mathcal{C}} = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$
 7. $[p(x)]_{\mathcal{B}} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, [p(x)]_{\mathcal{C}} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, P_{\mathcal{C} \leftarrow \mathcal{B}} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}, P_{\mathcal{B} \leftarrow \mathcal{C}} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$

9. $[A]_B = \begin{bmatrix} 4 \\ 2 \\ 0 \\ -1 \end{bmatrix}$, $[A]_C = \begin{bmatrix} \frac{5}{2} \\ 0 \\ -3 \\ \frac{9}{2} \end{bmatrix}$, $P_{C \leftarrow B} = \begin{bmatrix} \frac{1}{2} & 0 & -1 & -\frac{1}{2} \\ 0 & 0 & 1 & 0 \\ -1 & 1 & 1 & 1 \\ \frac{3}{2} & -1 & -2 & -\frac{1}{2} \end{bmatrix}$, $P_{B \leftarrow C} = \begin{bmatrix} 1 & 2 & 1 & 1 \\ 2 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 1 & 1 \end{bmatrix}$
11. $[f(x)]_B = \begin{bmatrix} 2 \\ -5 \end{bmatrix}$, $[f(x)]_C = \begin{bmatrix} -1/2 \\ 5/2 \end{bmatrix}$, $P_{C \leftarrow B} = \begin{bmatrix} 1 & 1/2 \\ 0 & -1/2 \end{bmatrix}$, $P_{B \leftarrow C} = \begin{bmatrix} 1 & 1 \\ 0 & -2 \end{bmatrix}$
13. (a) $\begin{bmatrix} (3 - 2\sqrt{3})/2 \\ (-3\sqrt{3} + 2)/2 \end{bmatrix} \approx \begin{bmatrix} 3.232 \\ -1.598 \end{bmatrix}$
(b) $\begin{bmatrix} 2 + 2\sqrt{3} \\ 2\sqrt{3} - 2 \end{bmatrix} \approx \begin{bmatrix} 5.464 \\ 1.464 \end{bmatrix}$
15. $B = \left\{ \begin{bmatrix} -1 \\ -1 \end{bmatrix}, \begin{bmatrix} 3 \\ 4 \end{bmatrix} \right\}$
17. $-2 - 8(x-1) - 5(x-1)^2$
19. $-1 + 3(x+1) - 3(x+1)^2 + (x+1)^3$

Exercises 6.4

1. Linear transformation 3. Linear transformation
5. Linear transformation
7. Not a linear transformation
9. Linear transformation
11. Not a linear transformation
13. We have

$$\begin{aligned} S(p(x) + q(x)) &= S((p+q)(x)) = x((p+q)(x)) \\ &= x(p(x) + q(x)) = xp(x) + xq(x) \\ &= S(p(x)) + S(q(x)) \end{aligned}$$

$$\text{and } S(cp(x)) = S((cp)(x)) = x((cp)(x)) = x(cp(x)) = cxp(x) = cS(p(x))$$

Therefore, S is linear. Similarly,

$$\begin{aligned} T\left(\begin{bmatrix} a \\ b \end{bmatrix} + \begin{bmatrix} c \\ d \end{bmatrix}\right) &= T\begin{bmatrix} a+c \\ b+d \end{bmatrix} \\ &= (a+c) + ((a+c) + (b+d))x \\ &= (a + (a+b)x) + (c + (c+d)x) \\ &= T\begin{bmatrix} a \\ b \end{bmatrix} + T\begin{bmatrix} c \\ d \end{bmatrix} \end{aligned}$$

$$\begin{aligned} \text{and } T\left(k\begin{bmatrix} a \\ b \end{bmatrix}\right) &= T\begin{bmatrix} ka \\ kb \end{bmatrix} = (ka) + (ka + kb)x \\ &= k(a + (a+b)x) = kT\left(\begin{bmatrix} a \\ b \end{bmatrix}\right) \end{aligned}$$

Therefore, T is linear.

15. $T\begin{bmatrix} -7 \\ 9 \end{bmatrix} = 5 - 14x - 8x^2$, $T\begin{bmatrix} a \\ b \end{bmatrix} = \left(\frac{a+3b}{4}\right) - \left(\frac{a+7b}{4}\right)x + \left(\frac{a-b}{2}\right)x^2$
17. $T(4 - x + 3x^2) = 4 + 3x + 5x^2$, $T(a + bx + cx^2) = a + cx + \left(\frac{3a-b-c}{2}\right)x^2$
19. Hint: Let $a = T(E_{11})$, $b = T(E_{12})$, $c = T(E_{21})$, $d = T(E_{22})$.
23. Hint: Consider the effect of T and D on the standard basis for \mathcal{P}_n .
25. $(S \circ T)\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 & -1 \\ 0 & 6 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2x & -y \\ 0 & 2x + 2y \end{bmatrix}$.
 $\begin{bmatrix} x \\ y \end{bmatrix}$ does not make sense.
27. $(S \circ T)(p(x)) = p'(x+1)$, $(T \circ S)(p(x)) = (p(x+1))' = p'(x+1)$
29. $(S \circ T)\begin{bmatrix} x \\ y \end{bmatrix} = S\left(T\begin{bmatrix} x \\ y \end{bmatrix}\right) = S\left(\begin{bmatrix} x-y \\ -3x+4y \end{bmatrix}\right) = \begin{bmatrix} 4(x-y) + (-3x+4y) \\ 3(x-y) + (-3x+4y) \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$
 $(T \circ S)\begin{bmatrix} x \\ y \end{bmatrix} = T\left(S\begin{bmatrix} x \\ y \end{bmatrix}\right) = T\left(\begin{bmatrix} 4x+y \\ 3x+y \end{bmatrix}\right) = \begin{bmatrix} (4x+y) - (3x+y) \\ -3(4x+y) + 4(3x+y) \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$

Therefore, $S \circ T = I$ and $T \circ S = I$, so S and T are inverses.

Exercises 6.5

1. (a) Only (ii) is in $\ker(T)$.
(b) Only (iii) is in $\text{range}(T)$.
(c) $\ker(T) = \left\{ \begin{bmatrix} 0 & b \\ c & 0 \end{bmatrix} \right\}$, $\text{range}(T) = \left\{ \begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix} \right\}$
3. (a) Only (iii) is in $\ker(T)$.
(b) All of them are in $\text{range}(T)$.
(c) $\ker(T) = \{a + bx + cx^2 : a = -c, b = -c\} = \{t + tx - tx^2\}$, $\text{range}(T) = \mathbb{R}^2$

5. A basis for $\ker(T)$ is $\left\{ \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \right\}$, and a basis for $\text{range}(T)$ is $\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$; $\text{rank}(T) = \text{nullity}(T) = 2$, and $\text{rank}(T) + \text{nullity}(T) = 4 = \dim M_{22}$.

7. A basis for $\ker(T)$ is $\{1 + x - x^2\}$, and a basis for $\text{range}(T)$ is $\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$; $\text{rank}(T) = 2$, $\text{nullity}(T) = 1$, and $\text{rank}(T) + \text{nullity}(T) = 3 = \dim \mathcal{P}_2$.

9. $\text{rank}(T) = \text{nullity}(T) = 2$

11. $\text{rank}(T) = \text{nullity}(T) = 2$

13. $\text{rank}(T) = 1$, $\text{nullity}(T) = 2$

15. One-to-one and onto

17. Neither one-to-one nor onto

19. One-to-one but not onto

$$21. \text{Isomorphic, } T \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

23. Not isomorphic

$$25. \text{Isomorphic, } T(a + bi) = \begin{bmatrix} a \\ b \end{bmatrix}$$

31. Hint: Define $T : \mathcal{C}[0, 1] \rightarrow \mathcal{C}[0, 2]$ by letting $T(f)$ be the function whose value at x is $(T(f))(x) = f(x/2)$ for x in $[0, 2]$.

33. (a) Let \mathbf{v}_1 and \mathbf{v}_2 be in V and let $(S \circ T)(\mathbf{v}_1) = (S \circ T)(\mathbf{v}_2)$. Then $S(T(\mathbf{v}_1)) = S(T(\mathbf{v}_2))$, so $T(\mathbf{v}_1) = T(\mathbf{v}_2)$, since S is one-to-one. But now $\mathbf{v}_1 = \mathbf{v}_2$, since T is one-to-one. Hence, $S \circ T$ is one-to-one.

35. (a) By the Rank Theorem, $\text{rank}(T) + \text{nullity}(T) = \dim V$. If T is onto, then $\text{range}(T) = W$, so $\text{rank}(T) = \dim(\text{range}(T)) = \dim W$. Therefore,

$\dim V + \text{nullity}(T) < \dim W + \text{nullity}(T)$
 $= \text{rank}(T) + \text{nullity}(T) = \dim V$
so $\text{nullity}(T) < 0$, which is impossible. Therefore, T cannot be onto.

Exercises 6.6

$$1. [T]_{C \leftarrow B} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, [T]_{C \leftarrow B}[4 + 2x]_B = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 4 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ -4 \end{bmatrix} = [2 - 4x]_C = [T(4 + 2x)]_C$$

$$3. [T]_{C \leftarrow B} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, [T]_{C \leftarrow B}[a + bx + cx^2]_B =$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \end{bmatrix} = [a + b(x + 2) + c(x + 2)^2]_C = [T(a + bx + cx^2)]_C$$

$$5. [T]_{C \leftarrow B} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix}, [T]_{C \leftarrow B}[a + bx + cx^2]_B = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} a \\ a + b + c \\ a + b \cdot 0 + c \cdot 0^2 \end{bmatrix} = [T(a + bx + cx^2)]_C$$

$$7. [T]_{C \leftarrow B} = \begin{bmatrix} 6 & 4 \\ -3 & -2 \\ 2 & -1 \end{bmatrix}, [T]_{C \leftarrow B} \begin{bmatrix} -7 \\ 7 \end{bmatrix}_B =$$

$$\begin{bmatrix} 6 & 4 \\ -3 & -2 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} 2 \\ -3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 7 \end{bmatrix} = \begin{bmatrix} 7 \\ 7 \end{bmatrix}_C = \begin{bmatrix} 7 \\ 7 \end{bmatrix}_C$$

$$\begin{bmatrix} T \begin{bmatrix} -7 \\ 7 \end{bmatrix} \end{bmatrix}_C$$

$$9. [T]_{C \leftarrow B} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, [T]_{C \leftarrow B}[A]_B =$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} a \\ c \\ b \\ d \end{bmatrix} = \begin{bmatrix} [a & c] \\ [b & d] \end{bmatrix}_C = [T(A)]_C$$

$$11. [T]_{C \leftarrow B} = \begin{bmatrix} 0 & -1 & 1 & 0 \\ -1 & 0 & 0 & 1 \\ 1 & 0 & 0 & -1 \\ 0 & 1 & -1 & 0 \end{bmatrix}, [T]_{C \leftarrow B}[A]_B =$$

$$\begin{bmatrix} 0 & -1 & 1 & 0 \\ -1 & 0 & 0 & 1 \\ 1 & 0 & 0 & -1 \\ 0 & 1 & -1 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} c - b \\ d - a \\ a - d \\ b - c \end{bmatrix} =$$

$$\begin{bmatrix} [c - b & d - a] \\ [a - d & b - c] \end{bmatrix}_C = [AB - BA]_C = [T(A)]_C$$

13. (b) $[D]_B = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

(c) $[D]_B [3 \sin x - 5 \cos x]_B = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 3 \\ -5 \end{bmatrix} = \begin{bmatrix} 5 \\ 3 \end{bmatrix} = [3 \cos x + 5 \sin x]_B = [D(3 \sin x - 5 \cos x)]_B$

15. (a) $[D]_B = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & -1 & 2 \end{bmatrix}$

17. $[S \circ T]_{\mathcal{D} \leftarrow \mathcal{B}} = \begin{bmatrix} -1 & -2 \\ 1 & -1 \end{bmatrix}$

19. Invertible, $T^{-1}(a + bx) = -b + ax$

21. Invertible, $T^{-1}(p(x)) = p(x - 2)$

23. Invertible, $T^{-1}(a + bx + cx^2) = (a - b + 2c) + (b - 2c)x + cx^2$ or $T^{-1}(p(x)) = p(x) - p'(x) + p''(x)$

25. Not invertible 27. $-3 \sin x - \cos x + C$

29. $\frac{4}{5}e^{2x} \cos x - \frac{3}{5}e^{2x} \sin x + C$

31. $\mathcal{C} = \left\{ \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \begin{bmatrix} -4 \\ 1 \end{bmatrix} \right\}$ 33. $\mathcal{C} = \{1 - x, 2 + x\}$

35. $\mathcal{C} = \{1, x\}$

37. $[T]_{\mathcal{E}} = \begin{bmatrix} (d_1^2 - d_2^2)/(d_1^2 + d_2^2) & 2d_1d_2/(d_1^2 + d_2^2) \\ 2d_1d_2/(d_1^2 + d_2^2) & (d_2^2 - d_1^2)/(d_1^2 + d_2^2) \end{bmatrix}$

Exercises 6.7

1. $y(t) = 2e^{3t}/e^3$

3. $y(t) = ((1 - e^4)e^{3t} + (e^3 - 1)e^{4t})/(e^3 - e^4)$

5. $f(t) = \left(\frac{e^{(\sqrt{5}-1)/2}}{e^{\sqrt{5}} - 1} \right) [e^{(1+\sqrt{5})t/2} - e^{(1-\sqrt{5})t/2}]$

7. $y(t) = e^t - (1 - e^{-1})te^t$

9. $y(t) = ((k+1)e^{kt} + (k-1)e^{-kt})/2k$

11. $y(t) = e^t \cos(2t)$

13. (a) $p(t) = 100e^{\ln(16)t/3} \approx 100e^{0.924t}$

(b) 45 minutes (c) In 9.968 hours

15. (a) $m(t) = 50e^{-ct}$, where $c = \ln 2/1590 \approx 4.36 \times 10^{-4}$; 32.33 mg remain after 1000 years.

(b) After 3691.9 years

17. $x(t) = \frac{5 - 10 \cos(10\sqrt{K})}{\sin(10\sqrt{K})} \sin(\sqrt{K}t) + 10 \cos(\sqrt{K}t)$

19. (b) No

Review Questions

1. (a) F (c) T (e) F (g) F (i) T

3. Subspace

7. Let $c_1A + c_2B = O$. Then $c_1A - c_2B = c_1A^T + c_2B^T = (c_1A + c_2B)^T = O$. Adding, we have $2c_1A = O$, so $c_1 = 0$ because A is nonzero. Hence $c_2B = O$, and so $c_2 = 0$. Thus, $\{A, B\}$ is linearly independent.

9. $\{1, x^2, x^4\}$, $\dim W = 3$

11. Linear transformation

13. Linear transformation

17. $\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \\ 1 & 0 & -1 \end{bmatrix}$

19. $S \circ T$ is the zero transformation.

Chapter 7

Exercises 7.1

1. (a) -10 (b) $\sqrt{14}$ (c) $\sqrt{93}$

3. Any nonzero scalar multiple of $\begin{bmatrix} 3 \\ 1 \end{bmatrix}$

5. (a) 1 (b) $\sqrt{13}$ (c) $\sqrt{14}$

7. x^2 is one possibility

9. (a) π (b) $\sqrt{\pi}$ (c) $\sqrt{\pi}$

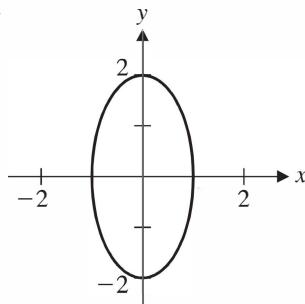
13. Axiom (4) fails: $\mathbf{u} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \neq \mathbf{0}$, but $\langle \mathbf{u}, \mathbf{u} \rangle = 0$.

15. Axiom (4) fails: $\mathbf{u} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \neq \mathbf{0}$, but $\langle \mathbf{u}, \mathbf{u} \rangle = 0$.

17. Axiom (4) fails: $p(x) = 1 - x$ is not the zero polynomial, but $\langle p(x), p(x) \rangle = 0$.

19. $A = \begin{bmatrix} 4 & 1 \\ 1 & 4 \end{bmatrix}$

21.



25. -8

27. $\sqrt{6}$

29. $\|\mathbf{u} + \mathbf{v} - \mathbf{w}\|^2 = \langle \mathbf{u} + \mathbf{v} - \mathbf{w}, \mathbf{u} + \mathbf{v} - \mathbf{w} \rangle = \langle \mathbf{u}, \mathbf{u} \rangle + \langle \mathbf{v}, \mathbf{v} \rangle + \langle \mathbf{w}, \mathbf{w} \rangle + 2\langle \mathbf{u}, \mathbf{v} \rangle - 2\langle \mathbf{u}, \mathbf{w} \rangle - 2\langle \mathbf{v}, \mathbf{w} \rangle = 1 + 3 + 4 + 2 - 10 - 0 = 0$

Therefore, $\|\mathbf{u} + \mathbf{v} - \mathbf{w}\| = 0$, so, by axiom (4),
 $\mathbf{u} + \mathbf{v} - \mathbf{w} = \mathbf{0}$ or $\mathbf{u} + \mathbf{v} = \mathbf{w}$.

31. $\langle \mathbf{u} + \mathbf{v}, \mathbf{u} - \mathbf{v} \rangle = \langle \mathbf{u}, \mathbf{u} \rangle - \langle \mathbf{u}, \mathbf{v} \rangle + \langle \mathbf{v}, \mathbf{u} \rangle - \langle \mathbf{v}, \mathbf{v} \rangle = \|\mathbf{u}\|^2 - \langle \mathbf{u}, \mathbf{v} \rangle + \langle \mathbf{u}, \mathbf{v} \rangle - \|\mathbf{v}\|^2 = \|\mathbf{u}\|^2 - \|\mathbf{v}\|^2$

33. Using Exercise 32 and a similar identity for $\|\mathbf{u} - \mathbf{v}\|^2$, we have

$$\begin{aligned} \|\mathbf{u} + \mathbf{v}\|^2 + \|\mathbf{u} - \mathbf{v}\|^2 &= \langle \mathbf{u} + \mathbf{v}, \mathbf{u} + \mathbf{v} \rangle + \langle \mathbf{u} - \mathbf{v}, \mathbf{u} - \mathbf{v} \rangle \\ &= \|\mathbf{u}\|^2 + 2\langle \mathbf{u}, \mathbf{v} \rangle + \|\mathbf{v}\|^2 \\ &\quad + \|\mathbf{u}\|^2 - 2\langle \mathbf{u}, \mathbf{v} \rangle + \|\mathbf{v}\|^2 \\ &= 2\|\mathbf{u}\|^2 + 2\|\mathbf{v}\|^2 \end{aligned}$$

Dividing by 2 yields the identity we want.

35. $\|\mathbf{u} + \mathbf{v}\| = \|\mathbf{u} - \mathbf{v}\| \Leftrightarrow \|\mathbf{u} + \mathbf{v}\|^2 = \|\mathbf{u} - \mathbf{v}\|^2$
 $\Leftrightarrow \|\mathbf{u}\|^2 + 2\langle \mathbf{u}, \mathbf{v} \rangle + \|\mathbf{v}\|^2 = \|\mathbf{u}\|^2 - 2\langle \mathbf{u}, \mathbf{v} \rangle + \|\mathbf{v}\|^2$
 $\Leftrightarrow 2\langle \mathbf{u}, \mathbf{v} \rangle = -2\langle \mathbf{u}, \mathbf{v} \rangle \Leftrightarrow \langle \mathbf{u}, \mathbf{v} \rangle = 0$

37. $\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$

39. $\{1, x, x^2\}$

41. (a) $1/\sqrt{2}, \sqrt{3}x/\sqrt{2}, \sqrt{5}(3x^2 - 1)/2\sqrt{2}$

(b) $\sqrt{7}(5x^3 - 3x)/2\sqrt{2}$

Exercises 7.2

1. $\|\mathbf{u}\|_E = \sqrt{42}$, $\|\mathbf{u}\|_s = 10$, $\|\mathbf{u}\|_m = 5$

3. $d_E(\mathbf{u}, \mathbf{v}) = \sqrt{70}$, $d_s(\mathbf{u}, \mathbf{v}) = 14$, $d_m(\mathbf{u}, \mathbf{v}) = 6$

5. $\|\mathbf{u}\|_H = 4$, $\|\mathbf{v}\|_H = 5$

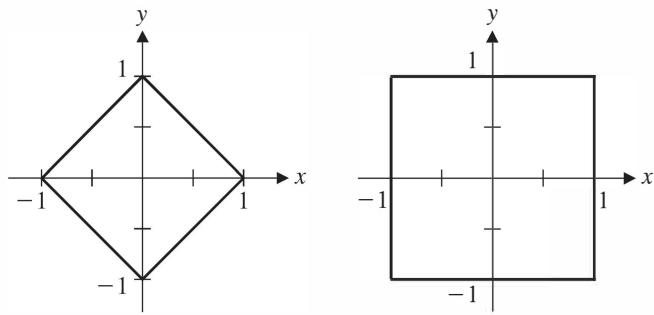
7. (a) At most one component of \mathbf{v} is nonzero.

9. Suppose $\|\mathbf{v}\|_m = |\nu_k|$. Then $\|\mathbf{v}\|_E = \sqrt{\nu_1^2 + \cdots + \nu_k^2 + \cdots + \nu_n^2} \geq \sqrt{\nu_k^2} = |\nu_k| = \|\mathbf{v}\|_m$.

11. Suppose $\|\mathbf{v}\|_m = |\nu_k|$. Then $|\nu_i| \leq |\nu_k|$ for $i = 1, \dots, n$, so

$$\begin{aligned} \|\mathbf{v}\|_s &= |\nu_1| + \cdots + |\nu_n| \leq |\nu_k| + \cdots + |\nu_k| \\ &= n|\nu_k| = n\|\mathbf{v}\|_m \end{aligned}$$

13.



21. $\|A\|_F = \sqrt{19}$, $\|A\|_1 = 4$, $\|A\|_\infty = 6$

23. $\|A\|_F = \sqrt{31}$, $\|A\|_1 = 6$, $\|A\|_\infty = 6$

25. $\|A\|_F = 2\sqrt{11}$, $\|A\|_1 = 7$, $\|A\|_\infty = 7$

27. $\mathbf{x} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \mathbf{y} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ 29. $\mathbf{x} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \mathbf{y} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

31. $\mathbf{x} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \mathbf{y} = \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}$

33. (a) By the definition of an operator norm, $\|I\| = \max_{\|\mathbf{x}\|=1} \|I\mathbf{x}\| = \max_{\|\mathbf{x}\|=1} \|\mathbf{x}\| = 1$.

35. $\text{cond}_1(A) = \text{cond}_\infty(A) = 21$; well-conditioned

37. $\text{cond}_1(A) = \text{cond}_\infty(A) = 400$; ill-conditioned

39. $\text{cond}_1(A) = 77$, $\text{cond}_\infty(A) = 128$; moderately ill-conditioned

41. (a) $\text{cond}_\infty(A) = (\max\{|k| + 1, 2\}) \cdot \left(\max\left\{\left|\frac{k}{k-1}\right| + \left|\frac{1}{k-1}\right|, \left|\frac{2}{k-1}\right|\right\} \right)$

43. (a) $\text{cond}_\infty(A) = 40$

(b) At most 400% relative change

45. Using Exercise 33(a), we have $\text{cond}(A) = \|A\| \|A^{-1}\| \geq \|AA^{-1}\| = \|I\| = 1$.

49. $k \geq 6$

51. $k \geq 10$

Exercises 7.3

1. $\|\mathbf{e}\| = \sqrt{2} \approx 1.414$ 3. $\|\mathbf{e}\| = \sqrt{6}/2 \approx 1.225$

5. $\|\mathbf{e}\| = \sqrt{7} \approx 2.646$

7. $y = -3 + \frac{5}{2}x$, $\|\mathbf{e}\| \approx 1.225$

9. $y = \frac{11}{3} - 2x$, $\|\mathbf{e}\| \approx 0.816$

11. $y = \frac{7}{10} + \frac{8}{25}x$, $\|\mathbf{e}\| \approx 0.447$

13. $y = -\frac{1}{5} + \frac{7}{5}x$, $\|\mathbf{e}\| \approx 0.632$

15. $y = 3 - \frac{18}{5}x + x^2$ 17. $y = \frac{18}{5} - \frac{17}{10}x - \frac{1}{2}x^2$

19. $\bar{\mathbf{x}} = \begin{bmatrix} \frac{1}{5} \\ \frac{7}{15} \end{bmatrix}$ 21. $\bar{\mathbf{x}} = \begin{bmatrix} \frac{4}{3} \\ -\frac{5}{6} \end{bmatrix}$

23. $\bar{\mathbf{x}} = \begin{bmatrix} 4+t \\ -5-t \\ -5-2t \\ t \end{bmatrix}$ 25. $\begin{bmatrix} \frac{42}{11} \\ \frac{19}{11} \\ \frac{42}{11} \end{bmatrix}$

27. $\bar{x} = \begin{bmatrix} \frac{5}{3} \\ -2 \end{bmatrix}$

29. $y = 0.92 + 0.73x$

31. (a) If we let the year 1920 correspond to $t = 0$, then $y = 56.6 + 2.9t$; 79.9 years

33. (a) $p(t) = 150e^{0.131t}$

35. 139 days

37. $\begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}, \begin{bmatrix} \frac{7}{2} \\ \frac{7}{2} \end{bmatrix}$

39. $\begin{bmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix}$

41. $\begin{bmatrix} \frac{5}{6} & \frac{1}{3} & -\frac{1}{6} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ -\frac{1}{6} & \frac{1}{3} & \frac{5}{6} \end{bmatrix}, \begin{bmatrix} \frac{5}{6} \\ \frac{1}{3} \\ -\frac{1}{6} \end{bmatrix}$

45.

47. $A^+ = \begin{bmatrix} \frac{1}{3} & -\frac{2}{3} & -\frac{1}{6} \\ \frac{1}{6} & \frac{1}{6} & \frac{1}{6} \end{bmatrix}$

49. $A^+ = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$

51. $A^+ = \begin{bmatrix} \frac{2}{3} & 0 & -\frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & -1 & \frac{1}{3} & \frac{2}{3} \\ -\frac{2}{3} & 1 & \frac{1}{3} & -\frac{1}{3} \end{bmatrix}$

53. (a) If A is invertible, so is A^T , and we have $A^+ = (A^T A)^{-1} A^T = A^{-1} (A^T)^{-1} A^T = A^{-1}$.

Exercises 7.4

1. 2, 3

3. $\sqrt{2}, 0$

5. 5

7. 2, 3

9. $\sqrt{5}, 2, 0$

11. $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \sqrt{2} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix}$

13. $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$

15. $A = \begin{bmatrix} \frac{3}{5} & -\frac{4}{5} \\ \frac{4}{5} & \frac{3}{5} \end{bmatrix} \begin{bmatrix} 5 \\ 0 \end{bmatrix} [1]$

17. $A = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 0 & 2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

19. $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \sqrt{5} & 0 & 0 \\ 0 & 2 & 0 \end{bmatrix} \begin{bmatrix} 2/\sqrt{5} & 0 & 1/\sqrt{5} \\ 0 & 1 & 0 \\ 1/\sqrt{5} & 0 & -2/\sqrt{5} \end{bmatrix}$

21. $A = \sqrt{2} \begin{bmatrix} 1 \\ 0 \end{bmatrix} [1/\sqrt{2} \quad 1/\sqrt{2}] + 0 \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$[-1/\sqrt{2} \quad 1/\sqrt{2}]$ (Exercise 3)

23. (Exercise 7) $A = 3 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} [0 \quad 1] + 2 \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix} [1 \quad 0]$

33. The line segment $[-1, 1]$

35. The solid ellipse $\frac{y_1^2}{5} + \frac{y_2^2}{4} \leq 1$

37. (a) $\|A\|_2 = \sqrt{2}$ (b) $\text{cond}_2(A) = \infty$

39. (a) $\|A\|_2 = 1.95$ (b) $\text{cond}_2(A) = 38.11$

41. $A^+ = \begin{bmatrix} \frac{1}{2} & 0 \\ \frac{1}{2} & 0 \end{bmatrix}$

43. $A^+ = \begin{bmatrix} \frac{2}{5} & 0 \\ 0 & \frac{1}{2} \\ \frac{1}{5} & 0 \end{bmatrix}$

45. $A^+ = \begin{bmatrix} \frac{1}{25} & \frac{2}{25} \\ \frac{2}{25} & \frac{4}{25} \end{bmatrix}, \bar{x} = \begin{bmatrix} 0.52 \\ 1.04 \end{bmatrix}$

47. $A^+ = \begin{bmatrix} \frac{1}{6} & \frac{1}{6} & \frac{1}{6} \\ \frac{1}{6} & \frac{1}{6} & \frac{1}{6} \end{bmatrix}, \bar{x} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

61. $\begin{bmatrix} \sqrt{2} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$

63. $\begin{bmatrix} 2 & -1 \\ -1 & 3 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$

Exercises 7.5

1. $g(x) = \frac{1}{3}$

3. $g(x) = \frac{3}{5}x$

5. $g(x) = \frac{3}{16} + \frac{15}{16}x^2$

7. $\{1, x - \frac{1}{2}\}$

9. $g(x) = x - \frac{1}{6}$

11. $g(x) = (4e - 10) + (18 - 6e)x \approx 0.87 + 1.69x$

13. $g(x) = \frac{1}{20} - \frac{3}{5}x + \frac{3}{2}x^2$

15. $g(x) = 39e - 105 + (588 - 216e)x + (210e - 570)x^2 \approx 1.01 + 0.85x + 0.84x^2$

21. $\frac{\pi}{2} - \frac{4}{\pi} \left(\cos x + \frac{\cos 3x}{9} \right)$

23. $a_0 = \frac{1}{2}, a_k = 0, b_k = \frac{1 - (-1)^k}{k\pi}$

25. $a_0 = \pi, a_k = 0, b_k = \frac{2(-1)^k}{k}$

Review Questions

1. (a) T (c) F (e) T (g) T (i) T

3. Inner product

5. $\sqrt{3}$

7. $\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}, \begin{bmatrix} -\frac{1}{2} \\ \frac{1}{2} \end{bmatrix}$

9. Not a norm

11. $\text{cond}_\infty(A) \approx 2432$

13. $y = 1.7x$

$$\mathbf{15.} \begin{bmatrix} \frac{7}{3} \\ \frac{2}{3} \\ \frac{5}{3} \end{bmatrix}$$

17. (a) $\sqrt{2}, \sqrt{2}$

$$\mathbf{(b)} \quad A = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & 0 & 1 \\ 1/\sqrt{2} & -1/\sqrt{2} & 0 \end{bmatrix} \begin{bmatrix} \sqrt{2} & 0 \\ 0 & \sqrt{2} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\mathbf{(c)} \quad A^+ = \begin{bmatrix} \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & 0 & -\frac{1}{2} \end{bmatrix}$$

19. The singular values of PAQ are the square roots of the eigenvalues of $(PAQ)^T(PAQ) = Q^T A^T P^T PAQ = Q^T (A^T A) Q$. But $Q^T (A^T A) Q$ is similar to $A^T A$ because $Q^T = Q^{-1}$, and hence it has the same eigenvalues as $A^T A$. Thus, PAQ and A have the same singular values.

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