Answers to Selected Odd-Numbered Exercises

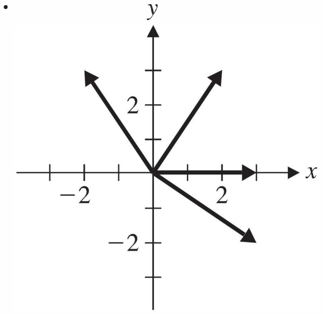
Answers are easy. It's asking
the right questions [that's] hard.

—Doctor Who
"The Face of Evil,"
By Chris Boucher
BBC, 1977

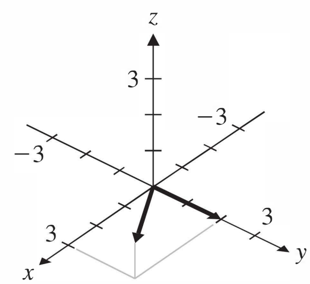
Chapter 1

Exercises 1.1

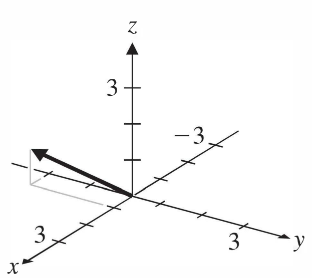
1.



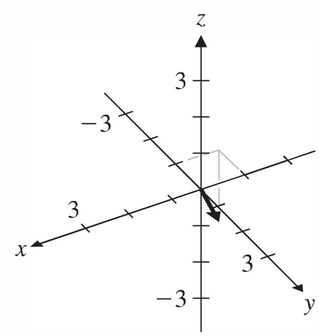
3. (a), (b)



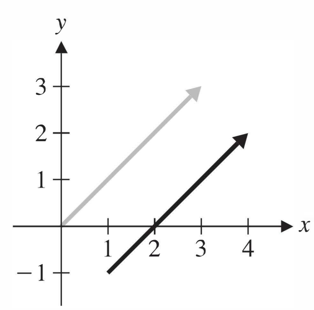
(c)



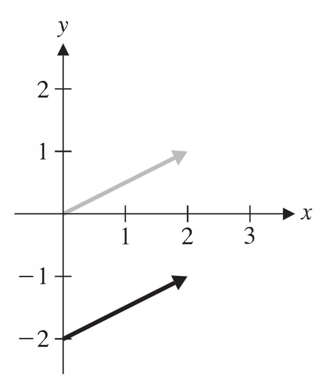
(d)



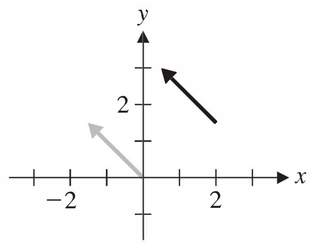
5. (a)



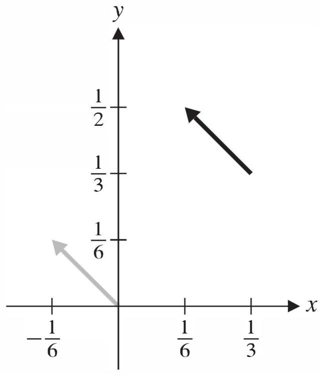
(b)



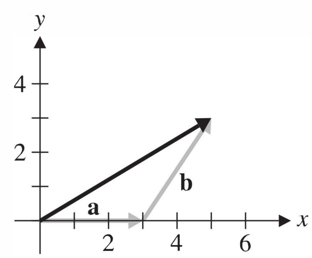
(c)



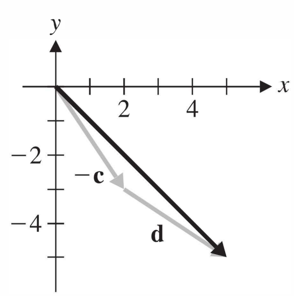
(d)



7. $\mathbf{a} + \mathbf{b} = [5, 3]$



9. $\mathbf{d} - \mathbf{c} = [5, -5]$



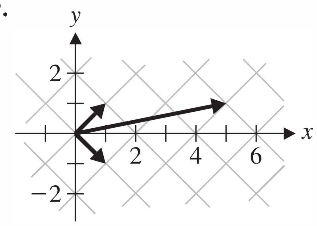
11. $[3, -2, 3]$

13. $\mathbf{u} = \begin{bmatrix} 1/2 \\ \sqrt{3}/2 \end{bmatrix}, \mathbf{v} = \begin{bmatrix} -\sqrt{3}/2 \\ -1/2 \end{bmatrix}, \mathbf{u} + \mathbf{v} = \begin{bmatrix} (1 - \sqrt{3})/2 \\ (\sqrt{3} - 1)/2 \end{bmatrix}, \mathbf{u} - \mathbf{v} = \begin{bmatrix} (1 + \sqrt{3})/2 \\ (1 + \sqrt{3})/2 \end{bmatrix}$

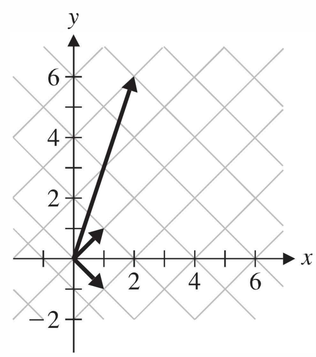
15. $5\mathbf{a}$

17. $\mathbf{x} = 3\mathbf{a}$

19.



21. $\mathbf{w} = -2\mathbf{u} + 4\mathbf{v}$



25. $\mathbf{u} + \mathbf{v} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

27. $\mathbf{u} + \mathbf{v} = [0, 1, 0, 0]$

29.

+	0	1	2	3
0	0	1	2	3
1	1	2	3	0
2	2	3	0	1
3	3	0	1	2

·	0	1	2	3
0	0	0	0	0
1	0	1	2	3
2	0	2	0	2
3	0	3	2	1

31. 0 **33.** 1

35. 0 **37.** 2, 0, 3

39. 5 **41.** [1, 1, 0]

43. [0, 0, 2, 2], [2, 3, 1, 1] **45.** $x = 2$

47. No solution **49.** $x = 3$

51. No solution **53.** $x = 2$

55. $x = 1$, or $x = 5$

57. **(a)** All $a \neq 0$ **(b)** $a = 1, 5$

(c) a and m can have no common factors other than 1 [i.e., the greatest common divisor (gcd) of a and m is 1].

Exercises 1.2

1. -1 **3.** 11 **5.** 2

7. $\sqrt{5}, \begin{bmatrix} -1/\sqrt{5} \\ 2/\sqrt{5} \end{bmatrix}$ **9.** $\sqrt{14}, \begin{bmatrix} 1/\sqrt{14} \\ 2/\sqrt{14} \\ 3/\sqrt{14} \end{bmatrix}$

11. $\sqrt{6}$, $[1/\sqrt{6}, 1/\sqrt{3}, 1/\sqrt{2}, 0]$

13. $\sqrt{17}$ **15.** $\sqrt{6}$

17. (a) $\mathbf{u} \cdot \mathbf{v}$ is a scalar, not a vector.

(c) $\mathbf{v} \cdot \mathbf{w}$ is a scalar and $\mathbf{u}$ is a vector.

19. Acute **21.** Acute **23.** Acute

25. 60° **27.** ≈88.10° **29.** ≈14.34°

31. Since $\overrightarrow{AB} \cdot \overrightarrow{AC} = \begin{bmatrix} -4 \\ 1 \\ -1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ -3 \end{bmatrix} = 0$, $\angle BAC$ is a right angle.

33. If we take the cube to be a unit cube (as in Figure 1.34), the four diagonals are given by the vectors

$$\mathbf{d}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \mathbf{d}_2 = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}, \mathbf{d}_3 = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}, \mathbf{d}_4 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

Since $\mathbf{d}_i \cdot \mathbf{d}_j \neq 0$ for all $i \neq j$ (six possibilities), no two diagonals are perpendicular.

35. $D = (-2, 1, 1)$

37. 5 mi/h at an angle of ≈ 53.13° to the bank

39. 60°

41. $\begin{bmatrix} -\frac{3}{5} \\ \frac{4}{5} \end{bmatrix}$ **43.** $\begin{bmatrix} \frac{3}{2} \\ -\frac{3}{2} \\ \frac{3}{2} \\ -\frac{3}{2} \end{bmatrix}$ **45.** $\begin{bmatrix} -0.301 \\ 0.033 \\ -0.252 \end{bmatrix}$

47. $A = \sqrt{45}/2$ **49.** $k = -2, 3$

51. $\mathbf{v}$ is of the form $k\begin{bmatrix} b \\ -a \end{bmatrix}$, where k is a scalar.

53. The Cauchy-Schwarz Inequality would be violated.

Exercises 1.3

1. (a) $\begin{bmatrix} 3 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = 0$ **(b)** $3x + 2y = 0$

3. (a) $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} -1 \\ 3 \end{bmatrix}$

(b) $x = 1 - t$
$y = 3t$

5. (a) $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = t\begin{bmatrix} 1 \\ -1 \\ 4 \end{bmatrix}$ **(b)** $x = t$, $y = -t$, $z = 4t$

7. (a) $\begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 2$ **(b)** $3x + 2y + z = 2$

9. (a) $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = s\begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} + t\begin{bmatrix} -3 \\ 2 \\ 1 \end{bmatrix}$

(b) $x = 2s - 3t$
$y = s + 2t$
$z = 2s + t$

11. $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \end{bmatrix} + t\begin{bmatrix} 2 \\ 2 \end{bmatrix}$

13. $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + s\begin{bmatrix} 3 \\ -1 \\ 1 \end{bmatrix} + t\begin{bmatrix} -1 \\ 0 \\ -2 \end{bmatrix}$

15. (a) $x = t$, $y = -1 + 3t$; $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \end{bmatrix} + t\begin{bmatrix} 1 \\ 3 \end{bmatrix}$

17. Direction vectors for the two lines are given by $\mathbf{d}_1 = \begin{bmatrix} 1 \\ m_1 \end{bmatrix}$ and $\mathbf{d}_2 = \begin{bmatrix} 1 \\ m_2 \end{bmatrix}$. The lines are perpendicular if and only if $\mathbf{d}_1$ and $\mathbf{d}_2$ are orthogonal. But $\mathbf{d}_1 \cdot \mathbf{d}_2 = 0$ if and only if $1 + m_1m_2 = 0$ or, equivalently, $m_1m_2 = -1$.

19. (a) Perpendicular **(b)** Parallel

(c) Perpendicular **(d)** Perpendicular

21. $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \end{bmatrix} + t\begin{bmatrix} 3 \\ 2 \end{bmatrix}$

23. $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 3 \end{bmatrix} + t\begin{bmatrix} -1 \\ 3 \\ -1 \end{bmatrix}$

25. (a) $x = 0, x = 1, y = 0, y = 1, z = 0, z = 1$

(b) $x - y = 0$ **(c)** $x + y - z = 0$

27. $3\sqrt{2}/2$ **29.** $2\sqrt{3}/3$ **31.** $(\frac{1}{2}, \frac{1}{2})$

33. $(\frac{4}{3}, \frac{4}{3}, \frac{8}{3})$ **35.** $18\sqrt{13}/13$ **37.** $\frac{5}{3}$

43. ≈78.9° **45.** ≈80.4°

Exercises 1.4

1. 13 N at approx N 67.38 E

3. $8\sqrt{3}$ N at an angle of 30° to $\mathbf{f}_1$

5. 4 N at an angle of 60° to $\mathbf{f}_2$

7. 5 N at an angle of 60° to the given force, $5\sqrt{3}$ N perpendicular to the 5 N force

9. $750\sqrt{2}$ N

11. 980 N

13. ≈ 117.6 N in the 15 cm wire, ≈ 88.2 N in the 20 cm wire

Review Questions

1. **(a)** T **(c)** F **(e)** T **(g)** F **(i)** T

3. $\mathbf{x} = \begin{bmatrix} 8 \\ 11 \end{bmatrix}$ **5.** 120° **7.** $\begin{bmatrix} -2/\sqrt{5} \\ 1/\sqrt{5} \\ 0 \end{bmatrix}$

9. $2x + 3y - z = 7$ **11.** $\sqrt{6}/2$

13. The Cauchy-Schwarz Inequality would be violated.

15. $2\sqrt{6}/3$ **17.** $x = 2$ **19.** 3

Chapter 2

Exercises 2.1

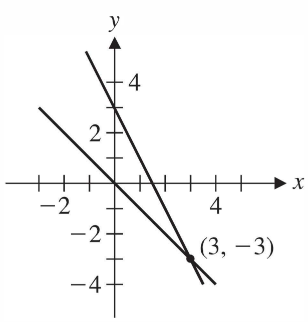
1. Linear **3.** Not linear because of the x^{-1} term

5. Not linear **7.** $2x + 4y = 7$

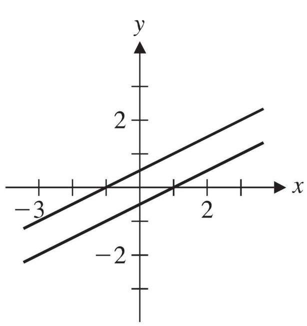
9. $x + y = 4(x, y \neq 0)$

11. $\left\{\begin{bmatrix} 2t \\ t \end{bmatrix}\right\}$ **13.** $\left\{\begin{bmatrix} 4 - 2s - 3t \\ s \\ t \end{bmatrix}\right\}$

15. Unique solution, $x = 3, y = -3$



17. No solution



19. [7, 3] **21.** $[\frac{2}{3}, \frac{1}{3}, -\frac{1}{3}]$

23. [5, −2, 1, 1] **25.** [2, −7, −32]

27. $\left[\begin{array}{rr|r} 1 & -1 & 0 \\ 2 & 1 & 3 \end{array}\right]$ **29.** $\left[\begin{array}{rr|r} 1 & 5 & -1 \\ -1 & 1 & -5 \\ 2 & 4 & 4 \end{array}\right]$

31. $\begin{aligned} y + z &= 1 \\ x - y \quad &= 1 \\ 2x - y + z &= 1 \end{aligned}$ **33.** [1, 1]

35. [4, −1] **37.** No solution

39. **(a)** $\begin{aligned} 2x + y &= 3 \\ 4x + 2y &= 6 \end{aligned}$ **(b)** $\begin{aligned} x &= \frac{3}{2} - \frac{1}{2}s \\ y &= s \end{aligned}$

41. Let $u = \dfrac{1}{x}$ and $v = \dfrac{1}{y}$. The solution is $x = \frac{1}{3}, y = -\frac{1}{2}$.

43. Let $u = \tan x$, $v = \sin y$, $w = \cos z$. One solution is $x = \pi/4, y = -\pi/6, z = \pi/3$. (There are infinitely many solutions.)

Exercises 2.2

1. No **3.** Reduced row echelon form

5. No **7.** No

9. **(a)** $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$ **11.** **(b)** $\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$

13. **(b)** $\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$

15. Perform elementary row operations in the order $R_4 + 29R_3$, $8R_3$, $R_4 - 3R_2$, $R_2 \leftrightarrow R_3$, $R_4 - R_1$, $R_3 + 2R_1$, and, finally, $R_2 + 2R_1$.

17. One possibility is to perform elementary row operations on A in the order $R_2 - 3R_1$, $\frac{1}{2}R_2$, $R_1 + 2R_2$, $R_2 + 3R_1$, $R_1 \leftrightarrow R_2$.

19. *Hint:* Pick a random 2×2 matrix and try this—carefully!

21. This is really two elementary row operations combined: $3R_2$ and $R_2 - 2R_1$.

23. Exercise 1: 3; Exercise 3: 2; Exercise 5: 2; Exercise 7: 3

25. $\begin{bmatrix} 2 \\ 5 \\ 1 \end{bmatrix}$ **27.** $t\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$ **29.** $\begin{bmatrix} 2 \\ -1 \end{bmatrix}$

31. $\begin{bmatrix} 24 \\ -10 \\ 0 \\ 0 \\ 0 \end{bmatrix} + r\begin{bmatrix} 6 \\ -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + s\begin{bmatrix} 0 \\ 6 \\ 0 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} 12 \\ -6 \\ 0 \\ 0 \\ 1 \end{bmatrix}$

33. No solution

35. Unique solution

37. Infinitely many solutions

39. *Hint:* Show that if $ad - bc \neq 0$, the rank of $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is 2. (There are two cases: $a = 0$ and $a \neq 0$.) Use the Rank Theorem to deduce that the given system must have a unique solution.

41. (a) No solution if $k = -1$
(b) A unique solution if $k \neq \pm 1$
(c) Infinitely many solutions if $k = 1$

43. (a) No solution if $k = 1$
(b) A unique solution if $k \neq -2, 1$
(c) Infinitely many solutions if $k = -2$

45. $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} + t\begin{bmatrix} 9 \\ -10 \\ -7 \end{bmatrix}$

49. No intersection

51. The required vectors $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ are the solutions of the homogeneous system with augmented matrix

$$\left[\begin{array}{ccc|c} u_1 & u_2 & u_3 & 0 \\ v_1 & v_2 & v_3 & 0 \end{array}\right]$$

By Theorem 3, there are infinitely many solutions. If $u_1 \neq 0$ and $u_1v_2 - u_2v_1 \neq 0$, the solutions are given by

$$t\begin{bmatrix} u_2v_3 - u_3v_2 \\ u_3v_1 - u_1v_3 \\ u_1v_2 - u_2v_1 \end{bmatrix}$$

But a direct check shows that these are still solutions even if $u_1 = 0$ and/or $u_1v_2 - u_2v_1 = 0$.

53. $\begin{bmatrix} 0 \\ 2 \end{bmatrix}$ **55.** $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ **57.** $\begin{bmatrix} 3 \\ 3 \end{bmatrix}$

Exercises 2.3

1. Yes **3.** No **5.** Yes **7.** Yes

9. We need to show that the vector equation $x\begin{bmatrix} 1 \\ 1 \end{bmatrix} + y\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} a \\ b \end{bmatrix}$ has a solution for all values of a and b. This vector equation is equivalent to the linear system whose augmented matrix is $\left[\begin{array}{cc|c} 1 & 1 & a \\ 1 & -1 & b \end{array}\right]$. Row reduction yields $\left[\begin{array}{cc|c} 1 & 1 & a \\ 0 & -2 & b - a \end{array}\right]$, from which we can see that there is a (unique) solution. [Further row operations yield $x = (a + b)/2$, $y = (a - b)/2$.] Hence, $\mathbb{R}^2 = \text{span}\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix}\right)$.

11. We need to show that the vector equation $x\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + y\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + z\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ has a solution for all values of a, b, and c. This vector equation is equivalent to the linear system whose augmented matrix is $\left[\begin{array}{ccc|c} 1 & 1 & 0 & a \\ 0 & 1 & 1 & b \\ 1 & 0 & 1 & c \end{array}\right]$. Row reduction yields $\left[\begin{array}{ccc|c} 1 & 1 & 0 & a \\ 0 & 1 & 1 & b \\ 0 & 0 & 2 & b + c - a \end{array}\right]$, from which we can see that there is a (unique) solution. [Further row operations yield $x = (a - b + c)/2$, $y = (a + b - c)/2$, $z = (-a + b + c)/2$.]

Hence, $\mathbb{R}^3 = \text{span}\left(\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}\right)$.

13. (a) The line through the origin with direction vector $\begin{bmatrix} -1 \\ 2 \end{bmatrix}$
(b) The line with general equation $2x + y = 0$

15. (a) The plane through the origin with direction vectors $\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \\ -1 \end{bmatrix}$
(b) The plane with general equation $2x - y + 4z = 0$

17. Substitution yields the linear system

$$\begin{aligned} a \qquad + 3c &= 0 \\ -a + b - 3c &= 0 \end{aligned}$$

whose solution is $t\begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}$. It follows that there are infinitely many solutions, the simplest perhaps being $a = -3$, $b = 0$, $c = 1$.

19. $\mathbf{u} = \mathbf{u} + 0(\mathbf{u} + \mathbf{v}) + 0(\mathbf{u} + \mathbf{v} + \mathbf{w})$
$\mathbf{v} = (-1)\mathbf{u} + (\mathbf{u} + \mathbf{v}) + 0(\mathbf{u} + \mathbf{v} + \mathbf{w})$
$\mathbf{w} = 0\mathbf{u} + (-1)(\mathbf{u} + \mathbf{v}) + (\mathbf{u} + \mathbf{v} + \mathbf{w})$

21. **(c)** We must show that span($\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$) = span($\mathbf{e}_1, \mathbf{e}_1 + \mathbf{e}_2, \mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3$). We know that span($\mathbf{e}_1, \mathbf{e}_1 + \mathbf{e}_2, \mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3$) $\subseteq \mathbb{R}^3$ = span($\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$). From Exercise 19, $\mathbf{e}_1$, $\mathbf{e}_2$, and $\mathbf{e}_3$ all belong to span($\mathbf{e}_1, \mathbf{e}_1 + \mathbf{e}_2, \mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3$). Therefore, by Exercise 21(b), span($\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$) = span($\mathbf{e}_1, \mathbf{e}_1 + \mathbf{e}_2, \mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3$).

23. Linearly independent

25. Linearly dependent, $-\begin{bmatrix}0\\1\\2\end{bmatrix} + \begin{bmatrix}2\\1\\3\end{bmatrix} = \begin{bmatrix}2\\0\\1\end{bmatrix}$

27. Linearly dependent, since the set contains the zero vector

29. Linearly independent

31. Linearly dependent, $\begin{bmatrix}3\\-1\\1\\-1\end{bmatrix} + \begin{bmatrix}-1\\3\\1\\-1\end{bmatrix} + \begin{bmatrix}-1\\-1\\1\\3\end{bmatrix} = \begin{bmatrix}1\\1\\3\\1\end{bmatrix}$

43. (a) Yes **(b)** No

Exercises 2.4

1. $x_1 = 160, x_2 = 120, x_3 = 160$

3. two small, three medium, four large

5. 65 bags of house blend, 30 bags of special blend, 45 bags of gourmet blend

7. $4FeS_2 + 11O_2 \longrightarrow 2Fe_2O_3 + 8SO_2$

9. $2C_4H_{10} + 13O_2 \longrightarrow 8CO_2 + 10H_2O$

11. $2C_5H_{11}OH + 15O_2 \longrightarrow 12H_2O + 10CO_2$

13. $Na_2CO_3 + 4C + N_2 \longrightarrow 2NaCN + 3CO$

15. **(a)** $\begin{aligned} f_1 &= 30 - t \\ f_2 &= -10 + t \\ f_3 &= t \end{aligned}$ **(b)** $f_1 = 15, f_3 = 15$

(c) $\begin{aligned} 0 &\le f_1 \le 20 \\ 0 &\le f_2 \le 20 \\ 10 &\le f_3 \le 30 \end{aligned}$

(d) Negative flow would mean that water was flowing backward, against the direction of the arrow.

17. **(a)** $\begin{aligned} f_1 &= -200 + s + t \\ f_2 &= 300 - s - t \\ f_3 &= s \\ f_4 &= 150 - t \\ f_5 &= t \end{aligned}$ **(b)** $200 \le f_3 \le 300$

(c) If $f_3 = s = 0$, then $f_5 = t \ge 200$ (from the f_1 equation), but $f_5 = t \le 150$ (from the f_4 equation). This is a contradiction.

(d) $50 \le f_3 \le 300$

19. $I_1 = 3$ amps, $I_2 = 5$ amps, $I_3 = 2$ amps

21. **(a)** $I = 10$ amps, $I_1 = I_5 = 6$ amps, $I_2 = I_4 = 4$ amps, $I_3 = 2$ amps

(b) $R_{eff} = \frac{7}{5}$ ohms

(c) Yes; change it to 4 ohms.

23. Farming : Manufacturing = 2 : 3

25. The painter charges \$39/hr, the plumber \$42/hr, the electrician \$54/hr.

27. **(a)** Coal should produce \$100 million and steel \$160 million.

(b) Coal should reduce production by ≈ \$4.2 million and steel should increase production by ≈ \$5.7 million.

29. **(a)** Yes; push switches 1, 2, and 3 or switches 3, 4, and 5.

(b) No

31. The states that can be obtained are represented by those vectors

$$\begin{bmatrix}x_1\\x_2\\x_3\\x_4\\x_5\end{bmatrix}$$

in $\mathbb{Z}_2^5$ for which $x_1 + x_2 + x_4 + x_5 = 0$. (There are 16 such possibilities.)

33. If 0 = off, 1 = light blue, and 2 = dark blue, then the linear system that arises has augmented matrix

$$\left[\begin{array}{ccccc|c} 1&1&0&0&0&2\\ 1&1&1&0&0&1\\ 0&1&1&1&0&2\\ 0&0&1&1&1&1\\ 0&0&0&1&1&2 \end{array}\right]$$

which reduces over $\mathbb{Z}_3$ to

$$\left[\begin{array}{ccccc|c} 1&0&0&0&1&1\\ 0&1&0&0&2&1\\ 0&0&1&0&0&2\\ 0&0&0&1&1&2\\ 0&0&0&0&0&0 \end{array}\right]$$

This yields the solutions

$$\begin{bmatrix}x_1\\x_2\\x_3\\x_4\\x_5\end{bmatrix} = \begin{bmatrix}1\\1\\2\\2\\0\end{bmatrix} + t\begin{bmatrix}2\\1\\0\\2\\1\end{bmatrix}$$

where t is in $\mathbb{Z}_3$. Hence, there are exactly three solutions:

$$\begin{bmatrix}1\\1\\2\\2\\0\end{bmatrix}, \begin{bmatrix}0\\2\\2\\1\\1\end{bmatrix}, \begin{bmatrix}2\\0\\2\\0\\2\end{bmatrix}$$

where each entry indicates the number of times the corresponding switch should be pushed.

35. (a) Push squares 3 and 7.

(b) The 9×9 coefficient matrix A is row equivalent to $\mathbb{Z}_2$, so for any $\mathbf{b}$ in $\mathbb{Z}_2^9$, $A\mathbf{x} = \mathbf{b}$ has a unique solution.

37. Grace is 15, and Hans is 5.

39. 1200 and 600 square yards

41. (a) $a = 4 - d, b = 5 - d, c = -2 + d, d$ is arbitrary

(b) No solution

43. (a) No solution

(b) $[a, b, c, d, e, f] = [4, 5, 6, -3, -1, 0] + f[-1, -1, -1, 1, 1, 1]$

45. (a) $y = x^2 - 2x + 1$ **(b)** $y = x^2 + 6x + 10$

47. $A = 1, B = 2$

49. $A = -\frac{1}{5}, B = \frac{1}{3}, C = 0, D = -\frac{2}{15}, E = -\frac{1}{5}$

51. $a = \frac{1}{2}, b = \frac{1}{2}, c = 0$

Exercises 2.5

1.

n	0	1	2	3	4	5
x_1	0	0.8571	0.9714	0.9959	0.9991	0.9998
x_2	0	0.8000	0.9714	0.9943	0.9992	0.9998

Exact solution: $x_1 = 1, x_2 = 1$

3.

n	0	1	2	3	4	5	6
x_1	0	0.2222	0.2539	0.2610	0.2620	0.2622	0.2623
x_2	0	0.2857	0.3492	0.3582	0.3603	0.3606	0.3606

Exact solution (to four decimal places): $x_1 = 0.2623$, $x_2 = 0.3606$

5.

n	0	1	2	3	4	5	6	7	8
x_1	0	0.3333	0.2500	0.3055	0.2916	0.3009	0.2986	0.3001	0.2997
x_2	0	0.2500	0.0834	0.1250	0.0972	0.1042	0.0996	0.1008	0.1000
x_3	0	0.3333	0.2500	0.3055	0.2916	0.3009	0.2986	0.3001	0.2997

Exact solution: $x_1 = 0.3, x_2 = 0.1, x_3 = 0.3$

7.

n	0	1	2	3	4
x_1	0	0.8571	0.9959	0.9998	1.0000
x_2	0	0.9714	0.9992	1.0000	1.0000

After three iterations, the Gauss-Seidel method is within 0.001 of the exact solution. Jacobi's method took four iterations to reach the same accuracy.

9.

n	0	1	2	3	4
x_1	0	0.2222	0.2610	0.2622	0.2623
x_2	0	0.3492	0.3603	0.3606	0.3606

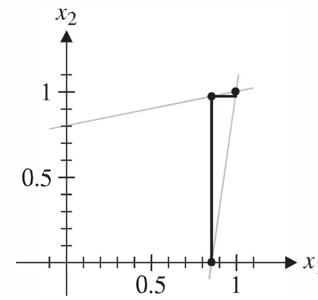
After three iterations, the Gauss-Seidel method is within 0.001 of the exact solution. Jacobi's method took four iterations to reach the same accuracy.

11.

n	0	1	2	3	4	5	6
x_1	0	0.3333	0.2777	0.2962	0.2993	0.2998	0.3000
x_2	0	0.1667	0.1112	0.1020	0.1004	0.1000	0.1000
x_3	0	0.2777	0.2962	0.2993	0.2998	0.3000	0.3000

After four iterations, the Gauss-Seidel method is within 0.001 of the exact solution. Jacobi's method took seven iterations to reach the same accuracy.

13.



15.

n	0	1	2	3	4
x_1	0	3	−5	19	−53
x_2	0	−4	8	−28	80

If the equations are interchanged and the Gauss-Seidel method is applied to the equivalent system

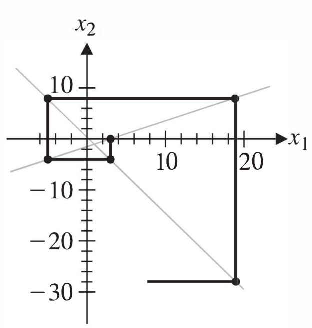
$$3x_1 + 2x_2 = 1$$
$$x_1 - 2x_2 = 3$$

we obtain

n	0	1	2	3	4	5	6	7	8
x_1	0	0.3333	1.2222	0.9260	1.0247	0.9918	1.0027	0.9991	1.0003
x_2	0	−1.3333	−0.8889	−1.0370	−0.9876	−1.0041	−0.9986	−1.0004	−0.9998

After seven iterations, the process has converged to within 0.001 of the exact solution $x_1 = 1, x_2 = -1$.

17.



19.

n	0	1	2	3	4	5	6
x_1	0	−1.6	14.97	8.550	10.740	9.839	10.120
x_2	0	25.9	11.408	14.051	11.615	11.718	11.249
x_3	0	−10.35	−9.311	−11.200	−11.322	−11.721	−11.816

n	7	8	9	10	11	12
x_1	9.989	10.022	10.002	10.005	10.001	10.001
x_2	11.187	11.082	11.052	11.026	11.015	11.008
x_3	−11.912	−11.948	−11.973	−11.985	−11.992	−11.996

After 12 iterations, the Gauss-Seidel method has converged to within 0.01 of the exact solution $x_1 = 10, x_2 = 11, x_3 = -12$.

21.

n	13	14	15	16
x_1	10.0004	10.0003	10.0001	10.0001
x_2	11.0043	11.0023	11.0014	11.0007
x_3	−11.9976	−11.9986	−11.9993	−11.9996

23. The Gauss-Seidel method produces

n	0	1	2	3	4	5	6	7	8	9
x_1	0	0	12.5	21.875	24.219	24.805	24.951	24.988	24.997	24.999
x_2	0	0	18.75	21.438	24.609	24.902	24.976	24.994	24.998	24.999
x_3	0	50	68.75	73.438	74.609	74.902	74.976	74.994	74.998	74.999
x_4	0	62.5	71.875	74.219	74.805	74.951	74.988	74.997	74.999	75.000

The exact solution is $x_1 = 25, x_2 = 25, x_3 = 75, x_4 = 75$.

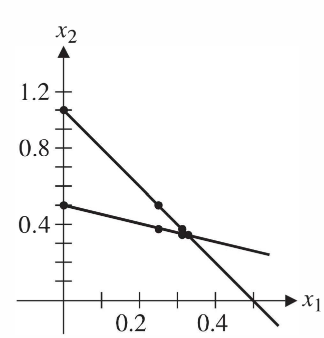
25. The Gauss-Seidel method produces the following iterates:

n	0	1	2	3	4	5	6
t_1	0	20	21.25	22.8125	23.3301	23.6596	23.7732
t_2	0	5	11.25	13.3203	14.6386	15.0926	15.2732
t_3	0	21.25	24.6094	26.9873	27.7303	27.9626	28.0352
t_4	0	2.5	5.8594	8.2373	8.9804	9.2126	9.2852
t_5	0	7.1875	14.6289	16.2829	16.7578	16.9036	16.9491
t_6	0	23.0469	24.9072	25.3207	25.4394	25.4759	25.4873

n	7	8	9	10	11	12
t_1	23.8093	23.8206	23.8242	23.8252	23.8256	23.8257
t_2	15.2824	15.2966	15.3010	15.3024	15.3029	15.3029
t_3	28.0579	28.0650	28.0671	28.0678	28.0681	28.0681
t_4	9.3079	9.3150	9.3172	9.3178	9.3181	9.3181
t_5	16.9633	16.9677	16.9690	16.9695	16.9696	16.9696
t_6	25.4908	25.4919	25.4922	25.4924	25.4924	25.4924

27. (a)

n	0	1	2	3	4	5	6
x_1	0	0	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{5}{16}$	$\frac{5}{16}$	$\frac{21}{64}$
x_2	1	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{11}{32}$	$\frac{11}{32}$



(b) $2x_1 + x_2 = 1$
$x_1 + 2x_2 = 1$

(c)

n	0	1	2	3	4	5	6	7
x_1	0	0	0.25	0.3125	0.3281	0.3320	0.3330	0.3332
x_2	1	0.5	0.375	0.3438	0.3360	0.3340	0.3335	0.3334

[Columns 1, 2, and 3 of this table are the *odd-numbered* columns 1, 3, and 5 from the table in part (a).] The iterates are converging to $x_1 = x_2 = 0.3333$.

(d) $x_1 = x_2 = \frac{1}{3}$

Review Questions

1. (a) F **(c)** F **(e)** T **(g)** T **(i)** F

3. $\begin{bmatrix} 0 \\ 2 \\ -1 \end{bmatrix}$ **5.** $\begin{bmatrix} 6 \\ 2 \end{bmatrix}$ **7.** $k = -1$ **9.** (0, 3, 1)

11. $x - 2y + z = 0$ **13. (a)** Yes **15.** 1 or 2

17. If $c_1(\mathbf{u} + \mathbf{v}) + c_2(\mathbf{u} - \mathbf{v}) = \mathbf{0}$, then $(c_1 + c_2)\mathbf{u} + (c_1 - c_2)\mathbf{v} = \mathbf{0}$. Linear independence of **u** and **v** implies $c_1 + c_2 = 0$ and $c_1 - c_2 = 0$. Solving this system, we get $c_1 = c_2 = 0$. Hence $\mathbf{u} + \mathbf{v}$ and $\mathbf{u} - \mathbf{v}$ are linearly independent.

19. Their ranks must be equal.

Chapter 3

Exercises 3.1

1. $\begin{bmatrix} 3 & -6 \\ -5 & 7 \end{bmatrix}$ **3.** Not possible

5. $\begin{bmatrix} 12 & -6 & 3 \\ -4 & 12 & 14 \end{bmatrix}$ **7.** $\begin{bmatrix} 3 & 3 \\ 19 & 27 \end{bmatrix}$

9. [10] **11.** $\begin{bmatrix} -4 & -2 \\ 8 & 4 \end{bmatrix}$

13. $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ **15.** $\begin{bmatrix} 27 & 0 \\ -49 & 125 \end{bmatrix}$

17. $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$

19. $B = \begin{bmatrix} 1.50 & 1.00 & 2.00 \\ 1.75 & 1.50 & 1.00 \end{bmatrix}$, $BA = \begin{bmatrix} 650.00 & 462.50 \\ 675.00 & 406.25 \end{bmatrix}$

Column i corresponds to warehouse i, row 1 contains the costs of shipping by truck, and row 2 contains the costs of shipping by train.

21. $\begin{bmatrix} 1 & -2 & 3 \\ 2 & 1 & -5 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 4 \end{bmatrix}$

23. $AB = [2\mathbf{a}_1 + \mathbf{a}_2 - \mathbf{a}_3 \quad 3\mathbf{a}_1 - \mathbf{a}_2 + 6\mathbf{a}_3 \quad \mathbf{a}_2 + 4\mathbf{a}_3]$ (where $\mathbf{a}_i$ is the ith column of A)

25. $\begin{bmatrix} 2 & 3 & 0 \\ -6 & -9 & 0 \\ 4 & 6 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 1 & -1 & 1 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 2 & -12 & -8 \\ -1 & 6 & 4 \\ 1 & -6 & -4 \end{bmatrix}$

27. $BA = \begin{bmatrix} 2\mathbf{A}_1 + 3\mathbf{A}_2 \\ \mathbf{A}_1 - \mathbf{A}_2 + \mathbf{A}_3 \\ -\mathbf{A}_1 + 6\mathbf{A}_2 + 4\mathbf{A}_3 \end{bmatrix}$ (where $\mathbf{A}_i$ is the ith row of A)

29. If $\mathbf{b}_i$ is the ith column of B, then $A\mathbf{b}_i$ is the ith column of AB. If the columns of B are linearly dependent, then there are scalars $c_1, \ldots, c_n$ (not all zero) such that $c_1\mathbf{b}_1 + \cdots + c_n\mathbf{b}_n = \mathbf{0}$. But then $c_1(A\mathbf{b}_1) + \cdots + c_n(A\mathbf{b}_n) = A(c_1\mathbf{b}_1 + \cdots + c_n\mathbf{b}_n) = A\mathbf{0} = \mathbf{0}$, so the columns of AB are linearly dependent.

31. $\left[\begin{array}{cc|c} 3 & 2 & 0 \\ -1 & 1 & 0 \\ \hline 0 & 0 & 5 \end{array}\right]$ **33.** $\left[\begin{array}{cc|cc} 1 & 2 & 2 & 0 \\ 3 & 4 & 5 & 3 \\ \hline 1 & 0 & 1 & 2 \\ 0 & 1 & 0 & -1 \end{array}\right]$

35. (a) $A^2 = \begin{bmatrix} -1 & 1 \\ -1 & 0 \end{bmatrix}$, $A^3 = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$,

$A^4 = \begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix}$, $A^5 = \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix}$, $A^6 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$,

$A^7 = \begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix}$

(b) $A^{2001} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$

37. $A^n = \begin{bmatrix} 1 & n \\ 0 & 1 \end{bmatrix}$

39. (a) $\begin{bmatrix} 1 & -1 & 1 & -1 \\ -1 & 1 & -1 & 1 \\ 1 & -1 & 1 & -1 \\ -1 & 1 & -1 & 1 \end{bmatrix}$ **(c)** $\begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 2 & 4 & 8 & 16 \\ 3 & 9 & 27 & 81 \end{bmatrix}$

Exercises 3.2

1. $X = \begin{bmatrix} 5 & 4 \\ 3 & 5 \end{bmatrix}$ **3.** $X = \begin{bmatrix} -\frac{2}{3} & \frac{4}{3} \\ \frac{10}{3} & 4 \end{bmatrix}$

5. $B = 2A_1 + A_2$ **7.** Not possible

9. $\text{span}(A_1, A_2) = \left\{ \begin{bmatrix} c_1 & 2c_1 + c_2 \\ -c_1 + 2c_2 & c_1 + c_2 \end{bmatrix} \right\} = \left\{ \begin{bmatrix} w & x \\ 2x - 5w & x - w \end{bmatrix} \right\}$

11. $\text{span}(A_1, A_2, A_3) = \left\{ \begin{bmatrix} c_1 - c_2 + c_3 & 2c_2 + c_3 & -c_1 + c_3 \\ 0 & c_1 + c_2 & 0 \end{bmatrix} \right\} = \left\{ \begin{bmatrix} -3b + 4c + 5e & b & c \\ 0 & e & 0 \end{bmatrix} \right\}$

13. Linearly independent **15.** Linearly independent

23. $a = d, c = 0$ **25.** $3b = 2c, a = d - c$

27. $a = d, b = c = 0$

29. Let $A = [a_{ij}]$ and $B = [b_{ij}]$ be upper triangular $n \times n$ matrices and let $i > j$. Then, by the definition of an upper triangular matrix,

$$a_{i1} = a_{i2} = \cdots = a_{i,i-1} = 0 \quad \text{and}$$
$$b_{ij} = b_{i+1,j} = \cdots = b_{nj} = 0$$

Now let $C = AB$. Then

$$\begin{aligned} c_{ij} &= a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots a_{i,i-1}b_{i-1,j} + a_{ii}b_{ij} \\ &\quad + a_{i,i+1}b_{i+1,j} + \cdots + a_{in}b_{nj} \\ &= 0 \cdot b_{1j} + 0 \cdot b_{2j} + \cdots 0 \cdot b_{i-1,j} + a_{ii} \cdot 0 \\ &\quad + a_{i,i+1} \cdot 0 + \cdots + a_{in} \cdot 0 = 0 \end{aligned}$$

from which it follows that C is upper triangular.

35. (a) A, B symmetric $\Rightarrow (A + B)^T = A^T + B^T = A + B \Rightarrow A + B$ is symmetric

37. Matrices (b) and (c) are skew-symmetric.

41. Either A or B (or both) must be the zero matrix.

43. (b) $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} = \begin{bmatrix} 1 & 3 & 5 \\ 3 & 5 & 7 \\ 5 & 7 & 9 \end{bmatrix} + \begin{bmatrix} 0 & -1 & -2 \\ 1 & 0 & -1 \\ 2 & 1 & 0 \end{bmatrix}$

47. *Hint:* Use the trace.

Exercises 3.3

1. $\begin{bmatrix} 2 & -7 \\ -1 & 4 \end{bmatrix}$ **3.** Not invertible

5. Not invertible **7.** $\begin{bmatrix} -1.6 & -2.8 \\ 0.\overline{3} & 1 \end{bmatrix}$

9. $\begin{bmatrix} a/(a^2 + b^2) & b/(a^2 + b^2) \\ -b/(a^2 + b^2) & a/(a^2 + b^2) \end{bmatrix}$ **11.** $\begin{bmatrix} -5 \\ 9 \end{bmatrix}$

13. (a) $\mathbf{x}_1 = \begin{bmatrix} 4 \\ -\frac{1}{2} \end{bmatrix}, \mathbf{x}_2 = \begin{bmatrix} -5 \\ 2 \end{bmatrix}, \mathbf{x}_3 = \begin{bmatrix} 6 \\ -2 \end{bmatrix}$

(c) The method in part (b) uses fewer multiplications.

17. (b) $(AB)^{-1} = A^{-1}B^{-1}$ if and only if $AB = BA$

21. $X = A^{-1}(BA)^2B^{-1}$

23. $X = (AB)^{-1}BA + A$

25. $E = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$ **27.** $E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$

29. $E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$ **31.** $\begin{bmatrix} \frac{1}{3} & 0 \\ 0 & 1 \end{bmatrix}$

33. $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ **35.** $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$ **37.** $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/c & 0 \\ 0 & 0 & 1 \end{bmatrix}$

39. $A = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix}, A^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & -\frac{1}{2} \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$

43. (a) If A is invertible, then $BA = CA \Rightarrow (BA)A^{-1} = (CA)A^{-1} \Rightarrow B(AA^{-1}) = C(AA^{-1}) \Rightarrow BI = CI \Rightarrow B = C$.

45. *Hint:* Rewrite $A^2 - 2A + I = O$ as $A(2I - A) = I$.

47. If AB is invertible, then there exists a matrix X such that $(AB)X = I$. But then $A(BX) = I$ too, so A is invertible (with inverse BX).

49. $\begin{bmatrix} \frac{1}{10} & \frac{2}{5} \\ \frac{3}{10} & \frac{1}{5} \end{bmatrix}$ **51.** $\begin{bmatrix} 1/(a^2 + 1) & -a/(a^2 + 1) \\ a/(a^2 + 1) & 1/(a^2 + 1) \end{bmatrix}$

53. Not invertible

55. $\begin{bmatrix} 1/a & 0 & 0 \\ -1/a^2 & 1/a & 0 \\ 1/a^3 & -1/a^2 & 1/a \end{bmatrix}, a \neq 0$

57. $\begin{bmatrix} -11 & -2 & 5 & -4 \\ 4 & 1 & -2 & 2 \\ 5 & 1 & -2 & 2 \\ 9 & 2 & -4 & 3 \end{bmatrix}$

59. $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -a/d & -b/d & -c/d & 1/d \end{bmatrix}, d \neq 0$

61. Not invertible **63.** $\begin{bmatrix} 4 & 6 & 4 \\ 5 & 3 & 2 \\ 0 & 6 & 5 \end{bmatrix}$

69. $\left[\begin{array}{cc|cc} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \hline -2 & -3 & 1 & 0 \\ -1 & -2 & 0 & 1 \end{array}\right]$ **71.** $\left[\begin{array}{cc|cc} -1 & 0 & 1 & 1 \\ 0 & 1 & -1 & 0 \\ \hline 0 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \end{array}\right]$

Exercises 3.4

1. $\begin{bmatrix} -2 \\ 1 \end{bmatrix}$ **3.** $\begin{bmatrix} -3/2 \\ -2 \\ -1 \end{bmatrix}$ **5.** $\begin{bmatrix} -7 \\ -15 \\ -2 \\ 2 \end{bmatrix}$

7. $\begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 0 & 5 \end{bmatrix}$

9. $\begin{bmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 8 & 3 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \\ 0 & 0 & 3 \end{bmatrix}$

11. $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 0 & 3 & 1 & 0 \\ -1 & 0 & -2 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 & 3 & -1 \\ 0 & 2 & -3 & 2 \\ 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

13. $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 1 & -2 \\ 0 & 3 & 3 & 1 \\ 0 & 0 & 0 & 5 \end{bmatrix}$

15. $L^{-1} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$, $U^{-1} = \begin{bmatrix} -\frac{1}{2} & \frac{1}{12} \\ 0 & \frac{1}{6} \end{bmatrix}$,

$A^{-1} = \begin{bmatrix} -5/12 & 1/12 \\ 1/6 & 1/6 \end{bmatrix}$

19. $\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

21. $\begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

23. $\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 5 & 1 \end{bmatrix}\begin{bmatrix} -1 & 2 & 1 \\ 0 & 1 & 4 \\ 0 & 0 & -16 \end{bmatrix}$

25. $\begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix}\begin{bmatrix} -1 & 1 & 1 & 2 \\ 0 & -1 & 1 & 3 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 4 \end{bmatrix}$

27. $\begin{bmatrix} 4 \\ -1 \\ -2 \end{bmatrix}$ **31.** $\begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} -2 & 0 \\ 0 & 6 \end{bmatrix}\begin{bmatrix} 1 & -\frac{1}{2} \\ 0 & 1 \end{bmatrix}$

Exercises 3.5

1. Subspace **3.** Subspace

5. Subspace **7.** Not a subspace

11. $\mathbf{b}$ is in col(A), $\mathbf{w}$ is not in row(A).

15. No

17. $\{[1 \;\; 0 \;\; -1], [0 \;\; 1 \;\; 2]\}$ is a basis for row(A); $\left\{\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right\}$ is a basis for col(A); $\left\{\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}\right\}$ is a basis for null(A).

19. $\{[1 \;\; 0 \;\; 1 \;\; 0], [0 \;\; 1 \;\; -1 \;\; 0], [0 \;\; 0 \;\; 0 \;\; 1]\}$ is a basis for row(A); $\left\{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}\right\}$ is a basis for col(A); $\left\{\begin{bmatrix} -1 \\ 1 \\ 1 \\ 0 \end{bmatrix}\right\}$ is a basis for null(A).

21. $\{[1 \;\; 0 \;\; -1], [1 \;\; 1 \;\; 1]\}$ is a basis for row(A); $\left\{\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right\}$ is a basis for col(A)

23. $\{[1 \;\; 1 \;\; 0 \;\; 1], [0 \;\; 1 \;\; -1 \;\; 1], [0 \;\; 1 \;\; -1 \;\; -1]\}$ is a basis for row(A); $\left\{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right\}$ is a basis for col(A)

25. Both $\{[1 \;\; 0 \;\; -1], [0 \;\; 1 \;\; 2]\}$ and $\{[1 \;\; 0 \;\; -1], [1 \;\; 1 \;\; 1]\}$ are linearly independent spanning sets for row(A) $= \{[a \;\; b \;\; -a + 2b]\}$. Both $\left\{\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right\}$ and $\left\{\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right\}$ are linearly independent spanning sets for col(A) $= \mathbb{R}^2$.

27. $\left\{\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}\right\}$

29. $\{[1 \;\; 0 \;\; 0], [0 \;\; 1 \;\; 0], [0 \;\; 0 \;\; 1]\}$

31. $\{[2 \;\; -3 \;\; 1], [1 \;\; -1 \;\; 0], [4 \;\; -4 \;\; 1]\}$

35. rank(A) $= 2$, nullity(A) $= 1$

37. rank(A) $= 3$, nullity(A) $= 1$

39. If A is 3×5, then rank(A) ≤ 3, so there cannot be more than three linearly independent columns.

41. nullity(A) $= 2, 3, 4,$ or 5

43. If $a = -1$, then $\operatorname{rank}(A) = 1$; if $a = 2$, then $\operatorname{rank}(A) = 2$; for $a \neq -1, 2$, $\operatorname{rank}(A) = 3$.

45. Yes **47.** Yes **49.** No

51. $\mathbf{w}$ is in $\operatorname{span}(\mathcal{B})$ if and only if the linear system with augmented matrix $[\mathcal{B} \mid \mathbf{w}]$ is consistent, which is true in this case, since

$$[\mathcal{B} \mid \mathbf{w}] = \left[\begin{array}{rr|r} 1 & 1 & 1 \\ 2 & 0 & 6 \\ 0 & -1 & 2 \end{array}\right] \longrightarrow \left[\begin{array}{rr|r} 1 & 0 & 3 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{array}\right]$$

From this reduced row echelon form, it is also clear that $[\mathbf{w}]_{\mathcal{B}} = \begin{bmatrix} 3 \\ -2 \end{bmatrix}$.

53. $\operatorname{rank}(A) = 2$, $\operatorname{nullity}(A) = 1$

55. $\operatorname{rank}(A) = 3$, $\operatorname{nullity}(A) = 1$

57. Let $\mathbf{A}_1, \ldots, \mathbf{A}_m$ be the row vectors of A so that $\operatorname{row}(A) = \operatorname{span}(\mathbf{A}_1, \ldots, \mathbf{A}_m)$. If $\mathbf{x}$ is in $\operatorname{null}(A)$, then, since $A\mathbf{x} = \mathbf{0}$, we also have $\mathbf{A}_i \cdot \mathbf{x} = 0$ for $i = 1, \ldots, m$, by the row-column definition of matrix multiplication. If $\mathbf{r}$ is in $\operatorname{row}(A)$, then $\mathbf{r}$ is of the form $\mathbf{r} = c_1\mathbf{A}_1 + \cdots + c_m\mathbf{A}_m$. Therefore,

$$\begin{aligned} \mathbf{r} \cdot \mathbf{x} &= (c_1\mathbf{A}_1 + \cdots + c_m\mathbf{A}_m) \cdot \mathbf{x} \\ &= c_1(\mathbf{A}_1 \cdot \mathbf{x}) + \cdots + c_m(\mathbf{A}_m \cdot \mathbf{x}) = 0 \end{aligned}$$

59. (a) If a set of columns of AB is linearly independent, then the corresponding columns of B are linearly independent (by an argument similar to that needed to prove Exercise 29 in Section 3.1). It follows that the *maximum* number k of linearly independent columns of AB [i.e., $k = \operatorname{rank}(AB)$] is not more than the *maximum* number r of linearly independent columns of B [i.e., $r = \operatorname{rank}(B)$]. In other words, $\operatorname{rank}(AB) \leq \operatorname{rank}(B)$.

61. (a) From Exercise 59(a), $\operatorname{rank}(UA) \leq \operatorname{rank}(A)$ and $\operatorname{rank}(A) = \operatorname{rank}((U^{-1}U)A) = \operatorname{rank}(U^{-1}(UA)) \leq \operatorname{rank}(UA)$. Hence, $\operatorname{rank}(UA) = \operatorname{rank}(A)$.

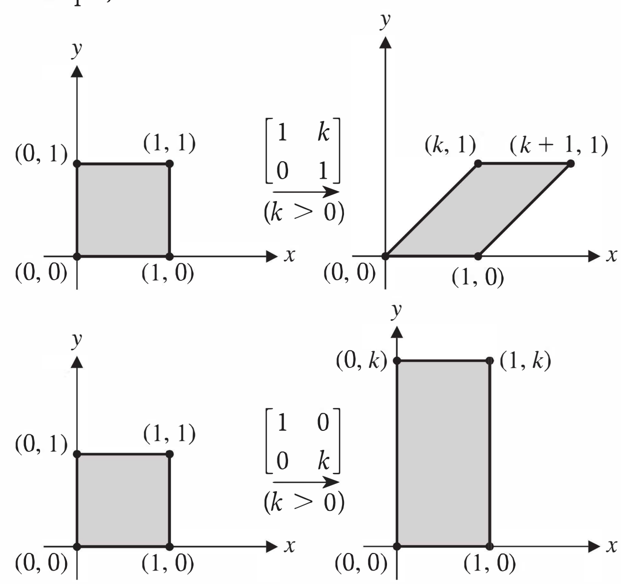
Exercises 3.6

1. $T(\mathbf{u}) = \begin{bmatrix} 0 \\ 11 \end{bmatrix}$, $T(\mathbf{v}) = \begin{bmatrix} 8 \\ 1 \end{bmatrix}$

11. $\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$ **13.** $\begin{bmatrix} 1 & -1 & 1 \\ 2 & 1 & -3 \end{bmatrix}$

15. $[F] = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$ **17.** $[D] = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$

19. $\begin{bmatrix} k & 0 \\ 0 & 1 \end{bmatrix}$ stretches or contracts in the x-direction (combined with a reflection in the y-axis if $k < 0$); $\begin{bmatrix} 1 & 0 \\ 0 & k \end{bmatrix}$ stretches or contracts in the y-direction (combined with a reflection in the x-axis if $k < 0$); $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ is a reflection in the line $y = x$; $\begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}$ is a *shear* in the x-direction; $\begin{bmatrix} 1 & 0 \\ k & 1 \end{bmatrix}$ is a *shear* in the y-direction. For example,



21. $\begin{bmatrix} \sqrt{3}/2 & 1/2 \\ -1/2 & \sqrt{3}/2 \end{bmatrix}$ **23.** $\begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix}$

25. $\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$ **27.** $\begin{bmatrix} -\frac{3}{5} & \frac{4}{5} \\ \frac{4}{5} & \frac{3}{5} \end{bmatrix}$

31. $[S \circ T] = \begin{bmatrix} -8 & 5 \\ 4 & 1 \end{bmatrix}$

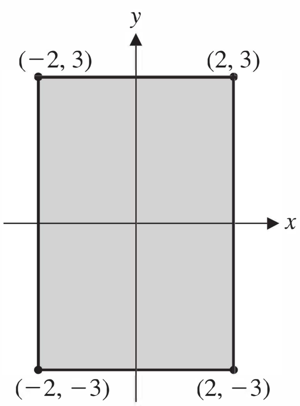
33. $[S \circ T] = \begin{bmatrix} 0 & 6 & -6 \\ 1 & -2 & 2 \end{bmatrix}$

35. $[S \circ T] = \begin{bmatrix} 1 & 0 & -1 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$

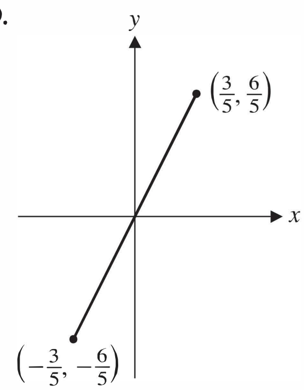
37. $\begin{bmatrix} -\sqrt{3}/2 & 1/2 \\ 1/2 & \sqrt{3}/2 \end{bmatrix}$ **39.** $\begin{bmatrix} -\sqrt{3}/2 & -1/2 \\ 1/2 & -\sqrt{3}/2 \end{bmatrix}$

45. In vector form, let the parallel lines be given by $\mathbf{x} = \mathbf{p} + t\mathbf{d}$ and $\mathbf{x}' = \mathbf{p}' + t\mathbf{d}$. Their images are $T(\mathbf{x}) = T(\mathbf{p} + t\mathbf{d}) = T(\mathbf{p}) + tT(\mathbf{d})$ and $T(\mathbf{x}') = T(\mathbf{p}' + t\mathbf{d}) = T(\mathbf{p}') + tT(\mathbf{d})$. Suppose $T(\mathbf{d}) \neq \mathbf{0}$. If $T(\mathbf{p}') - T(\mathbf{p})$ is parallel to $T(\mathbf{d})$, then the images represent the same line; otherwise the images represent distinct parallel lines. On the other hand, if $T(\mathbf{d}) = \mathbf{0}$, then the images represent two distinct points if $T(\mathbf{p}') \neq T(\mathbf{p})$ and single point otherwise.

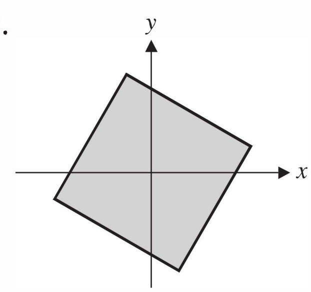
47.



49.



51.



Exercises 3.7

1. $\mathbf{x}_1 = \begin{bmatrix} 0.4 \\ 0.6 \end{bmatrix}, \mathbf{x}_2 = \begin{bmatrix} 0.38 \\ 0.62 \end{bmatrix}$

3. 64%

5. $\mathbf{x}_1 = \begin{bmatrix} 150 \\ 120 \\ 120 \end{bmatrix}, \mathbf{x}_2 = \begin{bmatrix} 155 \\ 120 \\ 115 \end{bmatrix}$

7. $\frac{5}{18}$

9. (a) $P = \begin{bmatrix} 0.662 & 0.250 \\ 0.338 & 0.750 \end{bmatrix}$

(b) 0.353

(c) 42.5% wet, 57.5% dry

11. (a) $P = \begin{bmatrix} 0.08 & 0.09 & 0.11 \\ 0.07 & 0.11 & 0.05 \\ 0.85 & 0.80 & 0.84 \end{bmatrix}$

(b) 0.08, 0.1062, 0.1057, 0.1057, 0.1057

(c) 10.6% good, 5.5% fair, 83.9% poor

13. The entries of the vector $\mathbf{j}P$ are just the column sums of the matrix P. So P is stochastic if and only if $\mathbf{j}P = \mathbf{j}$.

15. 4

17. 9.375

19. Yes, $\mathbf{x} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

21. No

23. No

25. Yes, $\mathbf{x} = \begin{bmatrix} 10 \\ 27 \\ 35 \end{bmatrix}$

27. Productive

29. Not productive

31. $\mathbf{x} = \begin{bmatrix} 10 \\ 16 \end{bmatrix}$

33. Yes, $\mathbf{x} = \begin{bmatrix} 10 \\ 6 \\ 8 \end{bmatrix}$

37. $\mathbf{x}_1 = \begin{bmatrix} 45 \\ 6 \end{bmatrix}, \mathbf{x}_2 = \begin{bmatrix} 120 \\ 27 \end{bmatrix}, \mathbf{x}_3 = \begin{bmatrix} 375 \\ 72 \end{bmatrix}$

39. $\mathbf{x}_1 = \begin{bmatrix} 500 \\ 70 \\ 50 \end{bmatrix}, \mathbf{x}_2 = \begin{bmatrix} 720 \\ 350 \\ 35 \end{bmatrix}, \mathbf{x}_3 = \begin{bmatrix} 1175 \\ 504 \\ 175 \end{bmatrix}$

41. (a) For L_1, we have $\mathbf{x}_1 = \begin{bmatrix} 50 \\ 8 \end{bmatrix}, \mathbf{x}_2 = \begin{bmatrix} 40 \\ 40 \end{bmatrix},$

$\mathbf{x}_3 = \begin{bmatrix} 200 \\ 32 \end{bmatrix}, \mathbf{x}_4 = \begin{bmatrix} 160 \\ 160 \end{bmatrix}, \mathbf{x}_5 = \begin{bmatrix} 800 \\ 128 \end{bmatrix}, \mathbf{x}_6 = \begin{bmatrix} 640 \\ 640 \end{bmatrix},$

$\mathbf{x}_7 = \begin{bmatrix} 3200 \\ 512 \end{bmatrix}, \mathbf{x}_8 = \begin{bmatrix} 2560 \\ 2560 \end{bmatrix}, \mathbf{x}_9 = \begin{bmatrix} 12800 \\ 2048 \end{bmatrix},$

$\mathbf{x}_{10} = \begin{bmatrix} 10240 \\ 10240 \end{bmatrix}.$

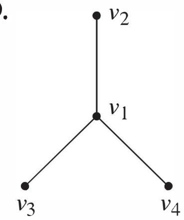
(b) The first population oscillates between two states,while the second approaches a steady state.

43. The population oscillates through a cycle of three states (for the relative population): If $0.1 < s \leq 1$, the actual population is growing; if $s = 0.1$, the actual population goes through a cycle of length 3; and if $0 \leq s < 0.1$, the actual population is declining (and will eventually die out).

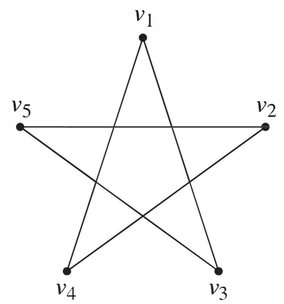
45. $A = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}$

47. $A = \begin{bmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 \end{bmatrix}$

49.



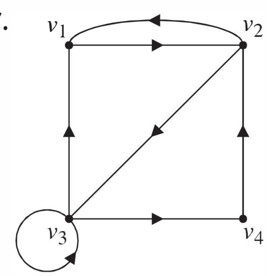
51.



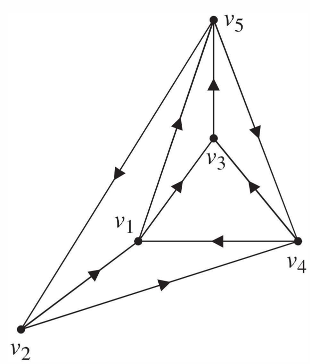
53. $A = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}$

55. $A = \begin{bmatrix} 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}$

57.



59.

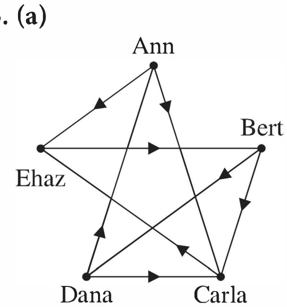


61. 2 **63.** 3 **65.** 0 **67.** 3

69. (a) Vertex i is not adjacent to any other vertices.

71. If we use direct wins only, P_2 is in first place; P_3, P_4, and P_6 tie for second place; and P_1 and P_5 tie for third place. If we combine direct and indirect wins, the players rank as follows: P_2 in first place, followed by P_6, P_4, P_3, P_5, and P_1.

73. (a)



$A = \begin{bmatrix} 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{bmatrix}$

(b) two steps; all of the off-diagonal entries of the second row of $A + A^2$ are nonzero.

(d) If the graph has n vertices, check the (i, j) entry of the powers A^k for $k = 1, \ldots, n - 1$. Vertex i is connected to vertex j by a path of length k if and only if $(A^k)_{ij} \neq 0$.

75. $(AA^T)_{ij}$ counts the number of vertices adjacent to *both* vertex i and vertex j.

77. Bipartite **79.** Bipartite

Review Questions

1. (a) T **(c)** F **(e)** T **(g)** T **(i)** T

3. Impossible

5. $\begin{bmatrix} \frac{17}{83} & -\frac{1}{83} \\ -\frac{1}{83} & \frac{5}{166} \end{bmatrix}$

7. $\begin{bmatrix} 1 & 3 \\ 3 & 9 \end{bmatrix} + \begin{bmatrix} 4 & 10 \\ 10 & 25 \end{bmatrix}$

9. $\begin{bmatrix} 0 & -9 \\ 2 & 4 \\ 1 & -6 \end{bmatrix}$

11. Because $(I - A)(I + A + A^2) = I - A^3 = I - O = I$, $(I - A)^{-1} = I + A + A^2$.

13. A basis for row(A) is $\{[1, -2, 0, -1, 0], [0, 0, 1, 2, 0], [0, 0, 0, 0, 1]\}$; a basis for col($A$) is $\left\{ \begin{bmatrix} 2 \\ 1 \\ 4 \end{bmatrix}, \begin{bmatrix} 5 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 5 \\ 1 \\ 6 \end{bmatrix} \right\}$ (or the standard basis for $\mathbf{R}^3$); and a basis for null(A) is

$$\left\{ \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -2 \\ 1 \\ 0 \end{bmatrix} \right\}.$$

15. An invertible matrix has a trivial (zero) null space. If A is invertible, then so is A^T, and so both A and A^T have trivial null spaces. If A is not invertible, then A and A^T need not have the same null space. For example, take $A = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$.

17. Because A has n linearly independent columns, rank(A) $= n$. Hence rank(A^TA) $= n$ by Theorem 3.28. Because A^TA is $n \times n$, this implies that A^TA is invertible, by the Fundamental Theorem of Invertible Matrices. AA^T need not be invertible. For example, take $A = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

19. $\begin{bmatrix} -1/5\sqrt{2} & -3/5\sqrt{2} \\ 2/5\sqrt{2} & 6/5\sqrt{2} \end{bmatrix}$

Chapter 4

Exercises 4.1

1. $A\mathbf{v} = \begin{bmatrix} 3 \\ 3 \end{bmatrix} = 3\mathbf{v}, \lambda = 3$

3. $A\mathbf{v} = \begin{bmatrix} -3 \\ 6 \end{bmatrix} = -3\mathbf{v}, \lambda = -3$

5. $A\mathbf{v} = \begin{bmatrix} 6 \\ -3 \\ 3 \end{bmatrix} = 3\mathbf{v}, \lambda = 3$

7. $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ **9.** $\begin{bmatrix} 4 \\ 1 \end{bmatrix}$ **11.** $\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$

13. $\lambda = 1, E_1 = \text{span}\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right); \lambda = -1, E_{-1} = \text{span}\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right)$

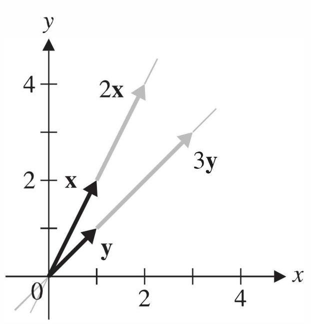
15. $\lambda = 0, E_0 = \text{span}\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right); \lambda = 1, E_1 = \text{span}\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right)$

17. $\lambda = 2, E_2 = \text{span}\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right); \lambda = 3, E_3 = \text{span}\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right)$

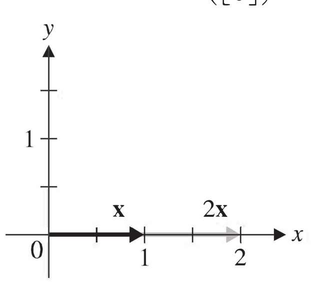
19. $\mathbf{v} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \lambda = 1; \mathbf{v} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \lambda = 2$

21. $\mathbf{v} = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}, \lambda = 2; \mathbf{v} = \begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}, \lambda = 0$

23. $\lambda = 2, E_2 = \text{span}\left(\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right); \lambda = 3, E_3 = \text{span}\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right)$



25. $\lambda = 2, E_2 = \text{span}\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right)$



27. $\lambda = 1 + i, E_{1+i} = \text{span}\left(\begin{bmatrix} 1 \\ i \end{bmatrix}\right); \lambda = 1 - i, E_{1-i} = \text{span}\left(\begin{bmatrix} 1 \\ -i \end{bmatrix}\right)$

29. $\lambda = 1 + i, E_{1+i} = \text{span}\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right); \lambda = 1 - i, E_{1-i} = \text{span}\left(\begin{bmatrix} 1 \\ -1 \end{bmatrix}\right)$

31. $\lambda = 1, 2$ **33.** $\lambda = 4$

Exercises 4.2

1. 16 **3.** 0 **5.** -18 **7.** 6
9. -12 **11.** $a^2b + ab^2$ **13.** 4 **15.** $abdg$
17. 0 **25.** 2 **27.** -24 **29.** 0
31. 0 **33.** -24 **35.** 8 **37.** -4
39. -8 **45.** $k \neq 0, 2$ **47.** -6 **49.** $-\frac{3}{2}$
51. $(-2)3^n$
53. $\det(AB) = (\det A)(\det B) = (\det B)(\det A) = \det(BA)$
55. 0, 1 **57.** $x = \frac{3}{2}, y = -\frac{1}{2}$

59. $x = -1, y = 0, z = 1$ **61.** $\begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{bmatrix}$

63. $\begin{bmatrix} \frac{1}{2} & -\frac{1}{2} & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$

Exercises 4.3

1. (a) $\lambda^2 - 7\lambda + 12$ **(b)** $\lambda = 3, 4$

(c) $E_3 = \text{span}\left(\begin{bmatrix} 3 \\ 2 \end{bmatrix}\right); E_4 = \text{span}\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right)$

(d) The algebraic and geometric multiplicities are all 1.

3. (a) $-\lambda^3 + 2\lambda^2 + 5\lambda - 6$

(b) $\lambda = -2, 1, 3$

(c) $E_{-2} = \text{span}\left(\begin{bmatrix} 1 \\ -3 \\ 0 \end{bmatrix}\right); E_1 = \text{span}\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right);$

$E_3 = \text{span}\left(\begin{bmatrix} 1 \\ 2 \\ 10 \end{bmatrix}\right)$

(d) The algebraic and geometric multiplicities are all 1.

5. (a) $-\lambda^3 + \lambda^2$ **(b)** $\lambda = 0, 1$

(c) $E_0 = \text{span}\left(\begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}\right); E_1 = \text{span}\left(\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}\right)$

(d) $\lambda = 0$ has algebraic multiplicity 2 and geometric multiplicity 1; $\lambda = 1$ has algebraic and geometric multiplicity 1.

7. (a) $-\lambda^3 + 9\lambda^2 - 27\lambda + 27$

(b) $\lambda = 3$

(c) $E_3 = \text{span}\left(\begin{bmatrix}-1\\0\\1\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}\right)$

(d) $\lambda = 3$ has algebraic multiplicity 3 and geometric multiplicity 2.

9. (a) $\lambda^4 - 6\lambda^3 + 9\lambda^2 + 4\lambda - 12$

(b) $\lambda = -1, 2, 3$

(c) $E_{-1} = \text{span}\left(\begin{bmatrix}0\\0\\-2\\1\end{bmatrix}\right)$; $E_2 = \text{span}\left(\begin{bmatrix}1\\-1\\0\\0\end{bmatrix}\right)$;

$E_3 = \text{span}\left(\begin{bmatrix}0\\0\\2\\1\end{bmatrix}\right)$

(d) $\lambda = -1$ and $\lambda = 3$ have algebraic and geometric multiplicity 1; $\lambda = 2$ has algebraic multiplicity 2 and geometric multiplicity 1.

11. (a) $\lambda^4 - 4\lambda^3 + 2\lambda^2 + 4\lambda - 3$

(b) $\lambda = -1, 1, 3$

(c) $E_{-1} = \text{span}\left(\begin{bmatrix}0\\0\\0\\1\end{bmatrix}\right)$;

$E_1 = \text{span}\left(\begin{bmatrix}-2\\0\\1\\3\end{bmatrix}, \begin{bmatrix}-2\\2\\0\\3\end{bmatrix}\right)$;

$E_3 = \text{span}\left(\begin{bmatrix}0\\0\\2\\1\end{bmatrix}\right)$

(d) $\lambda = -1$ and $\lambda = 3$ have algebraic and geometric multiplicity 1; $\lambda = 1$ has algebraic and geometric multiplicity 2.

15. $\begin{bmatrix}2^{-9} + 3 \cdot 2^{10}\\-2^{-9} + 3 \cdot 2^{10}\end{bmatrix}$ **17.** $\begin{bmatrix}2\\(2 \cdot 3^{20} - 1)/3^{20}\\2\end{bmatrix}$

23. (a) $\lambda = -2$, $E_{-2} = \text{span}\left(\begin{bmatrix}2\\-5\end{bmatrix}\right)$; $\lambda = 5$, $E_5 =$ $\text{span}\left(\begin{bmatrix}1\\1\end{bmatrix}\right)$

(b) (i) $\lambda = -\frac{1}{2}$, $E_{-1/2} = \text{span}\left(\begin{bmatrix}2\\-5\end{bmatrix}\right)$; $\lambda = \frac{1}{5}$, $E_{1/5} =$ $\text{span}\left(\begin{bmatrix}1\\1\end{bmatrix}\right)$

(iii) $\lambda = 0$, $E_0 = \text{span}\left(\begin{bmatrix}2\\-5\end{bmatrix}\right)$; $\lambda = 7$, $E_7 = \text{span}\left(\begin{bmatrix}1\\1\end{bmatrix}\right)$

27. $\begin{bmatrix}-3 & 4 & -12\\1 & 0 & 0\\0 & 1 & 0\end{bmatrix}$, $-\lambda^3 - 3\lambda^2 + 4\lambda - 12$

35. $A^2 = 4A - 5I$, $A^3 = 11A - 20I$
$A^4 = 24A - 55I$

37. $A^{-1} = -\frac{1}{5}A + \frac{4}{5}I$, $A^{-2} = -\frac{4}{25}A + \frac{11}{25}I$

Exercises 4.4

1. The characteristic polynomial of A is $\lambda^2 - 5\lambda + 1$, but that of B is $\lambda^2 - 2\lambda + 1$.

3. The eigenvalues of A are $\lambda = 2$ and $\lambda = 4$, but those of B are $\lambda = 1$ and $\lambda = 4$.

5. $\lambda_1 = 4$, $E_4 = \text{span}\left(\begin{bmatrix}1\\1\end{bmatrix}\right)$; $\lambda_2 = 3$, $E_3 = \text{span}\left(\begin{bmatrix}1\\2\end{bmatrix}\right)$

7. $\lambda_1 = 6$, $E_6 = \text{span}\left(\begin{bmatrix}3\\2\\3\end{bmatrix}\right)$; $\lambda_2 = -2$, $E_{-2} =$ $\text{span}\left(\begin{bmatrix}0\\1\\-1\end{bmatrix}, \begin{bmatrix}1\\0\\-1\end{bmatrix}\right)$

9. Not diagonalizable

11. $P = \begin{bmatrix}1 & 1 & -1\\1 & 1 & 1\\1 & -2 & 0\end{bmatrix}$, $D = \begin{bmatrix}2 & 0 & 0\\0 & -1 & 0\\0 & 0 & 1\end{bmatrix}$

13. Not diagonalizable

15. $P = \begin{bmatrix}1 & 0 & 0 & -1\\0 & 1 & 0 & 0\\0 & 0 & 1 & 0\\0 & 0 & 0 & 1\end{bmatrix}$, $D = \begin{bmatrix}2 & 0 & 0 & 0\\0 & 2 & 0 & 0\\0 & 0 & -2 & 0\\0 & 0 & 0 & -2\end{bmatrix}$

17. $\begin{bmatrix}35839 & -69630\\-11605 & 24234\end{bmatrix}$

19. $\begin{bmatrix}(3^k + 3(-1)^k)/4 & (3^{k+1} - 3(-1)^k)/4\\(3^k - (-1)^k)/4 & (3^{k+1} + (-1)^k)/4\end{bmatrix}$

21. $\begin{bmatrix} 1 & 1 & 1 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$

23.

$$\begin{bmatrix} (5 + 2^{k+2} + (-3)^k)/10 & (2^k - (-3)^k)/5 & (-5 + 2^{k+2} + (-3)^k)/10 \\ (2^{k+1} - 2(-3)^k)/5 & (2^k + 4(-3)^k)/5 & (2^{k+1} - 2(-3)^k)/5 \\ (-5 + 2^{k+2} + (-3)^k)/10 & (2^k - (-3)^k)/5 & (5 + 2^{k+2} + (-3)^k)/10 \end{bmatrix}$$

25. $k = 0$ **27.** $k = 0$

29. All real values of k

37. If $A \sim B$, then there is an invertible matrix P such that $B = P^{-1}AP$. Therefore, we have

$$\begin{aligned} \text{tr}(B) &= \text{tr}(P^{-1}AP) = \text{tr}(P^{-1}(AP)) = \text{tr}((AP)P^{-1}) \\ &= \text{tr}(APP^{-1}) = \text{tr}(AI) = \text{tr}(A) \end{aligned}$$

using Exercise 45 in Section 3.2.

39. $P = \begin{bmatrix} 7 & -2 \\ 10 & -3 \end{bmatrix}$

41. $P = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} & 0 \\ -\frac{3}{2} & -\frac{3}{2} & 1 \\ -\frac{5}{2} & -\frac{3}{2} & 0 \end{bmatrix}$

51. (b) $\dim E_{-1} = 1$, $\dim E_1 = 2$, $\dim E_2 = 3$

Exercises 4.5

1. (a) $\begin{bmatrix} 1 \\ 2.5 \end{bmatrix}$, 6.000

(b) $\lambda_1 = 6$

3. (a) $\begin{bmatrix} 1 \\ 0.618 \end{bmatrix}$, 2.618

(b) $\lambda_1 = (3 + \sqrt{5})/2 \approx 2.618$

5. (a) $m_5 = 11.001$, $\mathbf{y}_5 = \begin{bmatrix} -0.333 \\ 1.000 \end{bmatrix}$

7. (a) $m_8 = 10.000$, $\mathbf{y}_8 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$

9.

k	0	1	2	3	4	5
$\mathbf{x}_k$	$\begin{bmatrix} 1 \\ 1 \end{bmatrix}$	$\begin{bmatrix} 26 \\ 8 \end{bmatrix}$	$\begin{bmatrix} 17.692 \\ 5.923 \end{bmatrix}$	$\begin{bmatrix} 18.018 \\ 6.004 \end{bmatrix}$	$\begin{bmatrix} 17.999 \\ 6.000 \end{bmatrix}$	$\begin{bmatrix} 18.000 \\ 6.000 \end{bmatrix}$
$\mathbf{y}_k$	$\begin{bmatrix} 1 \\ 1 \end{bmatrix}$	$\begin{bmatrix} 1 \\ 0.308 \end{bmatrix}$	$\begin{bmatrix} 1 \\ 0.335 \end{bmatrix}$	$\begin{bmatrix} 1 \\ 0.333 \end{bmatrix}$	$\begin{bmatrix} 1 \\ 0.333 \end{bmatrix}$	$\begin{bmatrix} 1 \\ 0.333 \end{bmatrix}$
m_k	1	26	17.692	18.018	17.999	18.000

Therefore, $\lambda_1 \approx 18$, $\mathbf{v}_1 \approx \begin{bmatrix} 1 \\ 0.333 \end{bmatrix}$.

11.

k	0	1	2	3	4	5	6
$\mathbf{x}_k$	$\begin{bmatrix} 1 \\ 0 \end{bmatrix}$	$\begin{bmatrix} 7 \\ 2 \end{bmatrix}$	$\begin{bmatrix} 7.571 \\ 2.857 \end{bmatrix}$	$\begin{bmatrix} 7.755 \\ 3.132 \end{bmatrix}$	$\begin{bmatrix} 7.808 \\ 3.212 \end{bmatrix}$	$\begin{bmatrix} 7.823 \\ 3.234 \end{bmatrix}$	$\begin{bmatrix} 7.827 \\ 3.240 \end{bmatrix}$
$\mathbf{y}_k$	$\begin{bmatrix} 1 \\ 0 \end{bmatrix}$	$\begin{bmatrix} 1 \\ 0.286 \end{bmatrix}$	$\begin{bmatrix} 1 \\ 0.377 \end{bmatrix}$	$\begin{bmatrix} 1 \\ 0.404 \end{bmatrix}$	$\begin{bmatrix} 1 \\ 0.411 \end{bmatrix}$	$\begin{bmatrix} 1 \\ 0.413 \end{bmatrix}$	$\begin{bmatrix} 1 \\ 0.414 \end{bmatrix}$
m_k	1	7	7.571	7.755	7.808	7.823	7.827

Therefore, $\lambda_1 \approx 7.827$, $\mathbf{v}_1 \approx \begin{bmatrix} 1 \\ 0.414 \end{bmatrix}$.

13.

k	0	1	2	3	4	5
$\mathbf{x}_k$	$\begin{bmatrix}1\\1\\1\end{bmatrix}$	$\begin{bmatrix}21\\15\\13\end{bmatrix}$	$\begin{bmatrix}16.809\\12.238\\10.714\end{bmatrix}$	$\begin{bmatrix}17.011\\12.371\\10.824\end{bmatrix}$	$\begin{bmatrix}16.999\\12.363\\10.818\end{bmatrix}$	$\begin{bmatrix}17.000\\12.363\\10.818\end{bmatrix}$
$\mathbf{y}_k$	$\begin{bmatrix}1\\1\\1\end{bmatrix}$	$\begin{bmatrix}1\\0.714\\0.619\end{bmatrix}$	$\begin{bmatrix}1\\0.728\\0.637\end{bmatrix}$	$\begin{bmatrix}1\\0.727\\0.636\end{bmatrix}$	$\begin{bmatrix}1\\0.727\\0.636\end{bmatrix}$	$\begin{bmatrix}1\\0.727\\0.636\end{bmatrix}$
m_k	1	21	16.809	17.011	16.999	17.000

Therefore, $\lambda_1 \approx 17$, $\mathbf{v}_1 \approx \begin{bmatrix}1\\0.727\\0.636\end{bmatrix}$.

15. $\lambda_1 \approx 5$, $\mathbf{v}_1 \approx \begin{bmatrix}1\\0\\0.333\end{bmatrix}$

17.

k	0	1	2	3	4	5	6
$\mathbf{x}_k$	$\begin{bmatrix}1\\0\end{bmatrix}$	$\begin{bmatrix}7\\2\end{bmatrix}$	$\begin{bmatrix}7.571\\2.857\end{bmatrix}$	$\begin{bmatrix}7.755\\3.132\end{bmatrix}$	$\begin{bmatrix}7.808\\3.212\end{bmatrix}$	$\begin{bmatrix}7.823\\3.234\end{bmatrix}$	$\begin{bmatrix}7.827\\3.240\end{bmatrix}$
$R(\mathbf{x}_k)$	7	7.755	7.823	7.828	7.828	7.828	7.828
$\mathbf{y}_k$	$\begin{bmatrix}1\\0\end{bmatrix}$	$\begin{bmatrix}1\\0.286\end{bmatrix}$	$\begin{bmatrix}1\\0.377\end{bmatrix}$	$\begin{bmatrix}1\\0.404\end{bmatrix}$	$\begin{bmatrix}1\\0.411\end{bmatrix}$	$\begin{bmatrix}1\\0.413\end{bmatrix}$	$\begin{bmatrix}1\\0.414\end{bmatrix}$

19.

k	0	1	2	3	4	5
$\mathbf{x}_k$	$\begin{bmatrix}1\\1\\1\end{bmatrix}$	$\begin{bmatrix}21\\15\\13\end{bmatrix}$	$\begin{bmatrix}16.809\\12.238\\10.714\end{bmatrix}$	$\begin{bmatrix}17.011\\12.371\\10.824\end{bmatrix}$	$\begin{bmatrix}16.999\\12.363\\10.818\end{bmatrix}$	$\begin{bmatrix}17.000\\12.363\\10.818\end{bmatrix}$
$R(\mathbf{x}_k)$	16.333	16.998	17.000	17.000	17.000	17.000
$\mathbf{y}_k$	$\begin{bmatrix}1\\1\\1\end{bmatrix}$	$\begin{bmatrix}1\\0.714\\0.619\end{bmatrix}$	$\begin{bmatrix}1\\0.728\\0.637\end{bmatrix}$	$\begin{bmatrix}1\\0.727\\0.636\end{bmatrix}$	$\begin{bmatrix}1\\0.727\\0.636\end{bmatrix}$	$\begin{bmatrix}1\\0.727\\0.636\end{bmatrix}$

21.

k	0	1	2	3	4	5	6	7	8
$\mathbf{x}_k$	$\begin{bmatrix}1\\1\end{bmatrix}$	$\begin{bmatrix}5\\4\end{bmatrix}$	$\begin{bmatrix}4.8\\3.2\end{bmatrix}$	$\begin{bmatrix}4.667\\2.667\end{bmatrix}$	$\begin{bmatrix}4.571\\2.286\end{bmatrix}$	$\begin{bmatrix}4.500\\2.000\end{bmatrix}$	$\begin{bmatrix}4.444\\1.778\end{bmatrix}$	$\begin{bmatrix}4.400\\1.600\end{bmatrix}$	$\begin{bmatrix}4.364\\1.455\end{bmatrix}$
$\mathbf{y}_k$	$\begin{bmatrix}1\\1\end{bmatrix}$	$\begin{bmatrix}1\\0.8\end{bmatrix}$	$\begin{bmatrix}1\\0.667\end{bmatrix}$	$\begin{bmatrix}1\\0.571\end{bmatrix}$	$\begin{bmatrix}1\\0.500\end{bmatrix}$	$\begin{bmatrix}1\\0.444\end{bmatrix}$	$\begin{bmatrix}1\\0.400\end{bmatrix}$	$\begin{bmatrix}1\\0.364\end{bmatrix}$	$\begin{bmatrix}1\\0.333\end{bmatrix}$
m_k	1	5	4.8	4.667	4.571	4.500	4.444	4.400	4.364

Since $\lambda_1 = \lambda_2 = 4$, $\mathbf{v}_1 = \begin{bmatrix}1\\0\end{bmatrix}$, m_k is converging slowly to the exact answer.

23.

k	0	1	2	3	4	5	6	7	8
$\mathbf{x}_k$	$\begin{bmatrix}1\\1\\1\end{bmatrix}$	$\begin{bmatrix}5\\4\\1\end{bmatrix}$	$\begin{bmatrix}4.2\\3.2\\0.2\end{bmatrix}$	$\begin{bmatrix}4.048\\3.048\\0.048\end{bmatrix}$	$\begin{bmatrix}4.012\\3.012\\0.012\end{bmatrix}$	$\begin{bmatrix}4.003\\3.003\\0.003\end{bmatrix}$	$\begin{bmatrix}4.001\\3.001\\0.001\end{bmatrix}$	$\begin{bmatrix}4.000\\3.000\\0.000\end{bmatrix}$	$\begin{bmatrix}4.000\\3.000\\0.000\end{bmatrix}$
$\mathbf{y}_k$	$\begin{bmatrix}1\\1\\1\end{bmatrix}$	$\begin{bmatrix}1\\0.8\\0.2\end{bmatrix}$	$\begin{bmatrix}1\\0.762\\0.048\end{bmatrix}$	$\begin{bmatrix}1\\0.753\\0.012\end{bmatrix}$	$\begin{bmatrix}1\\0.751\\0.003\end{bmatrix}$	$\begin{bmatrix}1\\0.750\\0.001\end{bmatrix}$	$\begin{bmatrix}1\\0.750\\0\end{bmatrix}$	$\begin{bmatrix}1\\0.750\\0\end{bmatrix}$	$\begin{bmatrix}1\\0.750\\0\end{bmatrix}$
m_k	1	5	4.2	4.048	4.012	4.003	4.001	4.000	4.000

In this case, $\lambda_1 = \lambda_2 = 4$ and $E_4 = \text{span}\left(\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}\right)$.

Clearly, m_k is converging to 4 and $\mathbf{y}_k$ is converging to a vector in the eigenspace E_4—namely, $\begin{bmatrix}1\\0\\0\end{bmatrix} + 0.75\begin{bmatrix}0\\1\\0\end{bmatrix}$.

25.

k	0	1	2	3	4	5
$\mathbf{x}_k$	$\begin{bmatrix}1\\1\end{bmatrix}$	$\begin{bmatrix}1\\0\end{bmatrix}$	$\begin{bmatrix}-1\\-1\end{bmatrix}$	$\begin{bmatrix}1\\0\end{bmatrix}$	$\begin{bmatrix}-1\\-1\end{bmatrix}$	$\begin{bmatrix}1\\0\end{bmatrix}$
$\mathbf{y}_k$	$\begin{bmatrix}1\\1\end{bmatrix}$	$\begin{bmatrix}1\\0\end{bmatrix}$	$\begin{bmatrix}1\\1\end{bmatrix}$	$\begin{bmatrix}1\\0\end{bmatrix}$	$\begin{bmatrix}1\\1\end{bmatrix}$	$\begin{bmatrix}1\\0\end{bmatrix}$
m_k	1	1	-1	1	-1	1

The exact eigenvalues are complex (i and $-i$), so the power method cannot possibly converge to either the dominant eigenvalue or the dominant eigenvector if we start with a *real* initial iterate. Instead, the power method oscillates between two sets of real vectors.

27.

k	0	1	2	3	4	5
$\mathbf{x}_k$	$\begin{bmatrix}1\\1\\1\end{bmatrix}$	$\begin{bmatrix}3\\4\\3\end{bmatrix}$	$\begin{bmatrix}2.500\\4.000\\2.500\end{bmatrix}$	$\begin{bmatrix}2.250\\4.000\\2.250\end{bmatrix}$	$\begin{bmatrix}2.125\\4.000\\2.125\end{bmatrix}$	$\begin{bmatrix}2.063\\4.000\\2.063\end{bmatrix}$
$\mathbf{y}_k$	$\begin{bmatrix}1\\1\\1\end{bmatrix}$	$\begin{bmatrix}0.750\\1\\0.750\end{bmatrix}$	$\begin{bmatrix}0.625\\1\\0.625\end{bmatrix}$	$\begin{bmatrix}0.562\\1\\0.562\end{bmatrix}$	$\begin{bmatrix}0.531\\1\\0.531\end{bmatrix}$	$\begin{bmatrix}0.516\\1\\0.516\end{bmatrix}$
m_k	1	4	4	4	4	4

The eigenvalues are $\lambda_1 = -12$, $\lambda_2 = 4$, $\lambda_3 = 2$, with corresponding eigenvectors

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}.$$

Since $\mathbf{x}_0 = \frac{1}{2}\mathbf{v}_2 + \frac{1}{2}\mathbf{v}_3$, the initial vector $\mathbf{x}_0$ has a zero component in the direction of the dominant eigenvector, so the power method cannot converge to the dominant eigenvalue/eigenvector. Instead, it converges to a *second* eigenvalue/eigenvector pair, as the calculations show.

29. Apply the power method to $A - 18I = \begin{bmatrix} -4 & 12 \\ 5 & -15 \end{bmatrix}$.

k	0	1	2	3
$\mathbf{x}_k$	$\begin{bmatrix} 1 \\ 1 \end{bmatrix}$	$\begin{bmatrix} 8 \\ -10 \end{bmatrix}$	$\begin{bmatrix} 15.2 \\ -19 \end{bmatrix}$	$\begin{bmatrix} 15.2 \\ -19 \end{bmatrix}$
$\mathbf{y}_k$	$\begin{bmatrix} 1 \\ 1 \end{bmatrix}$	$\begin{bmatrix} -0.8 \\ 1 \end{bmatrix}$	$\begin{bmatrix} -0.8 \\ 1 \end{bmatrix}$	$\begin{bmatrix} -0.8 \\ 1 \end{bmatrix}$
m_k	1	-10	-19	-19

Thus, -19 is the dominant eigenvalue of $A - 18I$, and $\lambda_2 = -19 + 18 = -1$ is the second eigenvalue of A.

31. Apply the power method to $A - 17I = \begin{bmatrix} -8 & 4 & 8 \\ 4 & -2 & -4 \\ 8 & -4 & -8 \end{bmatrix}$.

k	0	1	2	3
$\mathbf{x}_k$	$\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$	$\begin{bmatrix} 4 \\ -2 \\ -4 \end{bmatrix}$	$\begin{bmatrix} -18 \\ 9 \\ 18 \end{bmatrix}$	$\begin{bmatrix} -18 \\ 9 \\ 18 \end{bmatrix}$
$\mathbf{y}_k$	$\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$	$\begin{bmatrix} 1 \\ -0.5 \\ -1 \end{bmatrix}$	$\begin{bmatrix} 1 \\ -0.5 \\ -1 \end{bmatrix}$	$\begin{bmatrix} 1 \\ -0.5 \\ -1 \end{bmatrix}$
m_k	1	4	-18	-18
$R(\mathbf{x}_k)$	-0.667	-18	-18	-18

In this case, there is no dominant eigenvalue. (We could choose either 18 or -18 for m_k, $k \geq 2$.) However, the Rayleigh quotient method (Exercises 17–20) converges to -18. Thus, -18 is the dominant eigenvalue of $A - 17I$, and $\lambda_2 = -18 + 17 = -1$ is the second eigenvalue of A.

33.

k	0	1	2	3	4	5
$\mathbf{x}_k$	$\begin{bmatrix} 1 \\ 1 \end{bmatrix}$	$\begin{bmatrix} 0.5 \\ -0.5 \end{bmatrix}$	$\begin{bmatrix} -0.833 \\ 1.056 \end{bmatrix}$	$\begin{bmatrix} 0.798 \\ -0.997 \end{bmatrix}$	$\begin{bmatrix} 0.800 \\ -1.000 \end{bmatrix}$	$\begin{bmatrix} 0.800 \\ -1.000 \end{bmatrix}$
$\mathbf{y}_k$	$\begin{bmatrix} 1 \\ 1 \end{bmatrix}$	$\begin{bmatrix} 1 \\ -1 \end{bmatrix}$	$\begin{bmatrix} -0.789 \\ 1 \end{bmatrix}$	$\begin{bmatrix} -0.801 \\ 1 \end{bmatrix}$	$\begin{bmatrix} -0.800 \\ 1 \end{bmatrix}$	$\begin{bmatrix} -0.800 \\ 1 \end{bmatrix}$
m_k	1	0.5	1.056	-0.997	-1.000	-1.000

Thus, the eigenvalue of A that is smallest in magnitude is $1/(-1) = -1$.

35.

k	0	1	2	3	4	5
$\mathbf{x}_k$	$\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$	$\begin{bmatrix} -0.500 \\ 0.000 \\ 0.500 \end{bmatrix}$	$\begin{bmatrix} 0.500 \\ 0.333 \\ -0.500 \end{bmatrix}$	$\begin{bmatrix} 0.500 \\ 0.111 \\ -0.500 \end{bmatrix}$	$\begin{bmatrix} 0.500 \\ 0.259 \\ -0.500 \end{bmatrix}$	$\begin{bmatrix} 0.500 \\ 0.160 \\ -0.500 \end{bmatrix}$
$\mathbf{y}_k$	$\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$	$\begin{bmatrix} -1.000 \\ 0.000 \\ 1.000 \end{bmatrix}$	$\begin{bmatrix} -1.000 \\ -0.667 \\ 1.000 \end{bmatrix}$	$\begin{bmatrix} -1.000 \\ -0.222 \\ 1.000 \end{bmatrix}$	$\begin{bmatrix} -1.000 \\ -0.518 \\ 1.000 \end{bmatrix}$	$\begin{bmatrix} -1.000 \\ -0.321 \\ 1.000 \end{bmatrix}$
m_k	1	-0.500	-0.500	-0.500	-0.500	-0.500

Clearly, m_k converges to -0.5, so the smallest eigenvalue of A is $1/(-0.5) = -2$.

37. The calculations are the same as for Exercise 33.

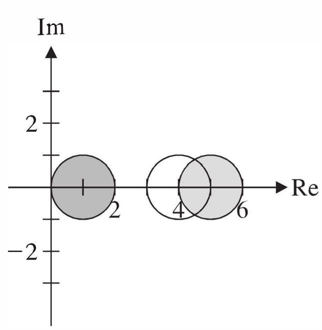
39. We apply the inverse power method to $A - 5I = \begin{bmatrix} -1 & 0 & 6 \\ -1 & -2 & 1 \\ 6 & 0 & -1 \end{bmatrix}$. Taking $\mathbf{x}_0 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, we have

k	0	1	2	3
$\mathbf{x}_k$	$\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$	$\begin{bmatrix} 0.200 \\ -0.500 \\ 0.200 \end{bmatrix}$	$\begin{bmatrix} -0.080 \\ -0.500 \\ -0.080 \end{bmatrix}$	$\begin{bmatrix} 0.032 \\ -0.500 \\ 0.032 \end{bmatrix}$
$\mathbf{y}_k$	$\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$	$\begin{bmatrix} -0.400 \\ 1 \\ -0.400 \end{bmatrix}$	$\begin{bmatrix} 0.160 \\ 1 \\ 0.160 \end{bmatrix}$	$\begin{bmatrix} -0.064 \\ 1 \\ -0.064 \end{bmatrix}$
m_k	1	-0.500	-0.500	-0.500

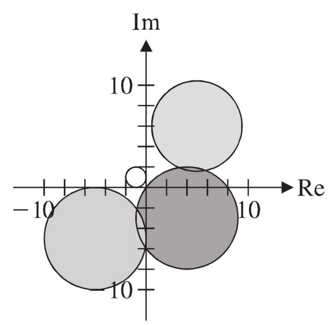
Clearly, m_k converges to -0.5, so the eigenvalue of A closest to 5 is $5 + 1/(-0.5) = 5 - 2 = 3$.

41. 0.732 **43.** -0.619

47.



49.



51. *Hint:* Show that 0 is not contained in any Gerschgorin disk and then apply Theorem 4.16.

53. Exercise 52 implies that $|\lambda|$ is less than or equal to all of the column sums of A for every eigenvalue λ. But for a stochastic matrix, all column sums are 1. Hence $|\lambda| \le 1$.

Exercises 4.6

1. Not regular **3.** Regular

5. Not regular **7.** $L = \begin{bmatrix} \frac{1}{5} & \frac{1}{5} \\ \frac{4}{5} & \frac{4}{5} \end{bmatrix}$

9. $L = \begin{bmatrix} 0.304 & 0.304 & 0.304 \\ 0.354 & 0.354 & 0.354 \\ 0.342 & 0.342 & 0.342 \end{bmatrix}$

11. 1, $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ **13.** 2, $\begin{bmatrix} 16 \\ 4 \\ 1 \end{bmatrix}$

15. The population is increasing, decreasing, and constant, respectively.

17.

$$P^{-1}LP = \begin{bmatrix} b_1 & b_2s_1 & b_3s_1s_2 & \cdots & b_{n-1}s_1s_2\cdots s_{n-2} & b_ns_1s_2\cdots s_{n-1} \\ 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 1 & 0 \end{bmatrix}.$$

The characteristic polynomial of L is $(\lambda^n - b_1\lambda^{n-1} - b_2s_1\lambda^{n-2} - b_3s_1s_2\lambda^{n-3} - \cdots - b_ns_1s_2\cdots s_{n-1})(-1)^n$.

19. $\lambda \approx 1.746$, $\mathbf{p} \approx \begin{bmatrix} 0.660 \\ 0.264 \\ 0.076 \end{bmatrix}$

21. $\lambda \approx 1.092$, $\mathbf{p} \approx \begin{bmatrix} 0.535 \\ 0.147 \\ 0.094 \\ 0.078 \\ 0.064 \\ 0.053 \\ 0.029 \end{bmatrix}$

25. (a) $h \approx 0.082$

29. 3, $\begin{bmatrix} \frac{3}{5} \\ \frac{2}{5} \end{bmatrix}$

31. 3, $\begin{bmatrix} \frac{1}{2} \\ \frac{1}{4} \\ \frac{1}{4} \end{bmatrix}$

33. Reducible

35. Irreducible

43. 1, 2, 4, 8, 16

45. 0, 1, 1, 0, -1

47. $x_n = 4^n - (-1)^n$

49. $y_n = (n - \frac{1}{2})2^n$

51. $b_n = \dfrac{1}{2\sqrt{3}}[(1 + \sqrt{3})^n - (1 - \sqrt{3})^n]$

57. (a) $d_1 = 1, d_2 = 2, d_3 = 3, d_4 = 5, d_5 = 8$

(b) $d_n = d_{n-1} + d_{n-2}$

(c) $d_n = \dfrac{1}{\sqrt{5}}\left[\left(\dfrac{1 + \sqrt{5}}{2}\right)^{n+1} - \left(\dfrac{1 - \sqrt{5}}{2}\right)^{n+1}\right]$

59. The general solution is $x(t) = -3C_1e^{-t} + C_2e^{4t}$, $y(t) = 2C_1e^{-t} + C_2e^{4t}$. The specific solution is $x(t) = -3e^{-t} + 3e^{4t}$, $y(t) = 2e^{-t} + 3e^{4t}$.

61. The general solution is $x_1(t) = (1 + \sqrt{2})C_1e^{\sqrt{2}t} + (1 - \sqrt{2})C_2e^{-\sqrt{2}t}$, $x_2(t) = C_1e^{\sqrt{2}t} + C_2e^{-\sqrt{2}t}$. The specific solution is $x_1(t) = (2 + \sqrt{2})e^{\sqrt{2}t}/4 + (2 - \sqrt{2})e^{-\sqrt{2}t}/4$, $x_2(t) = \sqrt{2}e^{\sqrt{2}t}/4 - \sqrt{2}e^{-\sqrt{2}t}/4$.

63. The general solution is $x(t) = -C_1 + C_3e^{-t}$, $y(t) = C_1 + C_2e^t - C_3e^{-t}$, $z(t) = C_1 + C_2e^t$. The specific solution is $x(t) = 2 - e^{-t}$, $y(t) = -2 + e^t + e^{-t}$, $z(t) = -2 + e^t$.

65. (a) $x(t) = -120e^{8t/5} + 520e^{11t/10}$, $y(t) = 240e^{8t/5} + 260^{11t/10}$. Strain X dies out after approximately 2.93 days; strain Y continues to grow.

67. $a = 10, b = 20$; $x(t) = 10e^t(\cos t + \sin t) + 10$, $y(t) = 10e^t(\cos t - \sin t) + 20$. Species Y dies out when $t \approx 1.22$.

71. $x(t) = C_1e^{2t} + C_2e^{3t}$

77. (a) $\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 3 \end{bmatrix}, \begin{bmatrix} 9 \\ 9 \end{bmatrix}, \begin{bmatrix} 27 \\ 27 \end{bmatrix}$ **(c)** Repeller

79. (a) $\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ **(c)** Neither

81. (a) $\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0.5 \\ -1 \end{bmatrix}, \begin{bmatrix} 1.75 \\ -0.5 \end{bmatrix}, \begin{bmatrix} 3.125 \\ -1.75 \end{bmatrix}$ **(c)** Saddle point

83. (a) $\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0.6 \\ 0.6 \end{bmatrix}, \begin{bmatrix} 0.36 \\ 0.36 \end{bmatrix}, \begin{bmatrix} 0.216 \\ 0.216 \end{bmatrix}$ **(c)** Attractor

85. $r = \sqrt{2}$, $\theta = 45°$, spiral repeller

87. $r = 2$, $\theta = -60°$, spiral repeller

89. $P = \begin{bmatrix} -1 & -1 \\ 1 & 0 \end{bmatrix}$, $C = \begin{bmatrix} 0.2 & -0.1 \\ 0.1 & 0.2 \end{bmatrix}$, spiral attractor

91. $P = \begin{bmatrix} 1/2 & -\sqrt{3}/2 \\ 1 & 0 \end{bmatrix}$, $C = \begin{bmatrix} 1/2 & -\sqrt{3}/2 \\ \sqrt{3}/2 & 1/2 \end{bmatrix}$, orbital center

Review Questions

1. (a) F **(c)** F **(e)** F **(g)** T **(i)** F

3. -18

5. Since $A^T = -A$, we have $\det A = \det(A^T) = \det(-A) = (-1)^n \det A = -\det A$ by Theorem 4.7 and the fact that n is odd. It follows that $\det A = 0$.

7. $A\mathbf{x} = \begin{bmatrix} 5 \\ 10 \end{bmatrix} = 5\mathbf{x}$, $\lambda = 5$

9. (a) $4 - 3\lambda^2 - \lambda^3$

(c) $E_1 = \text{span}\left(\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}\right)$, $E_{-2} = \text{span}\left(\begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}\right)$

11. $\begin{bmatrix} 162 \\ 158 \end{bmatrix}$

13. Not similar

15. Not similar

17. 0, 1, or -1

19. If $A\mathbf{x} = \lambda\mathbf{x}$, then $(A^2 - 5A + 2I)\mathbf{x} = A^2\mathbf{x} - 5A\mathbf{x} + 2\mathbf{x} = 3^2\mathbf{x} - 5(3\mathbf{x}) + 2\mathbf{x} = -4\mathbf{x}$.

Chapter 5

Exercises 5.1

1. Orthogonal **3.** Not orthogonal **5.** Orthogonal

7. $[\mathbf{w}]_{\mathcal{B}} = \begin{bmatrix} \frac{1}{2} \\ -1 \end{bmatrix}$ **9.** $[\mathbf{w}]_{\mathcal{B}} = \begin{bmatrix} 0 \\ \frac{2}{3} \\ \frac{1}{3} \end{bmatrix}$ **11.** Orthonormal

13. $\begin{bmatrix} 1/3 \\ 2/3 \\ 2/3 \end{bmatrix}, \begin{bmatrix} 2/\sqrt{5} \\ -1/\sqrt{5} \\ 0 \end{bmatrix}, \begin{bmatrix} 2/3\sqrt{5} \\ 4/3\sqrt{5} \\ -5/3\sqrt{5} \end{bmatrix}$

15. Orthonormal

17. Orthogonal, $\begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$

19. Orthogonal, $\begin{bmatrix} \cos\theta\sin\theta & \cos^2\theta & \sin\theta \\ -\cos\theta & \sin\theta & 0 \\ -\sin^2\theta & -\cos\theta\sin\theta & \cos\theta \end{bmatrix}$

21. Not orthogonal

27. $\cos(\angle(Q\mathbf{x}, Q\mathbf{y})) = \dfrac{Q\mathbf{x}\cdot Q\mathbf{y}}{\|Q\mathbf{x}\|\|Q\mathbf{y}\|} = \dfrac{\mathbf{x}\cdot\mathbf{y}}{\|\mathbf{x}\|\|\mathbf{y}\|}$

$= \cos(\angle(\mathbf{x}, \mathbf{y}))$ by Theorem 5.6

29. Rotation, $\theta = 45°$ **31.** Reflection, $y = \sqrt{3}x$

33. **(a)** $A(A^T + B^T)B = AA^TB + AB^TB = IB + AI = B + A = A + B$

(b) From part (a),

$$\begin{aligned}\det(A+B) &= \det(A(A^T + B^T)B)\\ &= \det A \det(A^T + B^T)\det B\\ &= \det A \det((A+B)^T)\det B\\ &= \det A \det(A+B)\det B\end{aligned}$$

Assume that $\det A + \det B = 0$ (so that $\det B = -\det A$) but that $A + B$ is invertible. Then $\det(A + B) \neq 0$, so $1 = \det A \det B = \det A(-\det A) = -(\det A)^2$. This is impossible, so we conclude that $A + B$ cannot be invertible.

Exercises 5.2

1. $W^\perp = \left\{\begin{bmatrix} x \\ y \end{bmatrix} : x + 2y = 0\right\}, \mathcal{B}^\perp = \left\{\begin{bmatrix} -2 \\ 1 \end{bmatrix}\right\}$

3. $W^\perp = \left\{\begin{bmatrix} x \\ y \\ z \end{bmatrix} : x = t, y = t, z = -t\right\},$

$\mathcal{B}^\perp = \left\{\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}\right\}$

5. $W^\perp = \left\{\begin{bmatrix} x \\ y \\ z \end{bmatrix} : x - y + 3z = 0\right\}, \mathcal{B}^\perp = \left\{\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 3 \\ 1 \end{bmatrix}\right\}$

7. row(A): $\{[1 \;\; 0 \;\; 1], [0 \;\; 1 \;\; -2]\}$, null($A$): $\left\{\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}\right\}$

9. col(A): $\left\{\begin{bmatrix} 1 \\ 5 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \\ 1 \\ -1 \end{bmatrix}\right\}$, null($A^T$): $\left\{\begin{bmatrix} 1 \\ 0 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} -5 \\ 1 \\ -7 \\ 0 \end{bmatrix}\right\}$

11. $\left\{\begin{bmatrix} 1 \\ -10 \\ -4 \end{bmatrix}\right\}$ **13.** $\left\{\begin{bmatrix} -4 \\ 1 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 1 \\ 0 \end{bmatrix}\right\}$

15. $\begin{bmatrix} \frac{3}{2} \\ \frac{3}{2} \end{bmatrix}$ **17.** $\begin{bmatrix} -\frac{1}{2} \\ \frac{1}{2} \\ 3 \end{bmatrix}$

19. $\mathbf{v} = \begin{bmatrix} -\frac{2}{5} \\ -\frac{6}{5} \end{bmatrix} + \begin{bmatrix} \frac{12}{5} \\ -\frac{4}{5} \end{bmatrix}$ **21.** $\mathbf{v} = \begin{bmatrix} \frac{7}{2} \\ -2 \\ \frac{7}{2} \end{bmatrix} + \begin{bmatrix} \frac{1}{2} \\ 0 \\ -\frac{1}{2} \end{bmatrix}$

25. No

Exercises 5.3

1. $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} -1/2 \\ 1/2 \end{bmatrix}; \mathbf{q}_1 = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}, \mathbf{q}_2 = \begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$

3. $\mathbf{v}_1 = \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}; \mathbf{q}_1 = \begin{bmatrix} 1/\sqrt{3} \\ -1/\sqrt{3} \\ -1/\sqrt{3} \end{bmatrix},$

$\mathbf{q}_2 = \begin{bmatrix} 2/\sqrt{6} \\ 1/\sqrt{6} \\ 1/\sqrt{6} \end{bmatrix}, \mathbf{q}_3 = \begin{bmatrix} 0 \\ -1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$

5. $\left\{\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -\frac{1}{2} \\ \frac{1}{2} \\ 2 \end{bmatrix}\right\}$ **7.** $\mathbf{v} = \begin{bmatrix} -\frac{2}{9} \\ \frac{2}{9} \\ \frac{8}{9} \end{bmatrix} + \begin{bmatrix} \frac{38}{9} \\ -\frac{38}{9} \\ \frac{19}{9} \end{bmatrix}$

9. $\left\{\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -\frac{1}{2} \\ \frac{1}{2} \end{bmatrix}, \begin{bmatrix} \frac{2}{3} \\ \frac{2}{3} \\ -\frac{2}{3} \end{bmatrix}\right\}$

11. $\left\{\begin{bmatrix}3\\1\\5\end{bmatrix}, \begin{bmatrix}-\frac{3}{35}\\ \frac{34}{35}\\ -\frac{1}{7}\end{bmatrix}, \begin{bmatrix}-\frac{15}{34}\\ 0\\ \frac{9}{34}\end{bmatrix}\right\}$

13. $Q = \begin{bmatrix}1/\sqrt{2} & 1/\sqrt{3} & 1/\sqrt{6}\\ 0 & 1/\sqrt{3} & -2/\sqrt{2}\\ -1/\sqrt{2} & 1/\sqrt{3} & 1/\sqrt{6}\end{bmatrix}$

15. $\begin{bmatrix}0 & 2/\sqrt{6} & 1/\sqrt{3}\\ 1/\sqrt{2} & -1/\sqrt{6} & 1/\sqrt{3}\\ 1/\sqrt{2} & 1/\sqrt{6} & -1/\sqrt{3}\end{bmatrix}\begin{bmatrix}\sqrt{2} & 1/\sqrt{2} & 1/\sqrt{2}\\ 0 & 3/\sqrt{6} & 1/\sqrt{6}\\ 0 & 0 & 2/\sqrt{3}\end{bmatrix}$

17. $R = \begin{bmatrix}3 & 9 & \frac{1}{3}\\ 0 & 6 & \frac{2}{3}\\ 0 & 0 & \frac{7}{3}\end{bmatrix}$

19. $A = AI$

21. $A^{-1} = (QR)^{-1} = R^{-1}Q^{-1} = R^{-1}Q^T =$

$\begin{bmatrix}1/\sqrt{2} & -1/\sqrt{6} & -1/2\sqrt{3}\\ 0 & 2/\sqrt{6} & -1/2\sqrt{3}\\ 0 & 0 & 3/2\sqrt{3}\end{bmatrix}\cdot$

$\begin{bmatrix}0 & 1/\sqrt{2} & 1/\sqrt{2}\\ 2/\sqrt{6} & -1/\sqrt{6} & 1/\sqrt{6}\\ 1/\sqrt{3} & 1/\sqrt{3} & -1/\sqrt{3}\end{bmatrix}$

23. Let $R\mathbf{x} = \mathbf{0}$. Then $A\mathbf{x} = QR\mathbf{x} = Q\mathbf{0} = \mathbf{0}$. Since $A\mathbf{x}$ represents a linear combination of the columns of A (which are linearly independent), we must have $\mathbf{x} = \mathbf{0}$. Hence, R is invertible, by the Fundamental Theorem.

Exercises 5.4

1. $Q = \begin{bmatrix}1/\sqrt{2} & 1/\sqrt{2}\\ 1/\sqrt{2} & -1/\sqrt{2}\end{bmatrix}, D = \begin{bmatrix}5 & 0\\ 0 & 3\end{bmatrix}$

3. $Q = \begin{bmatrix}2/\sqrt{6} & 1/\sqrt{3}\\ 1/\sqrt{3} & -2/\sqrt{6}\end{bmatrix}, D = \begin{bmatrix}2 & 0\\ 0 & -1\end{bmatrix}$

5. $Q = \begin{bmatrix}1 & 0 & 0\\ 0 & 1/\sqrt{2} & -1/\sqrt{2}\\ 0 & 1/\sqrt{2} & 1/\sqrt{2}\end{bmatrix}, D = \begin{bmatrix}5 & 0 & 0\\ 0 & 4 & 0\\ 0 & 0 & -2\end{bmatrix}$

7. $Q = \begin{bmatrix}-1/\sqrt{2} & 0 & 1/\sqrt{2}\\ 0 & 1 & 0\\ 1/\sqrt{2} & 0 & 1/\sqrt{2}\end{bmatrix}, D = \begin{bmatrix}2 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 0\end{bmatrix}$

9. $Q = \begin{bmatrix}1/\sqrt{2} & 0 & 1/\sqrt{2} & 0\\ 1/\sqrt{2} & 0 & -1/\sqrt{2} & 0\\ 0 & 1/\sqrt{2} & 0 & 1/\sqrt{2}\\ 0 & 1/\sqrt{2} & 0 & -1/\sqrt{2}\end{bmatrix},$

$D = \begin{bmatrix}2 & 0 & 0 & 0\\ 0 & 2 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{bmatrix}$

11. $Q^TAQ = \begin{bmatrix}1/\sqrt{2} & 1/\sqrt{2}\\ 1/\sqrt{2} & -1/\sqrt{2}\end{bmatrix}\begin{bmatrix}a & b\\ b & a\end{bmatrix}\cdot$

$\begin{bmatrix}1/\sqrt{2} & 1/\sqrt{2}\\ 1/\sqrt{2} & -1/\sqrt{2}\end{bmatrix} = \begin{bmatrix}a+b & 0\\ 0 & a-b\end{bmatrix} = D$

13. (a) If A and B are orthogonally diagonalizable, then each is symmetric, by the Spectral Theorem. Therefore, $A + B$ is symmetric, by Exercise 35 in Section 3.2, and so is orthogonally diagonalizable, by the Spectral Theorem.

15. If A and B are orthogonally diagonalizable, then each is symmetric, by the Spectral Theorem. Since $AB = BA$, AB is also symmetric, by Exercise 36 in Section 3.2. Hence, AB is orthogonally diagonalizable, by the Spectral Theorem.

17. $A = \begin{bmatrix}\frac{5}{2} & \frac{5}{2}\\ \frac{5}{2} & \frac{5}{2}\end{bmatrix} + \begin{bmatrix}\frac{3}{2} & -\frac{3}{2}\\ -\frac{3}{2} & \frac{3}{2}\end{bmatrix}$

19. $A = \begin{bmatrix}5 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix} + \begin{bmatrix}0 & 0 & 0\\ 0 & 2 & 2\\ 0 & 2 & 2\end{bmatrix} + \begin{bmatrix}0 & 0 & 0\\ 0 & -1 & 1\\ 0 & 1 & -1\end{bmatrix}$

21. $\begin{bmatrix}\frac{1}{2} & -\frac{3}{2}\\ -\frac{3}{2} & \frac{1}{2}\end{bmatrix}$ **23.** $\begin{bmatrix}\frac{5}{3} & -\frac{2}{3} & -\frac{1}{3}\\ -\frac{2}{3} & \frac{5}{3} & \frac{1}{3}\\ -\frac{1}{3} & \frac{1}{3} & \frac{8}{3}\end{bmatrix}$

Exercises 5.5

1. $2x^2 + 6xy + 4y^2$ **3.** 123 **5.** -5

7. $\begin{bmatrix}1 & 3\\ 3 & 2\end{bmatrix}$ **9.** $\begin{bmatrix}3 & -\frac{3}{2}\\ -\frac{3}{2} & -1\end{bmatrix}$ **11.** $\begin{bmatrix}5 & 1 & -2\\ 1 & -1 & 2\\ -2 & 2 & 2\end{bmatrix}$

13. $Q = \begin{bmatrix}2/\sqrt{5} & 1/\sqrt{5}\\ 1/\sqrt{5} & -2/\sqrt{5}\end{bmatrix}, y_1^2 + 6y_2^2$

15. $Q = \begin{bmatrix}2/\sqrt{5} & 2/3\sqrt{5} & -1/3\\ 0 & 5/3\sqrt{5} & 2/3\\ 1/\sqrt{5} & -4/3\sqrt{5} & 2/3\end{bmatrix}, 9y_1^2 + 9y_2^2 - 9y_3^2$

17. $Q = \begin{bmatrix}1/\sqrt{3} & 1/\sqrt{2} & 1/\sqrt{6}\\ -1/\sqrt{3} & 0 & 2/\sqrt{6}\\ -1/\sqrt{3} & 1/\sqrt{2} & -1/\sqrt{6}\end{bmatrix}, 2(x')^2 +$ $(y')^2 - (z')^2$

19. Positive definite

21. Negative definite

23. Positive definite

25. Indefinite

27. For any vector $\mathbf{x}$, we have $\mathbf{x}^T A\mathbf{x} = \mathbf{x}^T B^T B\mathbf{x} = (B\mathbf{x})^T(B\mathbf{x}) = \|B\mathbf{x}\|^2 \geq 0$. If $\mathbf{x}^T A\mathbf{x} = 0$, then $\|B\mathbf{x}\|^2 = 0$, so $B\mathbf{x} = \mathbf{0}$. Since B is invertible, this implies that $\mathbf{x} = \mathbf{0}$. Therefore, $\mathbf{x}^T A\mathbf{x} > 0$ for all $\mathbf{x} \neq \mathbf{0}$, and hence $A = B^T B$ is positive definite.

29. **(a)** Every eigenvalue of cA is of the form $c\lambda$ for some eigenvalue λ of A. By Theorem 5.24, $\lambda > 0$, so $c\lambda > 0$, since c is positive. Hence, cA is positive definite, by Theorem 5.24.

(c) Let $\mathbf{x} \neq \mathbf{0}$. Then $\mathbf{x}^T A\mathbf{x} > 0$ and $\mathbf{x}^T B\mathbf{x} > 0$, since A and B are positive definite. But then $\mathbf{x}^T(A + B)\mathbf{x} = \mathbf{x}^T A\mathbf{x} + \mathbf{x}^T B\mathbf{x} > 0$, so $A + B$ is positive definite.

31. The maximum value of $f(\mathbf{x})$ is 2 when $\mathbf{x} = \pm\begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix}$; the minimum value of $f(\mathbf{x})$ is 0 when $\mathbf{x} = \pm\begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$.

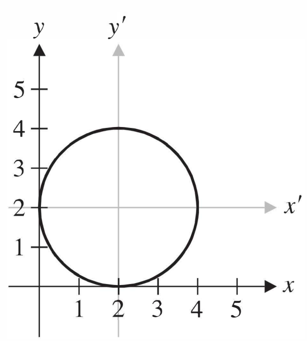
33. The maximum value of $f(\mathbf{x})$ is 4 when $\mathbf{x} = \pm\begin{bmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix}$; the minimum value of $f(\mathbf{x})$ is 1 when $\mathbf{x} = \pm\begin{bmatrix} 1/\sqrt{2} \\ 0 \\ -1/\sqrt{2} \end{bmatrix}$ or $\pm\begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{bmatrix}$.

35. Ellipse

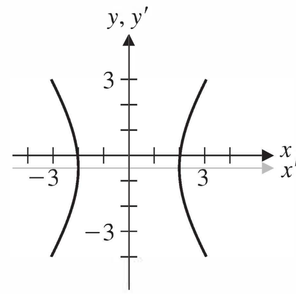
37. Parabola

39. Hyperbola

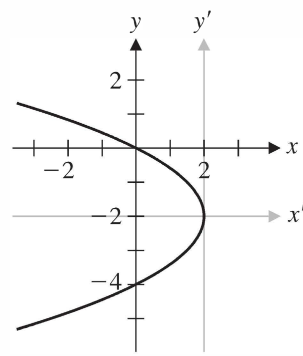
41. Circle, $x' = x - 2, y' = y - 2, (x')^2 + (y')^2 = 4$



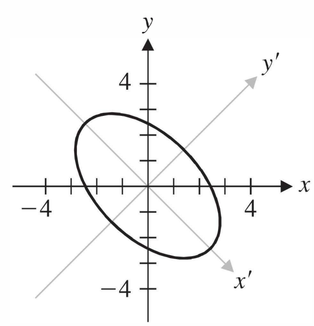
43. Hyperbola, $x' = x, y' = y + \frac{1}{2}, (x')^2/4 - (y')^2/9 = 1$



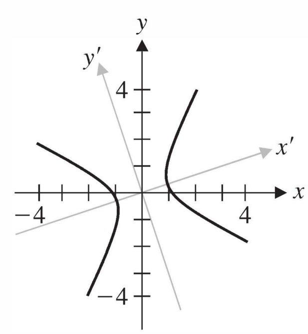
45. Parabola, $x' = x - 2, y' = y + 2, x' = -\frac{1}{2}(y')^2$



47. Ellipse, $(x')^2/4 + (y')^2/12 = 1$



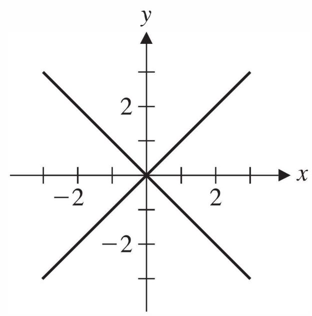
49. Hyperbola, $(x')^2 - (y')^2 = 1$



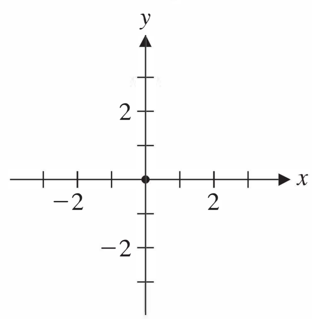
51. Ellipse, $(x'')^2/50 + (y'')^2/10 = 1$

53. Hyperbola, $(x'')^2 - (y'')^2 = 1$

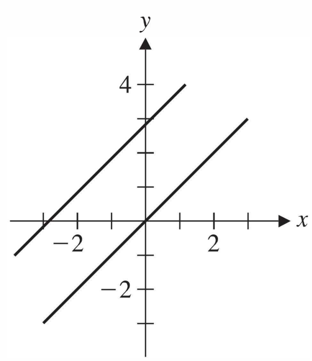
55. Degenerate (two lines)



57. Degenerate (a point)



59. Degenerate (two lines)



61. Hyperboloid of one sheet, $(x')^2 - (y')^2 + 3(z')^2 = 1$

63. Hyperbolic paraboloid, $z = -(x')^2 + (y')^2$

65. Hyperbolic paraboloid, $x' = -\sqrt{3}(y')^2 + \sqrt{3}(z')^2$

67. Ellipsoid, $3(x'')^2 + (y'')^2 + 2(z'')^2 = 4$

Review Questions

1. (a) T **(c)** T **(e)** F **(g)** F **(i)** F

3. $\begin{bmatrix} 9/2 \\ 2/3 \\ -11/6 \end{bmatrix}$

5. Verify that $Q^TQ = I$.

7. Theorem 5.6(c) shows that if $\mathbf{v}_i \cdot \mathbf{v}_j = 0$, then $Q\mathbf{v}_i \cdot Q\mathbf{v}_j = 0$. Theorem 5.6(b) shows that $\{Q\mathbf{v}_1, \ldots, Q\mathbf{v}_k\}$ consists of unit vectors, because $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ does. Hence, $\{Q\mathbf{v}_1, \ldots, Q\mathbf{v}_k\}$ is an orthonormal set.

9. $\left\{ \begin{bmatrix} 2 \\ -5 \end{bmatrix} \right\}$

11. $\left\{ \begin{bmatrix} -1 \\ 3 \\ 1 \end{bmatrix} \right\}$

13. row(A): $\{[1 \quad 0 \quad 2 \quad 3 \quad 4], [0 \quad 1 \quad 0 \quad 2 \quad 1]\}$

col(A): $\left\{ \begin{bmatrix} 1 \\ -1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \\ 1 \\ -5 \end{bmatrix} \right\}$

null(A): $\left\{ \begin{bmatrix} -2 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ -2 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -4 \\ -1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$

null(A^T): $\left\{ \begin{bmatrix} -5 \\ -3 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \\ 0 \\ 1 \end{bmatrix} \right\}$

15. (a) $\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} \frac{1}{4} \\ \frac{1}{4} \\ \frac{1}{4} \\ -\frac{3}{4} \end{bmatrix}, \begin{bmatrix} -\frac{2}{3} \\ \frac{1}{3} \\ \frac{1}{3} \\ 0 \end{bmatrix} \right\}$

17. $\left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} \frac{1}{3} \\ \frac{1}{3} \\ \frac{1}{3} \\ -1 \end{bmatrix} \right\}$

19. $\begin{bmatrix} -\frac{1}{2} & \frac{3}{2} & 0 \\ \frac{3}{2} & -\frac{1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Chapter 6

Exercises 6.1

1. Vector space

3. Not a vector space; axiom 1 fails.

5. Not a vector space; axiom 8 fails.

7. Vector space

9. Vector space

11. Vector space

15. Complex vector space

17. Not a complex vector space; axiom 6 fails.

19. Not a vector space; axioms 1, 4, and 6 fail.

21. Not a vector space; the operations of addition and multiplication are not even the same.

25. Subspace **27.** Not a subspace

29. Not a subspace **31.** Subspace

33. Subspace **35.** Subspace

37. Not a subspace **39.** Subspace

41. Subspace **43.** Not a subspace

45. Not a subspace

47. Take U to be the x-axis and W the y-axis, for example. Then $\begin{bmatrix}1\\0\end{bmatrix}$ and $\begin{bmatrix}0\\1\end{bmatrix}$ are in $U \cup W$, but $\begin{bmatrix}1\\1\end{bmatrix} = \begin{bmatrix}1\\0\end{bmatrix} + \begin{bmatrix}0\\1\end{bmatrix}$ is not.

51. No

53. Yes; $s(x) = (3 + 2t)p(x) + (1 + t)q(x) + tr(x)$ for any scalar t.

55. Yes; $h(x) = f(x) + g(x)$

57. No

59. No

61. Yes

Exercises 6.2

1. Linearly independent

3. Linearly dependent; $\begin{bmatrix}-1 & 0\\-1 & 7\end{bmatrix} = 4\begin{bmatrix}-1 & 1\\-2 & 2\end{bmatrix} + \begin{bmatrix}3 & 0\\1 & 1\end{bmatrix} - 2\begin{bmatrix}0 & 2\\-3 & 1\end{bmatrix}$

5. Linearly independent

7. Linearly dependent; $3x + 2x^2 = 7x - 2(2x - x^2)$

9. Linearly independent

11. Linearly dependent; $1 = \sin^2 x + \cos^2 x$

13. Linearly dependent; $\ln(x^2) = -2\ln 2 \cdot 1 + 2 \cdot \ln(2x)$

17. **(a)** Linearly independent

(b) Linearly dependent

19. Basis **21.** Not a basis

23. Not a basis **25.** Not a basis

27. $[A]_{\mathcal{B}} = \begin{bmatrix}-1\\-1\\-1\\4\end{bmatrix}$ **29.** $[p(x)]_{\mathcal{B}} = \begin{bmatrix}6\\-1\\3\end{bmatrix}$

35. $\dim V = 2$, $\mathcal{B} = \{1 - x, 1 - x^2\}$

37. $\dim V = 3$, $\mathcal{B} = \left\{\begin{bmatrix}1 & 0\\0 & 0\end{bmatrix}, \begin{bmatrix}0 & 1\\0 & 0\end{bmatrix}, \begin{bmatrix}0 & 0\\0 & 1\end{bmatrix}\right\}$

39. $\dim V = 2$, $\mathcal{B} = \left\{\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}, \begin{bmatrix}0 & 1\\0 & 0\end{bmatrix}\right\}$

41. $(n^2 - n)/2$

43. **(a)** $\dim(U \times V) = \dim U + \dim V$

(b) Show that if $\{\mathbf{w}_1, \ldots, \mathbf{w}_n\}$ is a basis for W, then $\{(\mathbf{w}_1, \mathbf{w}_1), \ldots, (\mathbf{w}_n, \mathbf{w}_n)\}$ is a basis for Δ.

45. $\{1 + x, 1 + x + x^2, 1\}$

47. $\left\{\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}, \begin{bmatrix}0 & 1\\1 & 0\end{bmatrix}, \begin{bmatrix}0 & -1\\1 & 0\end{bmatrix}, \begin{bmatrix}1 & 0\\0 & 0\end{bmatrix}\right\}$

49. $\{1, 1 + x\}$

51. $\{1 - x, x - x^2\}$

53. $\{\sin^2 x, \cos^2 x\}$

59. **(a)** $p_0(x) = \frac{1}{2}x^2 - \frac{5}{2}x + 3$, $p_1(x) = -x^2 + 4x - 3$, $p_2(x) = \frac{1}{2}x^2 - \frac{3}{2}x + 1$

61. **(c)** (i) $3x^2 - 16x + 19$ (ii) $x^2 - 4x + 5$

63. $(p^n - 1)(p^n - p)(p^n - p^2)\cdots(p^n - p^{n-1})$

Exercises 6.3

1. $[\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix}2\\3\end{bmatrix}$, $[\mathbf{x}]_{\mathcal{C}} = \begin{bmatrix}\frac{5}{2}\\-\frac{1}{2}\end{bmatrix}$, $P_{\mathcal{C}\leftarrow\mathcal{B}} = \begin{bmatrix}\frac{1}{2} & \frac{1}{2}\\\frac{1}{2} & -\frac{1}{2}\end{bmatrix}$, $P_{\mathcal{B}\leftarrow\mathcal{C}} = \begin{bmatrix}1 & 1\\1 & -1\end{bmatrix}$

3. $[\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix}1\\0\\-1\end{bmatrix}$, $[\mathbf{x}]_{\mathcal{C}} = \begin{bmatrix}1\\-1\\-1\end{bmatrix}$, $P_{\mathcal{C}\leftarrow\mathcal{B}} = \begin{bmatrix}1 & 0 & 0\\-1 & 1 & 0\\0 & -1 & 1\end{bmatrix}$,

$P_{\mathcal{B}\leftarrow\mathcal{C}} = \begin{bmatrix}1 & 0 & 0\\1 & 1 & 0\\1 & 1 & 1\end{bmatrix}$

5. $[p(x)]_{\mathcal{B}} = \begin{bmatrix}2\\-1\end{bmatrix}$, $[p(x)]_{\mathcal{C}} = \begin{bmatrix}-3\\2\end{bmatrix}$, $P_{\mathcal{C}\leftarrow\mathcal{B}} = \begin{bmatrix}-1 & 1\\1 & 0\end{bmatrix}$,

$P_{\mathcal{B}\leftarrow\mathcal{C}} = \begin{bmatrix}0 & 1\\1 & 1\end{bmatrix}$

7. $[p(x)]_{\mathcal{B}} = \begin{bmatrix}1\\-1\\1\end{bmatrix}$, $[p(x)]_{\mathcal{C}} = \begin{bmatrix}1\\0\\1\end{bmatrix}$, $P_{\mathcal{C}\leftarrow\mathcal{B}} = \begin{bmatrix}1 & 0 & 0\\1 & 1 & 0\\1 & 1 & 1\end{bmatrix}$,

$P_{\mathcal{B}\leftarrow\mathcal{C}} = \begin{bmatrix}1 & 0 & 0\\-1 & 1 & 0\\0 & -1 & 1\end{bmatrix}$

9. $[A]_{\mathcal{B}} = \begin{bmatrix} 4 \\ 2 \\ 0 \\ -1 \end{bmatrix}, [A]_{\mathcal{C}} = \begin{bmatrix} \frac{5}{2} \\ 0 \\ -3 \\ \frac{9}{2} \end{bmatrix}, P_{\mathcal{C}\leftarrow\mathcal{B}} =$

$\begin{bmatrix} \frac{1}{2} & 0 & -1 & -\frac{1}{2} \\ 0 & 0 & 1 & 0 \\ -1 & 1 & 1 & 1 \\ \frac{3}{2} & -1 & -2 & -\frac{1}{2} \end{bmatrix}, P_{\mathcal{B}\leftarrow\mathcal{C}} = \begin{bmatrix} 1 & 2 & 1 & 1 \\ 2 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 1 & 1 \end{bmatrix}$

11. $[f(x)]_{\mathcal{B}} = \begin{bmatrix} 2 \\ -5 \end{bmatrix}, [f(x)]_{\mathcal{C}} = \begin{bmatrix} -1/2 \\ 5/2 \end{bmatrix},$

$P_{\mathcal{C}\leftarrow\mathcal{B}} = \begin{bmatrix} 1 & 1/2 \\ 0 & -1/2 \end{bmatrix}, P_{\mathcal{B}\leftarrow\mathcal{C}} = \begin{bmatrix} 1 & 1 \\ 0 & -2 \end{bmatrix}$

13. **(a)** $\begin{bmatrix} (3 - 2\sqrt{3})/2 \\ (-3\sqrt{3} + 2)/2 \end{bmatrix} \approx \begin{bmatrix} 3.232 \\ -1.598 \end{bmatrix}$

(b) $\begin{bmatrix} 2 + 2\sqrt{3} \\ 2\sqrt{3} - 2 \end{bmatrix} \approx \begin{bmatrix} 5.464 \\ 1.464 \end{bmatrix}$

15. $\mathcal{B} = \left\{ \begin{bmatrix} -1 \\ -1 \end{bmatrix}, \begin{bmatrix} 3 \\ 4 \end{bmatrix} \right\}$

17. $-2 - 8(x - 1) - 5(x - 1)^2$

19. $-1 + 3(x + 1) - 3(x + 1)^2 + (x + 1)^3$

Exercises 6.4

1. Linear transformation **3.** Linear transformation

5. Linear transformation

7. Not a linear transformation

9. Linear transformation

11. Not a linear transformation

13. We have

$$\begin{aligned} S(p(x) + q(x)) &= S((p + q)(x)) = x((p + q)(x)) \\ &= x(p(x) + q(x)) = xp(x) + xq(x) \\ &= S(p(x)) + S(q(x)) \end{aligned}$$

and
$$\begin{aligned} S(cp(x)) &= S((cp)(x)) = x((cp)(x)) \\ &= x(cp(x)) = cxp(x) = cS(p(x)) \end{aligned}$$

Therefore, S is linear. Similarly,

$$\begin{aligned} T\left(\begin{bmatrix} a \\ b \end{bmatrix} + \begin{bmatrix} c \\ d \end{bmatrix}\right) &= T\begin{bmatrix} a + c \\ b + d \end{bmatrix} \\ &= (a + c) + ((a + c) + (b + d))x \\ &= (a + (a + b)x) + (c + (c + d)x) \\ &= T\left(\begin{bmatrix} a \\ b \end{bmatrix}\right) + T\left(\begin{bmatrix} c \\ d \end{bmatrix}\right) \end{aligned}$$

and
$$\begin{aligned} T\left(k\begin{bmatrix} a \\ b \end{bmatrix}\right) &= T\begin{bmatrix} ka \\ kb \end{bmatrix} = (ka) + (ka + kb)x \\ &= k(a + (a + b)x) = kT\left(\begin{bmatrix} a \\ b \end{bmatrix}\right) \end{aligned}$$

Therefore, T is linear.

15. $T\begin{bmatrix} -7 \\ 9 \end{bmatrix} = 5 - 14x - 8x^2, T\begin{bmatrix} a \\ b \end{bmatrix} = \left(\dfrac{a + 3b}{4}\right) - \left(\dfrac{a + 7b}{4}\right)x + \left(\dfrac{a - b}{2}\right)x^2$

17. $T(4 - x + 3x^2) = 4 + 3x + 5x^2, T(a + bx + cx^2) = a + cx + \left(\dfrac{3a - b - c}{2}\right)x^2$

19. *Hint:* Let $a = T(E_{11}), b = T(E_{12}), c = T(E_{21}), d = T(E_{22})$.

23. *Hint:* Consider the effect of T and D on the standard basis for $\mathscr{P}_n$.

25. $(S \circ T)\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 & -1 \\ 0 & 6 \end{bmatrix}, (S \circ T)\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2x & -y \\ 0 & 2x + 2y \end{bmatrix}.$

$(T \circ S)\begin{bmatrix} x \\ y \end{bmatrix}$ does not make sense.

27. $(S \circ T)(p(x)) = p'(x + 1), (T \circ S)(p(x)) = (p(x + 1))' = p'(x + 1)$

29. $(S \circ T)\begin{bmatrix} x \\ y \end{bmatrix} = S\left(T\begin{bmatrix} x \\ y \end{bmatrix}\right) = S\left(\begin{bmatrix} x - y \\ -3x + 4y \end{bmatrix}\right) = \begin{bmatrix} 4(x - y) + (-3x + 4y) \\ 3(x - y) + (-3x + 4y) \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$

$(T \circ S)\begin{bmatrix} x \\ y \end{bmatrix} = T\left(S\begin{bmatrix} x \\ y \end{bmatrix}\right) = T\left(\begin{bmatrix} 4x + y \\ 3x + y \end{bmatrix}\right) = \begin{bmatrix} (4x + y) - (3x + y) \\ -3(4x + y) + 4(3x + y) \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$

Therefore, $S \circ T = I$ and $T \circ S = I$, so S and T are inverses.

Exercises 6.5

1. **(a)** Only (ii) is in $\ker(T)$.
(b) Only (iii) is in range(T).
(c) $\ker(T) = \left\{ \begin{bmatrix} 0 & b \\ c & 0 \end{bmatrix} \right\}$, range$(T) = \left\{ \begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix} \right\}$

3. **(a)** Only (iii) is in $\ker(T)$.
(b) All of them are in range(T).
(c) $\ker(T) = \{a + bx + cx^2 : a = -c, b = -c\} = \{t + tx - tx^2\}$, range$(T) = \mathbb{R}^2$

5. A basis for ker(T) is $\left\{\begin{bmatrix}0&1\\0&0\end{bmatrix},\begin{bmatrix}0&0\\1&0\end{bmatrix}\right\}$, and a basis for range$(T)$ is $\left\{\begin{bmatrix}1&0\\0&0\end{bmatrix},\begin{bmatrix}0&0\\0&1\end{bmatrix}\right\}$; rank$(T)$ = nullity$(T) = 2$, and rank(T) + nullity$(T) = 4 =$ dim M_{22}.

7. A basis for ker(T) is $\{1 + x - x^2\}$, and a basis for range(T) is $\left\{\begin{bmatrix}1\\0\end{bmatrix},\begin{bmatrix}0\\1\end{bmatrix}\right\}$; rank$(T) = 2$, nullity$(T) = 1$, and rank$(T)$ + nullity$(T) = 3 = \dim \mathscr{P}_2$.

9. rank(T) = nullity(T) = 2

11. rank(T) = nullity(T) = 2

13. rank$(T) = 1$, nullity$(T) = 2$

15. One-to-one and onto

17. Neither one-to-one nor onto

19. One-to-one but not onto

21. Isomorphic, $T\begin{bmatrix}a&0&0\\0&b&0\\0&0&c\end{bmatrix}=\begin{bmatrix}a\\b\\c\end{bmatrix}$

23. Not isomorphic

25. Isomorphic, $T(a + bi) = \begin{bmatrix}a\\b\end{bmatrix}$

31. *Hint:* Define $T: \mathscr{C}[0, 1] \to \mathscr{C}[0, 2]$ by letting $T(f)$ be the function whose value at x is $(T(f))(x) = f(x/2)$ for x in $[0, 2]$.

33. (a) Let $\mathbf{v}_1$ and $\mathbf{v}_2$ be in V and let $(S \circ T)(\mathbf{v}_1) = (S \circ T)(\mathbf{v}_2)$. Then $S(T(\mathbf{v}_1)) = S(T(\mathbf{v}_2))$, so $T(\mathbf{v}_1) = T(\mathbf{v}_2)$, since S is one-to-one. But now $\mathbf{v}_1 = \mathbf{v}_2$, since T is one-to-one. Hence, $S \circ T$ is one-to-one.

35. (a) By the Rank Theorem, rank(T) + nullity(T) = dim V. If T is onto, then range$(T) = W$, so rank(T) = dim(range(T)) = dim W. Therefore,

$$\begin{aligned}\dim V + \text{nullity}(T) &< \dim W + \text{nullity}(T)\\ &= \text{rank}(T) + \text{nullity}(T) = \dim V\end{aligned}$$

so nullity $(T) < 0$, which is impossible. Therefore, T cannot be onto.

Exercises 6.6

1. $[T]_{\mathcal{C}\leftarrow\mathcal{B}} = \begin{bmatrix}0&1\\-1&0\end{bmatrix}$, $[T]_{\mathcal{C}\leftarrow\mathcal{B}}[4 + 2x]_{\mathcal{B}} =$

$\begin{bmatrix}0&1\\-1&0\end{bmatrix}\begin{bmatrix}4\\2\end{bmatrix} = \begin{bmatrix}2\\-4\end{bmatrix} = [2 - 4x]_{\mathcal{C}} = [T(4 + 2x)]_{\mathcal{C}}$

3. $[T]_{\mathcal{C}\leftarrow\mathcal{B}} = \begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}$, $[T]_{\mathcal{C}\leftarrow\mathcal{B}}[a + bx + cx^2]_{\mathcal{B}} =$

$\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}\begin{bmatrix}a\\b\\c\end{bmatrix} = \begin{bmatrix}a\\b\\c\end{bmatrix} = [a + b(x + 2) + c(x + 2)^2)]_{\mathcal{C}} = [T(a + bx + cx^2)]_{\mathcal{C}}$

5. $[T]_{\mathcal{C}\leftarrow\mathcal{B}} = \begin{bmatrix}1&0&0\\1&1&1\end{bmatrix}$, $[T]_{\mathcal{C}\leftarrow\mathcal{B}}[a + bx + cx^2]_{\mathcal{B}} =$

$\begin{bmatrix}1&0&0\\1&1&1\end{bmatrix}\begin{bmatrix}a\\b\\c\end{bmatrix} = \begin{bmatrix}a\\a+b+c\end{bmatrix}$

$= \begin{bmatrix}a + b\cdot 0 + c\cdot 0^2\\a + b\cdot 1 + c\cdot 1^2\end{bmatrix}_{\mathcal{C}} = [T(a + bx + cx^2)]_{\mathcal{C}}$

7. $[T]_{\mathcal{C}\leftarrow\mathcal{B}} = \begin{bmatrix}6&4\\-3&-2\\2&-1\end{bmatrix}$, $[T]_{\mathcal{C}\leftarrow\mathcal{B}}\begin{bmatrix}-7\\7\end{bmatrix}_{\mathcal{B}} =$

$\begin{bmatrix}6&4\\-3&-2\\2&-1\end{bmatrix}\begin{bmatrix}2\\-3\end{bmatrix} = \begin{bmatrix}0\\0\\7\end{bmatrix} = \begin{bmatrix}7\\7\\7\end{bmatrix}_{\mathcal{C}} = \begin{bmatrix}7\\7\\7\end{bmatrix}_{\mathcal{C}} =$

$\left[T\begin{bmatrix}-7\\7\end{bmatrix}\right]_{\mathcal{C}}$

9. $[T]_{\mathcal{C}\leftarrow\mathcal{B}} = \begin{bmatrix}1&0&0&0\\0&0&1&0\\0&1&0&0\\0&0&0&1\end{bmatrix}$, $[T]_{\mathcal{C}\leftarrow\mathcal{B}}[A]_{\mathcal{B}} =$

$\begin{bmatrix}1&0&0&0\\0&0&1&0\\0&1&0&0\\0&0&0&1\end{bmatrix}\begin{bmatrix}a\\b\\c\\d\end{bmatrix} = \begin{bmatrix}a\\c\\b\\d\end{bmatrix} = \left[\begin{bmatrix}a&c\\b&d\end{bmatrix}\right]_{\mathcal{C}} = [T(A)]_{\mathcal{C}}$

11. $[T]_{\mathcal{C}\leftarrow\mathcal{B}} = \begin{bmatrix}0&-1&1&0\\-1&0&0&1\\1&0&0&-1\\0&1&-1&0\end{bmatrix}$, $[T]_{\mathcal{C}\leftarrow\mathcal{B}}[A]_{\mathcal{B}} =$

$\begin{bmatrix}0&-1&1&0\\-1&0&0&1\\1&0&0&-1\\0&1&-1&0\end{bmatrix}\begin{bmatrix}a\\b\\c\\d\end{bmatrix} = \begin{bmatrix}c-b\\d-a\\a-d\\b-c\end{bmatrix} =$

$\left[\begin{bmatrix}c-b&d-a\\a-d&b-c\end{bmatrix}\right]_{\mathcal{C}} = [AB - BA]_{\mathcal{C}} = [T(A)]_{\mathcal{C}}$

13. (b) $[D]_{\mathcal{B}} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

(c) $[D]_{\mathcal{B}}[3 \sin x - 5 \cos x]_{\mathcal{B}} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 3 \\ -5 \end{bmatrix} = \begin{bmatrix} 5 \\ 3 \end{bmatrix} = [3 \cos x + 5 \sin x]_{\mathcal{B}} = [D(3 \sin x - 5 \cos x)]_{\mathcal{B}}$

15. (a) $[D]_{\mathcal{B}} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & -1 & 2 \end{bmatrix}$

17. $[S \circ T]_{\mathcal{D} \leftarrow \mathcal{B}} = \begin{bmatrix} -1 & -2 \\ 1 & -1 \end{bmatrix}$

19. Invertible, $T^{-1}(a + bx) = -b + ax$

21. Invertible, $T^{-1}(p(x)) = p(x - 2)$

23. Invertible, $T^{-1}(a + bx + cx^2) = (a - b + 2c) + (b - 2c)x + cx^2$ or $T^{-1}(p(x)) = p(x) - p'(x) + p''(x)$

25. Not invertible **27.** $-3 \sin x - \cos x + C$

29. $\frac{4}{5}e^{2x} \cos x - \frac{3}{5}e^{2x} \sin x + C$

31. $\mathcal{C} = \left\{ \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \begin{bmatrix} -4 \\ 1 \end{bmatrix} \right\}$ **33.** $\mathcal{C} = \{1 - x, 2 + x\}$

35. $\mathcal{C} = \{1, x\}$

37. $[T]_{\mathcal{E}} = \begin{bmatrix} (d_1^2 - d_2^2)/(d_1^2 + d_2^2) & 2d_1d_2/(d_1^2 + d_2^2) \\ 2d_1d_2/(d_1^2 + d_2^2) & (d_2^2 - d_1^2)/(d_1^2 + d_2^2) \end{bmatrix}$

Exercises 6.7

1. $y(t) = 2e^{3t}/e^3$

3. $y(t) = ((1 - e^4)e^{3t} + (e^3 - 1)e^{4t})/(e^3 - e^4)$

5. $f(t) = \left(\dfrac{e^{(\sqrt{5}-1)/2}}{e^{\sqrt{5}} - 1} \right)[e^{(1+\sqrt{5})t/2} - e^{(1-\sqrt{5})t/2}]$

7. $y(t) = e^t - (1 - e^{-1})te^t$

9. $y(t) = ((k + 1)e^{kt} + (k - 1)e^{-kt})/2k$

11. $y(t) = e^t \cos(2t)$

13. (a) $p(t) = 100e^{\ln(16)t/3} \approx 100e^{0.924t}$

(b) 45 minutes **(c)** In 9.968 hours

15. (a) $m(t) = 50e^{-ct}$, where $c = \ln 2/1590 \approx 4.36 \times 10^{-4}$; 32.33 mg remain after 1000 years.

(b) After 3691.9 years

17. $x(t) = \dfrac{5 - 10\cos(10\sqrt{K})}{\sin(10\sqrt{K})} \sin(\sqrt{K}t) + 10\cos(\sqrt{K}t)$

19. (b) No

Review Questions

1. (a) F **(c)** T **(e)** F **(g)** F **(i)** T

3. Subspace **5.** Subspace

7. Let $c_1A + c_2B = O$. Then $c_1A - c_2B = c_1A^T + c_2B^T = (c_1A + c_2B)^T = O$. Adding, we have $2c_1A = O$, so $c_1 = 0$ because A is nonzero. Hence $c_2B = O$, and so $c_2 = 0$. Thus, $\{A, B\}$ is linearly independent.

9. $\{1, x^2, x^4\}$, $\dim W = 3$

11. Linear transformation

13. Linear transformation **15.** $n^2 - 1$

17. $\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \\ 1 & 0 & -1 \end{bmatrix}$

19. $S \circ T$ is the zero transformation.

Chapter 7

Exercises 7.1

1. (a) -10 **(b)** $\sqrt{14}$ **(c)** $\sqrt{93}$

3. Any nonzero scalar multiple of $\begin{bmatrix} 3 \\ 1 \end{bmatrix}$

5. (a) 1 **(b)** $\sqrt{13}$ **(c)** $\sqrt{14}$

7. x^2 is one possibility

9. (a) π **(b)** $\sqrt{\pi}$ **(c)** $\sqrt{\pi}$

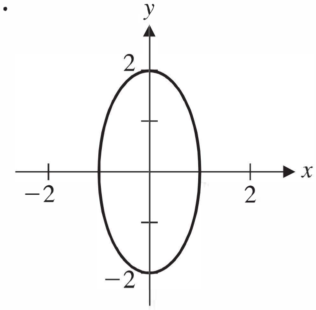
13. Axiom (4) fails: $\mathbf{u} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \neq \mathbf{0}$, but $\langle \mathbf{u}, \mathbf{u} \rangle = 0$.

15. Axiom (4) fails: $\mathbf{u} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \neq \mathbf{0}$, but $\langle \mathbf{u}, \mathbf{u} \rangle = 0$.

17. Axiom (4) fails: $p(x) = 1 - x$ is not the zero polynomial, but $\langle p(x), p(x) \rangle = 0$.

19. $A = \begin{bmatrix} 4 & 1 \\ 1 & 4 \end{bmatrix}$

21.



25. -8 **27.** $\sqrt{6}$

29. $\|\mathbf{u} + \mathbf{v} - \mathbf{w}\|^2 = \langle \mathbf{u} + \mathbf{v} - \mathbf{w}, \mathbf{u} + \mathbf{v} - \mathbf{w} \rangle$
$= \langle \mathbf{u}, \mathbf{u} \rangle + \langle \mathbf{v}, \mathbf{v} \rangle + \langle \mathbf{w}, \mathbf{w} \rangle + 2\langle \mathbf{u}, \mathbf{v} \rangle - 2\langle \mathbf{u}, \mathbf{w} \rangle - 2\langle \mathbf{v}, \mathbf{w} \rangle$
$= 1 + 3 + 4 + 2 - 10 - 0 = 0$

Therefore, $\|\mathbf{u} + \mathbf{v} - \mathbf{w}\| = 0$, so, by axiom (4), $\mathbf{u} + \mathbf{v} - \mathbf{w} = \mathbf{0}$ or $\mathbf{u} + \mathbf{v} = \mathbf{w}$.

31. $\langle \mathbf{u} + \mathbf{v}, \mathbf{u} - \mathbf{v}\rangle = \langle \mathbf{u}, \mathbf{u}\rangle - \langle \mathbf{u}, \mathbf{v}\rangle + \langle \mathbf{v}, \mathbf{u}\rangle - \langle \mathbf{v}, \mathbf{v}\rangle = \|\mathbf{u}\|^2 - \langle \mathbf{u}, \mathbf{v}\rangle + \langle \mathbf{u}, \mathbf{v}\rangle - \|\mathbf{v}\|^2 = \|\mathbf{u}\|^2 - \|\mathbf{v}\|^2$

33. Using Exercise 32 and a similar identity for $\|\mathbf{u} - \mathbf{v}\|^2$, we have

$$\begin{aligned}\|\mathbf{u} + \mathbf{v}\|^2 + \|\mathbf{u} - \mathbf{v}\|^2 &= \langle \mathbf{u} + \mathbf{v}, \mathbf{u} + \mathbf{v}\rangle + \langle \mathbf{u} - \mathbf{v}, \mathbf{u} - \mathbf{v}\rangle \\ &= \|\mathbf{u}\|^2 + 2\langle \mathbf{u}, \mathbf{v}\rangle + \|\mathbf{v}\|^2 \\ &\quad + \|\mathbf{u}\|^2 - 2\langle \mathbf{u}, \mathbf{v}\rangle + \|\mathbf{v}\|^2 \\ &= 2\|\mathbf{u}\|^2 + 2\|\mathbf{v}\|^2\end{aligned}$$

Dividing by 2 yields the identity we want.

35.
$$\begin{aligned}\|\mathbf{u} + \mathbf{v}\| = \|\mathbf{u} - \mathbf{v}\| &\Leftrightarrow \|\mathbf{u} + \mathbf{v}\|^2 = \|\mathbf{u} - \mathbf{v}\|^2 \\ &\Leftrightarrow \|\mathbf{u}\|^2 + 2\langle \mathbf{u}, \mathbf{v}\rangle + \|\mathbf{v}\|^2 \\ &\quad = \|\mathbf{u}\|^2 - 2\langle \mathbf{u}, \mathbf{v}\rangle + \|\mathbf{v}\|^2 \\ &\Leftrightarrow 2\langle \mathbf{u}, \mathbf{v}\rangle = -2\langle \mathbf{u}, \mathbf{v}\rangle \Leftrightarrow \langle \mathbf{u}, \mathbf{v}\rangle = 0\end{aligned}$$

37. $\left\{\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}\right\}$ **39.** $\{1, x, x^2\}$

41. (a) $1/\sqrt{2}, \sqrt{3}x/\sqrt{2}, \sqrt{5}(3x^2 - 1)/2\sqrt{2}$
(b) $\sqrt{7}(5x^3 - 3x)/2\sqrt{2}$

Exercises 7.2

1. $\|\mathbf{u}\|_E = \sqrt{42}$, $\|\mathbf{u}\|_s = 10$, $\|\mathbf{u}\|_m = 5$

3. $d_E(\mathbf{u}, \mathbf{v}) = \sqrt{70}$, $d_s(\mathbf{u}, \mathbf{v}) = 14$, $d_m(\mathbf{u}, \mathbf{v}) = 6$

5. $\|\mathbf{u}\|_H = 4$, $\|\mathbf{v}\|_H = 5$

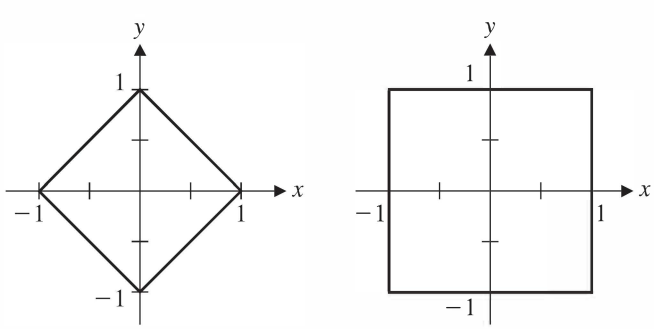
7. (a) At most one component of $\mathbf{v}$ is nonzero.

9. Suppose $\|\mathbf{v}\|_m = |v_k|$. Then $\|\mathbf{v}\|_E = \sqrt{v_1^2 + \cdots + v_k^2 + \cdots + v_n^2} \geq \sqrt{v_k^2} = |v_k| = \|\mathbf{v}\|_m$.

11. Suppose $\|\mathbf{v}\|_m = |v_k|$. Then $|v_i| \leq |v_k|$ for $i = 1, \ldots, n$, so

$$\begin{aligned}\|\mathbf{v}\|_s &= |v_1| + \cdots + |v_n| \leq |v_k| + \cdots + |v_k| \\ &= n|v_k| = n\|\mathbf{v}\|_m\end{aligned}$$

13.



21. $\|A\|_F = \sqrt{19}$, $\|A\|_1 = 4$, $\|A\|_\infty = 6$

23. $\|A\|_F = \sqrt{31}$, $\|A\|_1 = 6$, $\|A\|_\infty = 6$

25. $\|A\|_F = 2\sqrt{11}$, $\|A\|_1 = 7$, $\|A\|_\infty = 7$

27. $\mathbf{x} = \begin{bmatrix}0\\1\end{bmatrix}, \mathbf{y} = \begin{bmatrix}-1\\1\end{bmatrix}$ **29.** $\mathbf{x} = \begin{bmatrix}0\\0\\1\end{bmatrix}, \mathbf{y} = \begin{bmatrix}1\\1\\1\end{bmatrix}$

31. $\mathbf{x} = \begin{bmatrix}1\\0\\0\end{bmatrix}, \mathbf{y} = \begin{bmatrix}1\\-1\\-1\end{bmatrix}$

33. (a) By the definition of an operator norm, $\|I\| = \max\limits_{\|\mathbf{x}\|=1}\|I\mathbf{x}\| = \max\limits_{\|\mathbf{x}\|=1}\|\mathbf{x}\| = 1$.

35. $\text{cond}_1(A) = \text{cond}_\infty(A) = 21$; well-conditioned

37. $\text{cond}_1(A) = \text{cond}_\infty(A) = 400$; ill-conditioned

39. $\text{cond}_1(A) = 77$, $\text{cond}_\infty(A) = 128$; moderately ill-conditioned

41. (a) $\text{cond}_\infty(A) = (\max\{|k| + 1, 2\}) \cdot \left(\max\left\{\left|\frac{k}{k-1}\right| + \left|\frac{1}{k-1}\right|, \left|\frac{2}{k-1}\right|\right\}\right)$

43. (a) $\text{cond}_\infty(A) = 40$
(b) At most 400% relative change

45. Using Exercise 33(a), we have $\text{cond}(A) = \|A\|\|A^{-1}\| \geq \|AA^{-1}\| = \|I\| = 1$.

49. $k \geq 6$ **51.** $k \geq 10$

Exercises 7.3

1. $\|\mathbf{e}\| = \sqrt{2} \approx 1.414$ **3.** $\|\mathbf{e}\| = \sqrt{6}/2 \approx 1.225$

5. $\|\mathbf{e}\| = \sqrt{7} \approx 2.646$

7. $y = -3 + \frac{5}{2}x$, $\|\mathbf{e}\| \approx 1.225$

9. $y = \frac{11}{3} - 2x$, $\|\mathbf{e}\| \approx 0.816$

11. $y = \frac{7}{10} + \frac{8}{25}x$, $\|\mathbf{e}\| \approx 0.447$

13. $y = -\frac{1}{5} + \frac{7}{5}x$, $\|\mathbf{e}\| \approx 0.632$

15. $y = 3 - \frac{18}{5}x + x^2$ **17.** $y = \frac{18}{5} - \frac{17}{10}x - \frac{1}{2}x^2$

19. $\overline{\mathbf{x}} = \begin{bmatrix}\frac{1}{5}\\ \frac{7}{15}\end{bmatrix}$ **21.** $\overline{\mathbf{x}} = \begin{bmatrix}\frac{4}{3}\\ -\frac{5}{6}\end{bmatrix}$

23. $\overline{\mathbf{x}} = \begin{bmatrix}4 + t\\ -5 - t\\ -5 - 2t\\ t\end{bmatrix}$ **25.** $\begin{bmatrix}\frac{42}{11}\\ \frac{19}{11}\\ \frac{42}{11}\end{bmatrix}$

27. $\bar{\mathbf{x}} = \begin{bmatrix} \frac{5}{3} \\ -2 \end{bmatrix}$

29. $y = 0.92 + 0.73x$

31. (a) If we let the year 1920 correspond to $t = 0$, then $y = 56.6 + 2.9t$; 79.9 years

33. (a) $p(t) = 150e^{0.131t}$

35. 139 days

37. $\begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}, \begin{bmatrix} \frac{7}{2} \\ \frac{7}{2} \end{bmatrix}$

39. $\begin{bmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix}$

41. $\begin{bmatrix} \frac{5}{6} & \frac{1}{3} & -\frac{1}{6} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ -\frac{1}{6} & \frac{1}{3} & \frac{5}{6} \end{bmatrix}, \begin{bmatrix} \frac{5}{6} \\ \frac{1}{3} \\ -\frac{1}{6} \end{bmatrix}$

45. $A^+ = [\frac{1}{5} \quad \frac{2}{5}]$

47. $A^+ = \begin{bmatrix} \frac{1}{3} & -\frac{2}{3} & -\frac{1}{6} \\ \frac{1}{6} & \frac{1}{6} & \frac{1}{6} \end{bmatrix}$

49. $A^+ = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$

51. $A^+ = \begin{bmatrix} \frac{2}{3} & 0 & -\frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & -1 & \frac{1}{3} & \frac{2}{3} \\ -\frac{2}{3} & 1 & \frac{1}{3} & -\frac{1}{3} \end{bmatrix}$

53. (a) If A is invertible, so is A^T, and we have $A^+ = (A^TA)^{-1}A^T = A^{-1}(A^T)^{-1}A^T = A^{-1}$.

Exercises 7.4

1. 2, 3 **3.** $\sqrt{2}, 0$ **5.** 5 **7.** 2, 3

9. $\sqrt{5}, 2, 0$

11. $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} \sqrt{2} & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix}$

13. $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$

15. $A = \begin{bmatrix} \frac{3}{5} & -\frac{4}{5} \\ \frac{4}{5} & \frac{3}{5} \end{bmatrix}\begin{bmatrix} 5 \\ 0 \end{bmatrix}[1]$

17. $A = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix}\begin{bmatrix} 3 & 0 \\ 0 & 2 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

19. $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} \sqrt{5} & 0 & 0 \\ 0 & 2 & 0 \end{bmatrix}\begin{bmatrix} 2/\sqrt{5} & 0 & 1/\sqrt{5} \\ 0 & 1 & 0 \\ 1/\sqrt{5} & 0 & -2/\sqrt{5} \end{bmatrix}$

21. $A = \sqrt{2}\begin{bmatrix} 1 \\ 0 \end{bmatrix}[1/\sqrt{2} \quad 1/\sqrt{2}] + 0\begin{bmatrix} 0 \\ 1 \end{bmatrix}$
$[-1/\sqrt{2} \quad 1/\sqrt{2}]$ (Exercise 3)

23. (Exercise 7) $A = 3\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}[0 \quad 1] + 2\begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix}[1 \quad 0]$

33. The line segment $[-1, 1]$

35. The solid ellipse $\dfrac{y_1^2}{5} + \dfrac{y_2^2}{4} \le 1$

37. (a) $\|A\|_2 = \sqrt{2}$ **(b)** $\text{cond}_2(A) = \infty$

39. (a) $\|A\|_2 = 1.95$ **(b)** $\text{cond}_2(A) = 38.11$

41. $A^+ = \begin{bmatrix} \frac{1}{2} & 0 \\ \frac{1}{2} & 0 \end{bmatrix}$

43. $A^+ = \begin{bmatrix} \frac{2}{5} & 0 \\ 0 & \frac{1}{2} \\ \frac{1}{5} & 0 \end{bmatrix}$

45. $A^+ = \begin{bmatrix} \frac{1}{25} & \frac{2}{25} \\ \frac{2}{25} & \frac{4}{25} \end{bmatrix}, \bar{\mathbf{x}} = \begin{bmatrix} 0.52 \\ 1.04 \end{bmatrix}$

47. $A^+ = \begin{bmatrix} \frac{1}{6} & \frac{1}{6} & \frac{1}{6} \\ \frac{1}{6} & \frac{1}{6} & \frac{1}{6} \end{bmatrix}, \bar{\mathbf{x}} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

61. $\begin{bmatrix} \sqrt{2} & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$

63. $\begin{bmatrix} 2 & -1 \\ -1 & 3 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$

Exercises 7.5

1. $g(x) = \frac{1}{3}$ **3.** $g(x) = \frac{3}{5}x$

5. $g(x) = \frac{3}{16} + \frac{15}{16}x^2$ **7.** $\{1, x - \frac{1}{2}\}$

9. $g(x) = x - \frac{1}{6}$

11. $g(x) = (4e - 10) + (18 - 6e)x \approx 0.87 + 1.69x$

13. $g(x) = \frac{1}{20} - \frac{3}{5}x + \frac{3}{2}x^2$

15. $g(x) = 39e - 105 + (588 - 216e)x + (210e - 570)x^2 \approx 1.01 + 0.85x + 0.84x^2$

21. $\dfrac{\pi}{2} - \dfrac{4}{\pi}\left(\cos x + \dfrac{\cos 3x}{9}\right)$

23. $a_0 = \frac{1}{2}, a_k = 0, b_k = \dfrac{1 - (-1)^k}{k\pi}$

25. $a_0 = \pi, a_k = 0, b_k = \dfrac{2(-1)^k}{k}$

Review Questions

1. (a) T **(c)** F **(e)** T **(g)** T **(i)** T

3. Inner product **5.** $\sqrt{3}$ **7.** $\left\{\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right\}, \begin{bmatrix} -\frac{1}{2} \\ \frac{1}{2} \end{bmatrix}$

9. Not a norm

11. $\text{cond}_\infty(A) \approx 2432$

13. $y = 1.7x$

15. $\begin{bmatrix} \frac{7}{3} \\ \frac{2}{3} \\ \frac{5}{3} \end{bmatrix}$

17. (a) $\sqrt{2}, \sqrt{2}$

(b) $A = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & 0 & 1 \\ 1/\sqrt{2} & -1/\sqrt{2} & 0 \end{bmatrix} \begin{bmatrix} \sqrt{2} & 0 \\ 0 & \sqrt{2} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

(c) $A^+ = \begin{bmatrix} \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & 0 & -\frac{1}{2} \end{bmatrix}$

19. The singular values of PAQ are the square roots of the eigenvalues of $(PAQ)^T(PAQ) = Q^TA^TP^TPAQ = Q^T(A^TA)Q$. But $Q^T(A^TA)Q$ is similar to A^TA because $Q^T = Q^{-1}$, and hence it has the same eigenvalues as A^TA. Thus, PAQ and A have the same singular values.

Index