ANALOGUES OF THE LEBESGUE DENSITY THEOREM FOR FRACTAL SETS OF REALS AND INTEGERS

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ABSTRACT

We prove the following analogues of the Lebesgue density theorem for two types of fractal subsets of $\mathbb{R}$: cookie-cutter Cantor sets and the zero set of a Brownian path. Write C for the set, and μ for the positive finite Hausdorff measure on C. Then there exists a constant c (depending on the set C) such that for μ-almost every $x \in C$,

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T \frac{\mu(B(x, e^{-t}))}{(2e^{-t})^d}\,dt = c,$$

where $B(x, \varepsilon)$ is the ε-ball around x and d is the Hausdorff dimension of C. We also define analogues of Hausdorff dimension and Lebesgue density for subsets of the integers, and prove that a typical zero set of the simple random walk has dimension $\frac{1}{2}$ and density $\sqrt{(2/\pi)}$.

1. *Introduction*

In this paper we introduce a notion of density for fractal and fractal-like sets including certain kinds of Cantor sets and sparse sets of integers. This type of density is called order-two density, because it is based on the use of an order-two averaging method, in the sense of [**14**, **15**], to obtain a limit where the usual density of a measure or set does not exist.

The main examples considered here are the middle-third Cantor set, non-linear hyperbolic Cantor sets and the zero set of a Brownian path. Examples of fractal-like subsets of the integers which are considered are the integer middle-third Cantor set (for a definition see below) and the zero set of a simple random walk.

Hyperbolic 'cookie-cutter' Cantor sets (the terminology is due to Sullivan) were chosen because, in addition to their own intrinsic interest, the use of the basic tools (for example, the use of both the Gibbs and the conformal measures, the bounded distortion property, and the suspension to an ergodic flow over the Cantor set) is quite clear. This should enable an extension of the theory to more general situations where Bowen's Hausdorff dimension formula holds. This has already been done for certain hyperbolic Julia sets in conjunction with M. Urbański. For an overview of what is known about cookie-cutter Cantor sets, for complete references and for a self-contained development of the tools mentioned above, see [**4**].

For the zero set of a Brownian path, our result is related to an additive functional limit theorem of Brosamler ([**8**, Theorem 2.1]; there the limit at infinity is studied) and for the simple random walk, it can be seen as a special case of a beautiful but little known almost-sure limit theorem of Chung and Erdös [**10**, Theorem 6]. The proof we give here uses the ergodicity of the scaling flow plus Strassen's invariance principle and an almost-sure invariance principle of Révész for local time. Order-two density can also be proved to exist for times of return to

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a set of finite measure in a class of infinite ergodic measure-preserving transformations; this is joint work with J. Aaronson and M. Denker and will appear elsewhere. The theorem proved there is closely related to Chung and Erdös' work although the proof and interpretation are quite different.

The purposes of the present paper are several. Firstly we introduce the notion of order-two density and develop its basic properties: consistency with respect to usual density (which however diverges almost surely for all the examples mentioned above); a Radon–Nikodym-like result for absolutely continuous measures; and the comparison with a hierarchy of order-n densities based on the Hardy–Riesz log averages (see **[15]**) and on the averaging operators of **[14]**. Secondly we introduce the techniques needed to prove existence of the order-two density for the examples mentioned above; the existence of order-2 density for the Hausdorff measure on these sets can be considered to be an analogue (for Hausdorff measure) of the Lebesgue density theorem. Finally, we want to show that there is a deep underlying connection between all of the techniques we use—even though they may at first seem disparate. The middle-third Cantor set is dealt with in some detail because it is possible there to show these connections. The analogies that one sees between the different situations are not precise but seem to be very helpful in suggesting problems and methods.

The notion of order-two density is related to Mandelbrot's concept of lacunarity (see **[24]**, especially pp. 315–318, for an intuitive description and illustrations). The lacunarity of a fractal should describe the degree to which the structure is fractured; one wants a way of comparing different sets of the same dimension or related sets of different dimensions. Order-two density provides a possible tool for making such comparisons. In the physics literature Smith, Fournier and Spiegel **[33]** observe that estimates of fractal dimension (they consider in particular the correlation dimension) can show log-oscillatory behaviour. When such oscillations occur, this brings added difficulties to the problem of numerical estimation of dimension. Smith, Fournier and Spiegel are suggesting that one can however make use of this oscillation as a way of measuring the 'textural property of fractal objects that Mandelbrot calls lacunarity'. But as they point out, if the sets are not strictly self-similar then in general the oscillations can damp out for small radius R. In that case, apparently, one will not get a helpful definition of lacunarity by using the amplitude of the oscillation. Some examples where one would expect to see such damping are the non-linear sets studied in § 4 below.

Mandelbrot deals with the problem of oscillation in a different way. First, he considers the distribution of the values of mass $M(x, R)$ in a ball of fixed radius R about points x in the fractal (that is, integrating over x). The moments of this distribution are to provide parameters which measure the lacunarity. However (again for self-similar fractals) this distribution will, after normalization by R^d, oscillate log-periodically. Therefore he restricts attention to random fractal sets and takes the ensemble average. The resulting distribution will in nice cases now be R-independent.

What we are suggesting instead is to study the oscillations of $M(x, R)$ for fixed x as $R \to 0$, by means of ergodic theory. This produces, for the examples studied below, a limiting distribution (which one could call the *lacunarity distribution* at x) and which in these examples is in fact the same for almost all x in the fractal. This distribution has as its mean value (i.e. as first moment) our order-two density. In a later paper we will study this distribution, and its higher moments, more closely. But for the present we focus only on the order-two density, since it is the most basic of this class of measurements.

The authors would like to thank J. Aaronson, M. Denker, H. v.d. Weide, and especially M. Urbański for useful conversations. The work was partially carried out while the first author was visiting Göttingen, and while the second author was visiting Warwick and Delft.

2. *Definitions and properties of order-two density*

We wish to define d-dimensional analogues of the 'ordinary' densities, Lebesgue density (for subsets of $\mathbb{R}^n$) and Cesàro density (for subsets of $\mathbb{Z}$). For subsets of $\mathbb{R}^n$, Hausdorff dimension is considered; the corresponding notion of dimension in $\mathbb{Z}$ is explained below. In this paper $\mathbb{R}^n$ is always equipped with the usual Euclidean metric.

If the subset under consideration has Hausdorff dimension d smaller than the dimension of the ambient space then the most obvious analogue of Lebesgue density does not exist, because the sparseness of the set implies large fluctuations in the amount of mass in a neighbourhood of a point as the neighbourhood shrinks.

Order-two density for subsets of the reals

The outer d-dimensional Hausdorff measure of a set $C \subset \mathbb{R}^n$ is given by

$$H^d(C) = \lim_{\varepsilon\to 0} \inf_{U_i} \left\{ \sum_{i=1}^{\infty} \varphi_d(|U_i|)\colon\ |U_i| < \varepsilon, \bigcup U_i \supset C \right\},$$

where $\{U_i\}$ is a countable cover of C, $|U_i| = \operatorname{diam} U_i$ and $\varphi_d(t) = t^d$. The Hausdorff dimension of C is the unique d with the property that $d = \inf\{\delta\colon H^\delta(C) = 0\}$. A subset $C \subset \mathbb{R}^n$ is called a d-set by Falconer **[12]** if it is measurable with respect to d-dimensional Hausdorff measure H^d and $0 < H^d(C) < \infty$. We shall denote the restriction of H^d to C by μ.

DEFINITIONS. The *upper* and *lower densities* (in dimension d) of C at $x \in \mathbb{R}^n$ are respectively

$$\bar{D}(C, x) = \overline{\lim_{\varepsilon\to 0}} \frac{\mu(B(x, \varepsilon))}{(2\varepsilon)^d}, \quad \text{and} \quad \underline{D}(C, x) = \underline{\lim}_{\varepsilon\to 0} \frac{\mu(B(x, \varepsilon))}{(2\varepsilon)^d}.$$

If $\bar{D}(C, x) = \underline{D}(C, x)$, we call this common value the *density* of μ at x and denote it by $D(C, x)$; then one says that x is a *regular* point.

(We shall call this *ordinary* d-dimensional density when there is a likelihood of confusion.) One of the main theorems of geometric measure theory says (see **[12**, Theorem 4.12]) that for $d < n$ and non-integer, μ-almost every point is irregular. This should be compared with the Lebesgue density theorem which says that if one replaces μ with Lebesgue measure λ then for any λ-measurable set C the density with respect to λ exists at λ-almost all points of C and equals 1.

Now we shall define a new type of density, which does exist almost everywhere in the examples treated below. We wish to control the fluctuations of $\mu(B(x, \varepsilon))/(2\varepsilon)^d$ as ε converges to zero; what we do is to replace ε with e^{-t} and then apply the Cesàro average.

DEFINITIONS. The *upper* and *lower order-two densities* of C at x are

$$\bar{D}_2(C, x) = \overline{\lim_{T\to\infty}} \frac{1}{T} \int_0^T \frac{\mu(B(x, e^{-t}))}{2^d e^{-td}}\, dt,$$

and

$$\underline{D}_2(C, x) = \underline{\lim}_{T\to\infty} \frac{1}{T} \int_0^T \frac{\mu(B(x, e^{-t}))}{2^d e^{-td}}\, dt.$$

We similarly define $D_2(C, x)$, the *order-two density* (in dimension d) to be the common value if these are equal. In this case x is said to be *order-two regular.*

We choose the name 'order-two density' because the method being used to smooth out the fluctuations of $\mu(B(x, \varepsilon))/(2\varepsilon)^d$ can be seen to be an order-two averaging method composed with an inversion (using the terminology of Fisher **[14]**).

The exact connection of order-two density with the order-two averaging methods is as follows. Setting $f(t) = \mu(B(x, t))/2^d t^d$, we have $D_2(C, x)$ equal to $\lim_{t\to\infty} A_\psi^2(f(1/t))$, where $\psi(x) = \chi_{(-\infty,0]}(x)e^x$ and $(A_\psi^2 g)(t)$ is defined to be $(\psi * (g \circ \exp \circ \exp)) \circ \log \circ \log(t)$. By Wiener's Tauberian theorem, this is equivalent to A_φ^2 where $\varphi = \psi(-x)$, which can be written in the more familiar form

$$(A_\varphi^2 g)(t) = \frac{1}{\log t} \int_1^t g(x) \frac{1}{x}\, dx.$$

This is the Hardy–Riesz log average; see **[15]**. Based on this formula one can, if the order-two density fails to converge, apply, in place of the order-two average, higher-order averaging methods from an infinite, consistent hierarchy—the Hardy–Riesz higher log averages—and also ultimately one could apply an exponentially invariant mean, as in **[14]**. Thus, replacing A_ψ^2 by A_ψ^n in the equation above, for $n \geq 1$, defines the *order-n density* $D_n(C, x)$.

The definition of order-n density of a set extends in a natural way to the density of a Borel measure ν on a σ-compact metric space M. For a fixed positive d, we then write $D_n(\nu, x)$ for the order-n density in dimension d of ν at x; when μ is Hausdorff measure restricted to a d-dimensional set $C \subseteq M$, one has by definition $D_n(\mu, x) = D_n(C, x)$. The relationship between densities for absolutely continuous measures is given in Theorem 2.2 below; this is a Radon–Nikodym type of theorem. We use this in § 4 when comparing Gibbs measure with Hausdorff measure.

For sets C in $\mathbb{R}^1$, it is also natural to talk about right and left densities. These densities, which will be denoted by $\bar{D}^r$, $\bar{D}^l$ and so on, are defined as above by replacing $\mu(B(x, \varepsilon))/2^d\varepsilon^d$ with $\mu([x, x + \varepsilon))/\varepsilon^d$.

We now note some basic properties of density and the order-n densities. For the examples studied in this paper the order-two density always exists. The first two properties hold also for the order-n density of a finite regular Borel measure ν on a σ-compact metric space M.

(1) For all $x \in M$ and for $1 \leq n \leq m$,

$$\underline{D}(C, x) \leq \underline{D}_n(C, x) \leq \underline{D}_m(C, x) \leq \bar{D}_m(C, x) \leq \bar{D}_n(C, x) \leq \bar{D}(C, x).$$

The same is true, in $\mathbb{R}^1$, for the right and left densities.

(2) $\underline{D}(C, x)$ and $\bar{D}(C, x)$ are Borel-measurable functions of x. The same holds for the order-n densities, and for right and left ordinary densities in $\mathbb{R}$.

(3) $2^{-d} \leqslant \bar{D}(C, x) \leqslant 1$ at μ-almost all $x \in C$, so by (1) we have $\bar{D}_n(C, x) \leqslant 1$. For the case where $M = \mathbb{R}$,

$$2^{-d}(\underline{D}^r + \underline{D}^l) \leqslant \underline{D} \leqslant \bar{D} \leqslant 2^{-d}(\bar{D}^r + \bar{D}^l);$$

hence $\bar{D}^r(C, x) \leqslant 2^d$ (and similarly for $\bar{D}^l$). The same holds for the order-n densities.

(4) $\bar{D}(C, x) = 0$ at H^d-almost all x outside C. As above, by (1) this holds for order-n density also.

(5) Let $\tilde{C}$ be a μ-measurable subset of C; then

$$\underline{D}(\tilde{C}, x) = \underline{D}(C, x) \quad \text{and} \quad \bar{D}(\tilde{C}, x) = \bar{D}(C, x)$$

for μ-almost all $x \in \tilde{C}$. For order-n density, the same is true when $\bar{D}_n = \underline{D}_n$. That is, if $D_n(C, x)$ exists for almost every $x \in C$, then

$$D_n(\tilde{C}, x) = D_n(C, x)$$

for almost every $x \in \tilde{C} \subset C$. The same holds, in $M = \mathbb{R}$, for left and right order-n densities.

(6) More generally, let $C = \bigcup_{n=0}^{\infty} C_n$, a countable disjoint union of d-sets with $H^d(C) < \infty$. Then for any n,

$$\underline{D}(C_n, x) = \underline{D}(C, x) \quad \text{and} \quad \bar{D}(C_n, x) = \bar{D}(C, x)$$

for almost all $x \in C_n$. As in (5), this is true for D_n when $\bar{D}_n = \underline{D}_n$, and in $\mathbb{R}$ it holds also for D_n^r and D_n^l.

(7) Let ψ: $\mathbb{R}^n \to \mathbb{R}^n$ be conformal, that is, a C^1 diffeomorphism which in the tangent space sends circles to circles. Then

$$\underline{D}(\psi(C), \psi(x)) = \underline{D}(C, x) \quad \text{and} \quad \bar{D}(\psi(C), \psi(x)) = \bar{D}(C, x).$$

The same is true for the upper and lower order-n densities and in $\mathbb{R}$ for right and left ordinary and order-n densities.

Proof of properties (1)–(6) for ordinary density can be either found in Chapter 2 of **[12]** or proved using the methods described there. Consistency of D_m with D_n for $n \leqslant m$ (Property 1) will be proved elsewhere since we do not actually need it in this paper; the basic idea can be seen in Lemma 4.4 of **[15]**. To prove Properties (5) and (6) for D_n we need first this lemma, which has its origins in work of Besicovitch. It follows as a corollary of Theorem 2.9.8 of **[13]**.

LEMMA 2.1. (i) *Let ν be a regular Borel measure on a compact metric space M, and let μ be absolutely continuous with respect to ν, with Radon–Nikodym derivative $d\mu/d\nu = f(x)$. Assume that the collection of open balls forms a ν-Vitali relation (see* **[13]***). Then for ν-almost every $x \in M$,*

$$\lim_{\varepsilon \to 0} \frac{\mu(B(x, \varepsilon))}{\nu(B(x, \varepsilon))} = f(x).$$

(ii) *For $M = \mathbb{R}^1$, one also has for almost every x,*

$$\lim_{\varepsilon \to 0} \frac{\mu([x, x + \varepsilon))}{\nu([x, x + \varepsilon))} = f(x)$$

(and similarly for the left-sided limits).

The assumption that the collection of open balls forms a ν-Vitali relation holds for any Borel ν in finite-dimensional vector spaces, finite-dimensional Riemannian manifolds, and in shift spaces with the usual metric. In particular, we can apply it in the case of $\mathbb{R}^1$. We thank B. Kirchheim for pointing out to us that this assumption is needed in the above lemma. Property (7) is easily proved from the conformal transformation property (see § 4). Note that although the order-two density of a set is defined by means of the Euclidean metric on $\mathbb{R}$, by (7) it remains the same under diffeomorphic changes of metric. A consequence of (7) for $\mathbb{R}^n$ is that order-two density is unambiguously defined for subsets of conformal n-dimensional manifolds (via charts).

The proof of the next theorem then follows in a straightforward way, by use of L'Hôpital's Rule; we postpone details to a later paper.

THEOREM 2.2. (i) *With M, ν and μ as above, if the order-n density exists at ν-almost every x and equals $g(x)$, then the order-n density of μ exists ν-almost surely, and equals $g(x)\cdot d\mu/d\nu = g(x)\cdot f(x)$.*

(ii) *For $M=\mathbb{R}^1$, the same equations hold for right- and left-sided order-n density.*

To prove (5) and (6) above, note that for $\tilde{C}\subset C$, if one sets $\mu=\nu|_{\tilde{C}}$ then $d\mu/d\nu=\chi_{\tilde{C}}$, so that (5) and (6) now follow as corollaries of Theorem 2.2. In § 5 we shall extend the notion of order-two density to cover sets with positive finite Hausdorff φ-measures for functions $\varphi(t)\neq t^d$ which are regularly varying at the origin.

Dimension and order-two density for subsets of the integers

We will say a subset F of the integers is *sparse* if it has Cesàro density zero.

DEFINITION. Let F be a subset of the non-negative integers $\mathbb{Z}^+$ and define $N_0=0$, and

$$N_n=N_n(F)=\operatorname{card}(F\cap\{0,1,2,\ldots,n-1\}),\quad \text{for } n\geq 1.$$

The *upper and lower dimensions* of F are

$$\overline{\dim}(F)=\limsup_{n\to\infty}\log N_n/\log n$$

and

$$\underline{\dim}(F)=\liminf_{n\to\infty}\log N_n/\log n.$$

If $\overline{\dim}(F)=\underline{\dim}(F)$ then we call the common value the *dimension* of F, $\dim(F)$.

A useful equivalent definition is: $\dim(F)=d$ if and only if for every $\varepsilon>0$ there exists n_0 such that for every $n>n_0$,

$$n^{-\varepsilon}<N_n/n^d<n^{\varepsilon}.$$

Definitions of dimension for discrete sets appear in **[18]** and **[25]**, but these definitions have been designed for other purposes and generally take different values than our dimension.

We will call $F \subseteq \mathbb{N}$ *fractal* if $\overline{\dim}(F)$ is less than 1. Note that any fractal set is also sparse. We now give the examples of fractal subsets of $\mathbb{N}$ which originally motivated the definitions.

(1) $F = \{n^k \colon n \in \mathbb{N}\}$, for k a fixed integer greater than 1, has dimension $1/k$: this follows from the observation that $n^{1/k} - 1 \leqslant N_n \leqslant n^{1/k}$ for all n.

(2) The *integer Cantor set*,

$$[C] = \left\{\sum_{i=0}^{N} a_i 3^i \colon\ N \in \mathbb{N},\ a_i = 0 \text{ or } 2\right\}$$

$$= \{0, 2, 6, 8, 18, 20, 24, 26, \dots\},$$

has dimension $d = \log 2/\log 3$ (not surprisingly!?) which comes directly from the fact that $(n/2)^d \leqslant N_n \leqslant n^d$ for all n.

(3) Let Z_S be the set of zeros of a *simple random walk* ($S_0 = 0$, $S_n = \sum_{i=1}^n X_i$, where $X_i = \pm 1$ with independent probabilities $(\frac{1}{2}, \frac{1}{2})$), that is $Z_S = \{n \colon S_n = 0\}$. Then $\dim(Z_S) = \frac{1}{2}$ almost surely. This follows from bounds due to Chung and Erdös [**10**, Theorem 7], that for almost every S, given $\varepsilon > 0$ there exists n_0 such that for all $n > n_0$,

$$n^{\frac{1}{2}-\varepsilon} < N_n < n^{\frac{1}{2}+\varepsilon}.$$

Definition. Let $F \subset \mathbb{Z}^+$ have dimension d. The *order-two density* (in dimension d) is

$$\lim_{M \to \infty} \frac{1}{\log M} \sum_{k=1}^{M} (N_k/k^d) \frac{1}{k}$$

if the limit exists.

Note that this is the Hardy–Riesz log average applied to the sequence N_k/k^d. We mention that if, for instance, the (Cesàro) density of a set of integers exists and is positive, that is, $\lim_{n\to\infty} N_n/n = a > 0$, then the set has dimension equal to 1; this is straightforward to check.

For the examples we described above, the following are true.

(1) $F = \{n^k\}$ has order-two density 1 since in fact $N_n/n^{1/k} \to 1$.

(2) The integer Cantor set has order-two density which equals the right order-two density of the middle third Cantor set at 0 (see § 3). This can be proved by analogy with the proof given for the random-walk zeros in § 5.

(3) In § 5 we prove that the order-two density of Z_S exists almost surely and is equal to $\sqrt{(2/\pi)}$.

As in the real case, when the log average fails to converge, one can apply a higher-order averaging operator or an invariant mean. Details will appear in a later paper. The case of the integer Cantor set leads to some interesting ergodic theory; see [**16**]. Further i.i.d. random walk examples will also be treated in [**1**].

We suggest two possible interpretations for integer order-two density. First, it gives the density of a set F 'at the point $+\infty$' analogous to real order-two density of C at a point $x \in C$. Second, it is a sort of (finitely additive) d-dimensional Hausdorff measure on subsets of the integers. This second analogy is strengthened if one extends the definition to all subsets of $\mathbb{Z}^+$, by use of an appropriate invariant mean.

3. *The middle-third set*

In this section we will prove the existence of the order-two density at almost every point, and at every rational point, of the middle-third set C. This set has a very nice structure that makes the proof especially simple. We shall concentrate here on right order-two density, and prove that it is almost surely constant. From this we can determine the left and symmetric order-two densities using the symmetry of C.

The middle-third Cantor set is defined as $C = \{\sum_{i=1}^{\infty} a_i 3^{-i}\colon a_i = 0, 2\}$. It has Hausdorff dimension $d = \log 2/\log 3$, and its Hausdorff measure $H^d(C)$ is equal to one; see [**12**] for proofs. We let μ denote H^d restricted to C. The Cantor function (or Devil's Staircase) L (shown in Fig. 1), is defined by $L\colon [0, 1] \to [0, 1]$ with $L(y) = \mu([0, y])$; that is, L is the distribution function of μ and pushes μ forward to Lebesgue measure on $[0, 1]$. One can easily check the following explicit formula for L:

$$L\left(\sum_{i=1}^{\infty} a_i 3^{-i}\right) = \sum_{i=1}^{\infty} b_i 2^{-i},$$

where $b_i = 0$ when $a_i = 0$ and $b_i = 1$ when $a_i = 1, 2$. We use the letter L in analogy with P. Lèvy's local time for Brownian motion (see § 5). This important property of L that we shall use is its scaling structure: for any $y \in [0, 1]$,

$$L(\tfrac{1}{3}y) = \tfrac{1}{2}L(y).$$

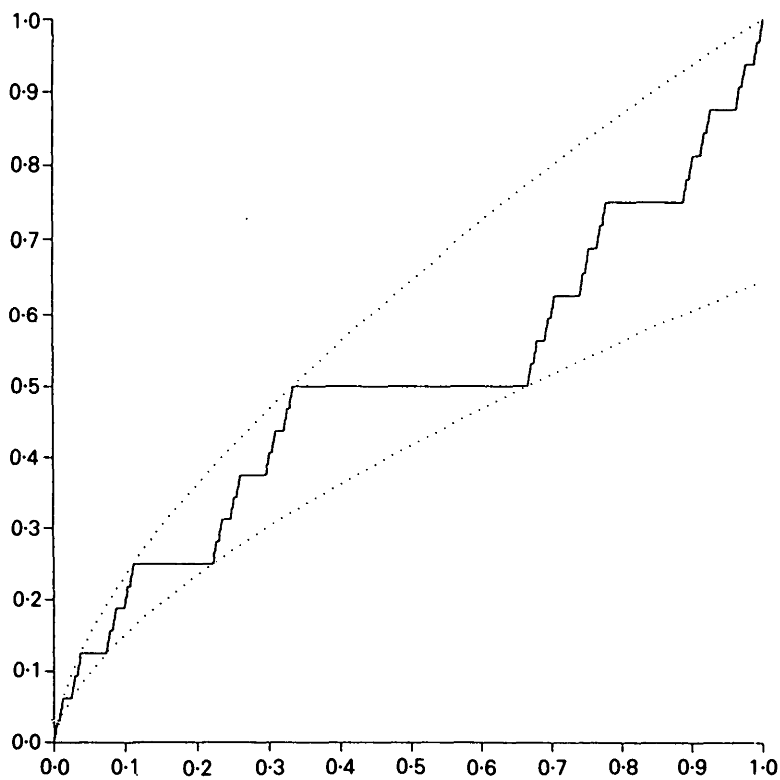

FIG. 1. The Devil's staircase function $L(y)$ with upper and lower envelopes y^d and $(\frac{1}{2}y)^d$.

This implies in particular that for any $t \geqslant 0$,

$$\frac{L(e^{-t-\log 3})}{e^{-dt-d\log 3}} = \frac{L(e^{-t})}{e^{-dt}}.$$

In other words the function $t \mapsto L(e^{-t})/e^{-td}$ is periodic with period $\log 3$ (see Fig. 2). This proves that the right order-two density of C at zero,

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T \frac{L(e^{-t})}{e^{-dt}}\, dt,$$

exists, because the Cesàro average of any periodic function converges.

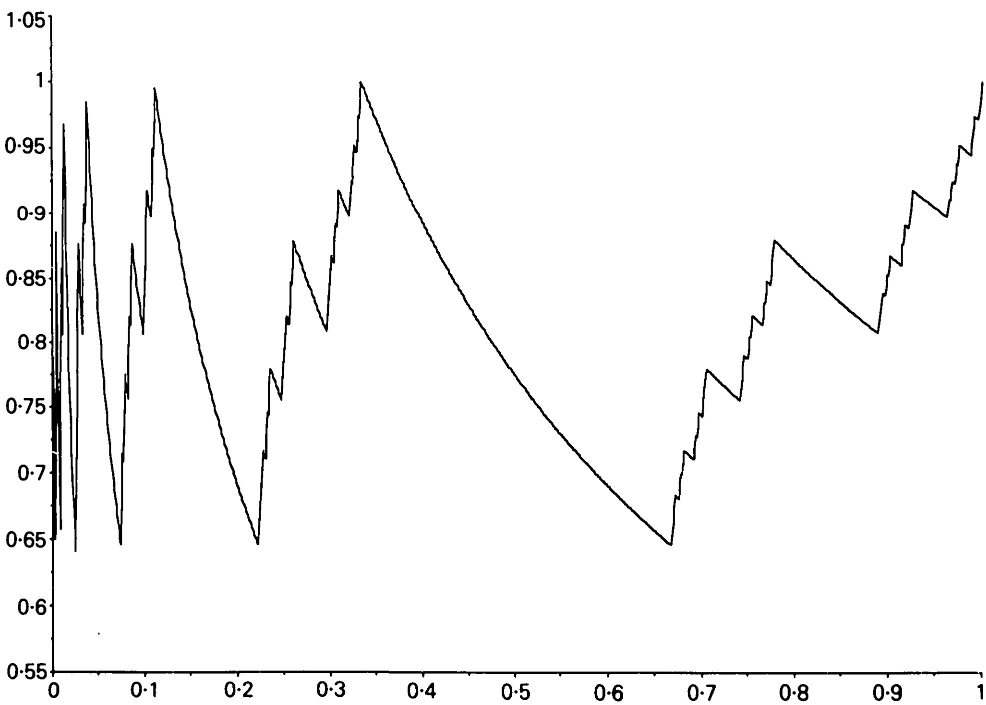

FIG. 2. The function $y \mapsto L(y)/y^d$.

It is clear that 0 is a very special point of the Cantor set, but there are also other points in C where the function

$$f(x, t) = \frac{\mu([x, x+e^{-t}))}{e^{-td}} = \frac{L(x+e^{-t}) - L(x)}{e^{-td}}$$

is periodic in t. Consider, for example, the points $x_1 = \frac{1}{4}$, $x_2 = \frac{3}{4}$. Figs 3a and 3b show the functions

$$y \mapsto L_{x_1}(y) = L(x_1 + y) - L(x_1) = \mu([x_1, x_1 + y))$$

and

$$y \mapsto L_{x_2}(y) = L(x_2 + y) - L(x_2) = \mu([x_2, x_2 + y)).$$

These functions satisfy

$$L_{x_1}(\tfrac{1}{3}y) = \tfrac{1}{2}L_{x_2}(y)$$

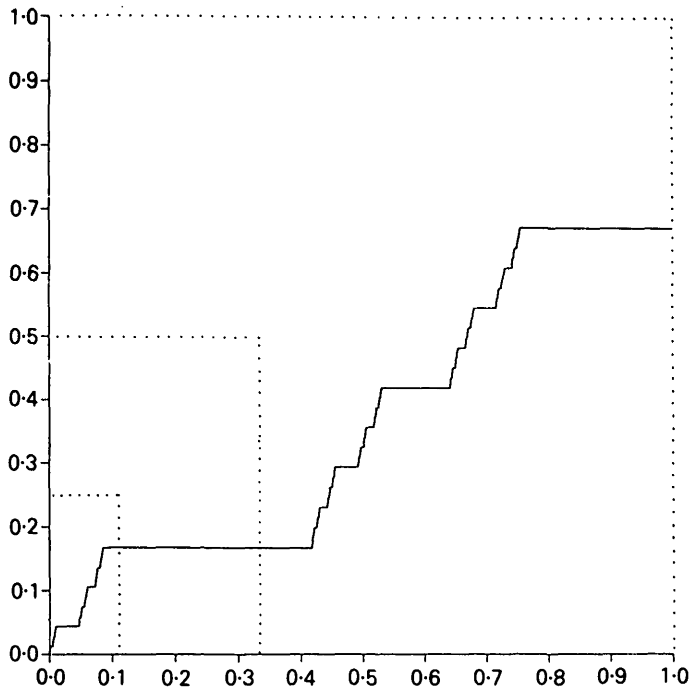

FIG. 3a. The function $y \mapsto L_{\frac{1}{4}}(y) = \mu([\frac{1}{4}, \frac{1}{4} + y))$.

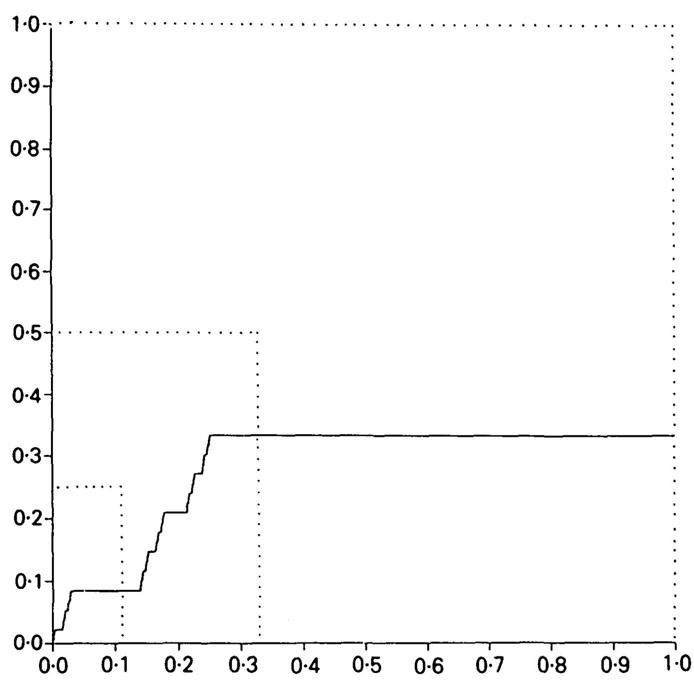

FIG. 3b. The function $y \mapsto L_{\frac{3}{4}}(y) = \mu([\frac{3}{4}, \frac{3}{4} + y))$.

and

$$L_{x_2}(\tfrac{1}{3}y) = \tfrac{1}{2}L_{x_1}(y)$$

(we shall see why in a moment) which, in particular, implies that $f(x_1, t)$ and $f(x_2, t)$ are periodic in t with period $2\log 3 = \log 9$. The right order-two densities at x_1 and x_2 therefore exist and since $f(x_1, t + \log 3) = f(x_2, t)$ for all t, the limits are equal. L_{x_1} and L_{x_2} are related because under the map S: $[0, 1] \to [0, 1]$ given by $S(x) = 3x \pmod 1$, the whole Cantor set is invariant, with $S(x_1) = x_2$ and $S(x_2) = x_1$. Now, for any small $y > 0$,

$$S[x_1, x_1 + \tfrac{1}{3}y) = [x_2, x_2 + y)$$

and, by the conformal transformation property of Hausdorff measure (see § 4),

$$\mu([x_1, x_1 + \tfrac{1}{3}y)) = \tfrac{1}{2}\mu([x_2, x_2 + y)),$$

which gives $L_{x_1}(\tfrac{1}{3}y) = \tfrac{1}{2}L_{x_2}(y)$. Similarly one gets $L_{x_2}(\tfrac{1}{3}y) = \tfrac{1}{2}L_{x_1}(y)$. These expressions generalise as follows. For each $x \in C$ define $L_x(t) = \mu[x, x + t)$ for $t \in [0, 1]$. We call this the *local time at* x. For each $x \in C$ we have

$$L_x(\tfrac{1}{3}t) = \tfrac{1}{2}L_{S(x)}(t).$$

This implies that

$$f(x, t + \log 3) = f(S(x), t) \quad \text{for all } x \in C,\ t \geq 0. \tag{3.1}$$

Now, for general points $x \in C$, the function $f(x, t)$ is not necessarily periodic in t, but enough statistical regularity exists at typical points for the order-two density to exist at μ-almost all points. The key to understanding this is the combination of (3.1) together with the observation

(3.2) μ is invariant and ergodic under the transformation S: $C \to C$.

Here (3.1) comes immediately from the conformal transformation property of μ, whilst (3.2) can either be checked directly or be seen from the fact that the system (S, μ) is naturally isomorphic to the one-sided $(\tfrac{1}{2}, \tfrac{1}{2})$ Bernoulli shift (ergodicity means that if $K \subset C$ is a Borel set such that $K = S^{-1}K$ then $\mu(K) = 0$ or 1).

We note that the S-periodic points are exactly the rational numbers in C, and that these are in fact the only points where f is a periodic function; this is not too hard to check.

THEOREM 3.1. *For any point $x \in C$ that is periodic with respect to S and also for μ-almost all $x \in C$ the order-two density $D_2(C, x)$ exists. Furthermore, for μ-almost all x,*

$$D_2^r(C, x) = D_2^l(C, x) = \frac{1}{\log 3}\int_0^{\log 3}\int_C f(z, t)\, d\mu(z)\, dt,$$

and

$$D_2(C, x) = 2^{1-d}D_2^r(C, x).$$

Proof. Define a function F: $C \to \mathbb{R}$ by

$$F(x) = \frac{1}{\log 3}\int_0^{\log 3} f(x, t)\, dt.$$

The function $f(x, t)$ is jointly continuous since L is continuous. This implies that F is also continuous. Note that

$$\sum_{i=0}^{n-1} F(S^i x) = \sum_{i=0}^{n-1} \frac{1}{\log 3} \int_0^{\log 3} f(S^i x, t)\, dt$$
$$= \frac{1}{\log 3} \int_0^{n \log 3} f(x, t)\, dt \quad \text{(from (3.1))}.$$

Averaging F along an S-orbit thus corresponds to averaging $f(x, t)$ over t; for, letting $n(T) = [T/\log 3]$, we have

$$\left| \frac{1}{n(T)} \sum_{i=0}^{n(T)-1} F(S^i x) - \frac{1}{T} \int_0^T f(x, t)\, dt \right|$$
$$= \left| \frac{1}{n(T) \log 3} \int_0^{n(T) \log 3} f(x, t)\, dt - \frac{1}{T} \int_0^T f(x, t)\, dt \right|$$
$$\leqslant \frac{(T - n(T) \log 3) 2\, \|f\|_\infty}{n(T) \log 3}$$
$$\leqslant \frac{2\, \|f\|_\infty}{n(T)},$$

which converges to zero as $T \to \infty$, using the fact that f is a jointly continuous function on a compact set and hence bounded ($\|\cdot\|_\infty$ denotes the sup-norm). Now the Birkhoff ergodic theorem says that, for μ-almost all x,

$$\frac{1}{n} \sum_{i=0}^{n-1} F(S^i x) \to \int_C F(z)\, d\mu(z),$$

and so we have for μ-almost all x that

$$\frac{1}{T} \int_0^T f(x, t)\, dt \to \int_C F(z)\, d\mu(z) = \frac{1}{\log 3} \int_0^{\log 3} \int_C f(z, t)\, d\mu(z)\, dt.$$

We mention two other ways in which one can prove the above result; these two ideas will be used for the work on hyperbolic Cantor sets in § 4 and on the Brownian zero sets in § 5. Define

$$M = \{(x, t) \colon x \in C, t \in [0, \log 3]\} / \equiv,$$

where $\equiv$ is the equivalence relation

$$(x, \log 3) \equiv (S(x), 0).$$

We can define a semi-flow on M by integrating the vector field $\dot{x} = 0$, $\dot{t} = 1$; that is, we flow up with unit speed in the constant-height suspension of S. Now the function $f(x, t)$ can be thought of as a function $f \colon M \to \mathbb{R}$ since it respects the identifications made in the definition of M, by property (3.1). Averaging $f(x, t)$ over t then corresponds to averaging f along the semiflow. Using ergodicity of the semiflow, one then gets convergence to a constant from almost all initial conditions for the semiflow. The second way to prove the result (which is the technique used for the Brownian motion example of § 5) is to take the space of paths $L_x(t)$ with the measure induced from μ on C. A scaling (semi-) flow on this space of functions can be defined such that the scaling flow does essentially the same as the flow induced on M above. One shows again that the flow is ergodic and that calculating the order-two density corresponds to taking an ergodic average of a certain function on the space of paths. In the corresponding construction for Brownian motion, the space of paths is the space of local times of the zero set for the Brownian motions.

4. *Hyperbolic Cantor sets*

In this section we show that for a class of Cantor sets in $\mathbb{R}^1$ the left, right and symmetric order-two densities of Hausdorff measure exist almost surely, and are each constant almost everywhere. One can see this as a version of the Lebesgue density theorem for Hausdorff measure on these Cantor sets, since almost every point has the same order-two density. We do not have an expression for this value in general, but J. Aaronson and T. Kamae have independently found ways to approximate the order-two density for the case of the middle-third Cantor set. As a corollary of the existence of order-two densities for Hausdorff measure, the order-two densities of the Gibbs measure also exist. We then prove that the densities exist at all periodic points, and show how the almost-sure value can be expressed in terms of the values at the periodic points. Further information on the techniques used here can be found in [**4**]. These techniques stem from Bowen's paper [**7**] which was the first to use the theory of Gibbs states to calculate Hausdorff dimension.

We now describe the construction of cookie-cutter Cantor sets.

Take a small neighbourhood $J \supset [0, 1]$ and two maps φ_0, φ_1: $J \to J$ satisfying the following hypotheses:

(1) $\varphi_0(0) = 0$, $\varphi_1(1) = 1$ and $\varphi_0(J) \cap \varphi_1(J) = \varnothing$;

(2) φ_0 and φ_1 are $\mathbf{C}^{1+\gamma}$ diffeomorphisms on their images;

(3) there exist $0 < \alpha < \beta < 1$ such that for all $x \in J$,

$$\alpha < |D\varphi_i(x)| < \beta \quad (i = 0, 1).$$

(Note that (1) implies that φ_0, φ_1 are orientation-preserving and thus that $D\varphi_0$, $D\varphi_1 > 0$. With minor changes to the proof of the existence of right order-two density, everything in this section can be done just as easily with orientation-reversing maps. For this reason we shall always use absolute value signs around $D\varphi_i$.)

Two such mappings φ_0, φ_1 uniquely determine a compact non-empty set $C = C(\varphi_0, \varphi_1)$ with the property that

$$C = \varphi_0(C) \cup \varphi_1(C)$$

(see [**21**]). Such a set will be called a hyperbolic Cantor set (the term hyperbolic is used since condition (3) is a hyperbolicity condition on φ_0 and φ_1). To see that $C = \varphi_0(C) \cup \varphi_1(C)$, first set $\Sigma = \{x_1x_2x_3 \ldots \mid x_n = 0 \text{ or } 1\}$; a point of Σ will be denoted $\underset{\sim}{x} = x_1x_2 \ldots$. Let $I = [0, 1]$ and for $\underset{\sim}{x} \in \Sigma$ let $I_{x_1 \ldots x_n} = \varphi_{x_1} \ldots \varphi_{x_n}(I)$, so that $I_{x_1 \ldots x_n} \supset I_{x_1 \ldots x_{n+1}}$. By (1) we have

$$I_0 = [0, \varphi_0(1)], \quad I_1 = [\varphi_1(0), 1]$$

and

$$I_0 \cap I_1 = \varnothing.$$

Inductively one sees that for any distinct finite sequences $x_1 \dots x_n$ and $y_1 \dots y_n$, the corresponding intervals $I_{x_1 \dots x_n}$ and $I_{y_1 \dots y_n}$ are disjoint. If we can show that $\mathrm{diam}(I_{x_1 \dots x_n}) \to 0$ as $n \to \infty$ then for any $\underline{x} \in \Sigma$, $\bigcap_{n=1}^{\infty} I_{x_1 \dots x_n}$ is a single point, and so

$$C = \bigcup_{\underline{x}} \bigcap_{n} I_{x_1 \dots x_n}$$

is a Cantor set (by which we mean it is homeomorphic to the middle-third set). Now

$$|I_{x_1 \dots x_n}| = |\varphi_{x_1} \dots \varphi_{x_n}(I)| \leq \beta^n \, |I| = \beta^n$$

so that $|I_{x_1 \dots x_n}| \to 0$ (in fact geometrically fast) as $n \to \infty$. We denote the map $\underline{x} \mapsto \bigcap_{n=1}^{\infty} I_{x_1 \dots x_n}$ by π: $\Sigma \to C$ and will use the notation $\pi(\underline{x}) = x$. We shall use the notation $J_{x_1 \dots x_n} = \varphi_{x_1} \dots \varphi_{x_n}(J)$.

The Cantor set C can be regarded as an invariant set of expanding dynamical system, with the map S: $J_0 \cup J_1 \to J$ defined by

$$S(x) = \begin{cases} \varphi_0^{-1}(x) & \text{if } x \in J_0, \\ \varphi_1^{-1}(x) & \text{if } x \in J_1. \end{cases}$$

(For the middle-third Cantor set one takes $\varphi_0(x) = \frac{1}{3}x$, $\varphi_1(x) = \frac{1}{3}x + \frac{2}{3}$ and $S(x) = 3x \pmod 1$.) The assumptions we made on φ_0, φ_1 then imply that S is a hyperbolic $\mathbf{C}^{1+\gamma}$ map with φ_0 and φ_1 as inverse branches, and with C as an invariant set. The condition from hypothesis (3) above implies that

$$\beta^{-1} < |DS(x)| < \alpha^{-1}. \tag{4.0}$$

Note that S^n maps $J_{x_1 \dots x_n}$ diffeomorphically onto J.

One can now apply the well-known argument of Bowen ([**7**]; see also, for example, [**2**, **4**]) to obtain an expression for the Hausdorff dimension d of C and to show that d-dimensional Hausdorff measure μ is positive and finite. The Hausdorff dimension is the unique real number d such that $P(-d \log |DS(x)|) = 0$, where P is the topological pressure. The concept of topological pressure is a part of the theory of equilibrium states (see [**6**]). We need only a few facts from this theory: there is a Borel probability measure ν on C which is invariant and ergodic with respect to S, and such that there exists $\eta \in (0, 1)$ with

$$\eta < \frac{\nu(I_{x_1 \dots x_n})}{\mu(I_{x_1 \dots x_n})} < \eta^{-1}, \tag{4.1}$$

for any $I_{x_1 \dots x_n}$ and

$$\eta < \nu(I_{x_1 \dots x_n}) \cdot |DS^n(x)|^d < \eta^{-1} \tag{4.2}$$

for any $x \in J_{x_1 \dots x_n}$ (the measure ν is actually the Gibbs—or equilibrium—state for the function $-d \log |DS(x)|$, and (4.2) is just a statement of the Gibbs property for our situation (see [**6**]) so we call ν the *Gibbs measure*). The reason for introducing ν is that the Ergodic Theorem can be used to obtain ν-almost everywhere results, which then automatically hold μ-almost everywhere because μ and ν are equivalent (with Radon–Nikodym derivative bounded below and above by η and η^{-1} respectively; this follows from (4.1)). In the case of the middle-third set, μ and ν are identical.

We shall make heavy use of two other facts. Firstly the *bounded distortion property* of S, which can be stated in this form: there exists $\hat{\eta} \in (0, 1)$ such that for all n,

$$\hat{\eta} < |I_{x_1 \dots x_n}| \cdot |DS^n(x)| < \hat{\eta}^{-1} \tag{4.3}$$

for any $x \in J_{x_1 \dots x_n}$ (to avoid too many constants we shall replace η by the minimum of η, $\hat{\eta}$ so that we can take $\eta = \hat{\eta}$ in the above inequality); see, for example, [**4**, **29**] for a proof. We also need the bounded distortion property in this slightly different form: there exists $\kappa > 0$ such that for $x, y \in J_{x_1 \dots x_n}$,

$$\big|\log |DS^n(x)| - \log |DS^n(y)|\big| < \kappa\, |S^n x - S^n y|^{\gamma}. \tag{4.4}$$

We mention that the bounded distortion property is proven in general for S^n restricted to an interval on which it is one-to-one; this is guaranteed by taking x, y to be in $J_{x_1 \dots x_n}$. Finally, we recall the fact that Hausdorff measure H^d on $\mathbb{R}^1$ satisfies the following *conformal transformation property*: for any one-to-one $\mathbf{C}^1$ map Φ: $\mathbb{R} \to \mathbb{R}$,

$$H^d(\Phi(E)) = \int_E |D\Phi|^d \, dH^d.$$

This is easily proved from the definition of Hausdorff outer measure. (In the higher-dimensional case $\mathbf{C}^1$ maps are replaced by conformal maps, which explains the terminology. Measures satisfying the conformal transformation property were first defined in the context of Fuchsian groups by Patterson [**27**], and for more general conformal transformations by Sullivan [**35**, **36**].) Now since S maps C to C, the measure μ (which is the restriction of H^d to C) satisfies

$$\mu(S(E)) = \int_E |DS(x)|^d \, d\mu(x), \tag{4.5}$$

for every E where $S|_E$ is one-to-one. Such a measure is known as *conformal measure* for the pair (C, S), so we shall refer to μ both as Hausdorff measure and conformal measure.

For a hyperbolic Cantor set we show that the order-two density and the right and left order-two densities exist μ-almost everywhere and are constant almost surely.

The arguments for left and right order-two densities are identical (up to confusion of left and right) and follow the argument for the symmetric case with a few obvious changes. We therefore give only the proof in the symmetric case.

Define a function f: $C \times \mathbb{R}^+ \to \mathbb{R}$,

$$f(x, t) = \frac{\mu((x - e^{-t}, x + e^{-t}))}{e^{-td}}.$$

We will show that the order-two density

$$D_2(C, x) = 2^{-d} \lim_{T \to \infty} \frac{1}{T} \int_0^T f(x, t)\, dt$$

exists μ-almost everywhere by comparing the function f to functions defined on C for which we can use the Ergodic Theorem to obtain averaging results. Define

$$M = \{(x, t)\colon\ x \in C,\ 0 \leqslant t \leqslant \log |DS(x)|\}/\equiv,$$

where $\equiv$ is the equivalence relation

$$(x, \log|DS(x)|) \equiv (S(x), 0).$$

There is a semi-flow Φ_t defined on M by flowing with unit speed in the t-direction. On M we define a function g_{t_0}: $M \to \mathbb{R}$ for each $t_0 \geq 0$ by

$$g_{t_0}(x, t) = f(x, t + t_0).$$

This function extends naturally to the domain $C \times \mathbb{R}^+$ by the equivalence relation $\equiv$; that is, it is extended so as to satisfy the equation

$$g_{t_0}(x, t + \log|DS(x)|) = g_{t_0}(Sx, t)$$

for all $t \geq 0$. We also have corresponding functions f_{t_0}: $C \times \mathbb{R}^+ \to \mathbb{R}$ given by

$$f_{t_0}(x, t) = f(x, t + t_0).$$

Our strategy is to show that f_{t_0} and g_{t_0} are close to each other uniformly in x and t, and then to use the Ergodic Theorem to show that $\lim_{T\to\infty} T^{-1} \int_0^T g_{t_0}\, dt$ exists. In the original version of this paper we estimated f_{t_0} and g_{t_0} via the ratios of certain quantities, in a way which necessitated separate considerations of the one-sided and symmetric cases. Following a suggestion of the referee and of M. Urbanski, however, we have replaced these estimates by difference-based estimates. This enables one to deal with the one-sided and symmetric arguments in the same way. The first step is to find a uniform bound on $f(x, t)$; note that *a fortiori* one then has the same bounds for f_{t_0} and g_{t_0}, for each $t_0 > 0$. The following lemma is well-known and holds in more general dynamical systems.

LEMMA 4.1. *The function $f(x, t)$ is bounded away from* 0 *and* ∞. *In fact for all x and t,*

$$\eta^{3+2d}\alpha^d \leq f(x, t) \leq 3\eta^{-3-2d}\alpha^{-d}.$$

In the next four lemmas, we prepare the ingredients for the proof of Proposition 4.6. We write $A_0 = B(y, \varepsilon)$ and $A_n = B(S^n y, |DS^n(y)| \cdot \varepsilon)$, and show that the following three quantities are almost equal: $\mu(A_0) \cdot |DS^n(y)|^d$, $\mu(S^n(A_0))$, and $\mu(A_n)$. Note that if (C, S) were a linear cookie-cutter (by this we mean that there exists an n such that DS is constant on each nth-level interval $I_{x_0 \ldots x_{n-1}}$) as for the middle third set, then these quantities are equal. We shall assume in Lemmas 4.2–4.5 that ε is small enough that $A_0 \subset J_{y_1 \ldots y_n}$; this is the hypothesis needed to apply the bounded distortion property (4.4) to S^n on A_0. First we need a preliminary lemma.

LEMMA 4.2. *There is a constant $k_0 > 0$ such that if $z \in A_0$ then*

$$\left|1 - \frac{|DS^n(y)|^d}{|DS^n(z)|^d}\right|, \left|1 - \frac{|DS^n(y)|}{|DS^n(z)|}\right| \leq k_0 |A_n|^\gamma.$$

Proof. By (4.4) we know that, since $A_0 \subset J_{y_1 \ldots y_n}$, we have

$$e^{-\kappa|J|^\gamma} < \frac{|DS^n(y)|}{|DS^n(z)|} < e^{\kappa|J|^\gamma},$$

and so $|DS^n(y)|/|DS^n(z)| \in U$ for some neighbourhood U of 1 bounded away from 0 and ∞. Now since the exponential function is Lipschitz on the domain

$\log U$, there is a $k'>0$ such that

$$|x-x'|\leqslant k'|\log x-\log x'| \quad \text{for } x, x' \in U.$$

Taking $x=1$ and $x'=|DS^n(y)|/|DS^n(z)|$ we have

$$\begin{aligned}\left|1-\frac{|DS^n(y)|}{|DS^n(z)|}\right| &\leqslant k'\,\big|\log|DS^n(y)|-\log|DS^n(z)|\big|\\ &\leqslant k'\kappa\,|S^n(y)-S^n(z)|^\gamma\\ &\leqslant k'\kappa\,|S^n(A_0)|^\gamma\\ &= k'\kappa\,|A_0|^\gamma\,|DS^n(x)|^\gamma \quad \text{(for some } x\in A_0 \\ &\qquad\qquad\qquad\qquad\qquad \text{by the Mean Value Theorem)}\\ &= k'\kappa\,|A_n|^\gamma\frac{|DS^n(x)|^\gamma}{|DS^n(y)|^\gamma} \quad \text{(by definition of } A_n)\\ &\leqslant k'\kappa e^{\gamma\kappa|J|^\gamma}|A_n|^\gamma \quad \text{(by the above inequality).}\end{aligned}$$

Setting $k_0=k'\kappa e^{\gamma\kappa|J|^\gamma}$ gives one of the claimed inequalities. A similar estimate holds for

$$\left|1-\frac{|DS^n(y)|^d}{|DS^n(z)|^d}\right|,$$

taking $x'=|DS^n(y)|^d/|DS^n(z)|^d$ in the argument. This gives the constant dk_0 and since $d<1$, k_0 works in both inequalities.

LEMMA 4.3. *There exists $k_0>0$ such that for any $y\in C$ and $\varepsilon>0$,*

$$\left|1-\frac{\mu(A_0)\,|DS^n(y)|^d}{\mu(S^nA_0)}\right|\leqslant k_0\,|A_n|^\gamma.$$

Proof. By the transformation property (4.5) of μ we have

$$\frac{\mu(A_0)\,|DS^n(y)|^d}{\mu(S^nA_0)}=\frac{\mu(A_0)\,|DS^n(y)|^d}{\int_{A_0}|DS^n(z)|^d\,d\mu(z)}=\frac{|DS^n(y)|^d}{|DS^n(z)|^d}$$

for some $z\in\bar{A}_0$, since DS^n is continuous. Applying Lemma 4.2 finishes the proof.

LEMMA 4.4. *There exists $k_1>0$ such that for any $y\in C$ and $\varepsilon>0$,*

$$|\mu(S^n(A_0))-\mu(A_n)|\leqslant k_1\,|A_n|^{\gamma d+d}.$$

Proof. We have

$$|\mu(S^n(A_0))-\mu(A_n)|\leqslant\mu(S^n(A_0)\,\Delta A_n).$$

This symmetric difference is a union of two intervals. We first estimate the measure of the right interval $A_n^r\equiv(A_n\,\Delta S^n(A_0))\cap[S^n(y),1]$. Now the length of A_n is exactly $|A_n|=|DS^n(y)|\cdot|A_0|$. Hence

$$\begin{aligned}|A_n^r| &= \big|\tfrac12|A_n|-|S^nA_0\cap[S^n(y),1]|\big|\\ &\leqslant \big|\tfrac12|A_n|-\tfrac12|DS^n(z)|\,|A_0|\big| \quad \text{(for some } z\in A_0 \text{ by}\\ &\qquad\qquad\qquad\qquad\qquad \text{the Mean Value Theorem)}\\ &= \tfrac12|A_n|\left|1-\frac{|DS^n(z)|}{|DS^n(y)|}\right| \quad \text{(by definition of } |A_n|)\\ &= \tfrac12 k_0\,|A_n|^{\gamma+1} \quad \text{(by Lemma 4.2).}\end{aligned}$$

Next, writing $k' = \sup f$ (which is finite by Lemma 4.1), we have

$$\mu(A_n^r) \leqslant k' \, |A_n^r|^d \leqslant k'(\tfrac{1}{2}k_0)^d \, |A_n|^{(\gamma+1)d}.$$

With the same estimate for the left interval, A_n^l, we have

$$\mu(A_n \, \Delta S^n A_0) = \mu(A_n^r) + \mu(A_n^l) \leqslant k_1 \, |A_n|^{\gamma d + d},$$

where $k_1 = 2k'(\frac{1}{2}k_0)^d$.

LEMMA 4.5. *There exists $k_2 > 0$ such that for any y and ε, with A_0 and A_n as above,*

$$\left|\mu(A_0) \, |DS^n(y)|^d - \mu(A_n)\right| \leqslant k_2 \, |A_n|^{\gamma d + d}.$$

Proof. From Lemmas 4.3 and 4.4,

$$\begin{aligned}
\left|\mu(A_0) \, |DS^n(y)|^d - \mu(A_n)\right| &\leqslant \left|\mu(A_0) \, |DS^n(y)|^d - \mu(S^n A_0)\right| + |\mu(S^n A_0) - \mu(A_n)| \\
&\leqslant k_0 \, |A_n|^\gamma \mu(S^n A_0) + k_1 \, |A_n|^{\gamma d + d} \\
&\leqslant k_0 \, |A_n|^\gamma \left(\mu(A_n) + k_1 \, |A_n|^{\gamma d + d}\right) + k_1 \, |A_n|^{\gamma d + d} \\
&\leqslant k_0 \, |A_n|^\gamma \left(k' \, |A_n|^d + k_1 \, |A_n|^d\right) + k_1 \, |A_n|^{\gamma d + d} \\
&\qquad\qquad (k' = \sup f) \\
&\leqslant k_0(k' + k_1) \, |A_n|^{\gamma + d} + k_1 \, |A_n|^{\gamma d + d} \\
&\leqslant k_2 \, |A_n|^{\gamma d + d},
\end{aligned}$$

where $k_2 = k_0(k' + k_1) + k_1$.

Before proving the principal estimate we introduce the convenient notion of reduction of $t \geqslant 0$ modulo x, mod_x.

DEFINITION. Given $x \in C$ define $r_0(x) = 0$ and $r_n(x)$ to be the nth return time of $(x, 0)$ to the Poincaré cross-section $C \times \{0\}$ under the flow Φ_t, that is,

$$r_n(x) = \log |DS^n(x)| = \sum_{i=0}^{n-1} \log |DS(S^i x)|.$$

Furthermore, define $\mathrm{int}_x(t)$ to be the unique integer n with

$$r_n \leqslant t < r_{n+1},$$

and define $\mathrm{mod}_x(t) = t - r_n$ where $n = \mathrm{int}_x(t)$.

PROPOSITION 4.6. *There are constants t^*, $k_3 > 0$ such that for any $x \in C$, setting $\delta = \gamma d$, then for all $t_0 > t^*$ and for all $t \geqslant 0$,*

$$|f_{t_0}(x, t) - g_{t_0}(x, t)| < k_3 e^{-\delta t_0}.$$

Proof. By definition of g_{t_0}, for $n = \mathrm{int}_x(t)$ and $t' = \mathrm{mod}_x(t)$ one has

$$|f_{t_0}(x, t) - g_{t_0}(x, t)| = |f_{t_0}(x, t) - g_{t_0}(S^n x, t')|.$$

Now if (C, S) were a linear cookie-cutter (as defined above) then the relation (3.1) from the last section would hold for large enough t_0, and we would have

$$f_{t_0}(x, t) = g_{t_0}(x, t)$$

for every $x \in C$ and $t \geqslant 0$. The first step of the proof is to note that

$$(4.6) \qquad \begin{aligned} |f_{t_0}(x, t) - g_{t_0}(x, t)| &= |f_{t_0}(x, t) - g_{t_0}(S^n x, t')| \\ &= |f_{t_0}(x, t) - f_{t_0}(S^n x, t')| \\ &\leqslant e^{(t_0+t')d} \, \big| |DS^n(x)|^d \mu(B(x, e^{-t_0-t})) \\ &\qquad - \mu(B(S^n x, e^{-t_0-t'}))\big|. \end{aligned}$$

We wish to apply Lemma 4.5, taking $A_0 = B(x, e^{-t_0-t})$, and $A_n = B(S^n(x), e^{-t_0-t'})$ (note that $e^{-t_0-t}|DS^n(x)| = e^{-t_0-t+r_n} = e^{-t_0-t'}$, that is, $|A_n| = |A_0|\,|DS^n(s)|$). However, to apply Lemma 4.5, we must check the assumption made before the statement of Lemma 4.2 that $A_0 \subset J_{x_1 \ldots x_n}$. Assuming for the moment that we can apply Lemma 4.5, we have

$$\begin{aligned} |f_{t_0}(x, t) - g_{t_0}(x, t)| &\leqslant e^{(t_0+t')d} \, \big| |DS^n(x)|^d \mu(A_0) - \mu(A_n)\big| \\ &\leqslant e^{(t_0+t')d} k_2 \, |A_n|^{d+\gamma d} \\ &\leqslant \frac{k_2}{2^{d+\gamma d}} e^{(t_0+t')d} e^{-(t_0+t')(d+d\gamma)} \\ &:= k_3 e^{-t_0 \delta} \end{aligned}$$

using the fact that $t' < \max r_1 < -\log \alpha$. To finish the proof we must verify the assumption stated above.

We claim that there exists t^* such that if $t_0 > t^*$ then one has that for any $x \in C$ and $t \geqslant 0$ that if $n = \mathrm{int}_x(t)$ then

$$B(x, e^{-t_0-t}) \subset J_{x_1 \ldots x_n}.$$

The idea of the proof is that $J_{x_1 \ldots x_n}$ has diameter approximately e^{-t} (by bounded distortion), and so one has to shrink the ball $B(x, e^{-t})$ only by a bounded amount, e^{-t_0}, to guarantee (again using bounded distortion) that $B(x, e^{-t_0-t}) \subset J_{x_1 \ldots x_n}$. First note that there is a $\delta > 0$ such that for any $y \in C$, $B(y, \delta) \subset J$ (the neighbourhood of I on which φ_0, φ_1 are defined). For $x \in C$ and t, n as above, take $t_0 > -\log \delta + \kappa\, |J|^\gamma \equiv t^*$. Choose z near x such that $S^n(z) = S^n(x) + \delta$. Then we have

$$\begin{aligned} |x - z| &= \frac{|S^n(x) - S^n(z)|}{|DS^n(y)|} \quad \text{(for some } y \in [x, z] \text{ by the Mean Value Theorem)} \\ &= \delta / |DS^n(y)| \\ &\geqslant e^{-t_0} e^{\kappa |J|^\gamma} / |DS^n(y)| \\ &= e^{-t_0-r_n} \frac{|DS^n(x)|}{|DS^n(y)|} e^{\kappa |J|^\gamma} \quad \text{(where } r_n = r_n(x)\text{)} \\ &\geqslant e^{-t_0-r_n} \quad \text{(by (4.4))} \\ &\geqslant e^{-t_0-t}. \end{aligned}$$

The same estimate holds if z is chosen so that $S^n(z) = S^n(x) - \delta$. Hence

$$B(x, e^{-t_0-t}) \subset \varphi_{x_1} \ldots \varphi_{x_n} B(x, \delta) \subset J_{x_1 \ldots x_n},$$

which was what we wanted.

In order to show that order-two density exists at μ-almost all $x \in C$, we shall show that

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T g_{t_0}(x, t)\, dt$$

exists (for μ-almost all x) for any t_0 and then compare $T^{-1}\int_0^T f(x, t)\, dt$ to this limit.

PROPOSITION 4.7. *There exists $h(t_0) \in \mathbb{R}$ such that, for μ-almost all $x \in C$,*

$$\frac{1}{T}\int_0^T g_{t_0}(x, t)\, dt \to h(t_0)$$

as $T \to \infty$.

Proof. The Cesàro average of g_{t_0} written above is just the ergodic average of g_{t_0} under the semiflow Φ_t on M from the initial point $(x, 0)$. One easily checks that Φ_t is an ergodic semi-flow with respect to the probability measure $\hat{\nu}$ which is locally $\nu \times \lambda$ (normalized) where λ is Lebesgue measure. The Birkhoff Ergodic Theorem for the semi-flow Φ_t then implies that the claimed limit exists and is constant for ν-almost all $x \in C$, and hence also for μ-almost all points in C.

We can now show that the order-two symmetric density exists.

THEOREM 4.8. *For a hyperbolic Cantor set C as above, the symmetric order-two density of μ exists at μ-almost all $x \in C$ and is constant almost surely: there is a number $D_2(\mu)$ such that, for μ-almost all $x \in C$,*

$$D_2(\mu) = D_2(C, x) := 2^{-d} \lim_{T\to\infty} \frac{1}{T}\int_0^T f(x, t)\, dt.$$

Proof. Take a sequence $t_k \to \infty$ such that $t_k > t^*$ for all k. By the last proposition there is a set $K(t_k)$ with $\nu(K(t_k)) = 1$ and

$$\frac{1}{T}\int_0^T g_{t_k}(x, t)\, dt \to h(t_k)$$

for $x \in K(t_k)$. Let $K = \bigcap_k K(t_k)$. This has ν-measure 1. Let $x \in K$ and fix $\varepsilon > 0$ while taking t_k large enough that $k_3 e^{-\delta t_k} < \varepsilon$. Choose also T_0 such that, for $T > T_0$,

$$\left|\frac{1}{T}\int_0^T g_{t_k}(x, t)\, dt - h(t_k)\right| < \varepsilon.$$

We then have

$$\begin{aligned}
\overline{\lim_{T\to\infty}} \frac{1}{T}\int_0^T f(x, t)\, dt &= \overline{\lim_{T\to\infty}} \frac{1}{T}\int_0^T f_{t_k}(x, t)\, dt \\
&< \frac{1}{T}\int_0^T f_{t_k}(x, t)\, dt + \varepsilon \quad \text{(for some } T > T_0) \\
&< \frac{1}{T}\int_0^T g_{t_k}(x, t)\, dt + 2\varepsilon \quad \text{(by Proposition 4.6)} \\
&< h(t_k) + 3\varepsilon.
\end{aligned}$$

Similarly we get

$$\underline{\lim}_{T\to\infty} \frac{1}{T} \int_0^T f(x, t)\, dt > h(t_k) - 3\varepsilon$$

and so

$$\left| \overline{\lim}_{T\to\infty} \frac{1}{T} \int_0^T f(x, t)\, dt - \underline{\lim}_{T\to\infty} \frac{1}{T} \int_0^T f(x, t)\, dt \right| < 6\varepsilon.$$

Letting $\varepsilon \to 0$ shows that the limit

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T f(x, t)\, dt$$

exists. The limit is clearly equal to the limit of $h(t_k)$ as $k\to\infty$, which is independent of x. This proves the theorem.

COROLLARY 4.9. *The order-two density of the Gibbs measure ν exists, and satisfies*

$$D_2(\nu, x) = \frac{d\nu}{d\mu} D_2(\mu),$$

for almost every x.

Proof. Apply Theorem 2.2.

THEOREM 4.10. *For any point $x \in C$ that is periodic with respect to S, the symmetric order-two density of μ exists.*

Proof. If x is periodic under S, then $g_{t_0}(x, t)$ is periodic in t so that the limit

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T g_{t_0}(x, t)\, dt$$

exists. Essentially the same argument as that used above then shows that $T^{-1} \int_0^T f(x, t)\, dt$ converges as $T \to \infty$.

The proof of existence of right order-two density is more or less the same as above. One defines a function f^r: $C \times \mathbb{R}^+ \to \mathbb{R}$ by

$$f^r(x, t) = \frac{\mu([x, x + e^{-t}))}{e^{-td}},$$

so that the right order-two density is given by

$$D_2^r(C, x) = \lim_{T\to\infty} \frac{1}{T} \int_0^T f^r(c, t)\, dt,$$

and then one works as above with intervals

$$A_0 = [y, y+\varepsilon), \quad A_n = [S^n y, S^n y + |DS^n y|\, \varepsilon).$$

Proceeding just as in the symmetric case (Proposition 4.6), one gets uniform estimates on $|f^r_{t_0} - g^r_{t_0}|$. This leads to

THEOREM 4.11. *The right order-two density of μ exists at μ-almost all $x \in C$ and is constant almost surely. For any point $x \in C$ that is periodic with respect to S, the right order-two density of μ exists at x.*

We remark that except where the Cantor set has an obvious symmetry (the middle-third set is an example) we do not yet know if the left and right order-two densities are equal. This seems to be quite a delicate problem. Our last result in this section shows that the almost sure order-two density value can be obtained from the order-two densities at periodic orbits.

THEOREM 4.12. *Let $B_\lambda = \{x \in C\colon\ x = S^n(x),\ |DS^n(x)|^d \leq \lambda\}$. Then*

$$\frac{1}{\operatorname{card} B_\lambda} \sum_{x \in B_\lambda} D_2(C, x) \to D_2(\mu) \quad \textit{as } \lambda \to \infty.$$

A similar statement holds for D_2^r and D_2^l.

One proves this by using the fact that

$$\frac{1}{\operatorname{card} B_\lambda} \sum_{x \in B_\lambda} \delta_x \to \nu \quad \text{as } \lambda \to \infty$$

in the weak topology (this is a consequence of a theorem of Bowen [**5**], and the fact that the measure of maximal entropy for the flow Φ_t is equal to ν times Lebesgue measure on the fibres of M).

5. *Zeros of Brownian motion and random walks*

In this section we will see that the notion of order-two density makes sense outside the narrow domain where it was defined in § 2. The examples we shall consider are the zero sets of Brownian motion and the simple random walk.

For a typical path of the one-dimensional Brownian motion $W(t)$, as is well known, the set of returns to zero

$$C_W = \{t \geq 0\colon\ W(t) = 0\}$$

is (almost surely with respect to Wiener measure) topologically a Cantor set, i.e. is homeomorphic to the middle-third set, and has Hausdorff dimension $\frac{1}{2}$. However the Hausdorff $\frac{1}{2}$-dimensional measure of C_W is zero. So instead one uses a more general kind of measure, which gives positive finite measure on the set: in the definition of Hausdorff measure given in § 2 one replaces the function $\varphi_d(t) = t^d$ by the function

$$\varphi(t) = (2t \log\log(1/t))^{\frac{1}{2}}$$

(for $0 < t < 1/e$). The resulting measure is known as Hausdorff φ-measure, and will be denoted H^φ.

The order-two density for C_W at x is, therefore, defined to be

$$D_2(C_W, x) = \lim_{T \to \infty} \frac{1}{T} \int_0^T \frac{H^\varphi(B(x, e^{-t}) \cap C_W)}{2^{\frac{1}{2}} e^{-t/2}}\, dt,$$

when this limit exists.

As in the proof of the existence of order-two density for the middle-third Cantor set, the proof here uses the ergodic theorem applied to a scaling flow on path space. The strategy of the proof is as follows: compare φ-measure with P. Lèvy's local time; compare local time with the maximum process of Brownian motion; then use ergodicity plus the strong Markov property to prove the theorem. Ergodicity for the scaling flow of the maximum process will follow from ergodicity of the scaling flow of Brownian motion.

For the case of random walk zeros, the proof is based on a dynamical interpretation of the almost-sure invariance principles of probability theory, given in **[15]**. There it is proved that having an almost-sure invariance principle of rate $o(t^{\frac{1}{2}})$ is equivalent to having a joining such that the paths are forward asymptotic in the scaling flow. Here we need to use two almost sure invariance principles, one for the random walk and one for its local time. Combining these allows us to pass the results for Brownian zeros over to the random walk.

A good introductory reference on Brownian motion is **[23]**; see also **[17, 19, 39, 22]**.

Scaling flow

Let $\Omega = \{f\colon \mathbb{R}^+ \to \mathbb{R} \mid f \text{ is continuous and } f(0)=0\}$, with the topology $\mathcal{T}$ given by uniform convergence on compact sets, and with $\mathcal{B}$ the Borel σ-algebra generated by $\mathcal{T}$. Define the *scaling maps* $\Delta_a\colon \Omega \to \Omega$ *of dimension* d by $(\Delta_a f)(t) = f(at)/a^d$, for $a>0$, and define the *scaling flow* τ_s on Ω by $\tau_s = \Delta_{\exp(s)}$ (where $s \in \mathbb{R}$). For this section we now fix $d=\frac{1}{2}$, so

$$(\tau_s f)(t) = f(e^s t)/e^{s/2}.$$

Note that $\tau_a \circ \tau_b = \tau_{a+b}$, that is, τ_s is a flow. We let ν denote Wiener measure on Ω, and write $\mathcal{B}_\nu$ for the ν-completion of $\mathcal{B}$. We recall from **[14, 15]** that $\mathcal{T}$ makes Ω into a Polish space (that is, a complete separable metric space) and that τ_s acting on $(\Omega, \mathcal{B}_\nu, \nu)$ is a Bernoulli flow of infinite entropy (on a Lebesgue space). In particular this is an ergodic, and mixing, flow.

φ-measure

The measure H^φ defined above has the following important *scaling property*: for any $a>0$, and any H^φ-measurable set E,

$$H^\varphi(aE) = a^{\frac{1}{2}} H^\varphi(E).$$

This is immediately seen from the definition of H^φ and the fact that $\lim_{t\to 0} \varphi(at)/\varphi(t) = a^d \ (=a^{\frac{1}{2}})$, in other words since $\varphi(t)$ is 'regularly varying at the origin' **[9**, p. 18**]**. We note that, more generally, such measures satisfy the conformal transformation property (see § 4), by the same argument used for Hausdorff measure; we will not however need that stronger version here.

The first goal of this section is to prove the existence of, and evaluate, the $\frac{1}{2}$-dimensional order-two density of H^φ on C_W, at H^φ-almost all $x \in C_W$. Since H^φ has the above scaling property and since the zero sets of a Brownian path are preserved by dilation (in the sense that the set aC_W for a fixed $a>0$ is the zero set of another path $(\Delta_a W)(t) = (1/\sqrt{a})W(t/a)$), one guesses that the average behaviour of mass around a point is governed by the function $t^{\frac{1}{2}}$ rather than by $\varphi(t)$. This guess is borne out by our result, that is, that the $\frac{1}{2}$-dimensional order-two density of H^φ on C_W, for H^φ-almost all $x \in C_W$, exists and is positive and finite.

We comment briefly on a basic difference between the geometry of the hyperbolic Cantor sets of § 3 and that of a Brownian zero set. There the average and extremal behaviours of mass in a ball of radius t were governed by the same function, t^d. Here, the average behaviour hovers around t^d, while the asymptotic upper envelope, for right density, is the larger function $\varphi(t)$. To prove this one uses Khinchine's law of the iterated logarithm. For symmetric density, an upper envelope of $c\varphi(t)$ for some constant c between $\sqrt{2}$ and 1 can be deduced from a purely geometric theorem of Wallin [**38**]. The point we wish to make here is that this extremal behaviour occurs infinitely often as $t \to 0$, but so rarely that it does not affect the time average which defines the order-two density.

Local time

P. Lèvy's *local time* of a Brownian path $W(t) \in \Omega$ is the function L_W: $\mathbb{R}^+ \to \mathbb{R}^+$ defined by

$$L_W(t) = \lim_{\varepsilon \to 0} \frac{1}{2\varepsilon} \int_0^t \chi_{[-\varepsilon, \varepsilon]}(W(s))\, ds$$

(when this limit exists). Some background references are [**11**, **39**, **22**, **32**].

We first prove these flow-invariant versions of two basic theorems concerning local time.

THEOREM 5.1. *There is a τ_s-invariant set, $\Omega_1 \subset \Omega$ with $\nu(\Omega_1) = 1$, such that for $W \in \Omega_1$, $L_W(t)$ is defined (for all $t \geq 0$) and is continuous (in t). Furthermore, the function $W \mapsto L_W$ from Ω_1 to Ω is $(\mathcal{B}_\nu, \mathcal{B})$-measurable.*

Proof. First, the fact that L_W is ν-almost surely defined and is continuous is a theorem of Lèvy; see, for example, [**11**] for the proof. Next we note that the definition of local time is scaling invariant. That is, if L_W exists for some $W \in \Omega$, then the limit for $\tau_s(W)$ also exists and $L_{\tau_s W} = \tau_s(L_W)$. (We call this the *scaling property of local time*). Finally, we check measurability. Let $\mathcal{F}$ be the algebra of finite cylinder sets in Ω; it is shown in [**11**] that $W \mapsto L_W$ is $(\mathcal{B}_\nu, \mathcal{F})$ measurable. This implies $(\mathcal{B}_\nu, \mathcal{B})$-measurability because $\mathcal{F}$ generates $\mathcal{B}$ in Ω (since a continuous path is determined by its values on a dense set of times).

Local time L_W is related to the Hausdorff measure of C_W by the next theorem, which is a corollary of work by Taylor and Wendell [**37**], Hawkes [**20**] and Perkins [**28**].

THEOREM 5.2. *There exists a τ_s-invariant set $\Omega_2 \subseteq \Omega_1$ of ν-measure 1 such that for any $W \in \Omega_2$, for all $t \geq 0$,*

$$L_W(t) = H^\varphi(C_W \cap [0, t]).$$

Proof. That the set Ω_2 on which the above property holds has ν-measure 1 follows from Hawkes' and Perkins' refinements of Taylor and Wendell's theorem. It suffices then to show that Ω_2 is flow-invariant. For $W \in \Omega_2$, we will show that $\Delta_a(W) \in \Omega_2$. Now given that for all t,

$$L_W(t) = H^\varphi(C_W \cap [0, t]),$$

we want to verify that for any $a > 0$,

$$L_{\Delta_a(W)}(t) = H^{\varphi}(C_{\Delta_a(W)} \cap [0, t]) \quad \text{for all } t.$$

We have

$$\begin{aligned} L_{\Delta_a(W)}(t) &= (\Delta_a(L_W))(t) \\ &= \Delta_a(H^{\varphi}(C_W \cap [0, t])) = H^{\varphi}(C_W \cap [0, at])/a^{\frac{1}{2}} \\ &= H^{\varphi}(a^{-1}C_W \cap [0, at]) = H^{\varphi}(C_{\Delta_a(W)} \cap [0, t]), \end{aligned}$$

where the first equality is the scaling property of local time and the next to last uses the scaling property of H^{φ}.

Now let ν_L denote the Borel measure on Ω which is the image of ν under the (measurable) map $W \mapsto L_W$. Let $\mathcal{B}_L$ denote the completion of $\mathcal{B}$. We call $(\Omega, \mathcal{B}_L, \nu_L, \tau_s)$ the *scaling flow for local time.*

We remark that the scaling flow for local time is a Bernoulli flow. One sees this as follows. As noted above, $L_{\tau_s W} = \tau_s(L_W)$. Therefore the map $W \mapsto L_W$ is a homomorphism of flows (it is, by definition of ν_L, measure-preserving). Thus since (as noted at the beginning of this section) the scaling flow for W is Bernoulli, one knows, by Ornstein's theory **[26]**, that this factor flow is also Bernoulli.

We shall show that $D_2^r(C_W, x) = \sqrt{(2/\pi)}$ for H^{φ}-almost every x and ν-almost every $W \in \Omega$. First we need:

DEFINITION. For $W \in \Omega$ write

$$M_W(t) = \sup_{s \in [0,t]} W(s).$$

This is the *maximum process* of Brownian motion.

Note that the map $W \mapsto M_W$ is continuous (since on any compact interval $[a, b]$, $\|W_1 - W_2\|_{[a,b]}^{\infty} < \varepsilon$ implies $\|M_{W_1} - M_{W_2}\|_{[a,b]}^{\infty} < \varepsilon$) and hence certainly $(\mathcal{B}_\nu, \mathcal{B})$-measurable. Thus ν pushes forward to a Borel measure ν_M on Ω.

In order to calculate the value of D_2, we will use a theorem of Lèvy which identifies the local time process as the maximum process of a different copy of Brownian motion. A rigorous statement of this is:

THEOREM 5.3. *For any Borel set* $A \subset \Omega$,

$$\nu_M(A) = \nu_L(A).$$

Proof. We begin with the statement usually given in the probability literature: that the two processes are equal *in distribution* (or *in law*), which means exactly that $\nu_M = \nu_L$ on the collection $\mathcal{F}$ of finite cylinder sets (see **[11]** or **[22]** for a proof). But this immediately extends to the Borel sets since $\mathcal{F}$ generates $\mathcal{B}$ in the space Ω.

In probability terminology one can also express this in the following way: given two copies W and $\hat{W}$ of Brownian motion, they can be redefined to live on the same probability space $(\tilde{\Omega}, \tilde{\nu})$, such that for ν-almost every $\omega \in \tilde{\Omega}$, with $W(t) = W(\omega, t)$ and $\hat{W}(t) = \hat{W}(\omega, t)$, we have that $L_W = M_{\hat{W}}$. This is therefore a close analogue of Rèvèsz' almost-sure invariance principle for random walk local time (Theorem 6 below).

To help explain this correspondence (between ν_M and ν_L) we note that one can see from the proof of (8.7) in **[11]** how it arises from an underlying isomorphism of Wiener space with itself, which is given by an explicit formula. Here sgn($\cdot$) is the sign function, taking the values, $+1$, -1, and 0, and the integral is a stochastic integral.

THEOREM 5.4. *The map*

$$W \mapsto \hat{W}(t) = -\int_0^t \operatorname{sgn}(W(s))\, dW(s)$$

is defined for ν-almost every $W \in \Omega$ and is an isomorphism of (Ω, ν, τ_s) with itself (in other words there are flow-invariant sets of full measure such that the map $W \mapsto \hat{W}$ is one-to-one surjective and measure-preserving). Furthermore, the following diagram commutes and is τ_s-invariant:

$$\begin{tikzcd} W \arrow[rr, mapsto] \arrow[dr] & & \hat{W} \arrow[dl] \\ & L_W = M_{\hat{W}} & \end{tikzcd}$$

We can now prove the existence of order-two density for the Brownian zero sets.

THEOREM 5.5. *For ν-almost every $W \in \Omega$, one has that for H^{φ}-almost every $x \in C_W$ the right and left and the symmetric $\frac{1}{2}$-dimensional densities exist and equal $\sqrt{(2/\pi)}$, $\sqrt{(2/\pi)}$ and $2\sqrt{(1/\pi)}$ respectively.*

Proof. One knows explicitly the distribution of the maximum process (for a good account see **[19]**); it is half of a Gaussian, i.e. has probability density function

$$\frac{2}{\sqrt{(2\pi)}} e^{-x^2/2} \text{ for } x \geq 0, \quad \text{and} \quad 0 \text{ for } x < 0.$$

This has expected value

$$\frac{2}{\sqrt{(2\pi)}} \int_0^\infty x e^{-x^2/2}\, dx = \sqrt{\frac{2}{\pi}}.$$

If we now define a function F: $\Omega \to \mathbb{R}$ to be evaluation (of the maximum process) at time 1,

$$F(f) = f(1),$$

then F is in $L_1(\Omega, \nu_M)$ and has expected value

$$\int_\Omega F\, d\nu_M = \sqrt{\frac{2}{\pi}}.$$

Now since (Ω, ν, τ_s) is an ergodic flow, so is its homomorphic image (Ω, ν_M, τ_s) (under the map $W \mapsto W_M$). The Birkhoff ergodic theorem for flows thus implies that for ν_M-almost every path $M \in \Omega$,

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T F(\tau_s(M))\, ds = \sqrt{\frac{2}{\pi}}. \tag{12}$$

The set of $M \in \Omega$ for which this holds meets the set for which $M_W = L_{\hat{W}}$ in a set of full measure. Hence (12) holds for ν_L-almost every L_W which means that

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T \frac{H^\varphi(C_W \cap [0, e^{-s}])}{e^{-s/2}}\, ds = \sqrt{\frac{2}{\pi}}$$

for ν-almost every W. This says that for a set of ν-measure 1, the right order-two density of C_W exists at zero and equals $\sqrt{(2/\pi)}$.

To extend this proof of the existence of D_2' at zero to give existence at H^φ-almost all $x \in C_W$ we will use the strong Markov property of Brownian motion plus a Fubini's Theorem argument.

For a fixed $W \in \Omega$ and $t \geq 0$, let $\mathbf{t}(t) = \inf\{s \in \mathbb{R}^+ \colon L_W(s) = t\}$. This is a *stopping time*, that is, it only depends on the path up to time $\mathbf{t}(t)$. Therefore by the strong Markov property, Brownian motion begins anew at time $\mathbf{t}(t)$ for each t. Hence for every fixed t, the right order-two density at the point $x = \mathbf{t}(t)$ exists ν-almost surely by what we have proved above for $x = 0$. Note that $\mathbf{t}$ is an inverse of L_W, that is (for ν-almost every W),

$$L_W(\mathbf{t}(s)) = s \quad \text{for all } s \in \mathbb{R}^+.$$

Note also that since H^φ of the zero set C_W gives local time, the image of $H^\varphi|_{C_W}$ under L_W is Lebesgue measure on $\mathbb{R}^+$.

Now let Ω_3 denote the subset of ν-measure 1 in Ω such that the right order-two density at zero exists and equals $\sqrt{(2/\pi)}$. Without loss of generality we assume that Ω_3 is a Borel set; we can do this since $\mathcal{B}_\nu$-measurable sets are exactly those whose inner and outer measures are equal **[31]**, so the set contains a Borel set of full measure. We need this to be a Borel set for a technical reason given below.

Define, for $s \geq 0$, $W_s(t) = W(s+t) - W(s)$, and set

$$A_T = \{(t, W) \in [0, T] \times \Omega \colon W_{\mathbf{t}(t)}(\cdot) \in \Omega_3\}.$$

We claim that A_T is $\mathcal{B}_m \times \mathcal{B}_\nu$-measurable, where m is Lebesgue measure. This is because the maps $\alpha\colon \mathbb{R}^+ \times \Omega \to \Omega$ and $\beta\colon \mathbb{R}^+ \times \Omega \to \mathbb{R}^+ \times \Omega$, defined by

$$\alpha\colon (s, W) \mapsto W_s(\cdot) \quad \text{and} \quad \beta\colon (t, W) \mapsto (\mathbf{t}(t, W), W),$$

are respectively jointly continuous and Borel measurable. Hence since Ω_3 is a Borel set, $A_T \equiv (\alpha \circ \beta)^{-1}(\Omega_3)$ is $\mathcal{B}_m \times \mathcal{B}_\nu$-measurable.

Now for each fixed $t \in [0, T]$ the set $\{W \colon (t, W) \in A_T\}$ has ν-measure 1. Therefore A_T has $\nu \times m$-measure T by Fubini's Theorem **[31]** (this is why we checked the measurability of A_T above). Also by Fubini's theorem, for ν-almost every fixed W, the set $\{t\colon (t, W) \in A_T\}$ has Lebesgue measure T. That says exactly that for H^φ-almost every $x \in [0, \mathbf{t}(T)]$, the right order-two density of C_W at x converges (since $\mathbf{t}$ pushes Lebesgue measure forward to H^φ restricted to C_W). This is true for each $T \in \mathbb{R}^+$ and so we have finished.

Convergence for the left order-two density is a consequence of the time symmetry for Brownian motion (that is, if $W(t)$ is Brownian motion on $\mathbb{R}$ with $W(0)=0$, then $W(t)\mapsto W(-t)$ is an isomorphism of the space (Ω, ν)). Hence the left and right order-two densities exist simultaneously and are equal. Therefore the symmetric order-two density also exists on a set of ν-measure 1, and (by (3) of § 2) equals $2\sqrt{(1/\pi)}$.

The simple random walk

We now turn our attention to the zero set of a simple random walk. The set-up is much like that in [15].

DEFINITIONS. Let X_i $(i=1,2,\ldots)$ be a sequence of i.i.d. random variables taking values ± 1 with probability $(\frac{1}{2},\frac{1}{2})$. Define $S_0=0$ and $S_n=\sum_{i=1}^n X_i$; this sequence of partial sums is commonly known as the *simple random walk.*

By the *polygonal random walk* we mean the functions $S(t)$ in continuous path space

$$\Omega=\{f\colon \mathbb{R}^+\to\mathbb{R} \mid f \text{ is continuous and } f(0)=0\}$$

such that $S(n)=S_n$ for $n\in\mathbb{N}$ and is linear in between. We write $S=(S_0, S_1, \ldots)$ for the sequence of partial sums, and also for the path in Ω it determines. The random walk gives a Borel probability measure on Ω which we call γ, the measure of the polygonal random walk.

The *zero set* of the random walk S is the set

$$C_S=\{n\colon S_n=0\}.$$

We define

$$N_n=N_n(C_S)=\operatorname{card}\{k\colon 0\leq k\leq n, S_k=0\}$$

and define the maximum process

$$M_n=\max_{1\leq k\leq n} S_k,$$

and interpolate linearly to consider M as an element of Ω.

We need the following theorem of Révész, which is a discrete time analogue of Theorem 4.3.

THEOREM 5.6 [30]. *For the simple random walk S_n, given any $\varepsilon>0$ the processes N_n and M_n can be redefined to live on the same probability space, so that for almost all ω in that space,*

$$|N_n(\omega)-M_n(\omega)|=o(n^{\frac{1}{4}+\varepsilon}).$$

We are now ready to prove our theorem.

THEOREM 5.7. *For γ-almost every path S of the simple random walk, the order-two $\frac{1}{2}$-dimensional density of its zero set exists and equals $\sqrt{(2/\pi)}$.*

Proof. We are to show that for $N_n=N_n(C_S)$, for γ-almost every S,

$$\lim_{K\to\infty}\frac{1}{\log K}\sum_{n=1}^{K}\frac{N_n}{n^{\frac{1}{2}}}\frac{1}{n}=\sqrt{\frac{2}{\pi}}.$$

Now as at the start of the proof of Theorem 5, let $F\colon \Omega \to \mathbb{R}$ denote evaluation at time 1, and (Ω, ν_M) the maximum process of Brownian motion. Using the Birkhoff ergodic theorem with negative time, we have that ν_M-almost surely,

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T F(\tau_{-s}(M))\, ds = \sqrt{\frac{2}{\pi}}. \tag{5.1}$$

By Strassen's almost-sure invariance principle (**[34]**; see also **[15]**), $S(t)$ and $W(t)$ can be redefined to live on the same space so that for almost every ω in that space

$$|W(\omega, t) - S(\omega, t)| = o(t^{\frac{1}{2}}).$$

Now notice that this implies the same estimate for the associated maximum processes. That is, for almost every ω,

$$\lim_{t\to\infty} |M_W(t) - M_S(t)|/t^{\frac{1}{2}} = 0. \tag{5.2}$$

Now notice that (5.1) can be written as:

$$\lim_{T\to\infty} \frac{1}{\log T} \int_1^T \frac{M_W(y)}{y^{\frac{1}{2}}} \frac{dy}{y} = \sqrt{\frac{2}{\pi}}.$$

From (5.2) one immediately sees that this is also true for M_S, and hence for γ-almost every M_S (technically speaking, one uses here Fubini's theorem on the joining given by the a.s.i.p. and the fact that Ω is a Lebesgue space; a basic theorem of Rochlin implies this—see, for example, **[14, 15]** for related details).

From the above, it easily follows that

$$\lim_{N\to\infty} \frac{1}{\log N} \sum_{k=1}^{N} \frac{M(n)}{n^{\frac{1}{2}}} \frac{1}{n} = \sqrt{\frac{2}{\pi}}.$$

Finally by Révész' theorem, the same is true for N_n, and we have finished.

Remark. Here is a more picturesque but less direct way of looking at the above proof. As in **[15]**, a $o(t^{\frac{1}{2}})$ a.s.i.p. is equivalent to the two paths being forward asymptotic in the scaling flow. Hence since $|M_W - M_S| = o(t^{\frac{1}{2}})$ and $|M_S - N| = o(t^{\frac{1}{2}})$, we have that there exists a joining of (Ω, ν_M) and (Ω, γ) such that M_W and N are forward asymptotic in the scaling flow. Hence the ergodic averages of F starting at these two points in Ω agree.

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