Algumas abordagens à geometria diferencial de espaços singulares

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Outline

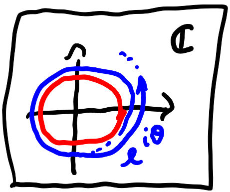
- All sorts of spaces with singularities
- Differentiable spaces.
- Lie groupoids and differentiable stacks
- Stratified spaces

All sorts of singular spaces

- Zeros of functions, intersections, ... (bad subspaces/fibred products)
- Quotients of actions, leaf spaces, ... (bad quotients).
- Gluings (bad pushouts)

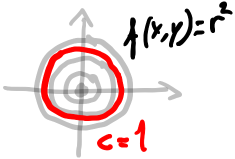
Smooth spaces

Many tools to describe them - Example - the circle:



- Parametrize

$$S^1 = \mathrm{Image}(\theta \mapsto e^{i\theta} \in \mathbb{C});$$

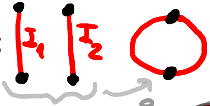


- Describe implicitly

$$S^1 = \mathrm{Zeros}(x^2 + y^2 - r^2) \subset \mathbb{R}^2;$$

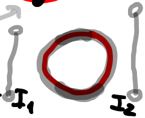
- Triangulate (or give CW-decomposition).

$$S^1 = I_1 \text{ glued to } I_2 \text{ at endpoints;}$$



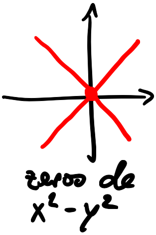
- Use atlas.

$$S^1 \text{ covered by open } I_1 \text{ and } I_2 \text{ glued on common opens.}$$



Singular spaces - Examples - Subspaces

If $X \subset M$ is closed, it is the zero set of a smooth function f.



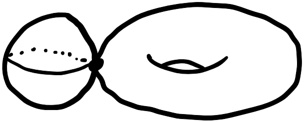
And then there is an **algebra of smooth functions**

$$C^\infty(X) = C^\infty(M)/(f).$$

But when studying zeros of functions, x and x^2 have the same zero-set in $\mathbb{R}$, for example.

We can still try to distinguish them via $C^\infty(M)/(x)$ and $C^\infty(M)/(x^2)$.

Examples - bad gluings

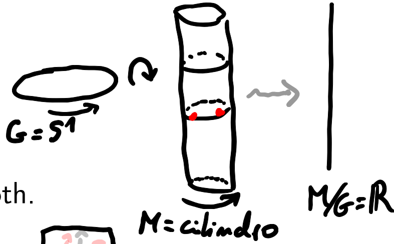


For example, the wedge sum of a sphere and a torus.

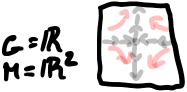
These do appear, as inertia groupoids / inertia stacks of orbit spaces = model for
$\{$elements of the free loop stack that vanish on the orbit space$\}$;

Useful in string topology.

Examples - Quotients

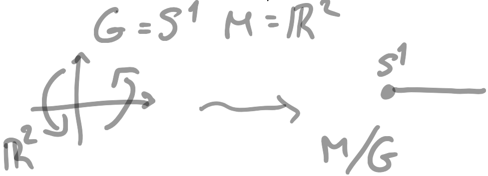


- $G \curvearrowright M$ free and proper $\Rightarrow M/G$ smooth.
- $G \curvearrowright M$ general $\Rightarrow M/G$ terrible.



- $G \curvearrowright M$ locally free, G compact $\Rightarrow M/G$ orbifold.

- $G \curvearrowright M$ proper $\Rightarrow M/G$ orbispace.



Examples - Quotients

- **Orbifolds** are locally modelled on $\mathbb{R}^n/G$, with G **finite**.

- **Orbispaces** are locally modelled on $\mathbb{R}^n/G$, with G **compact**.

- Both can be studied in terms of atlases - charts need to encode the local actions; "transitions" have to be *extra careful*!

- The orbit space X of a proper action is Hausdorff, second-countable, and locally compact, hence also paracompact.

Smooth functions on orbit spaces

Definition

The **algebra of smooth functions on** $X = M/G$ is defined as

$$C^\infty(X) := \{f : X \to \mathbb{R} \mid f \circ \pi \in C^\infty(M)\},$$

where $\pi : M \longrightarrow X$ denotes the canonical projection map.

Similarly define the **sheaf of smooth functions on** X,

$$\mathcal{C}^\infty_X(U) := C^\infty(\pi^{-1}(U)/(U \cap \text{orbits})).$$

The pullback map $\pi^* : C^\infty(X) \longrightarrow C^\infty(M)$ gives identification

$$C^\infty(X) \cong C^\infty(M)^{G-\text{inv}}$$

Embedding orbit spaces

Let G be compact and let $G \curvearrowright V$ be a representation.
Then V/G can be seen as closed subspace.

Theorem (Schwarz, 1975)

Let G be a compact Lie group and V a representation of G. Let $p_1, \ldots, p_k$ be generators of the algebra of invariant polynomials $\mathbb{R}[V]^G$.

Then $p : V \to \mathbb{R}^k$ defined by $p = (p_1, \ldots, p_k)$ induces an isomorphism

$$p^* C^\infty(\mathbb{R}^k) \cong C^\infty(V)^G.$$

Similarly for many non-linear $G \curvearrowright M$.

Examples - quotients of Euclidean plane

- $\mathbb{R}^2/G$, with $G \subset O(2)$ acting orthogonally.

Theorem (Leonardo da Vinci)

Finite subgroups of $O(2)$ (up to conjugation) are

1. $G = \{1\}$;
2. $G = \mathbb{Z}_2$ *generated by reflection on y-axis;*
3. $G = \mathbb{Z}_n$ *generated by rotation of order n;*
4. $G = D_n$ *dihedral group of order n.*

Examples - quotients of Euclidean plane

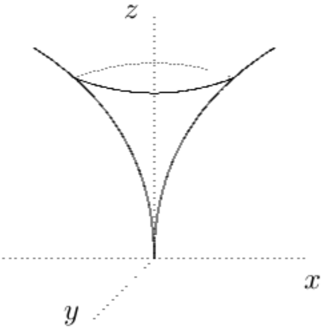
G generated by order n rotation

Generators of algebra of invariants:

$$p_1(x, y) := \mathrm{Re}(x + iy)^n$$
$$p_2(x, y) := \mathrm{Im}(x + iy)^n$$
$$p_3(x, y) := x^2 + y^2$$

Schwarz: $\mathbb{R}^2/G$ is isomorphic to image of $(p_1, p_2, p_3) : \mathbb{R}^2 \to \mathbb{R}^3$

Examples - quotients of Euclidean plane



$$z^n = x^2 + y^2$$
$$z \geq 0$$

Image from *González, Salas - C^∞-Differentiable spaces*.

Examples - quotients of Euclidean plane

$G = D_n$

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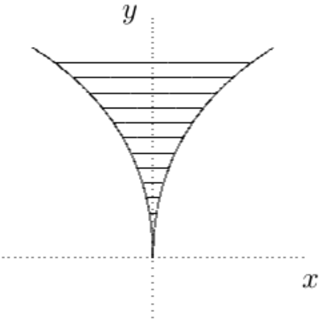

$$y^n \geq x^2$$
$$y \geq 0$$

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Recovering orbifolds from smooth functions

All these were orbifolds

These orbifolds $\mathbb{R}^2/G$ could be recovered from their algebra of smooth functions.

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These orbifolds $\mathbb{R}^2/G$ could be recovered from their algebra of smooth functions.

Theorem (Jordan Watts, 2017)

If X is an orbifold, the orbifold structure can be completely recovered from $C^\infty(X)$.

Other quotients - orbipaces

- $\mathbb{R}^n/O(n) \cong [0, \infty)$
- $C^\infty(\mathbb{R}^n/O(n))$ is the same for every n

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Possible to distinguish them as **diffeological spaces**

Even worse quotients - leaf spaces

Let $\mathcal{F}$ be a foliation on M.

$\mathcal{F}$ is decomposition of M into submanifolds, locally fitting together as a product.

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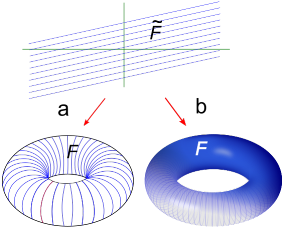

Image from Wikipedia, author - Lantonov

Even worse quotients - leaf spaces

Possible approaches:

• only work with nice foliations (holomorphic, Riemannian, or compact, etc.);

• Use diffeology;

• Use a different algebra of smooth functions, e.g. a **Non-commutative algebra of smooth functions**.

Quotients - Just don't take them!

Keep all the information by having a Lie groupoid around:

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2 - ????

3 - Profit!

Quotients - Just don't take them!

Keep all the information by having a Lie groupoid around:

1 - Model the problem by a groupoid (a sort of generalized equivalence relation):

2 - Do **Transverse differential geometry** on the Lie groupoid, which is an actual smooth space.

3 - Different Lie groupoids may model the same quotient. These are **Morita equivalent**

Idea: Doing **transverse geometry** on the groupoid, in a **Morita - invariant** way, corresponds to geometry on quotient.

How to do that?

1 - Modelling the problem by a (as nice as possible) Lie groupoid:

• Group actions - action groupoids

• Foliations - holonomy groupoids, étale groupoids
(André Haefliger's work, for example)

• Orbifolds - orbifold groupoids (proper étale groupoids)
(Ieke Moerdijk, Dorette Pronk, 1997)

• Orbispaces - proper groupoids
(Kirsten Wang 2018)

2 - What to do with the groupoid?

A - Transverse geometry, directly on the Lie groupoid:

- Compute cohomology and other invariants of quotient.
- Morphisms are much better defined!
- Transverse Riemannian metrics
- Transverse measures and integration
- Multiplicative vector fields, dynamics, etc.

And much more.

2 - What to do with the groupoid?

If the groupoid is nice, i.e. **proper**, then its quotient X is an orbispace, and we obtain.

- $C^\infty(X)$
- A **stratification** on X.

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If the groupoid is nice, i.e. **proper**, then its quotient X is an orbispace, and we obtain.

- $C^{\infty}(X)$
- A **stratification** on X.

These have been useful recently, for example for:

- Poisson manifolds of compact type
(Crainic, Fernandes, Martínez Torres, 2016)

- Proof of Molino's conjecture
(Alexandrino, Radeschi, 2016)

2 - What to do with the groupoid?

B - Build something additional out of the groupoid, study that instead.

For example, given any groupoid with quotient X, define a convolution algebra

$$\mathcal{NC}^{\infty}(X)$$

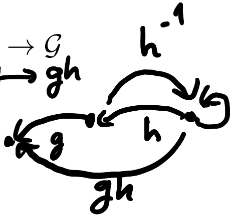
It works as substitute for $C^{\infty}(X)$.

Lie groupoids

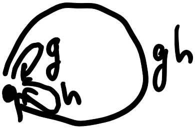
- A groupoid is a (small) category with all arrows invertible.
- A Lie groupoid is a "smooth" groupoid.

Explicitly: A Lie groupoid $\mathcal{G}$ over M consists of

- a manifold of **arrows** $\mathcal{G}$
- a manifold of **objects** M
- **source** and **target** submersions $s, t : \mathcal{G} \to M$
- a smooth **multiplication** $m : \mathcal{G}^{(2)} = \mathcal{G}_s \times_t \mathcal{G} \to \mathcal{G}$
  $(g, h) \mapsto gh$
- a **unit** embedding $u : M \to \mathcal{G}$
- an **inverse** diffeomorphism $i : \mathcal{G} \to \mathcal{G}$



Examples



Lie groups $G \rightrightarrows \{*\}$.

Submersion groupoids

Given any submersion $\pi : M \to B$ there is a groupoid

$$\mathcal{G}(\pi) = M \times_\pi M \rightrightarrows M$$

Arrows: pairs (x, y) such that $\pi(x) = \pi(y)$

$$s(x, y) = y, \quad t(x, y) = x, \quad (x, y) \cdot (y, z) = (x, z).$$

When $\pi = id_M \rightsquigarrow$ unit groupoid;
When B is a point, $\rightsquigarrow$ pair groupoid;

Examples

Let $G \curvearrowright M$.

Form the **action groupoid** $G \ltimes M \rightrightarrows M$.

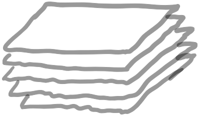
Objects = points of M,

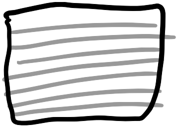
Arrows are pairs $(g, x) \in G \times M$.

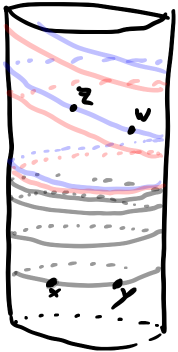
$$s(g, x) = x, \qquad t(g, x) = g \cdot x, \qquad (g, h \cdot x)(h, x) = (gh, x).$$

$$1_x = (e, x), \qquad (g, x)^{-1} = (g^{-1}, g \cdot x)$$

Examples







Let $\mathcal{F}$ be a foliation on M.

Form the **monodromy groupoid** $\Pi_1(\mathcal{F}) \rightrightarrows M$.

Objects = points of M,

Arrows = leafwise-homotopy classes of paths inside leaves

Composition is class of concatenation

Structure of Lie groupoids

Proposition

Let $\mathcal{G} \rightrightarrows M$ be a Lie groupoid and $x, y \in M$. Then:

1. *the set of arrows from x to y, $s^{-1}(x) \cap t^{-1}(y)$ is a Hausdorff submanifold of $\mathcal{G}$;*
2. *the isotropy group $\mathcal{G}_x$ is a Lie group;*
3. *the orbit $\mathcal{O}_x$ through x is an immersed submanifold of M;*
4. *the s-fibre of x is a principal $\mathcal{G}_x$-bundle over $\mathcal{O}_x$, with projection the target map t.*

Orbit spaces of Lie groupoids

The partition of the manifolds into connected components of the orbits forms a foliation, which is possibly singular, in the sense that different leaves might have different dimension.

Example

$S^1 \curvearrowright \mathbb{R}^2$ by rotations. Leaves of the associated action groupoid are the orbits, i.e., the origin and the concentric circles centred on it. Hausdorff orbit space;

$(\mathbb{R}_+, \times) \curvearrowright \mathbb{R}^2$ by scalar multiplication. Leaves are the origin and the radial open half-lines. There is a dense point in the orbit space.

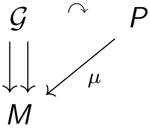
Next: When do two groupoids have "the same" orbit space?

Actions of Lie groupoids

Let $\mathcal{G} \rightrightarrows M$ be a Lie groupoid and consider a surjective smooth map $\mu : P \to M$. A **(left) action** of $\mathcal{G}$ on P along the map μ, which is called the **moment map**, is a smooth map

$$\mathcal{G} \times_M P = \{(g, p) \in \mathcal{G} \times P \mid s(g) = \mu(p)\} \to P,$$

denoted by $(g, p) \mapsto g \cdot p = gp$, such that $\mu(gp) = t(g)$, and satisfying $(gh)p = g(hp)$ and $1_{\mu(p)}p = p$. We then say that P is a **left $\mathcal{G}$-space**.



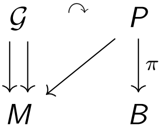
$g : x \to y$ maps the fibre over x onto the fibre over y

Principal $\mathcal{G}$-bundles

A **left $\mathcal{G}$-bundle** is a left $\mathcal{G}$-space P together with a $\mathcal{G}$-invariant surjective submersion $\pi : P \to B$.
A left $\mathcal{G}$-bundle is called **principal** if the map

$$\mathcal{G} \times_M P \to P \times_\pi P \ , \ (g, p) \mapsto (gp, p)$$

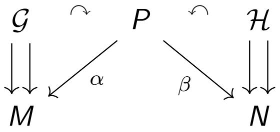
is a diffeomorphism.



So for a principal $\mathcal{G}$-bundle, each fibre of π is an orbit of the $\mathcal{G}$-action and all the stabilizers of the action are trivial.

Morita equivalence

A **Morita equivalence** between two Lie groupoids $\mathcal{G} \rightrightarrows M$ and $\mathcal{H} \rightrightarrows N$ is given by a **principal** $\mathcal{G}-\mathcal{H}$**-bibundle**, i.e.,



such that $\beta : P \to N$ is a left principal $\mathcal{G}$-bundle, $\alpha : P \to M$ is a right principal $\mathcal{H}$-bundle and the two actions commute:

$$g \cdot (p \cdot h) = (g \cdot p) \cdot h \text{ for any } g \in \mathcal{G},\ p \in P \text{ and } h \in \mathcal{H}.$$

Morita equivalence - Examples

[Isomorphisms]
If $f : \mathcal{G} \to \mathcal{H}$ is an isomorphism of Lie groupoids, then $\mathcal{G}$ and $\mathcal{H}$ are Morita equivalent.

Bibundle: $Graph(f) \subset \mathcal{G} \times \mathcal{H}$,

moment maps $t \circ pr_1$ and $s \circ pr_2$,

and the natural actions induced by the multiplications of $\mathcal{G}$ and $\mathcal{H}$.

Morita equivalence - Examples

Let $\mathcal{G}$ be a Lie groupoid over M and let $\alpha : P \to M$ be a surjective submersion.

pullback groupoid $\alpha^*\mathcal{G} \rightrightarrows P$:

Space of arrows $= P \times_M \mathcal{G} \times_M P$, i.e.
Arrows $=$ triples (p, g, q) with $\alpha(p) = t(g)$ and $s(g) = \alpha(q)$.

$$s(p, g, q) = q, \quad t(p, g, q) = p, \quad (p, g_1, q)(q, g_2, r) = (p, g_1 g_2, r).$$

$\mathcal{G}$ and $\alpha^*\mathcal{G}$ are Morita equivalent.
Bibundle given by $\mathcal{G} \times_M P$.

Morita equivalence - Examples

1. Two Lie groups are Morita equivalent if and only if they are isomorphic.
2. Any transitive Lie groupoid $\mathcal{G}$ is Morita equivalent to the isotropy group $\mathcal{G}_x$ of any point x in the base.
3. Let $\mathcal{G} \rightrightarrows M$ be a Lie groupoid, let $N \subset M$ be a submanifold that intersects transversely every orbit it meets and let $\langle N \rangle$ denote the saturation of N. Then $\mathcal{G}_N \rightrightarrows N$ is Morita equivalent to $\mathcal{G}_{\langle N \rangle} \rightrightarrows \langle N \rangle$. As a particular case, we can take N to be any open subset of M.
4. The groupoid $\mathcal{G}(\pi)$ associated to a submersion $\pi : M \to N$ is Morita equivalent to the unit groupoid $\pi(M)$.

Morita equivalence

Let $\mathcal{G}$ and $\mathcal{H}$ be Morita equivalent, with bibundle P
Since P is a principal bibundle, it is easy to check that

$$\alpha^*\mathcal{G} = P \times_M \mathcal{G} \times_M P \cong P \times_M P \times_N P \cong P \times_N \mathcal{H} \times_N P = \beta^*\mathcal{H},$$

as Lie groupoids over P.

This means that we can break any Morita equivalence between $\mathcal{G}$ and $\mathcal{H}$, using a bibundle P, into a chain of simpler Morita equivalences:

$\mathcal{G}$ is Morita equivalent to $\alpha^*\mathcal{G} \cong \beta^*\mathcal{H}$, which is Morita equivalent to $\mathcal{H}$.

Morita equivalences preserve transverse geometry

Let $\mathcal{G} \rightrightarrows M$ and $\mathcal{H} \rightrightarrows N$ be Morita equivalent Lie groupoids and let P be a bibundle realising the equivalence. Then P induces:

1. A homeomorphism between the orbit spaces of $\mathcal{G}$ and $\mathcal{H}$,
$$\Phi : M/\mathcal{G} \longrightarrow N/\mathcal{H};$$
2. isomorphisms $\phi : \mathcal{G}_x \longrightarrow \mathcal{H}_y$ between the isotropy groups at any $x \in M$ and $y \in N$;
3. isomorphisms $\tilde{\phi} : \mathcal{N}_x \longrightarrow \mathcal{N}_y$ between the normal representations at any points Φ-related points x and y, compatible with the isomorphism $\phi : \mathcal{G}_x \longrightarrow \mathcal{H}_y$.

Normal representations

Lie group action $G \curvearrowright M \overset{differentiate}{\rightsquigarrow} G \curvearrowright TM$ **tangent action**

$$g \cdot X = \frac{d}{d\epsilon}_{|\epsilon=0}(g \cdot x(\epsilon)),$$

where $X \in T_xM$ and $x(\epsilon)$ is a curve representing X.
Restrict this action to an action of an isotropy group G_x
$\Rightarrow$ obtain a representation of G_x on T_xM.

G_x leaves the tangent space to the orbit through x invariant, so we obtain an induced representation on the quotient $\mathcal{N}_x = T_xM/T_xO_x$, called the **isotropy representation** at x.

Differentiable stacks

Definition

A **differentiable stack atlas** on a topological space X is given by a Lie groupoid $\mathcal{G} \rightrightarrows M$ and a homeomorphism $f : M/\mathcal{G} \longrightarrow X$.

Atlases $(\mathcal{G}, f)$ and $(\mathcal{H}, f')$ are **equivalent** if $\mathcal{G} \rightrightarrows M$ and $\mathcal{H} \rightrightarrows N$ are Morita equivalent, and the homeomorphism $\Phi : N/\mathcal{H} \longrightarrow M/\mathcal{G}$ induced by the Morita equivalence satisfies $f \circ \Phi = f'$.

A **differentiable stack** is a topological space equipped with an equivalence class of differentiable stack atlases.

Given any Lie groupoid $\mathcal{G} \rightrightarrows M$, the differentiable stack associated to it by using the atlas $(\mathcal{G}, id_{M/\mathcal{G}})$ on $M/\mathcal{G}$ is denoted by $M//\mathcal{G}$.

Proper Lie groupoids

A Lie groupoid is called **proper** if is Hausdorff and $(s, t) : \mathcal{G} \to M \times M$ is a proper map.

Examples

1. If $\mathcal{G}$ is compact, it is proper
2. $M \times M$ is always proper
3. T^*G is proper for a compact Lie group G.

Étale Lie groupoids

A Lie groupoid is called **étale** if the source map is a local diffeomorphism.

Examples

1. some groupoids describing foliations

Nice differentiable stacks

Let $M//\mathcal{G}$ be a differentiable stack

1. if $\mathcal{G}$ is proper and étale, then $M//\mathcal{G}$ is a orbifold.
2. if $\mathcal{G}$ is proper, then $M//\mathcal{G}$ is a orbispace.

Proper Lie groupoids - Examples

1. A Lie group G is proper when seen as a Lie groupoid if and only if it is compact.
2. The submersion groupoid $\mathcal{G}(\pi)$ associated to a submersion $\pi : M \to B$ is always proper.
3. An action groupoid is proper if and only if it is associated to a proper Lie group action.
5. If $\mathcal{G} \rightrightarrows M$ is a proper Lie groupoid and $S \subset M$ a submanifold such that the restriction $\mathcal{G}_S \rightrightarrows S$ is a Lie groupoid, then $\mathcal{G}_S$ is proper as well.

Proper Lie groupoids - Simple structure

Proposition

Let $\mathcal{G} \rightrightarrows M$ be a proper Lie groupoid. Then the orbit space $M/\mathcal{G}$ is Hausdorff and the isotropy group $\mathcal{G}_x$ is compact for every $x \in M$.

Proof.

Since the map $(s, t) : \mathcal{G} \to M \times M$ is proper, it is closed, and has compact fibres. This automatically implies that the isotropy groups are compact and since the orbit space is the quotient of M by the closed relation $(s, t)(\mathcal{G}) \subset M \times M$, it is Hausdorff. □

Proper Lie groupoids - Morita Invariance

Proposition

Let $\mathcal{G}$ and $\mathcal{H}$ be Morita equivalent Lie groupoids. If one of them is proper, then the other one is proper as well.

Proof.

As mentioned before, in order to prove invariance of a property, we may assume that $\mathcal{H} \rightrightarrows N$ is equal to the pullback of $\mathcal{G} \rightrightarrows M$ via a surjective submersion $\alpha : N \to M$. But then we have a pullback diagram relating the maps $(s,t) : \mathcal{G} \longrightarrow M \times M$ and $(s',t') : \mathcal{H} \longrightarrow N \times N$. The result follows from stability of proper maps (with Hausdorff domain) under pullback. □

Local structure of proper Lie groupoids

Definition

Let $\mathcal{G}$ be a Lie groupoid over M and $x \in M$. A **slice** at x is an embedded submanifold $\Sigma \subset M$ of dimension complementary to $\mathcal{O}_x$ such that it is transverse to every orbit it meets and $\Sigma \cap \mathcal{O}_x = \{x\}$.

Information about the longitudinal (along the orbits) and the transverse structure of a proper groupoid $\mathcal{G}$:

Proposition

Let $\mathcal{G} \rightrightarrows M$ be a proper Lie groupoid. Then

1. *The orbit $\mathcal{O}_x$ is an embedded closed submanifold of M;*
2. *there is a slice Σ at x.*

Local structure of proper Lie groupoids

Theorem (Linearization theorem for proper groupoids)

Let $\mathcal{G} \rightrightarrows M$ be a Lie groupoid and let $\mathcal{O}$ be the orbit through $x \in M$. If $\mathcal{G}$ is proper at x, then there are neighbourhoods U and V of $\mathcal{O}$ such that $\mathcal{G}_U \cong \mathcal{N}(\mathcal{G}_{\mathcal{O}})_V$.

Local structure of proper Lie groupoids

Combining the Linearization theorem with the previous remarks on Morita equivalence:

Any orbit $\mathcal{O}_x$ of a proper groupoid $\mathcal{G}$ has an invariant neighbourhood such that the restriction of $\mathcal{G}$ to it is Morita equivalent to $\mathcal{G}_x \ltimes \mathcal{N}_x$.

This lets us apply Schwarz Theorem and embed $M/\mathcal{G}$ (in many cases) into an euclidean space.

Smooth functions on orbit spaces $X = M/\mathcal{G}$

Proposition

The algebra $\mathcal{C}(X)$ is ***normal****, i.e., for any disjoint closed subsets $A, B \subset X$ there is a function $f \in \mathcal{C}(X)$ with values in $[0, 1]$ such that $f_{|A} = 0$ and $f_{|B} = 1$.*

Proposition (Partitions of unity)

For any open cover $\mathcal{U}$ of X there is a smooth partition of unity subordinated to $\mathcal{U}$.

Proposition (Existence of proper functions)

Let X be the orbit space of a proper groupoid. There exists a smooth proper function $f : X \to \mathbb{R}$.

Morita stratification

Theorem

Let $\mathcal{G} \rightrightarrows M$ be a proper Lie groupoid. Then M and the orbit space $X = M/\mathcal{G}$, together with the canonical stratifications, are differentiable stratified spaces. Moreover, the canonical stratifications of M and X are Whitney stratifications.

Any Morita equivalence between two proper Lie groupoids induces an isomorphism of differentiable stratified spaces between their orbit spaces.

Stratifications

Definition

Let X be a Hausdorff second-countable paracompact space. A **stratification** of X is a locally finite partition $\mathcal{S} = \{X_i \mid i \in I\}$ of X such that its members satisfy:

1. Each X_i, endowed with the subspace topology, is a locally closed, *connected* subspace of X, carrying a given structure of a smooth manifold;
2. (frontier condition) the closure of each X_i is the union of X_i with members of $\mathcal{S}$ of strictly lower dimension.

The members $X_i \in \mathcal{S}$ are called the **strata** of the stratification.

Stratifications - Examples

1. Any connected manifold comes with the stratification by only one stratum.
2. A manifold with boundary can be stratified by its interior and the connected components of the boundary.
3. If M is compact, then the cone on M,

$$CM = [0,1) \times M \;/\; \{0\} \times M$$

comes with a stratification with two strata: the vertex point and $(0,1) \times M$.

Stratifications

Definition

Given a stratification $\mathcal{S}$ there is a natural partial order on the strata given by

$$S \leq T \;\Leftrightarrow\; S \subset \overline{T}.$$

The union of all maximal strata (with respect to this order) forms a subspace $M^{\mathcal{S}-\mathrm{reg}} \subset M$ called the $\mathcal{S}$**-regular part of** M.

The following lemma shows that maximality of a stratum is a local condition

Lemma

A stratum $S \in \mathcal{S}$ is maximal if and only if it is open. The regular part $M^{\mathcal{S}-\mathrm{reg}}$ is open and dense in M.

Stratifications

Lemma

Let $\mathcal{S}$ be a stratification on a smooth manifold M, with no strata of codimension 1. Then the $\mathcal{S}$-regular part of M, denoted by M^{reg}, is connected.

Proof.

Let $x, y \in M^{\mathrm{reg}}$ and consider a smooth curve $\gamma : [0,1] \to M$ connecting x and y (recall that by M is connected by assumption). The image of γ is compact, so it can be covered by a finite number of open subsets of M, each of which intersects finitely many strata. Let U be the union of those open subsets. By transversality, it is possible to find a map $\gamma' : [0,1] \to U$ homotopic to γ and transverse to all the finitely many strata of codimension greater than 1 in U, missing them. □

Proper group actions: the canonical stratification

Proper action of a Lie group G on a manifold M.

Definition

The **orbit type equivalence** is the equivalence relation on M given by

$$x \sim y \iff G_x \sim G_y \ \text{(i.e. } G_x \text{ and } G_y \text{ are conjugate in } G\text{)}.$$

The **partition by orbit types**, denoted by $\mathcal{P}_\sim(M)$, is the resulting partition (each member of $\mathcal{P}_\sim(M)$ is called an **orbit type**).

$x \sim y$ is equivalent to the fact that the orbits through x and y are diffeomorphic as G-manifolds.

Proper groupoids: the canonical stratification

$\mathcal{G}$ proper Lie groupoid over M.

Definition

The **Morita type equivalence** is the equivalence relation on M given by

$$x \sim y \iff \mathcal{O}_x \text{ and } \mathcal{O}_y \text{ have same local linear model.}$$

The **partition by Morita types**, denoted by $\mathcal{P}_\sim(M)$, is the resulting partition.

Application: Proper symplectic groupoids are stratified by regular proper symplectic groupoids

Thank you!

References

I added some references, sorted by topics. They include references for what appeared in the talk, for some of the things people asked questions about, and some extra ones that I just happen to like.

In a few cases they're not original references, just the ones from where I learned things, or references with good introductions/reference overviews themselves.

References - Textbooks

An introduction to scheme theory (of nice enough schemes) for differential geometry:

J. A. Navarro González and J. B. Sancho de Salas. C^∞-differentiable spaces, volume 1824 of Lecture Notes in Mathematics. Springer-Verlag, Berlin, 2003

Differential geometry of smooth objects, but treated in a more algebraic way:

Jet Nestruev. Smooth manifolds and observables, volume 220 of Graduate Texts in Mathematics. Springer-Verlag, New York, 2003

References - Textbooks

My favourite book on Lie groups. Maybe a little bit of a tough read (seems to me) if it's the first time one is seeing Lie groups. But it's full of good ideas. Section 2 is on orbit spaces of proper actions:

J. J. Duistermaat and J. A. C. Kolk. Lie groups. Universitext. Springer-Verlag, Berlin, 2000

An introduction to stratifications, with lots of examples:

M. J. Pflaum. Analytic and geometric study of stratified spaces, volume 1768 ofLecture Notes in Mathematics. Springer-Verlag, Berlin, 2001.

References - Functions on orbit spaces

G. W. Schwarz. Smooth functions invariant under the action of a compact Lie group. Topology,14:63–68, 1975

J. Watts. The differential structure of an orbifold. Rocky Mountain J. Math., 47:289–327,2017.

To upgrade Schwarz's theorem from vector spaces to manifolds, one uses the Mostow-Palais equivariant embedding:

G. D. Mostow. Equivariant embeddings in Euclidean space. Annals of Mathematics, Second Series, 65: 432–446, 1957

R. S. Palais. Imbedding of compact, differentiable transformation groups in orthogonal representations. Journal of Mathematics and Mechanics, 6: 673–678, 1957

References - Groupoids and foliations

This is a textbook reference, and it points to other references too:

I. Moerdijk, J. Mrčun, Introduction to foliations and Lie groupoids. Cambridge Studies in Advanced Mathematics, 91. Cambridge Univ. Press, 2003.

References - Groupoids and orbifolds

I. Moerdijk and D. A. Pronk. Orbifolds, sheaves and groupoids. K-Theory, 12(1):3–21, 1997.

I. Moerdijk, Orbifolds as groupoids: an introduction. Orbifolds in mathematics and physics (Madison, WI, 2001), 205–222, Contemp. Math., 310, Amer. Math. Soc., Providence, RI, 2002.

Same textbook reference:

I. Moerdijk, J. Mrčun, Introduction to foliations and Lie groupoids. Cambridge Studies in Advanced Mathematics, 91. Cambridge Univ. Press, 2003.

References - Proper groupoids and orbispaces

I'd say any of the first 3 is a nice introduction, the 4th reference may be a bit more technical:

M. del Hoyo. Lie groupoids and their orbispaces. Port. Math., 70(2):161–209, 2013.

K. J. L. Wang, Proper Lie groupoids and their orbit spaces, PhD Thesis. University of Amsterdam, 2018.

M. Crainic and J. N. Mestre. Orbispaces as differentiable stratified spaces. Letters in mathematical physics 108 (3), 805–859, 2018.

M. J. Pflaum, H. Posthuma, and X. Tang. Geometry of orbit spaces of proper Lie groupoids. J. Reine Angew. Math., 694:49–84, 2014.

References - Other approaches

Very short paper (a talk in print, really) comparing approaches to leaf spaces, using groupoids - via convolution algebras, classifying spaces, and topos theory:

Moerdijk I. (2001) Models for the Leaf Space of a Foliation. In: Casacuberta C., Miró-Roig R.M., Verdera J., Xambó-Descamps S. (eds) European Congress of Mathematics. Progress in Mathematics, vol 201. Birkhäuser, Basel.

`https://www.math.uni-bielefeld.de/~rehmann/ECM/cdrom/3ecm/pdfs/pant3/moerd.pdf`

References - Other approaches

Good introduction to diffeological spaces and how they relate to Lie groupoids:

Nesta van der Schaaf. Diffeological Morita equivalence. arXiv:2007.09901, 2020.

And here diffeologies and other approaches (using the algebra of smooth functions) are compared:

J. Watts. Diffeologies, Differential Spaces, and Symplectic Geometry. PhD thesis, University of Toronto, 2012.

References - Other approaches

How to deal with proper actions in infinite dimensions:

T Diez, G Rudolph , Slice theorem and orbit type stratification in infinite dimensions, Differential Geometry and its Applications Volume 65, August 2019, Pages 176-211

References - Other approaches - Noncommutative geometry

The standard reference here is:

A. Connes. Noncommutative geometry. Academic Press, Inc., San Diego, CA, 1994

But for a first introduction I prefer this, it is a very nice survey, and goes right to the use of Lie groupoids:

C. Debord and G. Skandalis. Lie groupoids, pseudodifferential calculus and index theory. arXiv:1907.05258, 2019

References - Applications - cohomology

There are many papers dealing with cohomology theories related to groupoids. Two recent ones that I like, and that also give a good overview of the literature are:

L. Accornero and M. Crainic. Haefliger's differentiable cohomology. arXiv:2012.07777, 2020

E. Meinrenken and M.A. Salazar. Van Est differentiation and integration. Math. Ann. 376, 1395–1428 2020

References - Applications - Cohomology

Some more classical references:

Raoul Bott, Herbert Shulman, and James Stasheff. On the de Rham theory of certain classifying spaces. Adv. Math., 20:43–56, 1976.

Marius Crainic. Differentiable and algebroid cohomology, van Est isomorphisms, and characteristic classes. Commentarii Mathematici Helvetici, 78(4):681–721, 2003.

References - Applications

Vector fields on stacks:

Richard Hepworth. Vector fields and flows on differentiable stacks.Theory Appl. Categ.,22:542–587, 2009.

C. Ortiz and J. Waldron. On the Lie 2-algebra of sections of an LA-groupoid. J. Geom. Phys. 145, 103474, 2019

D. Berwick-Evans and E. Lerman. Lie 2-algebras of vector fields. Pacific J. Math. 309, no. 1, 1–34, 2020.

References - Applications

Integration on stacks:

Alan Weinstein. The volume of a differentiable stack. Lett. Math. Phys., 90(1-3):353–371, 2009.

Marius Crainic and João Nuno Mestre, Measures on differentiable stacks, J. Noncommut. Geom 13, no. 4, 1235–1294, 2019.

References - Applications

Riemannian geometry on stacks:

M. del Hoyo and R. L. Fernandes. Riemannian metrics on Lie groupoids.J. Reine Angew. Math., 735:143–173, 2018.

M. del Hoyo and R. L. Fernandes. Riemannian metrics on differentiable stacks.Math. Z., 292(1-2):103–132, 2019.

M. del Hoyo, M. de Melo. Geodesics on Riemannian stacks, arXiv:1906.03459, 2019

References - Applications

Symplectic proper groupoids have rich orbit spaces:

M. Crainic, R. L. Fernandes, and D. Martínez Torres. Poisson manifolds of compact types (PMCT 1). J. Reine Angew. Math., 756:101–149, 2019.

M. Crainic, R. L. Fernandes, and D. Martínez Torres. Regular Poisson manifolds of compact types. Astérisque, (413):viii + 154, 2019.

References - Applications

Lie groupoids and stacks used to study dynamical systems:

Cabrera Alejandro, del Hoyo Matias, Pujals Enrique: Discrete dynamics and differentiable stacks. Rev. Mat. Iberoam. 36, 2020

References - Applications

Not using Lie groupoids, but using geometry of orbit spaces (stratifications, and flows on orbit spaces, for example) to prove Molino's conjecture:

M. M. Alexandrino and M. Radeschi. Smoothness of isometric flows on orbit spaces and applications to the theory of foliations, Transf. Groups, 1–26, 2016.

M. M. Alexandrino and M. Radeschi. Closure of singular foliations: The proof of Molino's conjecture. Compositio Mathematica, 153 (12), 2577-2590, 2017.