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THE WEIBULL NEGATIVE BINOMIAL DISTRIBUTION

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Abstract

We propose the Weibull negative binomial distribution that is a quite flexible model to analyze positive data, and includes as special sub-models the Weibull, Weibull Poisson and Weibull geometric distributions. Some of its structural properties follow from the fact that its density function can be expressed as a mixture of Weibull densities. We provide explicit expressions for moments, generating function, mean deviations, Bonferroni and Lorenz curves, quantile function, reliability and entropy. The density of the Weibull negative binomial order statistics can be expressed in terms of an infinite linear combination of Weibull densities. We obtain two alternative expressions for the moments of order statistics. The method of maximum likelihood is investigated for estimating the model parameters and the observed information matrix is calculated. We propose a new regression model based on the logarithm of the new

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distribution. The usefulness of the new models is illustrated in three applications to real data.

1. Introduction

The Weibull distribution is a very popular model that has been extensively used over the past decades for analyzing data in survival analysis, reliability engineering and failure analysis, industrial engineering to represent manufacturing and delivery times, extreme value theory, weather forecasting to describe wind speed distributions, wireless communications and insurance to predict the size of reinsurance claims. In hydrology, it is applied to extreme events such as annually maximum one-day rainfalls and river discharges. The need for extended forms of the Weibull distribution arises in many applied areas but the emergence of such extensions in the statistics literature is only very recent. Following an idea due to Adamidis and Loukas [1] for a mixing procedure of distributions, we define the Weibull negative binomial (WNB) distribution and study several of its mathematical properties. The Weibull distribution represents only a special sub-model of the new distribution.

Let $W_1, \ldots, W_Z$ be a random sample from a Weibull density function with scale parameter $a > 0$ and shape parameter $b > 0$, namely $g_{a,b}(\omega) = ab\omega^{b-1}e^{-a\omega^b}$ $(\text{for } \omega > 0)$. We assume that the random variable Z has a zero truncated negative binomial distribution with probability mass function

$$P(z;\, s,\, \beta) = \beta^z \binom{s + z - 1}{z} [(1-\beta)^{-s} - 1]^{-1}, \quad z \in N,\ s \in \mathbb{R}_+$$

and $\beta \in (0, 1)$. Here Z and W are assumed independent random variables. Let $X = \min(W_1, \ldots, W_z)$. Then $f(x \mid z;\, a,\, b) = abzx^{b-1}e^{-azx^b}$ and the marginal probability density function (pdf) of X with four parameters reduces to

$$f(x;\, \theta) = \frac{abs\beta}{[(1-\beta)^{-s} - 1]} x^{b-1}e^{-ax^b}(1 - \beta e^{-ax^b})^{-(s+1)}, \quad x,\, s > 0, \quad \beta \in (0, 1), \quad (1)$$

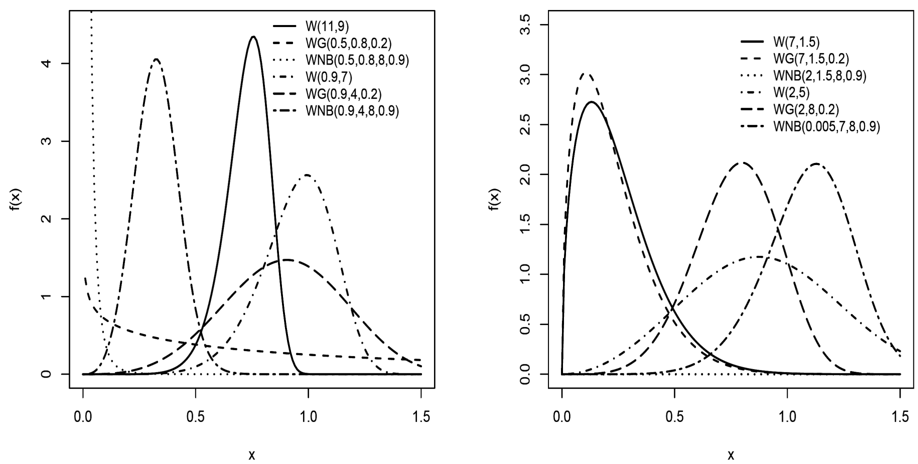
where $\boldsymbol{\theta} = (a,\, b,\, s,\, \beta)$. In the sequel, (1) is refereed to as the WNB density function which is customary for such names to be given to models arising via the operation of compounding (mixing) distributions.

By integrating (1) leads to its cumulative distribution function (cdf) given by

$$F(x;\,\theta) = \frac{[(1-\beta)^{-s} - (1-\beta e^{-ax^{b}})^{-s}]}{[(1-\beta)^{-s} - 1]}, \quad x > 0. \tag{2}$$

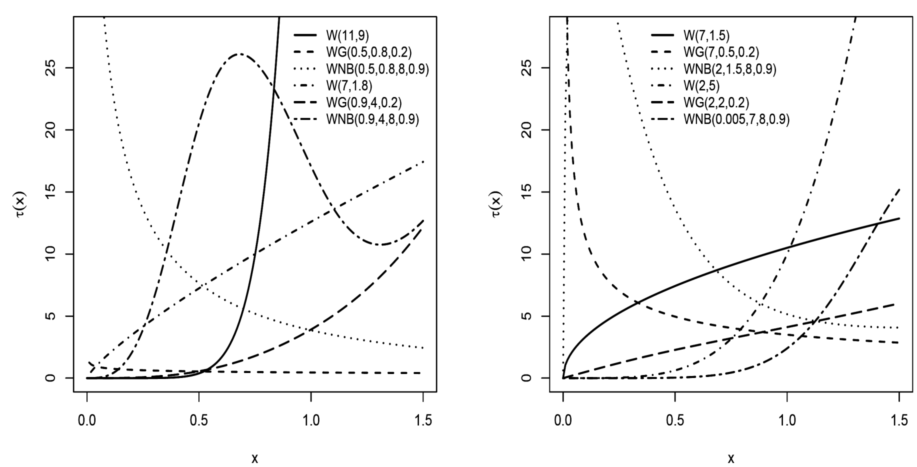
The hazard rate function corresponding to (2) is

$$\tau(x;\,\theta) = \frac{abx^{b-1}\beta e^{-ax^{b}} s(1-\beta e^{-ax^{b}})^{-(s+1)}}{[(1-\beta e^{-ax^{b}})^{-s} - 1]}. \tag{3}$$



(a) (b)

Figure 1. Plots of the WNB density for some parameter values.



(a) (b)

Figure 2. Plots of the WNB hazard rate function for some parameter values.

If X is a random variable with density function (1), then we write $X \sim \mathrm{WNB}(a, b, s, \beta)$. It extends several distributions previously considered in the literature and we study some structural properties of (1). In fact, the Weibull distribution with parameters a and b is clearly a special sub-model when $s = 1$ and $\beta \to 0$. For $s = 1$ and $b = 1$ in addition to $\beta \to 0$, it yields the exponential distribution. The WNB distribution also contains the exponential Poisson (EP) (Kus [11]) and Weibull Poisson (WP) distributions (Bereta et al. [5]) as sub-models when $\beta = \lambda/s$, $s \to \infty$ and, in the second case, in addition to $b = 1$. For $s = 1$, equation (1) reduces to the Weibull geometric (WG) density function (Barreto-Souza et al. [4]). In Figure 1, we plot the WNB density for selected parameter values. For all values of the parameters, the density function (1) tends to zero as $x \to \infty$. In Figure 2, we plot the WNB hazard rate function for selected parameter values, showing its flexibility. The WNB density can be widely applied in many areas of engineering and biology.

The rest of the article is organized as follows: In Section 2, we demonstrate that the WNB density function can be expressed as a mixture of Weibull densities. This result is important to provide some mathematical properties of the new model directly from those properties of the Weibull distribution. A range of mathematical properties is considered in Sections 3 to 6. These include moments, skewness and kurtosis, quantile function, generating function, mean deviations and Bonferroni and Lorenz curves. In Section 7, we show that the density of the WNB order statistics is a linear combination of Weibull densities. Two explicit expressions for the moments of the WNB order statistics are obtained in this section while the reliability and the Rényi entropy are derived in Sections 8 and 9, respectively. Maximum likelihood estimation and inference are discussed in Section 10. A log-Weibull negative binomial regression model is proposed in Section 11. Three applications to real data in Section 12 illustrate the importance of the new models. Concluding remarks are addressed in Section 13.

2. Expansion for the Density Function

Equations (1) and (2) are straightforward to compute using any statistical software. However, we obtain expansions for $f(x)$ and $F(x)$ in terms of an infinite weighted sum of cdf's and pdf's of Weibull distributions, respectively. Using the

Lagrange expansion (Consul and Famoye [7, Section 1.2.6]) for $(1 - \beta e^{-ax^b})^{-(s+1)}$,

$$\sum_{k=0}^{\infty} \frac{(s+1)}{[(s+1)+k]} \binom{(s+1)+k}{k} \{(\beta e^{-ax^b})[1 - (\beta e^{-ax^b})]^{1-1}\}^k = \sum_{k=0}^{\infty} \binom{s+k}{k} (\beta e^{-ax^b})^k, \tag{4}$$

equation (1) can be further expanded as

$$f(x;\, \theta) = \frac{abs}{[(1-\beta)^{-s} - 1]} \sum_{k=0}^{\infty} \binom{s+k}{k} \beta^{k+1} x^{b-1} e^{-a(k+1)x^b}.$$

Hence

$$f(x;\, \theta) = \sum_{k=0}^{\infty} \omega_k g_{a(k+1),\, b}(x), \tag{5}$$

where

$$\omega_k = \frac{s\beta^{k+1} \binom{s+k}{k}}{(k+1)[(1-\beta)^{-s} - 1]}, \tag{6}$$

and $g_{a(k+1),\, b}(x)$ is the Weibull density function with scale parameter $a(k+1)$ and shape parameter b. Clearly, $\sum_{k=0}^{\infty} \omega_k = 1$.

Equation (5) reveals that the WNB density function is a mixture of Weibull densities that holds for any parameter values. This result is important to obtain some of its mathematical properties from those of the Weibull distribution. The formulas related with the WNB distribution turn out manageable, as it is shown in the rest of this article, and with the use of modern computer resources with analytic and numerical capabilities, may turn into adequate tools comprising the arsenal of applied statisticians. Elementary integration of (5) gives the WNB cumulative distribution

$$F(x;\, \theta) = \sum_{k=0}^{\infty} \omega_k G_{a(k+1),\, b}(x), \tag{7}$$

where $G_{a(k+1),\, b}(x) = 1 - e^{-a(k+1)x^b}$ denotes the Weibull cumulative function with scale parameter $a(k+1)$ and shape parameter b.

3. Moments

From now on, let $X \sim \text{WNB}(a, b, s, \beta)$. The ordinary, central, inverse and factorial moments of the WNB distribution can be obtained from an infinite weighted linear combination of those quantities for Weibull distributions. For example, the sth moment of the Weibull distribution with parameters a and b is $\tau_s' = a^{-s/b}\Gamma(s/b + 1)$, where $\Gamma(\alpha) = \int_0^\infty w^{\alpha-1}e^{-w}dw$ is the gamma function. The sth generalized moment of the WNB distribution immediately comes from (5) as

$$\mu_s' = E(X^s) = \sum_{k=0}^{\infty} \omega_k \frac{\Gamma(s/b + 1)}{[a(k+1)]^{s/b}}. \tag{8}$$

Various closed form expressions can be obtained from (8) as particular cases. The central moments (μ_s) and the cumulants (κ_s) of X are easily obtained from the ordinary moments by

$$\mu_p = \sum_{r=0}^{p} (-1)^r \binom{p}{r} \mu_1'^p \mu_{p-r}' \quad \text{and} \quad \kappa_p = \mu_p' - \sum_{r=1}^{p-1} \binom{p-1}{r-1} \kappa_r \mu_{p-r}',$$

respectively. Here $\kappa_1 = \mu_1'$, $\kappa_2 = \mu_2' - \mu_1'^2$, $\kappa_3 = \mu_3' - 3\mu_2'\mu_1' + 2\mu_1'^3$, $\kappa_4 = \mu_4' - 4\mu_3'\mu_1' - 3\mu_2'^2 + 12\mu_2'\mu_1'^2 - 6\mu_1'^4$, etc., respectively. The skewness and kurtosis measures can be obtained from the classical relationships involving cumulants: $\text{Skewness}(X) = \kappa_3/\kappa_2^{3/2}$ and $\text{Kurtosis}(X) = \kappa_4/\kappa_2^2$, namely

$$\text{Skewness}(X) = \frac{E(X^3) - 3E(X)E(X^2) + 2E^3(X)}{\text{Var}^{3/2}(X)}$$

and

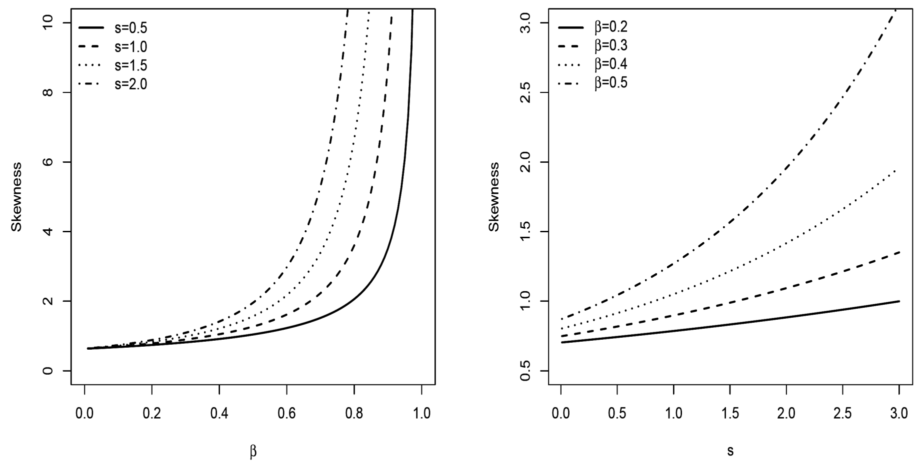
$$\text{Kurtosis}(X) = \frac{E(X^4) - 4E(X)E(X^3) + 6E(X^2)E^2(X) - 3E^4(X)}{\text{Var}^2(X)}.$$

Plots of the skewness and kurtosis of X as functions of s and β are given in Figures 3 and 4. Both skewness and kurtosis increase with s for fixed β and increase with β for fixed s.

Finally, the nth descending factorial moment of X is given by

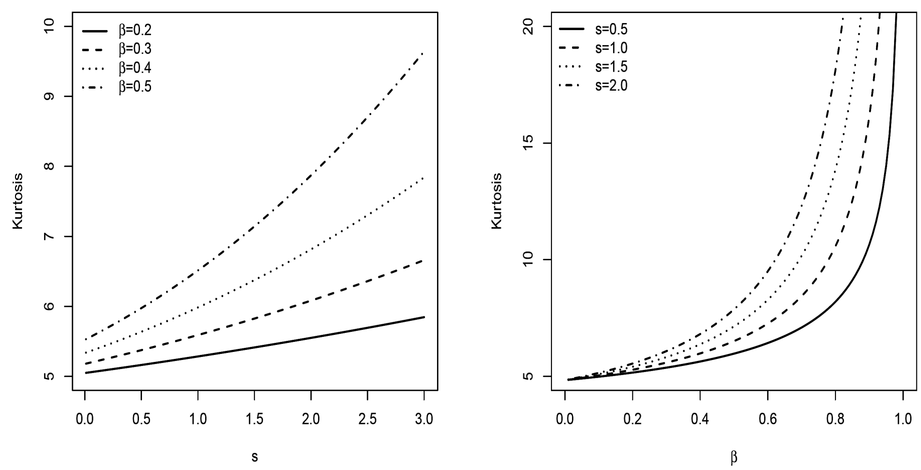
$$E[X^{(n)}] = E[X(X-1)(X-2)\cdots(X-n+1)] = \sum_{r=0}^{n} s(n, r)\mu_r',$$

where $s(n, r) = (r!)^{-1}[D^r x^{(n)}]_{x=0}$ are the Stirling numbers of the first kind. The WNB factorial moments are obtained from (8).



(a) (b)

Figure 3. The WNB skewness (for $a = 0.3$ and $b = 1.4$) as function of s (for fixed β) and as function of β (for fixed s).



(a) (b)

Figure 4. The WNB kurtosis (for $a = 0.3$ and $b = 1.4$) as function of s (for fixed β) and as function of β (for fixed s).

4. Quantile Measures

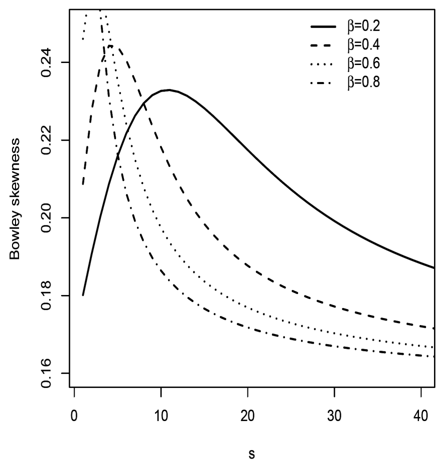
The WNB quantile function corresponds to the inverse of (2) given by

$$Q(u) = F^{-1}(u;\, \theta) = \{\log[\beta^{-1}(1 - \{[1-(1-\beta)^{-s}]u + (1-\beta)^{-s}\}^{-1/s})]^{-1/a}\}^{1/b}. \quad (9)$$

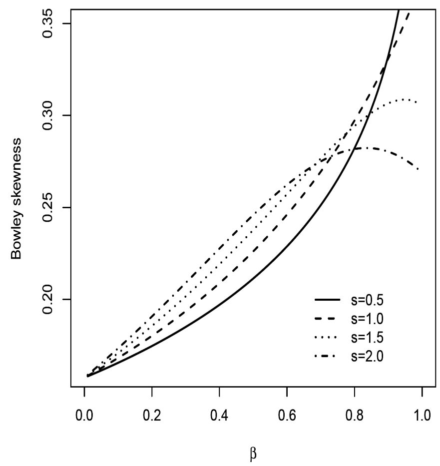
Then using (9), simulation from the WNB random variate is straightforward by

$$X = \{\log[\beta^{-1}(1 - \{[1-(1-\beta)^{-s}]V + (1-\beta)^{-s}\}^{-1/s})]^{-1/a}\}^{1/b},$$

where V is a uniformly distributed random variable over the interval (0, 1). We develop a script using the software R (R Development Core Team, 2008) to simulate X in Appendix A.



(a)



(b)

Figure 5. The Bowley skewness of the WNB distribution (for $a = 0.3$ and $b = 1.4$) as function of s (for fixed β) and as function of β (for fixed s).

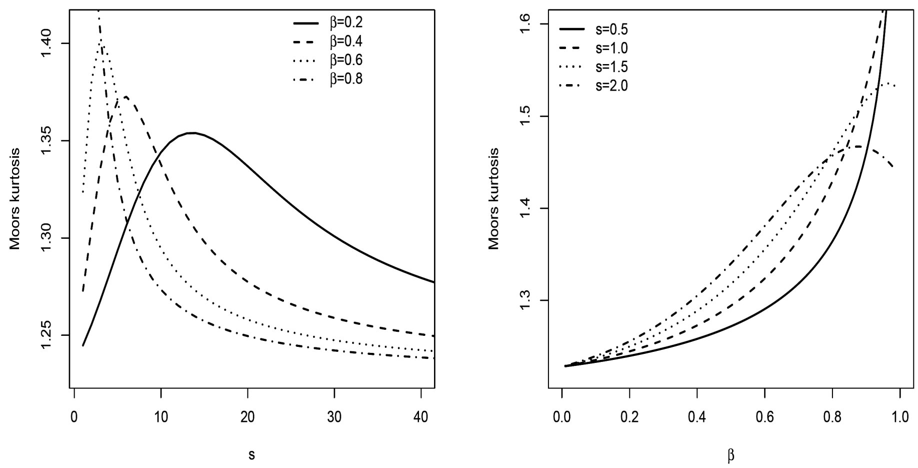
We now compute quantile measures for the skewness and kurtosis. The Bowley skewness (Kenney and Keeping [9]) is based on quartiles

$$B = \frac{Q(3/4) - 2Q(1/2) + Q(1/4)}{Q(3/4) - Q(1/4)}, \quad (10)$$

and the Moors kurtosis (Moors [13]) is based on octiles

$$M = \frac{Q(7/8) - Q(5/8) + Q(3/8) - Q(1/8)}{Q(6/8) - Q(2/8)}, \quad (11)$$

where $Q(u)$ is calculated from (9). These measures are less sensitive to outliers and they exist even for distributions without moments. Plots of (10) and (11) for selected parameter values are given in Figures 5 and 6, respectively. For fixed β, when *s* increases, the Bowley skewness and the Moors kurtosis first increase to a maximum and then decrease. The behavior of both measures when β increases depend upon the value fixed for *s*.



(a) (b)

Figure 6. The Moors kurtosis of the WNB distribution $(\text{for } a = 0.3 \text{ and } b = 1.4)$ as function of *s* (for fixed β) and as function of β (for fixed *s*).

5. Moment Generating Function

The moment generating function (mgf) $M(t) = E[\exp(tX)]$ follows from the power series expansion for the exponential function and (8),

$$M(t) = \sum_{s,\,k=0}^{\infty} \frac{\Gamma(s/b+1)\,\omega_k}{[a(k+1)]^{s/b}\,s!}\,t^s.$$

Now, we derive two explicit expressions for $M(t)$ using Meijer *G*-function and Wright generalized hypergeometric function. Let $A = abs\beta/[(1-\beta)^{-s}-1]$. First, we have

$$\begin{aligned} M(t) &= A\int_0^\infty \exp(tx)\,x^{b-1}\exp(-ax^b)\,[1-\beta\exp(-ax^b)]^{-(s+1)}dx \\ &= A\sum_{k=0}^{\infty}\beta^{k+1}\binom{s+k}{k}\int_0^\infty \exp(tx)\,x^{b-1}\exp[-a(k+1)\,x^b]\,dx. \qquad (12) \end{aligned}$$

From the Meijer G-function defined by

$$G_{p,q}^{m,n}\left(x \,\middle|\, \begin{matrix} a_1, \dots, a_p \\ b_1, \dots, b_q \end{matrix}\right) = \frac{1}{2\pi i}\int_L \frac{\prod_{j=1}^{m}\Gamma(b_j + t)\prod_{j=1}^{n}\Gamma(1 - a_j - t)}{\prod_{j=n+1}^{p}\Gamma(a_j + t)\prod_{j=m+1}^{p}\Gamma(1 - b_j - t)} x^{-t} dt$$

and the result $\exp\{-g(x)\} = G_{0,1}^{1,0}(g(x) \mid {}^{-}_{0})$ for $g(\cdot)$ an arbitrary function, we can write

$$K = \int_0^\infty x^{b-1}\exp[tx - a(k+1)x^b]dx$$

$$= \int_0^\infty x^{b-1}\exp(tx) G_{0,1}^{1,0}(a(k+1)x^b \mid {}^{-}_{0})dx.$$

If we assume that $b = p/q$, where $p \geq 1$ and $q \geq 1$ are co-prime integers, then equation (2.24.1.1) in Prudnikov et al. [15, Volume 3] yields

$$K = \frac{p^{b-1/2}(-t)^{-b}}{(2\pi)^{(p+q)/2-1}} G_{q,p}^{p,q}\left(\frac{\{[a(k+1)]^{b-1}b\}^q p^p}{(-t)^p q^q} \,\middle|\, \begin{matrix} \frac{1-b}{p}, \frac{2-b}{p}, \dots, \frac{p-b}{p} \\ 0, \frac{1}{q}, \dots, \frac{q-1}{q} \end{matrix}\right).$$

From (12) and the last two equations, we obtain

$$M(t) = \frac{absp^{b-1/2}(-t)^{-b}}{[(1-\beta)^{-s} - 1](2\pi)^{(p+q)/2-1}} \sum_{k=0}^{\infty} \beta^{k+1}\binom{s+k}{k}$$

$$\times G_{q,p}^{p,q}\left(\frac{\{[a(k+1)]^{b-1}b\}^q p^p}{(-t)^p q^q} \,\middle|\, \begin{matrix} \frac{1-b}{p}, \frac{2-b}{p}, \dots, \frac{p-b}{p} \\ 0, \frac{1}{q}, \dots, \frac{q-1}{q} \end{matrix}\right). \qquad (13)$$

The condition $b = p/q$ in (13) is not restrictive, since every real number can be approximated by a rational number.

A second representation for the integral M can be obtained using the Wright generalized hypergeometric function given by

$${}_p\Psi_q\left[\begin{matrix} (\alpha_1, A_1), \dots, (\alpha_p, A_p) \\ (\beta_1, B_1), \dots, (\beta_q, B_q) \end{matrix}; x\right] = \sum_{n=0}^{\infty} \frac{\prod_{j=1}^{p}\Gamma(\alpha_j + A_j n)}{\prod_{j=1}^{q}\Gamma(\beta_j + B_j n)} \frac{x^n}{n!}.$$

We assert that

$$K = \sum_{j=0}^{\infty} \frac{t^j}{j!} \int_0^{\infty} x^{j+b-1} \exp\{-[t/(a(k+1))^{b^{-1}} x]^b\} dx$$

$$= \frac{1}{ba(k+1)} \sum_{j=0}^{\infty} \frac{(l/[a(k+1)]^{b^{-1}})^j}{j!} \Gamma\left(\frac{j}{b}+1\right)$$

$$= \frac{1}{ba(k+1)} {}_1\Psi_0 \left[\begin{matrix} (1, 1/b) \\ - \end{matrix}; \frac{t}{[a(k+1)]^{b^{-1}}}\right] \tag{14}$$

provided that $b > 1$. Combining (12) and (14), we obtain the second representation for $M(t)$

$$M(t) = \frac{s}{[(1-\beta)^{-s} - 1]} \sum_{k=0}^{\infty} \frac{\beta^{k+1}}{(k+1)} \binom{s+k}{k} {}_1\Psi_0 \left[\begin{matrix} (1, 1/b) \\ - \end{matrix}; \frac{t}{[a(k+1)]^{b^{-1}}}\right] \tag{15}$$

provided that $b > 1$. Clearly, special formulas for the mgf of the Weibull, WG and WP distributions can be readily determined from equations (13) and (15) by substitution of known parameters.

6. Mean Deviations

The mean deviations about the mean $(\delta_1(X) = E(|X - \mu_1'|))$ and about the median $(\delta_2(X) = E(|X - m|))$ can be written as

$$\delta_1(X) = 2\mu_1' F(\mu_1') - 2T(\mu_1') \quad \text{and} \quad \delta_2(X) = \mu_1' - 2T(m), \tag{16}$$

respectively, where $\mu_1' = E(X)$ is given by (8), $F(\mu_1')$ comes from (2), $m =$ $\text{Median}(X) = Q(1/2)$ is computed from (9) and $T(q) = \int_0^q xf(x)dx.$

From (5) and using the incomplete gamma function $\gamma(\alpha, x) = \int_0^x w^{\alpha-1} e^{-\omega} dw,$ we obtain

$$T(q) = \sum_{k=0}^{\infty} \frac{\omega_k}{[a(k+1)]^{1/b}} \gamma(1 + b^{-1}, a(k+1)q^b). \tag{17}$$

Then the mean deviations can be calculated from (17). Further, we can obtain

Bonferroni and Lorenz curves from (17). These curves have applications in economics, reliability, demography, insurance and medicine and are defined by $B(p) = T(q)/(p\mu_1')$ and $L(p) = T(q)/\mu_1'$, respectively, where $q = \mathrm{Q}(p)$ is calculated by (9) for a given probability p.

7. Order Statistics

The density function $f_{i:n}(x)$ of the ith order statistic for $i = 1, \ldots, n$ corresponding to the random variables $X_1, \ldots, X_n$ following the WNB distribution, can be written as

$$f_{i:n}(x) = \frac{1}{B(i,\, n - i + 1)}\, f(x) \sum_{j=0}^{i-1} \binom{i-1}{j} (-1)^j \{1 - F(x)\}^{n+j-i},$$

where $f(x)$ is the pdf (1), $F(x)$ is the cdf (2) and $B(a, b) = \Gamma(a + b)/[\Gamma(a)\Gamma(b)]$ is the beta function. Setting $u = \exp(-ax^b)$, we can write from (5) and (7),

$$f_{i:n}(x) = \frac{ab}{B(i,\, n - i + 1)} \sum_{m=0}^{\infty} \omega_m (m+1) x^{b-1} u^{m+1}$$

$$\times \sum_{j=0}^{i-1} (-1)^j \binom{i-1}{j} \left(\sum_{k=0}^{\infty} \omega_k u^{k+1} \right)^{n+j-i}.$$

We use throughout an equation of Gradshteyn and Ryzhik [8, Section 0.314] for a power series raised to a positive integer j,

$$\left(\sum_{i=0}^{\infty} a_i y^i \right)^j = \sum_{i=0}^{\infty} c_{j,i}\, y^i, \tag{18}$$

whose coefficients $c_{j,i}$ (for $i = 1, 2, \ldots$) are easily obtained from the recurrence equation

$$c_{j,i} = (ia_0)^{-1} \sum_{m=1}^{i} (jm - i + m) a_m c_{j,i-m}, \tag{19}$$

where $c_{j,0} = a_0^j$. Hence, the coefficients $c_{j,i}$ come directly from $c_{j,0}, \ldots, c_{j,i-1}$

and, therefore, from $a_0, \ldots, a_i$. Using (18), it follows that

$$f_{i:n}(x) = \frac{ab}{B(i,\ n-i+1)} \sum_{m=0}^{\infty} \omega_m (m+1) x^{b-1} u^{m+1}$$

$$\times \sum_{j=0}^{i-1} (-1)^j \binom{i-1}{j} u^{n+j-1} \left(\sum_{k=0}^{\infty} c_{n+j-i,k} u^k \right).$$

Here the constants $c_{n+j-i,k}$ are determined from equations (6) and (19) as

$$c_{n+j-i,k} = (k\omega_0)^{-1} \sum_{l=1}^{k} [l(n+j-1) - k + l] \omega_l c_{n+j-i,k-l},$$

where $c_{n+j-i,0} = \omega_0^{n+j-i}$. Hence, $c_{n+j-i,k}$ can be calculated from $c_{n+j-i,0}, \ldots,$ $c_{n+j-i,k-1}$ and, therefore, from $\omega_0, \ldots, \omega_{k-1}$. By combining terms, we obtain

$$f_{i:n}(x) = \sum_{k,m=0}^{\infty} \sum_{j=0}^{i-1} \frac{(-1)^j (m+1) \binom{i-1}{j} \omega_m c_{n+j-i,k}}{(m+1+n+j-i+k) B(i,\ n-i+1)}$$

$$\times ab(m+1+n+j-i+k) x^{b-1} u^{m+1+n+j-i+k}.$$

Setting $\delta_{k,m,j} = a(m+1+n+j-i+k)$ in the above expression, we obtain

$$f_{i:n}(x) = \sum_{r,s=0}^{\infty} \sum_{j=0}^{i-1} \eta(k,\ m,\ j) g_{\delta_{k,m,j} b}(x), \tag{20}$$

whose coefficients $\eta(k,\ m,\ j)$ are easily obtained from

$$\eta(k,\ m,\ j) = \frac{(-1)^j (m+1) \binom{i-1}{j} \omega_m,\ c_{n+j-i,k}}{(m+1+n+j-i+k) B(i,\ n-i+1)}.$$

Equation (20) shows that the density function of the WNB order statistics can be expressed as an infinite weighted linear combination of Weibull densities. We can derive some mathematical measures of the WNB order statistics directly from those quantities of the Weibull distribution.

Using the linear combination representation (20), the moments of the WNB order statistics can be written directly in terms of the Weibull moments as

$$E(X_{i:n}^{s}) = \Gamma(s/b+1)\sum_{k,m=0}^{\infty}\sum_{j=0}^{i-1}\frac{\eta(k,\ m,\ j)}{\delta_{k,m,j}^{s/b}}, \tag{21}$$

where $\delta_{k,m,j}$ and $\eta(k,\ m,\ j)$ are given in Section 7.

Alternatively, we obtain another closed form expression for these moments using a result due to Barakat and Abdelkader [3] applied to the independent and identically distributed (i.i.d.) case

$$E(X_{i:n}^{r}) = r\sum_{j=n-i+1}^{n}(-1)^{j-n+i-1}\binom{j-1}{n-i}\binom{n}{j}I_j(r), \tag{22}$$

where

$$I_j(r) = \int_0^{\infty} x^{r-1}\{1-F(x)\}^j\,dx.$$

The integral can be written as

$$I_j(r) = \int_0^{\infty} x^{r-1}\left(\sum_{k=0}^{\infty}\omega_k u^{k+1}\right)^j dx = \sum_{k=0}^{\infty} c_{j,k}\int_0^{\infty} x^{r-1}\exp\{-a(k+j)x^b\}\,dx,$$

where

$$c_{j,k} = (k\omega_0)^{-1}\sum\nolimits_{m=1}^{k}(jm-k+m)\omega_m c_{j,k-m}$$

and $c_{j,0} = \omega_0^j$. Setting $y = a(j+k)x^b$, we obtain

$$I_j(r) = b^{-1}a^{-r/b}\Gamma(r/b)\sum_{k=0}^{\infty}\frac{c_{j,k}}{(j+k)^{r/b}}.$$

From equation (22), we have

$$E(X_{i:n}^{r}) = rb^{-1}a^{-r/b}\Gamma(r/b)\sum_{j=n-i+1}^{n}(-1)^{j-n+i-1}\binom{j-1}{n-i}\binom{n}{j}\sum_{k=0}^{\infty}\frac{c_{j,k}}{(j+k)^{r/b}}. \tag{23}$$

We can compute the moments of the WNB order statistics by three different ways: (i) from equation (21) that involves two infinite sums and one finite sum but no complicated function; (ii) from equation (23) that involves only two sums, one infinite and other finite; or (iii) by direct numerical integration. The moments of the WNB order statistics listed in Table 1 are in agreement using the three methods.

Table 1. Moments of the WNB order statistics for $n = 5$, $a = 4$, $b = 2$, $\beta = 0.8$ and $s = 3$

Order statistic	$X_{1:5}$	$X_{2:5}$	$X_{3:5}$	$X_{4:5}$	$X_{5:5}$
$r = 1$	0.294022	0.481234	0.674377	0.930862	1.451229
$r = 2$	0.112577	0.269695	0.513943	0.982929	2.558717
$r = 3$	0.052002	0.172134	0.439308	1.182069	5.581500
$r = 4$	0.027881	0.123392	0.419637	1.631341	15.08906
Variance	0.026128	0.038109	0.059160	0.116426	0.452652
Skewness	0.837768	0.762007	0.898139	1.267313	1.820482
Kurtosis	3.945583	4.059123	4.717588	6.474108	8.371872

8. Reliability

Here we derive the reliability $R = Pr(X_2 < X_1)$ when X_1 and X_2 have independent $\text{WNB}(a_1, b, s_1, \beta_1)$ and $\text{WNB}(a_2, b, s_2, \beta_2)$ distributions with the same shape parameter b. The density of X_1 and the cdf of X_2 are obtained from equations (5) and (7) as

$$f_1(x) = a_1 b x^{b-1} \sum_{r=0}^{\infty} (r+1) \omega_{1r} u_1^{r+1} \quad \text{and} \quad F_2(x) = 1 - \sum_{j=0}^{\infty} \omega_{2j} u_2^{j+1}, \tag{24}$$

where $u_i = \exp(-a_i x^b)$ for $i = 1, 2$ and the constants ω_{1r} and ω_{2j} are given by (8) with the corresponding parameters of the distributions of X_1 and X_2, respectively. We have

$$R = \int_0^{\infty} f_1(x) F_2(x) dx,$$

and then

$$R = \int_0^\infty bx^{b-1}\left(\sum_{r=0}^\infty \omega_{1r} a_1 (r+1) u_1^{r+1}\right)\left(1 - \sum_{j=0}^\infty \omega_{2j} u_2^{j+1}\right) dx$$

$$= 1 - \sum_{r,\, j=0}^\infty \omega_{1r}\omega_{2j} a_1 (r+1) \int_0^\infty bx^{b-1} \exp\{-[a_1(r+1) + a_2(j+1)] x^b\} dx.$$

By the application of $\int_0^\infty bx^{b-1} \exp(-\mu x^b) dx = \mu^{-1},$ we obtain

$$R = 1 - \sum_{r,\, j=0}^\infty \frac{(r+1) a_1 \omega_{1r} \omega_{2j}}{[a_1(r+1) + a_2(j+1)]}.$$

9. Entropy

The entropy of a random variable X with density $f(x)$ is a measure of variation of the uncertainty. Rényi entropy is defined by

$$I_R(\rho) = \frac{1}{1-\rho} \log\left\{\int f(x)^\rho dx\right\}, \tag{25}$$

where $f(x)$ is the pdf of X, $\rho > 0$ and $\rho \neq 1$. For a random variable X with a WNB distribution, we have

$$\int_0^\infty f(x)^\rho dx = \frac{a^\rho b^\rho s^\rho}{[(1-\beta)^{-s} - 1]^\rho} \int_0^\infty x^{\rho(b-1)} \beta^\rho e^{-ax^b \rho} (1 - \beta e^{-ax^b})^{-\rho(s+1)} dx.$$

Using the Lagrange expansion (4) in the last equation, we obtain

$$\int_0^\infty f(x)^\rho dx = \nu \sum_{k=0}^\infty \binom{\rho(s+1) + k - 1}{k} \frac{\beta^{\rho+k}}{(\rho+k)^{\rho(1-b^{-1}) + b^{-1}}}, \tag{26}$$

where

$$\nu = \frac{s^\rho a^{(\rho-1)/b} b^{\rho-1} \Gamma(\rho - \rho/b + 1/b)}{[(1-\beta)^{-s} - 1]^\rho}. \tag{27}$$

Substituting (26) and (27) into (25), we can write

$$I_R(\rho) = \frac{1}{1-\rho}\Bigg\{\rho\log(s) + (\rho - 1)[b^{-1}\log(a) + \log(b)] + \log[\Gamma(\rho - \rho/b + 1/b)]$$

$$- \rho\log[(1-\beta)^{-s} - 1] + \log\left[\sum_{k=0}^{\infty}\binom{\rho(s+1)+k-1}{k}\frac{\beta^{\rho+k}}{(\rho+k)^{\rho-\rho/b+1/b}}\right]\Bigg\}.$$

10. Estimation and Inference

Here we consider the estimation of the model parameters of the WNB distribution by the method of maximum likelihood. Suppose $X_1, \ldots, X_n$ is a random sample from (1) and $\boldsymbol{\theta} = (a, b, s, \beta)^T$ is the parameter vector. The log-likelihood (LL) function, say $\log L = \log\{L(a, b, s, \beta)\}$ for the four parameters is

$$\log\{L(a, b, s, \beta)\} = n\log(a) + n\log(b) + (b-1)\sum_{i=1}^{n}\log(x_i)$$

$$- a\sum_{i=1}^{n}x_i^b + n\log(\beta) + n\log(s)$$

$$- (s+1)\sum_{i=1}^{n}\log(1 - \beta e^{-ax_i^b}) - \log[(1-\beta)^{-s} - 1]. \tag{28}$$

It follows that the maximum likelihood estimators (MLEs) are the simultaneous solutions of the equations:

$$\sum_{i=1}^{n}x_i^b + (s+1)\sum_{i=1}^{n}\frac{x_i^b\beta e^{-ax_i^b}}{1-\beta e^{-ax_i^b}} = \frac{n}{a},$$

$$a\sum_{i=1}^{n}x_i^b\log(x_i) - \sum_{i=1}^{n}\log(x_i) + (s+1)\sum_{i=1}^{n}\frac{ax_i^b\log(x_i)\beta e^{-ax_i^b}}{1-\beta e^{-ax_i^b}} = \frac{n}{b},$$

$$\sum_{i=1}^{n}\log(1-\beta e^{-ax_i^b}) = \frac{n}{s} + \frac{(1-\beta)^{-s}\log(1-\beta)}{[(1-\beta)^{-s} - 1]}$$

and

$$(s+1)\sum_{i=1}^{n}\frac{e^{-ax_i^b}}{1-\beta e^{-ax_i^b}} = \frac{s(1-\beta)^{-(s+1)}}{[(1-\beta)^{-s}-1]} - \frac{n}{\beta}.$$

For interval estimation and hypothesis tests of the model parameters, we require the information matrix. The 4×4 observed information matrix is

$$\mathbf{K} = \mathbf{K}(\boldsymbol{\theta}) = \{\kappa_{i,j}\}, \quad i, j = a, b, s, \beta,$$

whose elements are given in Appendix B. Under conditions that are fulfilled for the parameter θ in the interior of the parameter space but not on the boundary, the asymptotic distribution of $(\hat{\boldsymbol{\theta}} - \boldsymbol{\theta})$ is $N_4(0, \mathbf{K}(\boldsymbol{\theta})^{-1})$. We can use the asymptotic multivariate normal $N_4(0, n^{-1}\mathbf{K}(\boldsymbol{\theta})^{-1})$ distribution of $\hat{\boldsymbol{\theta}}$ to construct approximate confidence regions for some parameters and for the hazard and survival functions. In fact, a $100(1-\gamma)$ asymptotic confidence interval for each parameter θ_i is given by

$$\mathrm{AIC}(\theta_i, (1-\gamma)) = (\hat{\theta}_i - z_{\gamma/2}\sqrt{\hat{\kappa}^{\theta_i,\theta_i}}, \hat{\theta}_i + z_{\gamma/2}\sqrt{\hat{\kappa}^{\theta_i,\theta_i}}),$$

where $\hat{\kappa}^{\theta_i,\theta_i}$ represents the (i, i) diagonal element of $n^{-1}\mathbf{K}(\boldsymbol{\theta})^{-1}$ for $i = 1, 2, 3, 4$ and $z_{\gamma/2}$ is the quantile $1 - \gamma/2$ of the standard normal distribution.

The asymptotic normality is also useful for testing goodness of fit of the WNB distribution and for comparing this distribution with some of its special sub-models using likelihood ratio (LR) statistics. We consider the partition $\boldsymbol{\theta} = (\boldsymbol{\theta}_1^T, \boldsymbol{\theta}_2^T)$, where $\boldsymbol{\theta}_1$ is a subset of parameters of interest of the WNB distribution and $\boldsymbol{\theta}_2$ is a subset of the remaining parameters. The LR statistic for testing the null hypothesis $H_0 : \boldsymbol{\theta}_1 = \boldsymbol{\theta}_1^{(0)}$ versus the alternative hypothesis $H_1 : \boldsymbol{\theta}_1 \neq \boldsymbol{\theta}_1^{(0)}$ is given by $\omega = 2\{l(\hat{\boldsymbol{\theta}} - \tilde{\boldsymbol{\theta}})\}$, where $\tilde{\boldsymbol{\theta}}$ and $\hat{\boldsymbol{\theta}}$ denote the MLEs under the null and alternative hypotheses, respectively. The statistic ω is asymptotically (as $n \to \infty$) distributed as χ_k^2, where k is the dimension of the subset $\boldsymbol{\theta}_1$ of interest. For example, we can compare the WG model against the WNB model by testing $H_0 : s = 1$ versus $H_1 : s \neq 1$.

11. The Log-Weibull Negative Binomial Model

Let X be a random variable having the WNB density function (1). The random variable $Y = \log(X)$ has a log-Weibull negative binomial (LWNB) distribution, whose density function (parameterized in terms of $a = \lambda^b$, $\sigma = b^{-1}$ and $\mu = -\log(\lambda))$ is

$$f(y) = \frac{s\beta}{\sigma[(1-\beta)^{-s}-1]} \exp\left\{\left(\frac{y-\mu}{\sigma}\right) - \exp\left(\frac{y-\mu}{\sigma}\right)\right\}$$

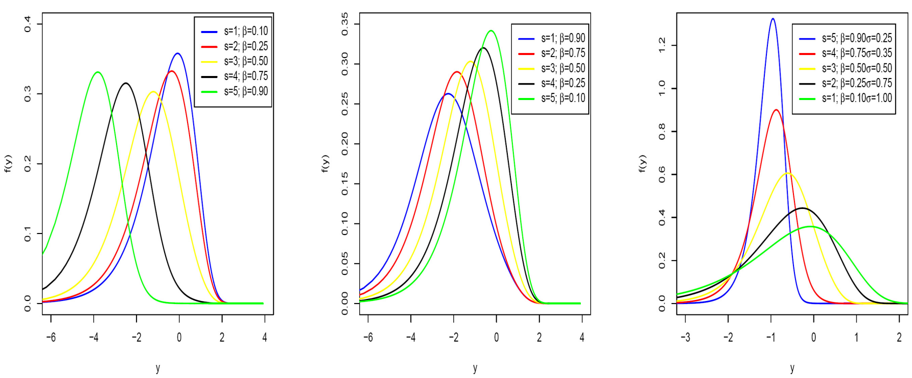
$$\times \left[1 - \beta \exp\left\{-\exp\left(\frac{y-\mu}{\sigma}\right)\right\}\right]^{-s-1}, \tag{29}$$

where $-\infty < y < \infty$, $\sigma > 0$ and $-\infty < \mu < \infty$. We refer to (29) as the (new) LWNB distribution, say $Y \sim \mathrm{LWNB}(s, \beta, \sigma, \mu)$, where μ is the location parameter, σ is the dispersion parameter and s and β are shape parameters. So

$$\text{if } X \sim \mathrm{WNB}(s, \beta, a, b), \text{ then } Y = \log(T) \sim \mathrm{LWNB}(s, \beta, \sigma, \mu).$$

The plots of (29) in Figure 7 for selected parameter values show great flexibility of the density function in terms of the shape parameters a and b. The survival function corresponding to (29) becomes

$$S(y) = 1 - \frac{(1-\beta)^{-s} - \left[1 - \beta \exp\left\{-\exp\left(\frac{y-\mu}{\sigma}\right)\right\}\right]^{-s}}{[(1-\beta)^{-s}-1]}. \tag{30}$$



(a) (b) (c)

Figure 7. Plots of the LWNB density for some parameter values: (a) $\mu = 0$, $\sigma = 1$, (b) $\mu = 0$, $\sigma = 1$ and (c) $\mu = 0$.

We define the standardized random variable $Z = (Y - \mu)/\sigma$ with density function

$$\pi(z;\, a,\, b) = \frac{s\beta}{[(1-\beta)^{-s} - 1]} \exp\{z - \exp(z)\} \times [1 - \beta \exp\{-\exp(z)\}]^{-(s+1)}, \quad z \in \Re. \tag{31}$$

The special case $s = 1$ corresponds to the (new) log-Weibull geometric (LWG) distribution.

The kth ordinary moment of the standardized distribution (31) is given by

$$\mu'_k = E(Z^k)$$
$$= \frac{s\beta}{[(1-\beta)^{-s} - 1]} \int_{-\infty}^{\infty} z^k \exp\{z - \exp(z)\}[1 - \beta \exp\{-\exp(z)\}]^{-(s+1)} dz.$$

By expanding the binomial term and setting $x = \exp(z)$, we obtain

$$\mu'_k = \frac{s\beta}{[(1-\beta)^{-s} - 1]} \sum_{j=0}^{\infty} (-1)^j \beta^j \binom{-(s+1)}{j} \int_0^{\infty} \log^k(x) \exp[-(j+1)x] dx.$$

The above integral can be calculated from Prudnikov et al. [15, Volume 1, equation 2.6.21.1] as

$$I(k,\, j) = \left(\frac{\partial}{\partial a}\right)^k [(j+1)^{-a}\Gamma(a)]\Big|_{a=1},$$

and then

$$\mu'_k = \frac{s\beta}{[(1-\beta)^{-s} - 1]} \sum_{j=0}^{\infty} (-1)^j \beta^j \binom{-(s+1)}{j} I(k,\, j).$$

In many practical applications, the lifetimes are affected by explanatory variables such as the cholesterol level, blood pressure, weight and several others. Parametric regression models to estimate univariate survival functions for censored data are widely used. A parametric model that provides a good fit to lifetime data tends to yield more precise estimates of the quantities of interest. Based on the LWNB density function, we propose a linear location-scale regression model for censored data linking the response variable y_i and the explanatory variable vector

$\mathbf{v}_i^T = (v_{i1}, \ldots, v_{ip})$ as follows:

$$y_i = \mathbf{v}_i^T \boldsymbol{\gamma} + \sigma z_i, \quad i = 1, \ldots, n, \tag{32}$$

where the random error z_i has density function (31), $\boldsymbol{\gamma} = (\gamma_1, \ldots, \lambda_p)^T$, $\sigma > 0$, $a > 0$ and $b > 0$ are unknown parameters. The parameter $\mu_i = \mathbf{v}_i^T \boldsymbol{\gamma}$ is the location of y_i. The location parameter vector $\boldsymbol{\mu} = (\mu_1, \ldots, \mu_n)^T$ is represented by a linear model $\boldsymbol{\mu} = \mathbf{V}\boldsymbol{\gamma}$, where $\mathbf{V} = (\mathbf{v}_1, \ldots, \mathbf{v}_n)^T$ is a known model matrix. The LWNB regression model (32) opens new possibilities for fitting many different types of censored data. It is an extension of an accelerated failure time model using the WNB distribution for censored data.

Consider a sample $(y_1, \mathbf{v}_1), \ldots, (y_n, \mathbf{v}_n)$ of n independent observations, where each random response is defined by $y_i = \min\{\log(x_i), \log(c_i)\}$. We assume non-informative censoring such that the observed lifetimes and censoring times are independent. Let F and C be the sets of individuals for which y_i is the log-lifetime or log-censoring, respectively. Conventional likelihood estimation techniques can be applied here. The log-likelihood function for the vector of parameters $\boldsymbol{\tau} = (s, \beta, \sigma, \boldsymbol{\gamma}^T)^T$ from model (32) has the form $l(\boldsymbol{\tau}) = \sum_{i \in F} l_i(\boldsymbol{\tau}) + \sum_{i \in C} l_i^{(c)}(\boldsymbol{\tau})$, where $l_i(\boldsymbol{\tau}) = \log[f(y_i \mid \mathbf{v}_i)]$, $l_i^{(c)}(\boldsymbol{\tau}) = \log[S(y_i \mid \mathbf{v}_i)]$, $f(y_i \mid \mathbf{v}_i)$ is the density (29) and $S(y_i \mid \mathbf{v}_i)$ is the survival function (30) of Y_i. The total log-likelihood function for $\boldsymbol{\tau}$ reduces to

$$\begin{aligned} l(\boldsymbol{\tau}) = {} & r \log\left\{\frac{s\beta}{\sigma[(1-\beta)^{-s} - 1]}\right\} + \sum_{i \in F}[z_i - \exp(z_i)] \\ & - (s+1)\sum_{i \in F} \log[1 - \beta \exp\{-\exp(z_i)\}] \\ & + \sum_{i \in C} \log\left\{1 - \frac{(1-\beta)^{-s} - [1 - \beta\exp\{-\exp(z_i)\}]^{-s}}{[(1-\beta)^{-s} - 1]}\right\}, \end{aligned} \tag{33}$$

where $z_i = (y_i - \mathbf{v}_i^T \boldsymbol{\gamma})/\sigma$ and r is the number of uncensored observations (failures).

The MLE $\hat{\boldsymbol{\tau}}$ of the vector of unknown parameters can be calculated by maximizing the log-likelihood (33). We use the subroutine NLMixed in SAS to obtain $\hat{\boldsymbol{\tau}}$. Initial values for β, σ and $\boldsymbol{\gamma}$ are taken from the fit of the LWG regression model with $s = 1$. The fit of the LWNB model yields the estimated survival function for y_i given by

$$S(y_i;\, \hat{s},\, \hat{\beta},\, \hat{\sigma},\, \hat{\boldsymbol{\gamma}}^T) = 1 - \frac{(1-\hat{\beta})^{-\hat{s}} - [1 - \hat{\beta}\exp\{-\exp(\hat{z}_i)\}]^{-\hat{s}}}{[(1-\hat{\beta})^{-\hat{s}} - 1]}, \tag{34}$$

where $\hat{z}_i = (y_i - \mathbf{v}_i^T \hat{\boldsymbol{\gamma}})/\hat{\sigma}$.

The asymptotic distribution of $(\hat{\boldsymbol{\tau}} - \boldsymbol{\tau})$ is multivariate normal $N_{p+3}(0,\, K(\boldsymbol{\tau})^{-1})$, where $K(\boldsymbol{\tau})$ is the information matrix. The asymptotic covariance matrix $K(\boldsymbol{\tau})^{-1}$ of $\hat{\boldsymbol{\tau}}$ can be approximated by the inverse of the $(p+3)\times(p+3)$ observed information matrix $-\ddot{\mathbf{L}}(\boldsymbol{\tau})$. The approximate multivariate normal distribution $N_{p+3}(0,\, -\ddot{\mathbf{L}}(\boldsymbol{\tau})^{-1})$ for $\hat{\boldsymbol{\tau}}$ can be used in the classical way to construct confidence regions for some parameters in $\boldsymbol{\tau}$. We can use LR statistics for comparing the LWNB model with some special sub-models.

12. Applications

In this section, we compare the results of fitting the WNB and LWNB distributions to three real data sets.

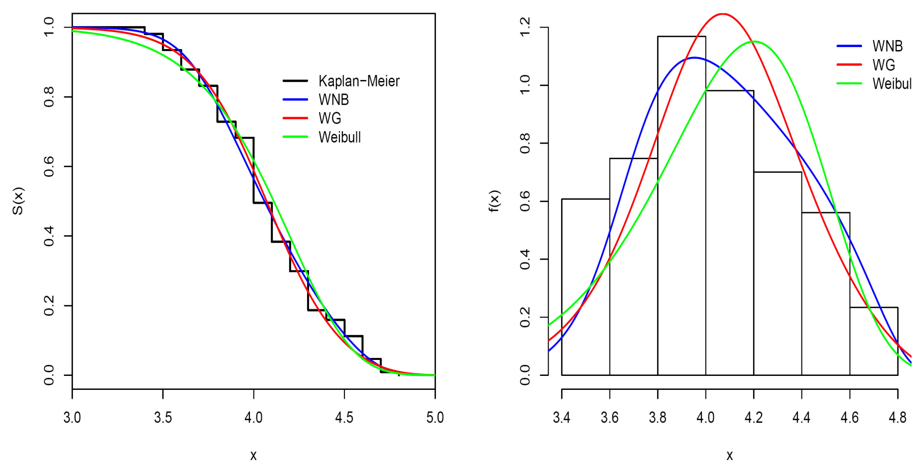
Clorpirifos data

A study conducted in Chile by Dra. Fernanda Cavieres (University of Valparaíso) established that clorpirifos likely causes congenital malformations, which can be avoided by folic acid. The response variable was the fetal height of a mouse (in millimeters) (Balakrishnan et al. [2]; and Leiva et al. [12]). We fitted the WNB, WG and Weibull distributions to the data using the maximum likelihood method for parameter estimation. The computations were performed using the subroutine NLMixed in SAS. The convergence was achieved using the re-parameterization $a = \lambda^b$ and $\beta = \exp(\beta_0)/[1 + \exp(\beta_0)]$ to guarantee the estimate of β in $(0, 1)$.

Table 2. MLEs of the model parameters for the clorpirifos data, the corresponding SEs (given in parentheses) and some statistical measures

Model	λ	b	s	β_0	AIC	CAIC	BIC
WNB	0.2160	29.8851	0.0975	6.6848	72.0	72.4	82.7
	(0.0034)	(9.3921)	(0.1618)	(3.4928)			
WG	0.2186	19.2622	2.0752	1	73.4	73.6	81.4
	(0.0062)	(2.0679)	-	(0.8898)			
Weibull	0.2366	0.0018			79.9	80.0	85.2
	(0.0018)	(0.9562)	-	-			

Table 2 lists the MLEs (the corresponding standard errors are in parentheses) of the parameters from the fitted WNB, WG and Weibull models and the values of the following statistics: AIC (Akaike Information Criterion), BIC (Bayesian Information Criterion) and CAIC (Consistent Akaike Information Criterion). These results indicate that the WNB and WG models have the lowest values for the AIC and BIC statistic, respectively, among the fitted models, and therefore they could be chosen as the best models. In order to assess if the model is appropriate, we plot in Figure 8(a) the empirical and estimated survival functions of the WNB, WG and Weibull distributions. Figure 8(b) gives the histogram of the data and the fitted WNB, WG and Weibull distributions. The plots indicate that the first two distributions provide good fits for these data.



(a) (b)

Figure 8. (a) Estimated survival function from the fitted WNB, WG and Weibull distributions and the empirical survival for clorpirifos data. (b) Estimated densities of the WNB, WG and Weibull models for clorpirifos data.

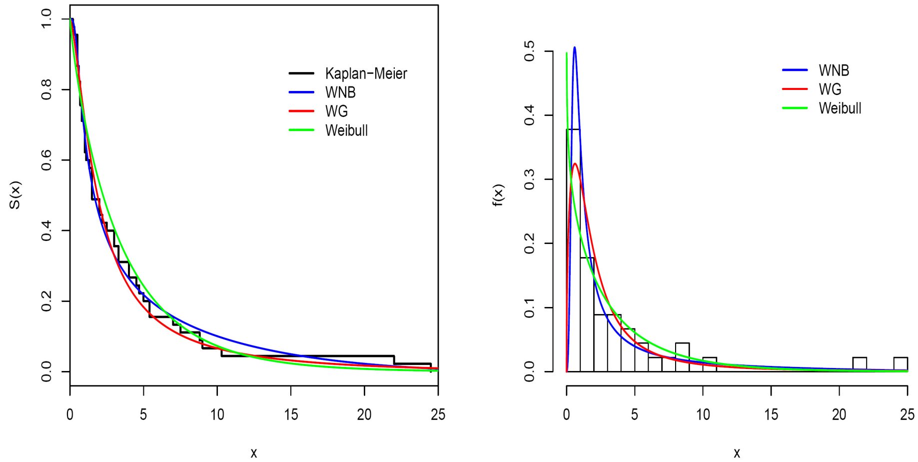
Airborne data

The data representing repair times (in h) for an airborne communication transceiver were first analyzed by Von Alven [16] using a two-parameter log-normal distribution. These data were reanalyzed by Chhikara and Folks [6] using a two-parameter inverse Gaussian distribution and by Koutrouvelis et al. [10] using the inverse Gaussian distribution with three parameters. We fitted the WNB, WG and Weibull distributions to the data using the maximum likelihood method for parameter estimation. The computations were performed using the subroutine NLMixed in SAS. Table 3 lists the MLEs (the corresponding standard errors are in parentheses) of the parameters from the fitted WNB, WG and Weibull models and the values of the AIC, BIC and CAIC statistics. These results indicate that the WNB model yields the lowest values for these statistics among the fitted models, and then it could be chosen as the best model.

Table 3. MLEs of the model parameters for the airborne data, the corresponding SEs (given in parentheses) and the AIC measures

Model	λ	b	s	β_0	AIC	CAIC	BIC
WNB	0.0445	3.4176	0.1295	12.9327	201.1	202.1	208.4
	(0.0131)	(1.4064)	(0.0951)	(6.4066)			
WG	0.0530	1.4669	0	3.3368	203.6	204.2	209.1
	(0.0468)	(0.2084)	-	(1.6851)			
Weibull	0.2952	0.8896			208.7	209.0	212.3
	(0.0525)	(0.0960)	-	-			

The LR statistic for testing the hypotheses $H_0 : s = 1$ versus $H_1 : H_0$ *is not true*, i.e., to compare the WNB and WG regression models, is $w = 2\{-96.55 - (-98.80)\} = 4.50$ $(p\text{-value} = 0.0339)$. It indicates that the proposed model is superior to the WG model in terms of model fitting. In order to assess if the model is appropriate, the empirical and estimated survival functions of the WNB, WG and Weibull distributions are plotted in Figure 9(a). In Figure 9(b), we plot the histogram of the data and the fitted WNB, WG and Weibull distributions. We conclude that the new distribution provides a good fit for these data.



(a) (b)

Figure 9. (a) Estimated survival function from the fitted WNB, WG and Weibull distributions and the empirical survival for airborne data. (b) Estimated densities of the WNB, WG and Weibull models for airborne data.

Multiply censored relay data

As an application of the LWNB regression model, we consider the data given in Table 4 analyzed by Nelson ([14, p. 160]) and concerning to "test data on a production relay (thousands of cycles)". The failure time of observation i, x_i, was defined as the thousands of cycles, and v_{i1} denotes the three levels of production (16 amps, 26 amps and 28 amps). The objective is to compare the levels of production in relation to thousands of cycles. The model considered in the analysis is described by

$$y_i = \gamma_0 + \gamma_1 v_{i1} + \sigma z_i,$$

where the random variable $y_i = \log(x_i)$ follows (29) for $i = 1, ..., 35$.

Table 4. Data on a production relay

Production	Thousands of cycles												
16 amps:	38^+	77^+	138	168^+	188	228	252	273	283^+	288	291	299	317
	374	527	529	559	567	656	873						
26 amps:	103	110	131	219	226^+								
28 amps:	84	92	121	138	191	206	254	267	308	313			

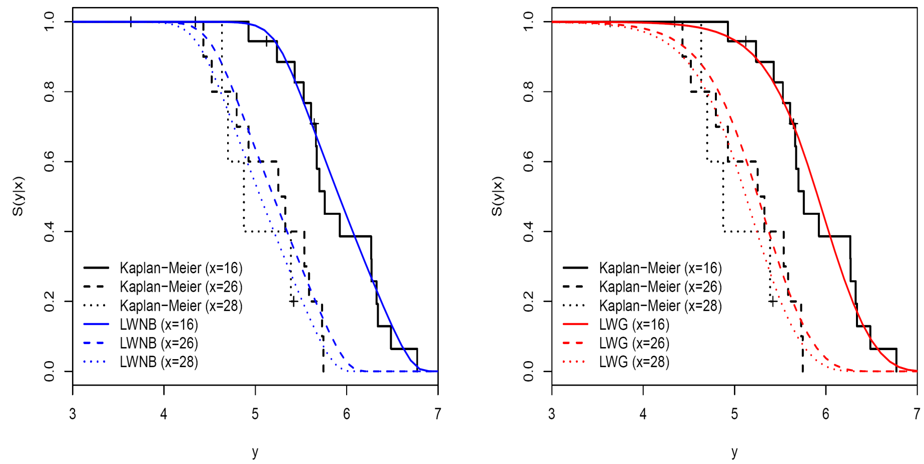
The symbol + indicates censoring

Table 5 lists the MLEs of the parameters for the LWNB and LWG regression models fitted to these data using the NLMixed procedure in SAS. The convergence was achieved using the re-parameterization $\beta = \exp(\beta_0)/[1+\exp(\beta_0)]$ to guarantee the estimate of β in $(0, 1)$. As initial values for the parameters γ and σ in the iterative maximization process of the log-likelihood function 33, we used the fitted values obtained with the log-Weibull regression model. These results indicate that the new regression model has the lowest values for the AIC, CAIC and BIC statistics among the fitted models, and then it could be chosen as the best model. For the fitted LWNB regression model, v_1 is significant at 1% and then there is a significant difference among the levels of the production for thousands of cycles. The LR statistic for testing the hypotheses $H_0 : s = 1$ versus $H_1 : H_0$ *is not true*, i.e., to compare the LWNB and LWG regression models, is $w = 2\{-19.50 - (-22.15)\} = 5.30$ (p-value $= 0.0213$) what indicates that the LWNB regression model is superior to the LWG regression model in terms of model fitting.

Table 5. MLEs of the parameters from the LWNB regression model fitted to the relay data, the corresponding SEs (given in parentheses), p-value in [.] and the AIC measure

Model	s	β_0	σ	v_0	v_1	AIC	CAIC	BIC
LWNB	0.0194	13.1187	0.1124	7.8528	–0.0719	49.0	51.1	56.8
	(0.00688)	(6.8980)	(0.0502)	(0.2301)	(0.0112)			
				[<0.0001]	[<0.0001]			
LWG	1	1.3464	0.3257	7.5061	–0.0656	52.3	53.6	58.5
	-	(1.7556)	(0.0689)	(0.3921)	(0.0147)			
				[<0.0001]	[<0.0001]			

A graphical comparison between the LWNB and LWG models is given in Figures 10(a) and 10(b). These plots provide the empirical survival function and the estimated survival functions given by (34). Based on these plots, it is evident that the LWNB model fits well to these data. From Figure 10, we note that there is a difference of the level 16 for the levels 26 and 28.



(a) (b)

Figure 10. Estimated survival functions and the empirical survival: (a) LWNB regression model versus K-M. (b) LWG regression model versus K-M.

13. Conclusions

For the first time, we introduce the Weibull negative binomial (WNB) distribution and study some of its structural properties. The new distribution generalizes some distributions studied recently in the literature. It is an important model for analysis of lifetime data because of the wide usage of the Weibull distribution and the fact that the current generalization provides means of its continuous extension to still more complex situations. We provide a comprehensive description of some structural properties of the proposed distribution with the hope that it will attract wider applications in several fields. The WNB density function can be expressed as a mixture of Weibull density functions. This result allows us to derive some expansions for the ordinary, factorial and inverse moments and moment generating function. The density function of the WNB order statistics can be written in terms of an infinite linear combination of Weibull density functions. We calculate mean deviations, Bonferroni and Lorenz curves, reliability and Rénvi entropy and obtain two representations for the moments of order statistics. The estimation of parameters is approached by the method of maximum likelihood and the observed information matrix is derived. We propose a new regression model based on the logarithm of the WNB distribution. The usefulness of the new models is illustrated in three real data sets using classical criterion. The proposed models provide a rather flexible mechanism for fitting a wide spectrum of positive real data sets.

Appendix A

R script to simulate data from the WNB distribution

First simulation

rm(list=ls(all=TRUE))

set.seed(45)

$a = 0.5$

$b = 1.5$

$s = 1$

$\text{beta} = 0.2$

$v < -runif\,(10000,\ 0,\ 1)$

$x < -\log(1/beta * (1 - 1/(1 - (1 - beta^{-s})) * v + (1 - beta)^{-s^{1/s}}))^{1/a^{1/b}}$

$f < -density(x)$

hist(x, freq=FALSE, main=” ”, ylab=”f(x)”, xlab=”x”)

lines(f, col=”red”)

Second simulation

set.seed(45)

$a = 0.9$

$b = 2$

$s = 4$

$\text{beta} = 0.3$

$v < -runif\,(10000,\ 0,\ 1)$

$x < -\log(1/beta * (1 - 1/(1 - (1 - beta^{-s})) * v + (1 - beta)^{-s^{1/s}}))^{1/a^{1/b}}$

$f < -density(x)$

hist(x, freq=FALSE, main=” ”, ylab=”f(x)”, xlab=”x”)

lines(f, col=”red”)

Appendix B

The elements of the observed information matrix

$$\kappa_{a,a} = -\frac{\partial^2 \ell}{\partial a^2} = \frac{n}{a^2} - (s+1)\sum_{i=1}^{n} \frac{[(x_i^b)^2 \beta e^{-ax_i^b}]}{(1-\beta e^{-ax_i^b})^2},$$

$$\kappa_{b,a} = -\frac{\partial^2 \ell}{\partial b \partial a} = -\sum_{i=1}^{n} x_i^b \log(x_i) - (s+1)$$

$$\times \sum_{i=1}^{n} x_i^b \log(x_i) \beta e^{-ax_i^b} \frac{(1-\beta e^{-ax_i^b} - ax_i^b)}{(1-\beta e^{-ax_i^b})^2},$$

$$\kappa_{s,a} = -\frac{\partial^2 \ell}{\partial s \partial a} = \sum_{i=1}^{n} \frac{x_i^b \beta e^{-ax_i^b}}{(1-\beta e^{-ax_i^b})},$$

$$\kappa_{\beta,a} = -\frac{\partial^2 \ell}{\partial \beta \partial a} = (s+1)\sum_{i=1}^{n} \frac{x_i^b e^{-ax_i^b}}{(1-\beta e^{-ax_i^b})^2},$$

$$\kappa_{b,b} = -\frac{\partial^2 \ell}{\partial b^2} = \frac{n}{b^2} + a\sum_{i=1}^{n} x_i^b [\log(x_i)]^2 + (s+1)$$

$$\times \sum_{i=1}^{n} \frac{[\log(x_i)]^2 a\beta x_i^b e^{-ax_i^b} [1 - ax_i^b - \beta e^{-ax_i^b}]}{(1-\beta e^{-ax_i^b})^2},$$

$$\kappa_{s,b} = -\frac{\partial^2 \ell}{\partial s \partial b} = \sum_{i=1}^{n} \frac{ax_i^b \log(x_i) \beta e^{-ax_i^b}}{(1-\beta e^{-ax_i^b})},$$

$$\kappa_{\beta,b} = -\frac{\partial^2 \ell}{\partial \beta \partial b} = (s+1)\sum_{i=1}^{n} \frac{ax_i^b \log(x_i) e^{-ax_i^b}}{(1-\beta e^{-ax_i^b})^2},$$

$$\kappa_{s,s} = -\frac{\partial^2 \ell}{\partial s^2} = \frac{n}{s^2} - \frac{[\log(1-\beta)]^2 (1-\beta)^{-s}}{[(1-\beta)^{-s} - 1]^2},$$

$$\kappa_{\beta, s} = -\frac{\partial^2 \ell}{\partial \beta \partial s} = -\sum_{i=1}^{n} \frac{e^{-ax_i^b}}{(1 - \beta e^{-ax_i^b})}$$

$$- \frac{(1-\beta)^{-(s+1)}[s \log(1-\beta) - 1][(1-\beta)^{-s} - 1] + [(1-\beta)^{-s} \log(1-\beta)]^2}{[(1-\beta)^{-s} - 1]^2},$$

$$\kappa_{\beta, \beta} = -\frac{\partial^2 \ell}{\partial \beta^2} = \frac{n}{\beta^2} - (s+1)$$

$$\times \sum_{i=1}^{n} \frac{(e^{-ax_i^b})^2}{(1 - \beta e^{-ax_i^b})^2} + \frac{s(1-\beta)^{-s-2}[-(s+1) + (1-\beta)^{-s}]}{[(1-\beta)^{-s} - 1]^2}.$$

References

[1] K. Adamidis and S. Loukas, A life time distribution with decreasing failure rate, Statist. Probab. Lett. 39 (1998), 35-42.

[2] N. Balakrishnan, V. Leiva, A. Sanhueza and E. Cabrera, Mixture inverse Gaussian distribution and its transformations, moments and applications, Statistics 43 (2009), 91-104.

[3] H. M. Barakat and Y. H. Abdelkader, Computing the moments of order statistics from nonidentical random variables, Stat. Methods Appl. 13 (2004), 15-26.

[4] W. Barreto-Souza, A. L. de Morais and G. M. Cordeiro, The Weibull-geometric distribution, J. Stat. Comput. Simul. 81(5) (2011), 645-657.

[5] E. M. P. Bereta, F. Louzada-Neto and M. A. P. Franco, A distribuição Weibull-Poisson, SINAPE – Simpósio Nacional de Probabilidade e Estatística, 19, São Pedro-SP, 2010.

[6] R. S. Chhikara and J. L. Folks, The inverse Gaussian distribution as a lifetime model, Technometrics 19 (1977), 461-468.

[7] P. C. Consul and F. Famoye, Lagrangian Probability Distributions, Birkhäuser, Boston, 2006.

[8] I. S. Gradshteyn and I. M. Ryzhik, Table of Integrals, Series, and Products, Academic Press, San Diego, 2000.

[9] J. F. Kenney and E. S. Keeping, Mathematics of Statistics, Part 1, 3rd ed., pp. 101-102, Princeton, New Jersey, 1962.

[10] I. A. Koutrouvelis, G. C. Canavos and S. G. Meintanis, Estimation in the three-parameter inverse Gaussian distribution, Comput. Statist. Data Anal. 49 (2005), 1132-1147.

[11] C. Kus, A new lifetime distribution, Comput. Statist. Data Anal. 51 (2007), 4497-4509.

[12] V. Leiva, M. Barros and G. A. Paula, Generalized Birnbaum-Saunders Models Using R, XI Escola de Modelos de Regressão, Recife, Brazil, 2009.

[13] J. J. A. Moors, A quantile alternative for kurtosis, J. Roy. Statist. Soc. Ser. D, The Statistician 37 (1998), 25-32.

[14] W. B. Nelson, Accelerated Testing Statistical Models, Test, Plans and Data Analysis, John Wiley & Sons, New Jersey, 2004.

[15] A. P. Prudnikov, Y. A. Brychkov and O. I. Marichev, Integrals and Series, Volumes 1, 2 and 3, Gordon and Breach Science Publishers, Amsterdam, 1986.

[16] W. H. Von Alven, Reliability Engineering by ARINC, Prentice-Hall, Inc., Englewood Cliffs, NJ, 1964.