MAT220 – Cálculo Diferencial e Integral IV – IAG
Prova 1 – 18/09/2008

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1. Use a definição para mostrar que a função $f(z) = \bar{z}$ não é derivável em nenhum ponto.

$$f'(z) = \lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \frac{\overline{z+\Delta z} - \bar{z}}{\Delta z} = \lim_{\Delta z \to 0} \frac{\overline{\Delta z}}{\Delta z}$$

Em particular, se $\Delta \cancel{z} = \Delta x$:

$$\lim_{\Delta z \to 0} \frac{\overline{\Delta z}}{\Delta z} = \lim_{\Delta x \to 0} \frac{\Delta x}{\Delta x} = 1$$

$\neq$

Se $\Delta z = i \Delta y$:

$$\lim_{\Delta z \to 0} \frac{\overline{\Delta z}}{\Delta z} = \lim_{\Delta y \to 0} \frac{-i\Delta y}{i\Delta y} = -1$$

Logo, $f'(z) = \lim_{\Delta z \to 0} \frac{\overline{\Delta z}}{\Delta z}$ não existe. //

2. Determine os pontos em que as funções indicadas são deriváveis e calcule a derivada nos pontos em que ela existir:

a. $f(x+iy) = x^2 - y^2 + i2xy$

b. $f(z) = z\Im z$

c. $f(x+iy) = x^3 + iy^3$

(a) $u = x^2 - y^2$

$v = 2xy$

$$\left.\begin{aligned}\frac{\partial u}{\partial x} &= 2x \\ \frac{\partial v}{\partial y} &= 2x\end{aligned}\right] = \qquad \left.\begin{aligned}\frac{\partial u}{\partial y} &= -2y \\ \frac{\partial v}{\partial x} &= 2y\end{aligned}\right) \text{ opostas}$$

As derivadas parciais de u, v são contínuas em $\mathbb{R}^2$ e vale C-R em $\mathbb{R}^2$ $\Rightarrow$ f é derivável em $\mathbb{C}$

e $f'(x+iy) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = 2x + i2y = 2(x+iy)$

ou $f'(z) = 2z$ //

(b) $f(x+iy) = (x+iy)\cdot y = xy + iy^2$

$u = xy$

$v = y^2$

$$\frac{\partial u}{\partial x} = y \qquad \frac{\partial u}{\partial y} = x$$

$$\frac{\partial v}{\partial y} = 2y \qquad \frac{\partial v}{\partial x} = 0$$

$$\begin{cases}\dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y} \\[2ex] \dfrac{\partial u}{\partial y} = -\dfrac{\partial v}{\partial x}\end{cases} \Leftrightarrow \begin{cases} y = 2y \\[2ex] x = 0\end{cases} \Leftrightarrow (x,y) = (0,0)$$

As derivadas parciais de u, v são contínuas em $\mathbb{R}^2$

Vale C-R apenas em $\{(0,0)\}$. $\Rightarrow$ f é derivável apenas em $z=0$ e $f'(0) = \frac{\partial u}{\partial x}(0,0) + i\frac{\partial v}{\partial x}(0,0) = 0$ //

(c) $u = x^3$

$v = y^3$

$\frac{\partial u}{\partial x} = 3x^2$

$\frac{\partial v}{\partial y} = 3y^2$

$\frac{\partial u}{\partial y} = 0$

$\frac{\partial v}{\partial x} = 0$

Vale C-R $\Leftrightarrow$ $3x^2 = 3y^2$ $\Leftrightarrow$ $x = \pm y$

$\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$ são contínuas em $\mathbb{R}^2$

Logo, f é derivável apenas em $x \pm ix$ e

$$f'(x \pm ix) = \frac{\partial u}{\partial x}(x, \pm x) + i\frac{\partial v}{\partial x}(x, \pm x)$$

$$= 3x^2$$

3. Calcular todos os valores de:

   a. $\log i$

   b. $(\sqrt{3}+i)^i$

   c. $\operatorname{arcsen} 0$

(a) $\log i = \overset{0}{\cancel{\log 1}} + i\left(\frac{\pi}{2} + 2k\pi\right)$ //

$|i| = 1$

$\arg i = \frac{\pi}{2} + 2k\pi, \ k \in \mathbb{Z}$

(b) $(\sqrt{3}+i)^i = \exp\left(i \log(\sqrt{3}+i)\right)$

$|\sqrt{3}+i| = \sqrt{3+1} = 2$

$\arg(\sqrt{3}+i) = \arctan\frac{1}{\sqrt{3}} + 2k\pi, \ k \in \mathbb{Z}$

$= \frac{\pi}{6} + 2k\pi$

$\log(\sqrt{3}+i) = \log 2 + i\left(\frac{\pi}{6} + 2k\pi\right)$

$(\sqrt{3}+i)^i = \exp\left(i \log 2 - \left(\frac{\pi}{6} + 2k\pi\right)\right)$

$= e^{-\left(\frac{\pi}{6}+2k\pi\right)}\left(\cos(\log 2) + i \operatorname{sen}(\log 2)\right)$

$k \in \mathbb{Z}$

(c) $z = \text{arc sen } 0 \iff \text{sen } z = 0$

$$\iff \frac{e^{iz} - e^{-iz}}{2i} = 0 \iff e^{iz} = e^{-iz}$$

$$\iff e^{2iz} = 1 \iff 2iz = \log 1 = i\,2k\pi,\ k \in \mathbb{Z}$$

$$\iff z = k\pi,\ k \in \mathbb{Z}\ //$$

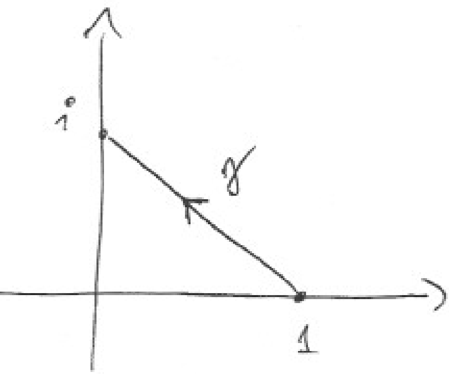
4. Calcular $\int_\gamma f(z)\,dz$ onde:

a. $f(z) = 2z$ e γ é o segmento de reta de 1 a i.

b. $f(z) = \frac{1}{z}$ e γ é o semi-círculo $|z| = 1$, $\Im z \geq 0$ de 1 a -1.

c. $f(z) = \dfrac{z^3 - 1}{z^2 + 4}$ e γ é o círculo $|z| = 1$ orientado no sentido anti-horário.

(a)



$$\gamma(t) = 1 + t(i - 1)$$
$$= (1-t) + ti$$
$$0 \leq t \leq 1$$

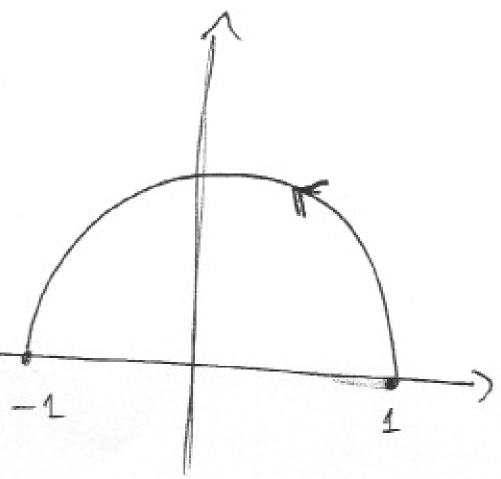
$$\int_\gamma f(z)\,dz = \int_0^1 f(\gamma(t)) . \gamma'(t)\,dt$$

$$= \int_0^1 2((1-t)+ti).(-1+i)\,dt = 2(-1+i)\int_0^1 (1-t)+ti\,dt$$

$$= 2(-1+i)\left[t - \frac{t^2}{2} + \frac{t^2}{2}i\right]_0^1 = 2(-1+i)\left(\frac{1}{2} + \frac{1}{2}i\right)$$

$$= (-1+i)(1+i) = i^2 - 1 = -2 \quad //$$

(b)



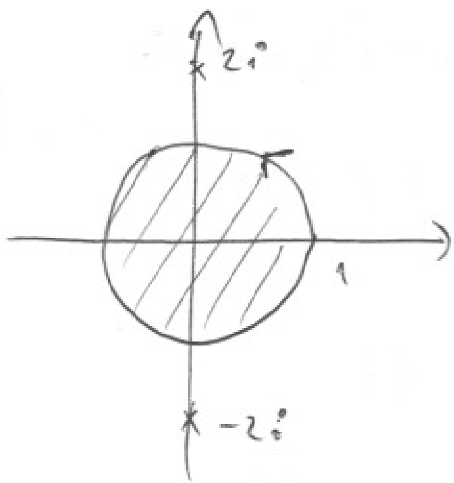
$$\gamma(t) = e^{it}, \quad 0 \leq t \leq \pi$$
$$\gamma'(t) = i e^{it}$$

4 (b), cont.

$$\int_\gamma f(z)\,dz = \int_0^\pi f(\gamma(t)) \cdot \gamma'(t)\,dt$$

$$= \int_0^\pi \frac{1}{e^{it}}\, i e^{it}\,dt = i\int_0^\pi dt = \pi i \ //$$

(c)



$$f(z) = \frac{z^3-1}{z^2+4}$$

$$z^2+4=0 \iff z = \pm 2i$$

f é analítica no interior e sobre γ

$$\Rightarrow \int_\gamma f(z)\,dz = 0 \ //$$