Isoparametric families of submanifolds

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Abstract

In this paper we present a generalization of the theory of isoparametric families of hypersurfaces in a Riemannian manifold of constant curvature, in that we consider families of submanifolds of codimension greater than one. Our starting point is a definition of isoparametric family of submanifolds which coincides with the classical definition formulated by É. Cartan [1] in the case of codimension 1. That definition is given in Section 1, where we also show that the mean curvature vector of the submanifolds that belong to an isoparametric family has constant length (cf. Proposition 1.4). In Section 2, we study the isoparametric families whose orthogonal distribution is integrable. In this case we prove that each leaf of the isoparametric family has flat normal bundle, the integral manifolds of the orthogonal distribution are totally geodesic and that the mean curvature vector of each submanifold in the isoparametric family is parallel with respect to the normal connection (Proposition 2.3). In Section 3 we study the parallel submanifolds. In this way we obtain an effective procedure to construct normal isoparametric families (i. e. whose orthogonal distributions are integrable) which, however, is not explored in the present paper (cf. Proposition 3.5). In Section 4 we show that the leaves of an isoparametric family of submanifolds in a Riemannian manifold of constant curvature have constant principal curvatures. Finally, in Section 5, we obtain a relation between these principal curvatures which generalizes the fundamental formula of Cartan to higher codimensions.

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Preliminaries. In this paper all considered manifolds and mappings are of class C^∞. Given a Riemannian manifold $(M^{n+q}, \langle , \rangle)$ of dimension $n+q$ with $n,\ q \geq 1$, we shall follow the convention that indices designated by small Latin letters range between 1 and n, by small Greek letters range between $n+1$ and $n+q$, and by big Latin letters between 1 and $n+q$. We denote with ∇ and Δ the covariant derivative and the Laplace operator of the Riemannian manifold.

For a given local orthonormal frame $\{e_A\}$ we have the corresponding differential forms ω_A, ω_{AB} which satisfy the equations:

$$\nabla e_A = \sum_B \omega_{BA} \otimes e_B.$$

Therefore the structure equations of that frame are:

$$d\omega_A + \sum_B \omega_{AB} \wedge \omega_B = 0,$$

$$d\omega_{AB} + \sum_S \omega_{AS} \wedge \omega_{SB} = \Omega_{AB},$$

where the Ω_{AB} are the curvature forms of the frame.

1 Isoparametric families of submanifolds

Definition 1.1 *Let M^{n+q} be a Riemannian manifold of dimension $n+q$, with $n,\, q \geq 1$, and let U be an open submanifold of M^{n+q}. We shall call isoparametric family of n-dimensional submanifolds any foliation of U which is locally defined as the inverse image of regular values of a mapping*

$$F : W \subset U \to \mathbf{R}^q, \quad F = (F_{n+1}, \ldots, F_{n+q}),$$

such that the functions

$$P_{\alpha\beta} = \langle grad\, F_\alpha, grad\, F_\beta \rangle, \quad \Delta F_\alpha,$$

are locally constant on any leaf.

Definition 1.2 *An orthonormal frame $e_1, \ldots, e_{n+q}$ is said to be distinguished if the vector fields $e_{n+1}, \ldots, e_{n+q}$ are obtained from the vector fields $grad\, F_{n+1}, \ldots, grad\, F_{n+q}$ by the usual Gram-Schmidt orthonormalization process.*

Lemma 1.3 *Let S be a leaf of a given isoparametric family of submanifolds and let H be the mean curvature vector of S. Then, for any distinguished frame (e_α), the functions $\langle H, e_\alpha \rangle$ are locally constant on S.*

Proof. From the definition of distinguished frame we get that:

$$\mathrm{grad}\, F_\alpha = \sum_{\beta=n+1}^{n+q} q_{\alpha\beta} e_\beta, \quad \alpha = n+1, \ldots, n+q,$$

where the functions $q_{\alpha\beta}$ are locally constant on S.

The above equations are equivalent to the following:

$$dF_\alpha = \sum_\beta q_{\alpha\beta} \omega_\beta, \quad \alpha = n+1, \ldots, n+q. \tag{1}$$

On the other hand, from the fact that the functions $q_{\alpha\beta}$ are locally constant on S we get:

$$dq_{\alpha\beta} = \sum_\gamma q_{\alpha\beta\gamma} dF_\gamma, \quad \alpha, \beta = n+1, \ldots, n+q. \tag{2}$$

Exterior differentiation of (2) gives:

$$\sum_{\gamma} dq_{\alpha\beta\gamma} \wedge dF_{\gamma} = 0, \quad \alpha, \beta = n+1, \ldots, n+q.$$

¿From Cartan's Lemma follows that the differential forms $dq_{\alpha\beta\gamma}$, α, β, $\gamma = n+1, \ldots, n+q$, are linear combinations of $dF_{n+1}, \ldots, dF_{n+q}$ and therefore the functions $q_{\alpha\beta\gamma}$ are also locally constant on S.

On the other hand, using exterior differentiation it follows from (1) that:

$$\sum_{\beta} dq_{\alpha\beta} \wedge \omega_{\beta} + q_{\alpha\beta} d\omega_{\beta} = 0, \quad \alpha = n+1, \ldots, n+q.$$

Using the structure equations we get from the above equations:

$$\sum_{\beta} (dq_{\alpha\beta} \wedge \omega_{\beta} - \sum_{A,\beta} q_{\alpha\beta}(\omega_{\beta A} \wedge \omega_{A})) = 0. \tag{3}$$

¿From the above relations we find, for each pair of vector fields e_{ν}, e_{μ}, with μ, $\nu \geq n+1$:

$$dq_{\alpha\mu}(e_{\nu}) - dq_{\alpha\nu}(e_{\mu}) = \sum_{\beta} q_{\alpha\beta}(\omega_{\beta\mu}(e_{\nu}) - \omega_{\beta\nu}(e_{\mu})).$$

Since the matrix $(q_{\alpha\beta})$ is invertible and the functions $dq_{\alpha\nu}(e_{\mu}) - dq_{\alpha\nu}(e_{\mu})$ are locally constant, we have that the functions $\omega_{\beta\nu}(e_{\mu}) - \omega_{\beta\mu}(e_{\nu})$ are locally constant, too. In particular, the functions $\omega_{\nu\mu}(e_{\nu})$ are locally constant on S.

In the sequel we analyse the Laplacian of each function F_{α} which is given by:

$$\Delta F_{\alpha} = \sum_{A} e_{A}(e_{A} F_{\alpha}) - (\nabla_{e_A} e_{A}) F_{\alpha}. \tag{4}$$

We observe that the functions $dF(e_i)$ vanish identically for $i = 1, \ldots, n$, and that, because of (1) and (2), we have:

$$e_{\beta}(e_{\beta} F_{\alpha}) = \sum_{\beta,\gamma} q_{\alpha\beta\gamma} q_{\gamma\beta},$$

so that equation (4) may be rewritten:

$$\Delta F_{\alpha} = \sum_{\beta,\gamma} q_{\alpha\beta\gamma} q_{\gamma\beta} - \sum_{A} (\nabla_{e_A} e_{A}) F_{\alpha}. \tag{5}$$

We have also that:

$$\sum_{A} dF_{\alpha}(\nabla_{e_A} e_{A}) = \sum_{i,\beta} \langle \nabla_{e_i} e_i, e_{\beta} \rangle q_{\alpha\beta} + \sum_{\gamma,\beta} \langle \nabla_{e_\gamma} e_{\gamma}, e_{\beta} \rangle q_{\alpha\beta}.$$

Recall that the mean curvature vector H is given by:

$$H = \frac{1}{n} \sum_{i,\beta} \langle \nabla_{e_i} e_i, e_{\beta} \rangle e_{\beta},$$

so that we obtain the expression

$$\sum_A dF_\alpha(\nabla_{e_A} e_A) = \sum_\beta q_{\alpha\beta}\langle nH, e_\beta\rangle + \sum_{\beta,\gamma} q_{\alpha\beta}\omega_{\beta\gamma}(e_\gamma). \tag{6}$$

¿From the definition of isoparametric family, we have that the functions ΔF_α are locally constant on each leaf S, and because of (5) and (6) we conclude that the functions $\langle H, e_\beta\rangle$ also have that property. □

An immediate consequence of the above lemma is the following proposition.

Proposition 1.4 *The mean curvature vector of each leaf of an isoparametric family of submanifolds has constant length.*

2 Normal isoparametric families

Definition 2.1 *An isoparametric family of submanifolds of a Riemannian manifold is called normal isoparametric family if the distribution defined by the normal spaces to the leaves is integrable*[1].

Lemma 2.2 *The connection forms relative to a distinguished frame of a normal isoparametric family of submanifolds satisfy:*

$$\omega_{\alpha i}(e_\beta) = 0, \quad \omega_{\alpha\beta}(e_i) = 0, \qquad i = 1, \ldots, n, \quad \alpha, \beta \geq n+1. \tag{1}$$

Proof. We start with the relations (3):

$$\sum_\beta (dq_{\alpha\beta} \wedge \omega_\beta - \sum_{A,\beta} q_{\alpha\beta}(\omega_{\beta A} \wedge \omega_A)) = 0,$$

and evaluate the above expression to the pair of vector fields e_i, e_ν in order to get

$$\sum_\beta q_{\alpha\beta}(\omega_{\beta\nu}(e_i) - \omega_{\beta i}(e_\nu)) = 0,$$

and then, since the matrix $(q_{\alpha\beta})$ is invertible, that:

$$\omega_{\beta\nu}(e_i) - \omega_{\beta i}(e_\nu) = 0. \tag{2}$$

On the other hand, the integrability of the orthogonal distribution means that:

$$\omega_{\alpha i}(e_\beta) = \omega_{\beta i}(e_\alpha). \tag{3}$$

Combining relations (2) and (3) we obtain (1). □

¿From Lemma 2.2 follows immediately the following result:

[1]Note of the translator: It is not true that every isoparametric family of submanifolds is automatically normal. For instance, the foliation of Euclidean space by orbits of an orthogonal representation of a compact connected Lie group is always an isoparametric family of submanifolds (G. Schwarz, *Smooth functions invariant under the action of a compact Lie group*, Topology **14** (1975), 63-68; see also Theorem 3 in Q.-M. Wang, *Isoparametric maps of Riemannian manifolds and their applications*, Advances in Science of China, Mathematics 2, Wiley-Interscience, New York (1986), 79-103), but it is a normal isoparametric family of submanifolds if and only if that representation is polar (R. S. Palais and C.-L. Terng, *Critical Point Theory and Submanifold Geometry*, Lect. Notes in Math. 1353, Springer-Verlag, 1988, Section 5.6).

Proposition 2.3 *The integral manifolds of the orthogonal distribution of a normal isoparametric family of submanifolds are totally geodesic submanifolds and the normal connection on each leaf of the isoparametric family has flat normal bundle.*

3 Parallel families of submanifolds

In this section we study the local structure of isoparametric families of submanifolds in a simply connected Riemannian manifold $M^{n+q}(c)$ of constant curvature c.

Lemma 3.1 *Let S be a leaf in a given normal isoparametric family of submanifolds of $M^{n+q}(c)$. Then for each point m in S there exist a simply-connected open neighbourhood U of m in S, an interval J of $\mathbf{R}$ centered at 0, and unit normal vector fields $\xi_{n+1}, \ldots, \xi_{n+q}$ defined on U and parallel with respect to the normal connection such that the mapping*

$$f : J \times S^{q-1} \times U \to M^{n+q}(c)$$

defined by

$$f(t, a, x) = \exp_x(t \sum_\alpha a_\alpha \xi_\alpha(x)), \quad a = (a_{n+1}, \ldots, a_{n+q}),$$

is a diffeomorphism, where S^{q-1} is the unit sphere in $\mathbf{R}^q$.

The proof of this lemma is simple and we shall omit it.

Definition 3.2 *A diffeomorphism f in the conditions of Lemma 3.1 is called normal trivialization of the given normal isoparametric family. Moreover, we consider the diffeomorphism*

$$f_t^a : U \to M^{n+q}(c), \quad f_t^a(x) = f(t, a, x), \quad x \in U.$$

Lemma 3.3 *For each $x \in U$ and $a \in S^{q-1}$, the subspace field $t \mapsto (f_t^a)_* T_x S$ is parallel along the geodesic $t \mapsto f_t^a(x)$.*

Proof. We shall consider separately each one of the cases $c = 0$, $c > 0$ and $c < 0$.

In the first case the ambient space is $\mathbf{R}^{n+q}$ and the mapping f can be written $f(t, a, x) = x + t \sum_\alpha a_\alpha \xi_\alpha(x)$ where x is identified with its position vector in $\mathbf{R}^{n+q}$.

We denote the covariant differentiation of $\mathbf{R}^{n+q}$ with D and write

$$(f_t^a)_* v = v + t \sum_\alpha a_\alpha D_v \xi_\alpha,$$

for every vector v tangent to S.

Since $D_v \xi_\alpha$ is still tangent to S, we have that the vector $(f_t^a)_* v$ is parallel to the tangent space $T_x S$ and thus the lemma is proved in the Euclidean case.

In the remaining cases, without loss of generality we can restrict to the cases $c = 1$ and $c = -1$.

In the elliptic case we have that the mapping f given by

$$f(t, a, x) = (\cos t) v + (\sin t) \sum_\alpha a_\alpha \xi_\alpha$$

has values on the unit sphere S^{n+q} of $\mathbf{R}^{n+q+1}$. For every vector v tangent to S we have

$$(f_t^a)_* v = (\cos t)v + (\sin t)\sum_\alpha a_\alpha D_v \xi_\alpha,$$

where D denotes the covariant derivative in $\mathbf{R}^{n+q+1}$. Since the vector fields are parallel in the normal connection of the leaf S, we get that $D_v\xi_\alpha$ is the sum of a vector tangent to S and the vector $\langle D_v\xi_\alpha, x\rangle x$. On the other hand, since $\langle D_v\xi_\alpha, x\rangle = 0$, we see that $(f_t^a)_* v$ is parallel to a vector in T_xS.

In the case $c = -1$ we can consider that the ambient space is $H^{n+q} \subset \mathbf{R}^{n+q+1}$ defined by

$$H^{n+q} = \{x \in \mathbf{R}^{n+q+1}, (x,x) = -1\},$$

where $(,)$ denotes the Lorentz metric in $\mathbf{R}^{n+q+1}$.

In this case we have the expression:

$$f(t,a,x) = (\cosh t)x + (\sinh t)\sum_\alpha a_\alpha \xi_\alpha(x),$$

and for every tangent vector $v \in T_xS$ holds that

$$(f_t^a)_* v = (\cosh t)x + (\sinh t)\sum_\alpha a_\alpha D_v\xi_\alpha(x).$$

Therefore we easily conclude that $(f_t^a)_* v$ is parallel (in the Euclidean sense) to a tangent vector in T_xS. With that we conclude the proof of Lemma 3.3. $\square$

In the following we state the following fact of a general character:

Lemma 3.4 *If N is a totally geodesic submanifold of a Riemannian manifold M^{n+q} and $\gamma = \gamma(t)$ is a curve in N, then the subspace field $t \mapsto T_{\gamma(t)}N$ is parallel in M^{n+q}.*

We are now in the position to prove the following:

Proposition 3.5 *Let f be a normal trivialization of a normal isoparametric family of submanifolds in $M^{n+q}(c)$. Then each of the submanifolds $f_t^a(U)$ is contained in some leaf of the given isoparametric family.*

Proof. Let N be a maximal integral manifold of the orthogonal distribution of the isoparametric family. Propostion 2.3 implies that N is totally geodesic. Because of Lemmas 3.3 and 3.4, we have also that the families of subspaces $T_{\gamma(t)}N$ and $(f_t^a)_*T_xS$ are parallel along the geodesic $t \mapsto f_t^a(x)$. Therefore those are orthogonal along that geodesic. By the uniqueness of the orthogonal complement we see that the subspaces $(f_t^a)_*T_xS$ are tangent to the leaves of the isoparametric family. $\square$

4 Submanifolds with constant principal curvatures

To begin with we state two results of technical flavor about tensor fields:

Lemma 4.1 *Let A be a field of symmetric tensors of type $(1,1)$ on a Riemannian manifold M^n. Then there exist n continuous functions $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n$ such that for each x in M^n the set $\{\lambda_i(x) : i = 1, 2, \ldots, n\}$ is the set of the eigenvalues of A_x.*

The proof of this lemma can be found in [4].
This result allows us to state the following:

Definition 4.2 *We say that a n-dimensional submanifold S of a Riemannian manifold M^{n+q} has constant principal curvatures if:*

a. The normal bundle of S is flat.

b. For each normal vector field ξ defined on a connected open submanifold of S which is parallel on the normal connection, the functions λ_i relative to the tensor field A^ξ which are given by Lemma 4.1 are constant, where A^ξ denotes the second fundamental form defined by ξ.

The main goal of this section is to prove that the leaves of a normal isoparametric family of submanifolds of $M^{n+q}(c)$ have constant principal curvatures.

Definition 4.3 *Let S be a submanifold of codimension $q \geq 1$ in a Riemannian manifold M^{n+q} whose normal bundle is flat. A point x in S is called a general point if there exists an orthonormal frame $\{e_A\}$ of M adapted to S such that the first n vector fields diagonalize the second fundamental forms $A^{e_{n+1}}, \ldots, A^{e_{n+q}}$.*

We will use the following result whose proof we omit.

Proposition 4.4 *The set of general points of a submanifold is open and dense.*

Definition 4.5 *Given a normal trivialization of a normal isoparametric family $f : J \times S^{q-1} \times U \to M^{n+q}(c)$, we call normal frame adapted to this trivialization the set of vector fields $e_{n+1}, \ldots, e_{n+q}$ defined along f as follows: each e_α is the vector field which coincides at the point $f_t^a(x)$, $x \in f_0^a(x)$, with the parallel translate of $\xi_\alpha(x)$ along the geodesic $t \mapsto f_t^a(x)$.*

Lemma 4.6 *The restriction of each vector field e_α to the submanifold $f_t^a(U)$ is a normal vector field parallel in the normal connection of that submanifold.*

Proof. The fact that the vector fields e_α are orthogonal to the submanifold $f_t^a(U)$ follows from Lemma 3.3. In order to show that the e_α are parallel in the normal connection of $f_t^a(U)$, we take an arbitrary curve in U, say $\mu = \mu(s)$, and consider the functions:

$$\mu(s,t) = f_t^a(\mu(s)), \quad \varphi_{\alpha\beta}(s,t) = \langle \nabla_{\frac{d}{ds}} e_\alpha, e_\beta \rangle.$$

¿From the definition of the e_α we have that $\nabla_{\frac{d}{dt}} e_\alpha$ is zero and therefore the derivatives $\frac{\partial \varphi_{\alpha\beta}}{\partial t}$ are also zero[2]. Therefore, the functions $\varphi_{\alpha\beta}$ do not depend on t and as these are zero for $t = 0$, they are always zero. Thus the lemma is proved. □

[2] N. T.: Use that the normal bundle is flat (Proposition 2.3).

Proposition 4.7 *Consider a normal isoparametric family of n-dimensional submanifolds of $M^{n+q}(c)$. Then every leaf has constant principal curvatures.*

Proof. Let S be a leaf and f a normal trivialization defined in a neighbourhood of a general point of S. From Lemma 3.3, if an orthonormal frame $e_1, \ldots, e_n$ of the leaf S diagonalizes the second fundamental forms then the frame $(f_t^a)_* e_i$, $i = 1, \ldots, n$, diagonalizes the second fundamental forms of $f_t^a(U)$.

Now we have n unit vector fields $\mathbf{e}$, pairwise orthogonal, tangent to the leaves of the isoparametric family. We will still denote them with $e_1, \ldots, e_n$. Finally, we complete these fields to an orthonormal frame by making use of the vector fields defined in 4.5. We denote with a_i the differentiable functions defined on the image of f by

$$a_i^\alpha = -\langle A^{e_\alpha} e_i, e_i \rangle, \quad i = 1, \ldots, n; \quad \alpha = n+1, \ldots, n+q.$$

Then we have the equations $\omega_{\alpha i} = a_i^\alpha \omega_i$, and by exterior differentiation, $d\omega_{\alpha i} = da_i^\alpha \wedge \omega_i + a_i^\alpha d\omega_i$.

We use the structure equations and the fact that the curvature forms satisfy $\Omega_{AB} = c\omega_A \wedge \omega_B$ in order to write:

$$c\omega_\alpha \wedge \omega_i - \sum_A \omega_{\alpha A} \wedge \omega_{Ai} = da_i^\alpha \wedge \omega_i - a_i^\alpha \sum_B \omega_{iB} \wedge \omega_B,$$

or,

$$c\omega_\alpha \wedge \omega_i - \sum_j \omega_{\alpha j} \wedge \omega_{ji} - \sum_\gamma \omega_{\alpha\gamma} \wedge \omega_{\gamma i} = da_i^\alpha \wedge \omega_i - a_i^\alpha \sum_j \omega_{ij} \wedge \omega_j - a_i^\alpha \sum_\gamma \omega_{i\gamma} \wedge \omega_\gamma.$$

If we evaluate this equation on the pair e_μ, e_i, we get:

$$da_i^\alpha(e_\mu) - a_i^\alpha a_i^\mu = c\delta_\mu^\alpha - \sum_\gamma a_i^\gamma \omega_{\alpha\gamma}(e_\mu).$$

In particular, for $\mu = \alpha$ we get:

$$da_i^\alpha(e_\alpha) - (a_i^\alpha)^2 = c - \sum_\gamma a_i^\gamma \omega_{\alpha\gamma}(e_\alpha).$$

We remark that the integral curves of $\mathbf{e}$ that start at the submanifold U are geodesics in $M^{n+q}(c)$, so that the restriction of the functions a_i to these geodesics satisfy the following differential equations:

$$\frac{da_i^\alpha}{dt} = c + (a_i^\alpha)^2, \quad a_i^\alpha = a_i^\alpha(t, x), \quad x \in U.$$

Lemma 1.3 gives the initial condition $a_i^\alpha(0, x) = \text{constant}$ and, therefore, from a result in [3] we conclude that the functions a_i are locally constant. Thus we have proved that the leaves of the isoparametric family have constant principal curvatures. $\square$

5 The fundamental formula

In this section we obtain a relation among the principal curvatures a_i of the leaves of an isoparametric family of submanifolds of $M^{n+q}(c)$. The arguments that allow us to arrive at that relation are strongly based on the ideas of É. Cartan ([1]).

With respect to a frame $e_1, \dots, e_n, e_{n+1}, \dots, e_{n+q}$, adapted to the given isoparametric family, and where the vector fields $e_1, \dots, e_n$ diagonalize the second fundamental form of each leaf, we have:

$$\omega_{\alpha i} = a_i^\alpha \omega_i, \quad i = 1, \dots, n; \qquad \alpha = n+1, \dots, n+q, \tag{1}$$

where the functions a_i^α are constant. We introduce in the following the functions:

$$\lambda_{ijk}^\alpha = (a_i^\alpha - a_j^\alpha)\omega_{ij}(e_k). \tag{2}$$

We exterior differentiate (1) and evaluate the result on the pair of vector fields e_j, e_k to obtain[3]

$$(a_i^\alpha - a_j^\alpha)\omega_{ij}(e_k) = (a_i^\alpha - a_k^\alpha)\omega_{ik}(e_j), \tag{3}$$

which shows the symmetry of the λ_{ijk}^α with respect to the lower indices. We get also, for distinct indices i, j, the relation:

$$\begin{aligned} c + \sum_\gamma a_i^\gamma a_j^\gamma = & \sum_k \omega_{ij}(e_k)(\omega_{ki}(e_j) - \omega_{kj}(e_i)) \\ & + \omega_{ik}(e_i)\omega_{kj}(e_j) - \omega_{ik}(e_j)\omega_{kj}(e_i) + (d(\omega_{ij}(e_j))(e_i) - d(\omega_{ij}(e_i))(e_j)). \end{aligned} \tag{4}$$

On the other hand, we see that for any indices i, j, k holds

$$(\lambda_{ijk}^\alpha)^2 = (a_k^\alpha - a_i^\alpha)(a_k^\alpha - a_j^\alpha)(\omega_{ij}(e_k)(\omega_{ki}(e_j) - \omega_{kj}(e_i))) \tag{5}$$

and

$$-(\lambda_{ijk}^\alpha)^2 = (a_k^\alpha - a_i^\alpha)(a_k^\alpha - a_j^\alpha)\omega_{ik}(e_j)\omega_{kj}(e_i). \tag{6}$$

In order to simplify the notation, we give the following

Definition 5.1 *Let $\alpha \in \{n+1, \dots, n+q\}$. We say that two indices i, $j \in \{1, \dots, n\}$ are α-essentially distinct if $a_i^\alpha \neq a_j^\alpha$. We denote with J_i^α the set of indices that are α-essentially distinct from i. Two indices are called essentially distinct if they are β-essentially distinct for some β.*

In that terminology, if i, j are essentially distinct then[4]

$$\omega_{ij}(e_i) = 0, \quad \omega_{ij}(e_j) = 0. \tag{7}$$

In the following we subdivide our exposition into a series of small lemmas:

[3]N. T.: Use also the structure equations.

[4]N. T.: This follows from (2) and the symmetry of λ_{ijk}^α.

Lemma 1 *If i, j are two essentially distinct indices, then for every k holds:*

$$\omega_{ik}(e_i)\omega_{kj}(e_j) = 0. \tag{8}$$

Proof. If[5] $a_k^\alpha = a_i^\alpha$, then $a_k^\alpha \neq a_j^\alpha$, and then $\omega_{kj}(e_j) = 0$. If $a_k^\alpha = a_j^\alpha$, then $a_k^\alpha \neq a_i^\alpha$, and then $\omega_{ki}(e_i) = 0$. If $a_k^\alpha \neq a_i^\alpha$, $a_k^\alpha \neq a_j^\alpha$, then $\omega_{ki}(e_i) = 0$, $\omega_{kj}(e_j) = 0$. □

Lemma 2 *If i, j are two α-essentially distinct indices and if k is an index such that $a_k^\alpha = a_i^\alpha$ or $a_k^\alpha = a_j^\alpha$, then hold:*

$$\omega_{ij}(e_k) = 0, \quad \omega_{ik}(e_j)\omega_{kj}(e_i) = 0. \tag{9}$$

Proof. We start with the relation (3) in the form:

$$(a_k^\alpha - a_i^\alpha)\omega_{ki}(e_j) = (a_k^\alpha - a_j^\alpha)\omega_{kj}(e_i). \tag{10}$$

If $a_k^\alpha = a_i^\alpha$, then $a_k^\alpha \neq a_j^\alpha$ and then, from the above relation, $\omega_{kj}(e_i) = 0$. Analogously, if $a_k^\alpha = a_j^\alpha$ we get $\omega_{ki}(e_j) = 0$, which proves the second relation (9).

In order to prove the first of the relations (9), we start with

$$(a_j^\alpha - a_i^\alpha)\omega_{ji}(e_k) = (a_i^\alpha - a_k^\alpha)\omega_{ik}(e_j), \tag{11}$$

and we observe that, if $a_k^\alpha = a_i^\alpha$, then $\omega_{ij}(e_k) = 0$. Finally[6], if $a_k^\alpha = a_j^\alpha$, then $\omega_{ji}(e_k) = 0$. □

Lemma 3 *If i, j are two essentially distinct indices, then holds:*

$$c + \sum_\gamma a_i^\gamma a_j^\gamma = \sum_{k \in J_i^\alpha \cap J_j^\alpha} \omega_{ij}(e_k)(\omega_{ki}(e_j) - \omega_{kj}(e_i)) - \omega_{ik}(e_j)\omega_{kj}(e_i). \tag{12}$$

Proof. If[7] $k \notin J_i^\alpha$ then $a_k^\alpha = a_i^\alpha$. Then, by Lemma 2, we have that the term corresponding to the index k in the sum above is zero. The same happens if $k \notin J_j^\alpha$. □

¿From the above follows immediately the following:

Corollary 5.4 *If i, j are two essentially distinct indices such that for some β we have $J_i^\beta \cap J_j^\beta = \emptyset$, then*

$$c + \sum_\gamma a_i^\gamma a_j^\gamma = 0. \tag{13}$$

Lemma 5.5 *If i, j are α-essentially distinct with $J_i^\alpha \cap J_j^\alpha \neq \emptyset$, then:*

$$c + \sum_\gamma a_i^\gamma a_j^\gamma = 2 \sum_{k \in J_i^\alpha \cap J_j^\alpha} \left(\frac{(\lambda_{ijk}^\alpha)^2}{(a_k^\alpha - a_i^\alpha)(a_k^\alpha - a_j^\alpha)} \right). \tag{14}$$

[5]N. T.: Suppose i, j are α-essentially distinct.

[6]N. T.: Note that (11) equals $(a_k^\alpha - a_j^\alpha)\omega_{kj}(e_i)$.

[7]N. T.: Start with (4) and (7).

This lemma follows immediately from (12) and from the definition[8] of the λ^{α}_{ijk}.

Proposition 5.6 (Cartan's generalized formula) *For each index $i = 1, \ldots, n$ and each index $\alpha = n+1, \ldots, n+q$ holds:*

$$\sum_{j \in J_i^\alpha; J_i^\alpha \cap J_j^\alpha \neq \emptyset} \left(\frac{c + \sum_\gamma a_i^\gamma a_j^\gamma}{a_i^\alpha - a_j^\alpha} \right) = 0. \tag{15}$$

Proof. It is enough to show that

$$\sum_{j \in J_i^\alpha; J_i^\alpha \cap J_j^\alpha \neq \emptyset} \sum_{k \in J_i^\alpha \cap J_j^\alpha} \left(\frac{(\lambda^\alpha_{ijk})^2}{(a_i^\alpha - a_j^\alpha)(a_k^\alpha - a_i^\alpha)(a_k^\alpha - a_j^\alpha)} \right) = 0. \tag{16}$$

In order to do that, we note that the term corresponding to the indices ijk contributes to the sum if and only if $J_i^\alpha \cap J_j^\alpha \neq \emptyset$ and $k \in J_i^\alpha \cap J_j^\alpha$. Also, we check that the term ikj also appears in the sum and with opposite sign, since $J_i^\alpha \cap J_k^\alpha \neq \emptyset$ and $j \in J_i^\alpha \cap J_k^\alpha$. □

We expect the formula (15) to play an important role in the theory of the isoparametric families of submanifolds.

We give now an application of this formula, namely:

Proposition 5.7 *Let M^n be a submanifold of $\mathbf{R}^{n+q}$ with constant principal curvatures and satisfying:*

a. $q \geq n$.

b. The dimension of the first normal space at each point is n.

Then M^n is flat.

Proof. Let $B_p : T_pM \times T_pM \to (T_pM)^\perp$ the second fundamental form of M^n at the point p. Suppose that $e_1, \ldots, e_n$ is a basis of T_pM that diagonalizes[9] B_p. From hypothesis b. we have that the vectors $B(e_1, e_1), \ldots, B(e_n, e_n)$ are linearly independent. Applying the customary orthonormalization process to the above basis, we obtain a basis of $(T_pM)^\perp$ such that the eigenvalues a_i^α, $i = 1, \ldots, n$, $\alpha = n+1, \ldots, n+q$, form a triangular matrix[10] of rank n:

$$\begin{pmatrix} a_1^{n+1} & a_2^{n+1} & \cdots & a_n^{n+1} \\ 0 & a_2^{n+2} & \cdots & a_n^{n+2} \\ 0 & 0 & & \vdots \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & a_n^{2n} \end{pmatrix}.$$

[8] N. T.: Namely, (5) and (6).

[9] N. T.: Such a basis exists because the normal bundle of M is flat, so Ricci's equation implies that the Weingarten operators at p pairwise commute.

[10] N. T.: Namely, the matrix of change of coordinates.

Here we should remark that all the diagonal elements a_1^{n+1}, a_2^{n+2},..., a_n^{2n} are different from zero.

We now apply the formula (13) for the pairs of indices $(1,n),(2,n),\dots,(n-1,n)$ and for the index $\beta = 2n$ in order to deduce that the elements $a_n^{n+1}, a_n^{n+2}, \dots, a_n^{2n-1}$ are all zero. Then an induction argument combined in each step with formula (13) shows that the matrix above is diagonal. From that we easily deduce that M^n has zero curvature[11].

References

[1] É. Cartan, *Familles de surfaces isoparamétriques dans les espaces à courbure constante*, Ann. di Mat. **17** (1938), 177–191.

[2] D. Ferus, *Notes on isoparametric hypersurfacers*, Esc. de Geom. Diferencial, Unicamp, 1980.

[3] K. Nomizu, *Élie Cartan's work on isoparametric families of hypersurfaces*, Proc. Sympos. in Pure Math., vol. 27, Amer. Math. Soc., 1975, pp. 191–200.

[4] P. J. Ryan, *Homogeneity and some curvature conditions for hypersurfaces*, Ph.D. thesis, Brown University, 1968.

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[11] N. T.: This can be seen from Gauss's equation.