Koszul's bracket and Jacobi structures

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2 Jacobi structures: definition and examples

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Motivations

- In his seminal paper entitled "crochet de Schouten-Nijenhuis et cohomologie" (Astérisque,1985), Koszul considered a triple $(\mathcal{A}, \wedge, d)$ consisting of a graded commutative algebra $(\mathcal{A} = \oplus_{n\in\mathbb{Z}}\mathcal{A}^n, \wedge)$ with unit element $\mathbb{1}$ together with an odd degree differential operator $\partial : \mathcal{A} \to \mathcal{A}$ satisfying $d(\mathbb{1}) = 0$ and $\partial \circ \partial = 0$.

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- He then constructed the bracket on $\mathcal{A}$ as follows:

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for all $a, b \in \mathcal{A}$. This bracket is nowadays called a Gerstenhaber bracket and $(\mathcal{A}, [\ ,\ ]_\partial)$ is a Batalin-Vilkovisky algebra (BV-algebra for short).

Motivations

- In the early nineties, there were a lot of interests in the study of BV-algebras due to their appearance in string theory.
- In 1995, Y. Kosmann-Schwarzbach, observed that BV-algebras already appeared in Koszul's work and she noted some connections with Lie algebroids and Loday algebras.
- In brief, Koszul's paper on the Schouten-Nijenhuis bracket has been the foundation of several important works on various algebraic and geometric structures.

Motivations

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- On the other hand, **Jacobi manifolds are natural generalizations of Poisson manifolds which can be viewed as a bridge between Poisson geometry and contact geometry.**

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Goals

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  1. Firstly, we will explain an extension Koszul's construction from Poisson structures to Jacobi structures.
  2. Secondly, we will discuss the modular class of a Jacobi structure.

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- A **contact structure** on a smooth manifold M is defined by a co-dimension 1 maximally non-integrable distribution $\xi \subseteq TM$. Let $L := TM/\xi$ be its associated line bundle. The distribution ξ can be equivalently defined by a line bundle-valued 1-form $\Theta : TM \to L$ which is viewed as the canonical projection.

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- More generally, a **Jacobi structure** on M is given by a line bundle $L \to M$ and a Lie bracket $\{\cdot,\cdot\} : \Gamma(L) \times \Gamma(L) \to \Gamma(L)$ which is a bi-derivation, that is a derivation with respect to each entry.

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Definition: By a derivation of L, we mean an $\mathbb{R}$-linear operation $\Delta : \Gamma(L) \to \Gamma(L)$ satisfying:

$$\Delta(fe) = f\Delta(e) + (\sigma(\Delta) \cdot f)e,$$

where $\sigma(\Delta)$ is the symbol of Δ. Derivations of L can be identified with infinitesimal isomorphisms of L.

Example 1

- **Trivial line bundle**
  Let M be a smooth manifold. Consider $L : M \times \mathbb{R} \to M$. The first jet bundle of L is $\mathfrak{J}^1 L = T^*M \oplus \mathbb{R}$. A Jacobi structure for the trivial line bundle is given by a pair (Π, E) formed by a bivector field Π and a vector field E satisfying the relations:

$$[\Pi, \Pi] = 2E \wedge \Pi \qquad \text{and} \quad [E, \Pi] = 0$$

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$$\{(df, f),\ (dg, g)\} = (d\{f, g\},\ \{f, g\}).$$

Example 2

Projective spaces.

- Let $\mathfrak{g}$ be a real Lie algebra. Consider $M = \mathbb{RP}(\mathfrak{g}^*)$ the projective space and let $L \to M$ be the tautological space.There is an inclusion $\iota : \mathfrak{g} \hookrightarrow \Gamma(L)$ such that:

$$\iota(v) = \lambda_v, \quad \text{with} \quad \lambda_v(r) = \ell_v|_r, \quad \forall r \subseteq \mathfrak{g}^*$$

  where $\ell_v : \mathfrak{g} \to \mathbb{R}$ is the linear function corresponding to v.
- In this case, the Jacobi bracket on $\Gamma(L)$ is given by

$$J(\lambda_v, \lambda_w) = \lambda_{[v,w]}.$$

Atiyah Lie algebroid and Jacobi brackets

In other words, derivations of L are sections of the Lie algebroid $DL \to M$ called the gauge (or Atiyah) Lie algebroid of L. Its anchor map is the symbol and its Lie bracket is the commutator of derivations and let $\mathfrak{J}^1 L$ be the first jet bundle of L. We have the vector bundle isomorphisms:

$$DL \simeq \mathrm{Hom}(\mathfrak{J}^1 L, L) \quad \text{and} \quad \mathfrak{J}^1 L \simeq \mathrm{Hom}(DL, L).$$

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Remark: A Jacobi manifold $(M, L, \{,\cdot,\cdot\})$ is completely defined by its associated 2-form $J : \Gamma(\Lambda^2(\mathfrak{J}^1 L)) \to \Gamma(L)$ given by:

$$\{\lambda, \mu\} = J(j^1\lambda, j^1\mu),$$

for all $\lambda, \mu \in \Gamma(L)$.

Koszul bracket for Jacobi manifolds

- For a line bundle $L \to M$, we know that DL is a transitive Lie algebroid. Let $(\Omega_L^\bullet(M) := (\Gamma(\wedge^\bullet(DL)^* \otimes L), d_{DL})$ be the de Rham complex (with coefficients in L) associated with DL.

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- For every $\eta \in \Omega_L^k(M)$, we denote the interior product by η as follows:

$$\iota_\eta \colon \Omega_L^n(M) \to \Omega_L^{n-k}(M), \qquad \iota_\eta(\alpha) = \eta \rfloor \alpha,$$

Using the Lie derivative $\partial_\eta = [d_{DL},\ \iota_\eta]$, one gets the Cartan formulas

$$[\partial_\eta,\ d_{DL}] = 0, \qquad [\iota_\alpha,\ \iota_\beta] = 0, \qquad [\partial_\alpha,\ \partial_\beta] = \partial_{[\alpha,\beta]}$$

Koszul bracket for Jacobi manifolds

Definition: The Koszul bracket asociated with a tensor $J \in \Gamma((\Lambda^2(\mathfrak{J}^1 L))^* \otimes L)$ is the bilinear operation: $[\cdot,\cdot]_J : \bigwedge^2 \Omega_L[1] \to \Omega_L[1]$ defined by: $\forall\ \alpha \in \Omega_L^p(M), \forall\ \beta \in \Omega_L^q(M)$

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$$\begin{aligned}[\alpha,\beta]_J = &-(-1)^p\Big(\iota_J d_{DL}(\alpha\wedge\beta) - d_{DL}\iota_J(\alpha\wedge\beta)\Big)\\ &+(-1)^p\Big(\iota_J d_{DL}(\iota_J\alpha)\wedge\beta - \iota_J(d\alpha)\wedge\beta\Big)\\ &-\alpha\wedge\iota_J(d\beta) + \alpha\wedge d_{DL}(\iota_J(\beta)).\end{aligned}$$

Koszul bracket for Jacobi manifolds

Lemma: Let (M, L, J) be a Jacobi manifold with its associated bundle map $J^\sharp : \mathfrak{J}^1 L \to DL$ defined by $\iota_{J^\sharp \alpha}(\beta) = J(\alpha, \beta)$, for all $\alpha, \beta \in \Gamma(\mathfrak{J}^1 L)$. Then, one has:

$$\iota_J(\alpha \wedge d_{DL}\beta) = \iota_{J^\sharp(\alpha)}(d_{DL}\beta) + \alpha \wedge \iota_J(d_{DL}\beta)$$

and the Koszul bracket in $\Omega^1_L(M)$ can be express as follows:

$$[\alpha, \beta]_J = \partial_{J^\sharp(\alpha)}\beta - \partial_{J^\sharp(\beta)}\alpha - d_{DL}\iota_J(\alpha \wedge \beta), \quad \forall\ \alpha, \beta \in \Omega^1_L(M)$$

Koszul bracket for Jacobi manifolds

Theorem: **Let (M, L, J) be a Jacobi manifold. Its algebra of Atiyah forms $(\Omega_L^\bullet(M), \wedge)$ equipped with the Koszul bracket is a Batalin-Vilkovisky (BV for short) algebra with generator $\partial_J = [d_{DL}, \ \iota_J]$.**

Modular vector field of a Jacobi manifold

- *If* (M, L, J) is a Jacobi manifold, there is an associated Lie algebroid $(\mathfrak{J}^1 L, [\cdot,\cdot]_J, \varrho)$, where the anchor map is $\varrho = \sigma \circ J^\sharp$. Moreover, we have the complex :

$$0 \xrightarrow{d_J} \Gamma(L) \xrightarrow{d_J} \Gamma(DL) \xrightarrow{d_J} \Gamma(\wedge^2 (\mathfrak{J}^1 L)^* \otimes L) \to \cdots$$

where

$$d_J(A) = [J, A]_{SN},$$

for any $A \in \Gamma(\wedge^k (\mathfrak{J}^1 L)^* \otimes L)$.

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- Here $[\cdot,\cdot]_{SN}$ is the Schouten Nijenhuis bracket on the graded commutative algebra $\oplus_{k=0}^{\infty}\Gamma(\bigwedge^k(\mathfrak{J}^1 L)^* \otimes L)$.
- **We have a cohomology operator since**

$$d_J \circ d_J(A) = [J, [J, A]_{SN}]_{SN} = 0$$

  for all $A \in \Gamma(\wedge^k(\mathfrak{J}^1 L)^* \otimes L)$.

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- Observe that the map $\Omega^\sharp : \Gamma(\wedge^k(\mathfrak{J}^1 L)^* \otimes L) \to \Omega_L^{n-k+1}(M)$ is an isomorphism and we denote its inverse by $\Omega^\flat : \Omega_L^{n-k+1}(M) \to \Gamma(\wedge^k(\mathfrak{J}^1 L)^* \otimes L)$.

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Theorem: The element $D_\Omega(J) \in \Gamma(L)$ is a 1-cocycle in the above cohomology complex, that is, $[J, D_\Omega(J)]_{SN} = 0$.

Definition: The modular vector field is the vector field is the symbol of the derivation $D_\Omega(J)$, that is, $\sigma(D_\Omega(J))$.

Future plans

We plan to use the Koszul bracket for Jacobi manifolds to study

1. Deformations of Jacobi structures
2. Formality of Koszul bracket for Jacobi manifolds (following Fiorenza and Manetti)

The modular class could be useful in the study of normal forms of Jacobi structures in low dimensions.

THANK YOU !