The extension problem for Lie algebroids on schemes

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Lie algebroids

X: a differentiable manifold, or complex manifold, or a noetherian separated scheme over an algebraically closed field $\Bbbk$ of characteristic zero.

Lie algebroid: a vector bundle/coherent sheaf $\mathscr{C}$ with a morphism of $\mathscr{O}_X$-modules $a\colon \mathscr{C} \to \Theta_X$ and a $\Bbbk$-linear Lie bracket on the sections of $\mathscr{C}$ satisfying

$$[s, ft] = f[s, t] + a(s)(f)\, t$$

for all sections s, t of $\mathscr{C}$ and f of $\mathscr{O}_X$.

- a is a morphism of sheaves of Lie $\Bbbk$-algebras
- $\ker a$ is a bundle of Lie $\mathscr{O}_X$-algebras

Examples

- A sheaf of Lie algebras, with $a = 0$
- Θ_X, with $a = \mathrm{id}$
- More generally, foliations, i.e., a is injective
- Poisson structures $\Omega^1_X \xrightarrow{\pi} \Theta_X$,

  Poisson-Nijenhuis bracket

$$\{\omega, \tau\} = \mathrm{Lie}_{\pi(\omega)}\tau - \mathrm{Lie}_{\pi(\tau)}\omega - d\pi(\omega, \tau)$$

  Jacobi identity $\Leftrightarrow [\![\pi, \pi]\!] = 0$

Lie algebroid morphisms

$f\colon \mathscr{C} \to \mathscr{C}'$ a morphism of $\mathscr{O}_X$-modules & sheaves of Lie k-algebras

$$\begin{array}{ccc} \mathscr{C} & \xrightarrow{\ f\ } & \mathscr{C}' \\ & {\scriptstyle a}\searrow & \downarrow{\scriptstyle a'} \\ & & \Theta_X \end{array}$$

$\Rightarrow$ $\ker f$ is a bundle of Lie algebras

Derived functors

$\mathfrak{A}$ an abelian category, $A \in \mathrm{Ob}(\mathfrak{A})$

$$\mathrm{Hom}(-, A)\colon \to \mathfrak{Ab}$$

is a (contravariant) left exact functor, i.e., if

$$0 \to B' \to B \to B'' \to 0 \qquad (*)$$

is exact, then

$$0 \to \mathrm{Hom}(B'', A) \to \mathrm{Hom}(B, A) \to \mathrm{Hom}(B', A)$$

is exact

Definition

$I \in \mathrm{Ob}(\mathfrak{A})$ is injective if $\mathrm{Hom}(-, I)$ is exact, i.e., for every exact sequence (*),

$$0 \to \mathrm{Hom}(B'', I) \to \mathrm{Hom}(B, I) \to \mathrm{Hom}(B', I) \to 0$$

is exact

Definition

The category $\mathfrak{A}$ has enough injectives if every object in $\mathfrak{A}$ has an injective resolution

$$0 \to A \to I^0 \to I^1 \to I^2 \to \dots$$

$\mathfrak{A}$ abelian category with enough injectives

$$F\colon \mathfrak{A} \to \mathfrak{B} \quad \text{left exact functor}$$

Derived functors $R^iF\colon \mathfrak{A} \to \mathfrak{B}$

$$R^iF(A) = H^i(F(I^\bullet))$$

Example: Sheaf cohomology. X topological space, $\mathfrak{A} = \mathfrak{Sh}_X$, $\mathfrak{B} = \mathfrak{Ab}$, $F = \Gamma$ (global sections functor)

$$R^i\Gamma(\mathscr{F}) = H^i(X, \mathscr{F})$$

From now on, X will be a scheme (with the previous hypotheses)

Given a Lie algebroid $\mathscr{C}$ there is a notion of enveloping algebra $\mathfrak{U}(\mathscr{C})$

It is a sheaf of associative $\mathscr{O}_X$-algebras with a $\Bbbk$-linear augmentation $\mathfrak{U}(\mathscr{C}) \to \mathscr{O}_X$

$$\boxed{\mathrm{Rep}(\mathscr{C}) \simeq \mathfrak{U}(\mathscr{C})\textbf{-mod}}$$

$\Rightarrow$ Rep($\mathscr{C}$) has enough injectives

A $\Bbbk$-algebra $\mathfrak{U}(L)$ with an algebra monomorphism $\imath \colon A \to \mathfrak{U}(L)$ and a $\Bbbk$-module morphism $\jmath \colon L \to \mathfrak{U}(L)$, such that

$$[\jmath(s), \jmath(t)] - \jmath([s,t]) = 0\,, \quad s,t \in L\,,$$
$$[\jmath(s), \imath(f)] - \imath(a(s)(f)) = 0\,, \quad s \in L, f \in A \qquad (*)$$

Construction: standard enveloping algebra $U(A \rtimes L)$ of the semi-direct product $\Bbbk$-Lie algebra $A \rtimes L$

$$\mathfrak{U}(L) = U(A \rtimes L)/V, \qquad V = \langle f(g,s) - (fg, fs)\rangle$$

- $\mathfrak{U}(L)$ is an A-module via the morphism $\imath$
- due to (*) the left and right A-module structures are different
- morphism $\varepsilon \colon \mathfrak{U}(L) \to \mathfrak{U}(L)/I = A$ (the augmentation morphism) where I is the ideal generated by $\jmath(L)$. Note that ε is a morphism of $\mathfrak{U}(L)$-modules but not of A-modules, as $\varepsilon(fs) = a(s)(f)$ when $f \in A$, $s \in L$.

Lie algebroid cohomology

Given a representation $(\rho, \mathscr{M})$ of $\mathscr{M}$ define

$$\mathscr{M}^{\mathscr{C}}(U) = \{m \in \mathscr{M}(U) \,|\, \rho(\mathscr{C})(m) = 0\}$$

and a left exact functor

$$\begin{aligned} I^{\mathscr{C}}\colon \ \mathrm{Rep}(\mathscr{C}) &\to \mathbb{k}\textbf{-mod} \\ \mathscr{M} &\mapsto \Gamma(X, \mathscr{M}^{\mathscr{C}}) \end{aligned}$$

Definition (B 2016[1])

$$\mathbb{H}^\bullet(\mathscr{C}; \mathscr{M}) \simeq R^\bullet I^{\mathscr{C}}(\mathscr{M})$$

([1]) J. of Algebra **483** (2017) 245–261

Grothendieck's thm about composition of derived functors

$$\mathfrak{A} \xrightarrow{F} \mathfrak{B} \xrightarrow{G} \mathfrak{C}$$

$\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$, abelian categories

$\mathfrak{A}$, $\mathfrak{B}$ with enough injectives

F and G left exact, F sends injectives to G-acyclics (i.e., $R^iG(F(I)) = 0$ for $i > 0$ when I is injective)

Theorem

For every object A in $\mathfrak{A}$ there is a spectral sequence abutting to $R^\bullet(G \circ F)(A)$ whose second page is

$$E_2^{pq} = R^pF(R^qG(A))$$

Local to global

$$\begin{array}{ccc} \mathrm{Rep}(\mathscr{C}) & \xrightarrow{(-)^{\mathscr{C}}} & \Bbbk_X\text{-}\mathbf{mod} \\ & \searrow^{I^{\mathscr{C}}} & \downarrow \Gamma \\ & & \Bbbk\text{-}\mathbf{mod} \end{array}$$

Grothendieck's theorem on the derived functors of a composition of functors implies:

Theorem (Local to global spectral sequence)

There is a spectral sequence, converging to $\mathbb{H}^\bullet(\mathscr{C};\mathscr{M})$, whose second term is

$$E_2^{pq} = H^p(X, \mathscr{H}^q(\mathscr{C};\mathscr{M}))$$

Hochschild-Serre

Extension of Lie algebroids

$$0 \to \mathscr{K} \to \mathscr{E} \to \mathscr{Q} \to 0$$

$\mathscr{K}$ is a sheaf of Lie $\mathscr{O}_X$-algebras

$$\begin{array}{ccc} \mathrm{Rep}(\mathscr{E}) & \xrightarrow{(-)^{\mathscr{K}}} & \mathrm{Rep}(\mathscr{Q}) \\ & {\scriptstyle I^{\mathscr{E}}} \searrow & \downarrow {\scriptstyle I^{\mathscr{Q}}} \\ & & \mathbb{k}\text{-}\mathbf{mod} \end{array}$$

Moreover, the sheaves $\mathscr{H}^q(\mathscr{K};\mathscr{M})$ are representations of $\mathscr{Q}$

Theorem (Hochschild-Serre type spectral sequence)

For every representation $\mathscr{M}$ of $\mathscr{E}$ there is a spectral sequence E converging to $\mathbb{H}^\bullet(\mathscr{E};\mathscr{M})$, whose second page is

$$E_2^{pq} = \mathbb{H}^p(\mathscr{Q};\mathscr{H}^q(\mathscr{K};\mathscr{M})).$$

The extension problem

An extension

$$0 \to \mathscr{K} \to \mathscr{E} \xrightarrow{\pi} \mathscr{Q} \to 0 \qquad (1)$$

defines a morphims

$$\begin{aligned} \alpha \colon \mathscr{Q} &\to \mathscr{O}ut(Z(\mathscr{K})) \\ \alpha(x)(y) &= \{y, x'\} \qquad \text{where} \quad \pi(x') = x \end{aligned} \qquad (2)$$

The extension problem is the following:

Given a Lie algebroid $\mathscr{Q}$, a coherent sheaf of Lie $\mathscr{O}_X$-algebras $\mathscr{K}$, and a morphism α as in (2), does there exist an extension as in (1) which induces the given α?

We assume $\mathscr{Q}$ is locally free

Abelian extensions

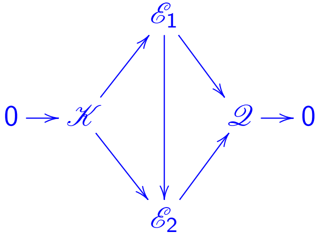
If $\mathscr{K}$ is abelian, $(\mathscr{K}, \alpha)$ is a representation of $\mathscr{Q}$ on $\mathscr{K}$, and one can form the semidirect product

$$\mathscr{E} = \mathscr{K} \rtimes_\alpha \mathscr{Q},$$

$$\mathscr{E} = \mathscr{K} \oplus \mathscr{Q} \quad \text{as } \mathscr{O}_X\text{-modules},$$
$$\{(\ell, x), (\ell', x')\} = (\alpha(x)(\ell') - \alpha(x')(\ell), \{x, x'\})$$

Theorem ([2])

If $\mathscr{K}$ is abelian, the extension problem is unobstructed; extensions are classified up to equivalence by the hypercohomology group $\mathbb{H}^2(\mathscr{Q}; \mathscr{K})^{(1)}_\alpha$



([2]) U.B., I. Mencattini, V. Rubtsov, and P. Tortella, Nonabelian holomorphic Lie algebroid extensions, Internat. J. Math. **26** (2015) 1550040

$\mathscr{M}$ a representation of a Lie algebroid $\mathscr{C}$. Sharp truncation of the Chevalley-Eilenberg complex $\sigma^{\geq 1}\Lambda^\bullet\mathscr{C}^*\otimes\mathscr{M}$ defined by

$$0 \longrightarrow 0 \longrightarrow \underset{\substack{\uparrow \\ \text{degree } 1}}{\mathscr{C}^*\otimes\mathscr{M}} \longrightarrow \Lambda^2\mathscr{C}^*\otimes\mathscr{M} \longrightarrow \cdots$$

We denote $\mathbb{H}^i(\mathscr{C};\mathscr{M})^{(1)} := \mathbb{H}^i(X,\sigma^{\geq 1}\Lambda^\bullet\mathscr{C}^*\otimes\mathscr{M})$

Derivation of $\mathscr{C}$ in $\mathscr{M}$: morphism $d\colon\mathscr{C}\to\mathscr{M}$ such that

$$d(\{x,y\}) = x(d(y)) - y(d(x))$$

Proposition

The functors $\mathbb{H}^i(\mathscr{C};-)^{(1)}$ *are, up to a shift, the derived functors of*

$$\begin{aligned} \mathrm{Der}(\mathscr{C};-)\colon\ \mathrm{Rep}(\mathscr{C}) &\to \Bbbk\text{-}\mathbf{mod} \\ \mathscr{M} &\mapsto \mathrm{Der}(\mathscr{C},\mathscr{M}) \end{aligned}$$

i.e.,

$$R^i\,\mathrm{Der}(\mathscr{C};-) \simeq \mathbb{H}^{i+1}(\mathscr{C};-)^{(1)}$$

Realize the hypercohomology using Čech cochains: if $\mathfrak{U}$ is an affine cover of X, and $\mathscr{F}^\bullet$ a complex of sheaves on X, then $\mathbb{H}^\bullet(X, \mathscr{F}^\bullet)$ is isomorphic to the cohomology of the total complex T of

$$K^{p,q} = \check{C}^p(\mathfrak{U}, \mathscr{F}^q)$$

$$0 \longrightarrow \mathscr{K}_{|U_i} \longrightarrow \mathscr{E}_{|U_i} \underset{s_i}{\overset{\pi}{\rightleftarrows}} \mathscr{Q}_{|U_i} \longrightarrow 0 \qquad (3)$$

If $U_i \in \mathfrak{U}$, $\mathrm{Hom}(\mathscr{Q}_{|U_i}, \mathscr{E}_{|U_i}) \to \mathrm{Hom}(\mathscr{Q}_{|U_i}, \mathscr{Q}_{|U_i})$ is surjective, so that one has splittings s_i, and one can define

$$\{\phi_{ij} = s_i - s_j\} \in \check{C}^1(\mathfrak{U}, \mathscr{K} \otimes \mathscr{Q}^*)$$

This is a 1-cocycle, which describes the extension only as an extension of $\mathscr{O}_X$-modules

$$0 \to \mathscr{K}(U_i) \to \mathscr{E}(U_i) \to \mathscr{Q}(U_i) \to 0$$

is an exact sequence of Lie-Rinehart algebras (over $(\Bbbk, \mathscr{O}_X(U_i))$)

which is described by a 2-cocycle ψ_i in the Chevalley-Eilenberg (-Rinehart) cohomology of $\mathscr{Q}(U_i)$ with coefficients in $\mathscr{K}(U_i)$

$$(\phi, \psi) \in \check{C}^1(\mathfrak{U}, \mathscr{K} \otimes \mathscr{Q}^*) \oplus \check{C}^0(\mathfrak{U}, \mathscr{K} \otimes \Lambda^2 \mathscr{Q}^*) = T^2$$

$$\delta\phi = 0, \qquad d\phi + \delta\psi = 0, \qquad d\psi = 0$$

$\Rightarrow$ cohomology class in $\mathbb{H}^2(\mathscr{Q}; \mathscr{K})^{(1)}_\alpha$

The nonabelian case

Theorem ([2,3])

If $\mathscr{K}$ is nonabelian, the extension problem is obstructed by a class $\mathbf{ob}(\alpha)$ in $\mathbb{H}^3(\mathscr{Q}; Z(\mathscr{K}))^{(1)}_\alpha$.
If $\mathbf{ob}(\alpha) = 0$, the space of equivalence classes of extensions is a torsor on $\mathbb{H}^2(\mathscr{Q}; Z(\mathscr{K}))^{(1)}_\alpha$.

Proof

$\mathscr{Q}$ can be written as a quotient of a free Lie algebroid $\mathscr{F}$

$$\begin{array}{ccccccccc} & & & & 0 & & & & \\ & & & & \downarrow & & & & \\ & & & & \mathscr{J} & & & & \\ & & & & \downarrow & & & & \\ 0 & \to & \mathscr{N} & \to & \mathfrak{U}(\mathscr{F}) & \to & \mathfrak{U}(\mathscr{Q}) & \to & 0 \\ & & & & \downarrow & & & & \\ & & & & \mathscr{O}_X & & & & \\ & & & & \downarrow & & & & \\ & & & & 0 & & & & \end{array}$$

([3]) E. Aldrovandi, U.B., V. Rubtsov, Lie algebroid cohomology and Lie algebroid extensions, J. of Algebra **505** (2018) 456–481

$$\widetilde{\mathscr{N}}^i = \mathscr{N}^i/\mathscr{N}^{i+1}, \qquad \widetilde{\mathscr{J}}^i = \mathscr{N}^i\mathscr{J}/\mathscr{N}^{i+1}\mathscr{J}, \quad \text{for } i = 0, \dots$$

Locally free resolution

$$\dots \to \widetilde{\mathscr{N}}^2 \to \widetilde{\mathscr{J}}^1 \to \widetilde{\mathscr{N}}^1 \to \widetilde{\mathscr{J}}^0 \to \mathscr{J} \to 0$$

As $\mathrm{Hom}_{\mathfrak{U}(\mathscr{Q})}(\mathscr{J}, Z(\mathscr{K})) \simeq \mathrm{Der}(\mathscr{Q}, Z(\mathscr{K}))$, applying the functor $\mathrm{Hom}_{\mathfrak{U}(\mathscr{Q})}(-, Z(\mathscr{K}))$ we obtain

$$0 \to \mathrm{Der}(\mathscr{Q}, Z(\mathscr{K})) \to \mathrm{Hom}_{\mathfrak{U}(\mathscr{Q})}(\widetilde{\mathscr{J}}^0, Z(\mathscr{K})) \xrightarrow{d_1} \mathrm{Hom}_{\mathfrak{U}(\mathscr{Q})}(\widetilde{\mathscr{K}}^1, Z(\mathscr{K})) \xrightarrow{d_2} \mathrm{Hom}_{\mathfrak{U}(\mathscr{Q})}(\widetilde{\mathscr{J}}^1, Z(\mathscr{K})) \xrightarrow{d_3} \mathrm{Hom}_{\mathfrak{U}(\mathscr{Q})}(\widetilde{\mathscr{K}}^2, Z(\mathscr{K})) \to \dots$$

The cohomology of this complex is isomorphic to $\mathbb{H}^{\bullet+1}(\mathscr{Q}; Z(\mathscr{K}))$.

Pick a lift $\tilde{\alpha}\colon \mathscr{F} \to \mathscr{D}er(\mathscr{K})$ of α and get commutative diagram

$$\begin{array}{ccccccccccc}
 & & 0 & \longrightarrow & \mathscr{T} & \longrightarrow & \mathscr{F} & \longrightarrow & \mathscr{Q} & \longrightarrow & 0 \\
 & & & & \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\tilde{\alpha}} & & \downarrow{\scriptstyle\alpha} & & \\
0 & \longrightarrow & Z(\mathscr{K}) & \longrightarrow & \mathscr{K} & \xrightarrow{\text{ad}} & \mathscr{D}er(\mathscr{K}) & \longrightarrow & \mathscr{O}ut(\mathscr{K}) & \longrightarrow & 0
\end{array}$$

where β is the induced morphism.

Define a morphism

$$o\colon \widetilde{\mathscr{J}}^{1} \to Z(\mathscr{K}) \qquad (4)$$

It is enough to define o on an element of the type yx, where x is a generator of $\mathscr{F}$, and y is a generator of $\mathscr{T}$

$$o(yx) = \beta(\{x, y\}) - \tilde{\alpha}(x)(\beta(y)).$$

Note that $o \in \mathrm{Hom}_{\mathfrak{U}(\mathscr{Q})}(\widetilde{\mathscr{J}}^{1}, Z(\mathscr{K}))$.

Lemma

$d_3(o) = 0$. Moreover, the cohomology class of $[o] \in \mathbb{H}^3(\mathscr{Q}; Z(\mathscr{K}))^{(1)}$ only depends on α.

Part I of the proof: if an extension exists consider the diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathscr{T} & \longrightarrow & \mathscr{F} & \longrightarrow & \mathscr{Q} & \longrightarrow & 0 \\
 & & \big\downarrow \beta & & \big\downarrow \gamma & & \big\| & & \\
0 & \longrightarrow & \mathscr{K} & \longrightarrow & \mathscr{E} & \longrightarrow & \mathscr{Q} & \longrightarrow & 0
\end{array}$$

Define

$$\tilde{\alpha} \colon \mathscr{F} \to \mathscr{D}er(\mathscr{K}, \mathscr{K}), \qquad \tilde{\alpha} = -\,\mathrm{ad} \circ \gamma$$

Then $\tilde{\alpha}$ is a lift of α, and for all sections t of $\mathscr{T}$ and x of $\mathscr{F}$

$$\beta(\{x, t\}) - \tilde{\alpha}(x)(\beta(t)) = 0 \tag{5}$$

so that the obstruction class $\mathbf{ob}(\alpha)$ vanishes.

Conversely, assume that $\mathbf{ob}(\alpha) = 0$, and take a lift $\tilde{\alpha} \colon \mathscr{F} \to \mathscr{D}er(\mathscr{K}, \mathscr{K})$. The corresponding cocycle lies in the image of the morphism d_2, so it defines a morphism $\beta \colon \mathscr{T} \to \mathscr{K}$, which satisfies the equation (5). Again, we consider the extension

$$0 \to \mathscr{T} \to \mathscr{F} \to \mathscr{Q} \to 0.$$

Note that $\mathscr{K}$ is an $\mathscr{F}$-module via $\mathscr{F} \to \mathscr{Q}$. The semidirect product $\mathscr{K} \rtimes \mathscr{F}$ contains the sheaf of Lie algebras

$$\mathscr{H} = \{(\ell, x) \,|\, x \in \mathscr{T}, \; \ell = \beta(x)\}.$$

The quotient $\mathscr{E} = \mathscr{K} \rtimes \mathscr{F} / \mathscr{H}$ provides the desired extension

Part II of the proof: reduction to the abelian case

Proposition

Once a reference extension $\mathscr{E}_0$ has been fixed, the equivalence classes of extensions of $\mathscr{Q}$ by $\mathscr{K}$ inducing α are in a one-to-one correspondence with equivalence classes of extensions of $\mathscr{Q}$ by $Z(\mathscr{K})$ inducing α, and are therefore in a one-to-one correspondence with the elements of the group $\mathbb{H}^2(\mathscr{Q}; Z(\mathscr{K}))^{(1)}$

$\mathscr{C}_1$, $\mathscr{C}_2$ Lie algebroids with surjective morphisms $f_i\colon \mathscr{C}_i \to \mathscr{Q}$.
Assuming $Z(\ker f_1) \simeq Z(\ker f_2) = \mathscr{Z}$ define

$$\mathscr{C}_1 \star \mathscr{C}_2 = \mathscr{C}_1 \times_{\mathscr{Q}} \mathscr{C}_2 / \mathscr{Z},$$

where $\mathscr{Z} \to \mathscr{C}_1 \times_{\mathscr{Q}} \mathscr{C}_2$ by $z \mapsto (z, -z)$

Fix a reference extension $\mathscr{E}_0$ of $\mathscr{Q}$ by $\mathscr{K}$

Lemma

(1) Any extension $\mathscr{E}$ of $\mathscr{Q}$ by $\mathscr{K}$ is equivalent to a product $\mathscr{E}_0 \star \mathscr{D}$ where $\mathscr{D}$ is an extension of $\mathscr{Q}$ by $Z(\mathscr{K})$

(2) Given two extensions $\mathscr{D}_1$, $\mathscr{D}_2$ of $\mathscr{Q}$ by $Z(\mathscr{K})$, the extensions $\mathscr{E}_1 = \mathscr{E}_0 \star \mathscr{D}_1$ and $\mathscr{E}_2 = \mathscr{E}_0 \star \mathscr{D}_2$ are equivalent if and only if $\mathscr{D}_1$ and $\mathscr{D}_2$ are equivalent

Muito obrigado pela atenção!!