2ND WORKSHOP OF THE SAO PAULO JOURNAL OF MATHEMATICAL SCIENCES:
JEAN-LOUS KOSZUL IN SAO PAULO, HIS WORK AND LEGACY
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MICHEL NGUIFFO BOYOM

1. PART A: SEMINAR OF GEOMETRY.
EXISTENCE DEFECTS OF GEOMETRIC-TOPOLOGICAL STRUCTURES IN DIFFERENTIAL MANIFOLDS
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2. EXISTENCE OF GEOMETRIC STRUCUTRES VERSUS GLOBAL ANALYSIS AND HOMOLOGY

2.1. *Some open problems*

In Finite dimensional Differential Geometry, Riemanniann structure (M, g) and gauge structure (M, ∇) are examples of Geometric Structure which exists in every differential manifold M. Here ∇ is a Koszul connection in the tangent bundle of M.
For many important Geometric structures the question whether a given differential manifold M does admit a given Geometric structure $\mathcal{S}$ is widely known to be an open difficult problem.
Examples of those open problems are.

(1.1) The existence of symplectic structures in a gven maniflod M.
(1.2) The existence of left invariant symplectic structure in a given Lie group G.
(1.3) The existence of two-sided invariant Riemannian structure in a given Lie group G.

Similar open problems are met in the gauge geometry of tangent vector bundles of smooth manifolds.

(2.1) The existence of locally flat Koszul connections in the tangent bundle of a given manifold M.
(2.2) The existence of left invariant locally flat connections in a given Lie group G.
(2.3) The existence of two-sided invariant Koszul connections in a given Lie group G.

Mutatis mutandis one faces open existence problems in the Differential Toplogy.

(3.1) The existence of regular Riemannian foliations in a given manifold M
(3.2) The existence of regular symplectic foliations in a given manifold M.
(3.3) The existence of foliations with a prescrbed structure for leaves.

2.2. *Motivations*

Throughout this talk a Riemannian structure in a smooth manifold M is a couple (M, g) formed of M and a non degnerate symmetric bilinear form g. A foliation is called regular if the dimension of leaves in constant.

I go to focus on the question whether a given manifold M does admit (eventually singular) foliations the leaves of which leaves carry a prescribed structures $\mathcal{S}$. To this aim, I go to overview some matherials which will be used.

3. Homological materials

Let us recall that a locaaly flat structure in a smooth manifold M is a couple (M, ∇) formed of M and a locally flat Koszul connection ∇.
The local flatness means the following identities

$$[X, Y] = \nabla_X Y - \nabla_Y X,$$
$$\nabla_X(\nabla_Y Z) - \nabla_Y(\nabla_X Z) = \nabla_{[X,Y]}Z.$$

Here X, Y, Z are smooth vector fields and $[X, Y]$ is the Poisson braacket.

The vector space of smooth vector in M and the vector associative algebra of real valued smooth functions in M are denoted by $\mathcal{A}$ and by $C^\infty(M)$ respectively.
For

$$\xi = X_1 \otimes .. \otimes X_{q+1}$$

one put

$$\partial_i \xi = .. \otimes \hat{X}_i \otimes ..$$
$$\nabla_{X_i}(\partial_i \xi) = \Sigma_{j \neq i} .. \otimes \hat{X}_i \otimes .. \otimes \nabla_{X_i} X_j \otimes ..$$

I go the involve the (positively) graded differential vector spaces

$$(\oplus_q C^q(\nabla), \delta_{KV}),$$
$$(\oplus_q C^q(\nabla), \delta_\tau).$$

Here

$$C^q(\nabla) = Hom_{\mathbb{R}}(\mathcal{A}^{\otimes q}, C^\infty(M)),$$

the differentials

$$\delta_{KV}; \delta_\tau : C^q(\nabla) \to C^{q+1}(\nabla)$$

are defined as it follows,
given $f \in C^q(\nabla)$ and ξ as above

$$(2.1) \quad \delta_{KV} f(\xi) = \Sigma_{i \leq q}(-1)^i [d(f(\partial_i \xi))(X_i) - f(\nabla_{X_i}(\partial_i \xi))]$$
$$(2.2) \quad \delta \tau f(\xi) = \Sigma_{i \leq q+1}(-1)^i [(d(f(\partial_i \xi))(X_i) - f(\nabla_{X_i} \partial_i \xi)]$$

The operators δ_{KV} and δ_τ satisfy the following identities

$$\delta \circ \delta = 0,$$
$$\delta_\tau \circ \delta_\tau = 0.$$

The derived cohomology spaces are denoted by

$$H_{KV}(\nabla) = \oplus_q H^q_{KV}(\nabla),$$
$$H_\tau(\nabla) = \oplus_q H^q_\tau(\nabla).$$

4. Fundamental Equations

To handle some between the open problems which have been raised, I go to assign two differential operators to every pair of Koszul Connections defined in the tangent bundle TM. Now (∇, ∇^*) is a pair of Koszul connections (defined in the same tangent bundle TM).

4.1. *The Hessian equation of* ∇

The Hessian differential operator of ∇ assigns a (2,1)-tensor to every vector field X, namely $\nabla^2 X$ which is defined by

$$(\nabla^2 X)(Y, Z) = \nabla_Y(\nabla_Z X) - \nabla_{\nabla_Y Z} X.$$

Let $x = (x_1, .., x_n)$ be local coordinate functions and let

$$X = \Sigma_1^m X^k \frac{\partial}{\partial x_i}$$

$$\nabla_{\frac{\partial}{\partial x_i}} \frac{\partial}{\partial x_j} = \Sigma_k \Gamma_{ij}^k \frac{\partial}{\partial x_k}$$

Let one evalue the principal symbol of $X \to \nabla^2 X$,

$$(\nabla^2 X)(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}) = \Sigma_\ell \Omega_{ij}^\ell \frac{\partial}{\partial x_\ell}.$$

Here

$$\Omega_{ij}^\ell = \frac{\partial^2 X^\ell}{\partial x_i \partial x_j} + \Sigma_k [\Gamma_{ik}^\ell \frac{\partial X^k}{\partial x_j} + \Gamma_{jk}^\ell \frac{\partial X^k}{\partial x_i} - \Gamma_{ij}^k \frac{\partial X^\ell}{\partial x_k}] + \Sigma_k [\frac{\partial \Gamma_{jk}^\ell}{\partial x_i} + \Sigma_m (\Gamma_{jk}^m \Gamma_{im}^k - \Gamma_{ij}^m \Gamma_{mk}^\ell)].$$

This expression looks awful, nevertheless from the viewpoint of both the Syernberg Geometry and the Spencer formalism, it allow to see that the dfferential operator $X \to \nabla^2 X$ is of type 2 and is involutive. Since the involutivity yields the formal integrability, the equation

$$\nabla^2 X = O$$

is formally integrable.

Lemma 4.1. *The sheaf* $\mathcal{J}(\nabla)$ *of solutions of the equation*

$$\nabla^2 X = 0$$

is a sheaf of real associative algbera whose product is defined by ∇

Let $\mathcal{KOSS}$ be the convex set of symmetric Koszul connections in TM.
At $x \in M$ let $\mathcal{J}_\nabla(x) \subset T_x M$ be the vector which spanned by the valued at x of sections of $\mathcal{J}(\nabla)$. One define the following numerical geometric invariants

$$r^b(M, \nabla) = \max_{x \in M} \{dim(M) - dim(\mathcal{J}_\nabla(x)\},$$

and

$$r^b(M) = \max_{\nabla \in \mathcal{KOSS}} r^b(\nabla)$$

4.2. *The Gauge equation*

The space (2,1)- tensors in a smooth manifold M is denoted by $\mathcal{T}_1^2(M)$.
The vector bundle of infinitesimal gauge transformations of TM is denoted by $\mathcal{G}(TM)$.
For every pair of Koszul connections, (∇, ∇^*), the $\mathcal{T}_1^2(M)$-valued differential operator

$$\mathcal{G}(TM) \ni \psi \to D^{\nabla\nabla^*}(\psi)$$

is defined by

$$D^{\nabla\nabla^*}\psi = \nabla^* \circ \psi - \psi \circ \nabla$$

Let X, Y be vector fields,

$$D^{\nabla}\nabla^*\psi(X,Y) = \nabla^*_X\psi(Y) - \psi(\nabla_X Y)$$

We denote by $\mathcal{J}(\nabla\nabla^*)$ the sheaf of solutions of the equation

$$D^{\nabla}\nabla^*\psi = 0.$$

4.3. *The Amari-Chentsov Formalism*

We have raised open (existence) problm in the differental topology. Remind that a Riemannian foliation in M is a couple (M, g) where g is a symmetric bilinear form subject to the following requirements.

(r.1) The rank of g is constant.
(r.2) If a vector filed X is a section of the kernel of g then

$$L_X g = 0,$$

Here $L_X g$ is the Lie derivative of g in the direction X.

Mutatis mutandis a symplectic foliation in M is a couple $(M?\omega)$ where ω is a closed differential 2-form subject to the following requirements

(s.1) The rank of ω is constant.
(s.2) If a vector field X is a section ofthe kernel of ω then

$$L_X\omega = 0.$$

According to the Amari-Rao-Chentsov formalism evry Riemannian metric tensor g is a symmetry of the affine space of the convex set of Koszul connections in TM. Given such a connection ∇ its image ∇^g under the metric tensor g is defined by

$$g(\nabla^g_X Y, Z) = Xg(Y,Z) - g(Y, \nabla_X Z)$$

We go to focus on global sections of the sheaf

$$\mathcal{J}(\nabla\nabla^g)$$

If both ∇ and ∇^g are symmetric, viz torsion free, the triple (M, g, ∇) is called a statistical manifold.

Let ψ be an infinitesimal gauge transformation of the vector bundle TM. To ψ one assigns two other infinitesimal gauge transformations of TM, namely Ψ and Ψ^* which are defined as it follow.

$$g(\Psi(X), Y) = \frac{1}{2}(g(\psi(X), Y) + g(X, \psi(Y))).$$
$$g(\Psi * (X), Y) = \frac{1}{2}(g(\psi(X), Y) - g(X, \psi(Y)))$$

Theorem 4.2. *If ψ is a section of the sheaf $\mathcal{J}(\nabla\nabla^g$ then so are Ψ and $\Psi *$. Further if g is positive definite one has the g-orthogonal decoposition*

$$TM = Ker(\Psi) \oplus Im(\Psi).$$
$$TM = Ker(\Psi*) \oplus Im(\Psi^*).$$

Now we involve global sections of the sheaf $\mathcal{J}(\nabla\nabla^g)$ to introduce new numerical invariants

$$(3.2.1) \quad (r^d)(g, \nabla) = \max_{\psi\in\mathcal{J}(\nabla\nabla^g)}[\max_{x\in M}\{dim(M) - rank(\Psi(x))\}]$$

$$(3.2.2) \quad r^d(\nabla) = \max_g\{r^d(\nabla, g)\}$$

$$(3.2.3) \quad s^d(g, \nabla) = \max_{\psi\in\mathcal{J}(\nabla\nabla^g)}[\max_{x\in M}\left\{dim(M) - rank(\Psi^*(x))\right\}]$$

$$(3.2.4) \quad s^d(\nabla) = \max_g\{s^d(g, \nabla)\}$$

$$(3.2.5) \quad s^d(M) = \max_{\nabla\in\mathcal{KOSS}(M)}\{s^d(\nabla)\}$$

4.4. *Links with the de Rham algebra*

Henceforth we will be concerned with global section of the sheaf $\mathcal{J}(\nabla\nabla^g$.
We go to point out some exact sequences which are linked with some between the open existence problems that I have listed.

Given a Koszul connection ∇ in TM the vector sheaf of ∇-parallel symmetric (2,0)-tensors is denoted by

$$\mathcal{S}_2^{\nabla}(M),$$

The sheaf of ∇-parallel skew symmetric (2,0)-tensors is denoted by

$$\Omega_2^{\nabla}(M)$$

Definition 4.3. A Hessian cocycle in a locally flat structure (M, ∇) is a non degenerate symmetric 2-cocyle in $C(\nabla, \delta_{KV})$
A compact locally flat structure (M, ∇) is called hyperbolic if $C(\nabla), \delta_{KV}$ contains a positive definite exact 2-cocycle g, viz $g = \delta_{KV}\partial$ with $\partial \in C^1(\nabla)$

4.5. *Some canonical sequences*

I go to focus on a few sequences and leur usefulness. The following notation is used:

$H^2_{dR}(M)$ is the 2nd space of de Rham cohomology of M.

In a locally flat structure (M, ∇), $H^2_{KVS}(\nabla)$ is the subspace of cohomology class $[g] \in H^2_{KV}(\nabla)$ which are represente by a symmetric cocycle g.

We consider the mapping Λ which sends every 2-cohcain $\partial \in C^2(\nabla)$ to its skew symmetry part

$$\Lambda(X, Y) = \frac{1}{2}(\partial(X, Y) - \partial(Y, X))$$

Assume that ∂ is a 2-cocycle of the cochain complex $C(\nabla), \delta_{KV})$, then Λ is a de Rham closed differential 2-form. That yields a canonical linear mapping

$$H^2_K V(\nabla) \to H^2_{dR}(M)$$

Thus in a locally flat structure (M, ∇) the following sequences are exact

$$(exs.1) \quad 0 \to H^2_{KVS}(\nabla) \to H^2_{KV}(\nabla) \to H^2_{dR}(M)$$
$$(exs.2) \quad 0 \to H^2_{dR}(M) \to H^2\tau(\nabla) \to \mathcal{S}^{\nabla}_2(M) \to 0.$$

Before pursuing I remind that a couple (M, ∇) where ∇ is Koszul connection in TM is called a gauge structure in M.

Proposition 4.4. *In a gauge structure (M, ∇) every Riemannian metric tensor g gives rise to a caninical splitting short exact sequence*

$$0 \to \Omega\nabla_2(M) \to \mathcal{J}(\nabla\nabla^g) \to \mathcal{S}^{\nabla}_2(M) \to 0.$$

4.6. *The canonical Koszul class of Riemannian foliations and symplectic foliations*

Let $\mathbf{G}$ be a Lie subalgebra of the Lie algebra of smooth vector fields in a manifold M. The vector space of (2,1)-tensors $T_1^2(M)$ is a left $\mathbf{G}$-module under the Lie derivative L_X, $X \in \mathbf{G}$.

If ∇ is a Koszul connection in TM the linear mapping

$$\mathbf{G} \ni X \to k^G(X) = L_X\nabla \in T_1^2(M)$$

is a Chevalley-Eilenrg cocycle whose cohomology class

$$[k^G] \in H^1_{CE}(\mathbf{G}, T_1^2(M))$$

does not depend on the choice of ∇.

Proposition 4.5. *Suppose that $[k^G]$ vanishes. Then either*

$$dim(\mathbf{G}) = 0$$

or

$$0 < dim(\mathbf{G}) < \infty.$$

I go implement Proposition 3.5 to Riemannian foliations and to symplectic foliations.

Let (M, g) be a Riemannian foliation and (M, Ω) be a symplectic foliation. Their kernels are denoted by K^g and by K^ω respectively. The Lie algebra of sections of those kernels are denoted by $\mathbf{G}^g$ and by $\mathbf{G}^\omega$ respectively. Thus we get the canonical Koszul classes

$$[k^g_\infty] \in H^1_{CE}(\mathbf{G}^g, T_1^2(M)),$$
$$[k^\omega_\infty] \in H^1_{CE}(\mathbf{G}^\omega, T_1^2(M))$$

Therefore the following statements are straightforward corallaries of Proposition 3.5

Corollary 4.6. *Given a Riemannian foliation (M, g) the following assertions are equivalent.*

$$(3.6.1) \quad [k^g_\infty] = 0.$$
$$(3.6.2) \quad \mathbf{G}^g = 0.$$

Mutatis mutandis we obtain

Corollary 4.7. *Given a symplectic foliation (M, ω) the following assertions are equivalent.*

$$(3.7.1) \quad [k^\omega_\infty] = 0.$$
$$(3.7.2) \quad \mathbf{G}^\omega = 0.$$

In the next I keep the notation as in Corollary 3.7 and I put

$$K^S_\infty(M) = \{(\omega, [k^\omega_\infty])\}$$

A couple $(\omega, [k^\omega_\infty])$ is called trivial if the cohomology class $[k^\omega_\infty]$ vanishes.

5. A few quantitative results

I go to impliment the materials we just introduced. The aim is to resolve the open existems problem for a few geometric structures.

Between the numerical invariants which are introduced in section 3 some are characteristic obstructions to the existence of a specific geometric structure. I go to list a few examples.

5.1. *Locally flat geometry*

Theorem 5.1. *In a finite dimensional smooth manifold M the following assertions are equivalents*
(a1.1) M admits affinely flat structures.
(a1.2) M admits locally flat structures.
(a1.3) $r^b(M) = 0$.

5.2. *Symplectic geometry*

Theorem 5.2. *In a even dimensional smooth manifold M the following assertion are equivalent.*
(a2.1) M admit symplectic strur// (a.2.2) $s^d(M) = 0$.

5.3. *The differential topology*

Proposition 5.3. *In every symmetric gauge structure (M, ∇)*
(a3.1) $\mathcal{S}_2^\nabla(M)$ is the sheaf of ∇-geodesic Riemannian foliations in M.
(a3.2) $\Omega_2^\nabla(M)$ is a sheaf of ∇-geodesic symplectic foliations in M.
(a3.3) Every Riemannian foliation is deduced from a short exact sequence

$$0 \to \Omega_2^\nabla(M) \to \mathcal{J}(\nabla\nabla^g) \to \mathcal{S}_2^\nabla(M)$$

By involving the canoncal Koszul classes of symplectic foliations one obtains the following stateent.

Theorem 5.4. *The following assertions are equivalent.*
(A1) $K_\infty^S(M)$ contains a trivial couple $(\omega, [k_\infty^\omega])$.
(A2) M admits symplectic structures.

5.4. *Riemannian geometry*

Here I am intersted in (eventually singular) foliations with prescrbed structure for leaves.

Proposition 5.5. *Let ∇ be the Levi-Civita connection of a geodesically complete positive Riemannian structure $(M, g$. Assume that the following inequalities hold*

$$0 < r^b(M, \nabla^) < dim(M).$$

Then M admits a foliations $\mathcal{F}$ with the following properties.

(p4.1) Up to finite covering, every n-dimensional leave endowed with the induced metric is isometric to the canonical flat cylinder over the flat torus

$$(\mathbb{T}^k \times \mathbb{R}^{n-k}, g_0)$$

The metric g_0 is given by the Euclidean metric of $\mathbb{R}^n$.

((p4.2) Further the leaves of $\mathcal{F}$ of orbits a locally effective action of a finite dimensional simply connected Lie group.

6. 2ND WORKSHOP OF THE SAO PAULO JOURNAL OF MATHEMATICAL SCIENCES. J-L KOSZUL IN SAO PAULO, HIS WORK AND LEGACY

UNIVERSITY OF SAO PAULO
13-14 NOVEMBER 2019

Ce qui reste par contre u mystère absolu pour moi c'est ce qui signifie au juste Gémométrie de l'Information. Et quand en plus elle est Hessienne, cela n'arrange rien. Notez que je suis habitué depuis longtemps à voir naître des terminologies bizzares et à assister à des détournements de sens audacieux, voire criminels.
J-L Koszul to M-N Boyoym, 3 February 2012

O que permanece, no entanto, um mistério absoluto para mim é o que significa ao certo Geometria da InformaÃğao. E quando, além disso, ela é Hessiana, isso nao ajuda em nada. Note que tenho o habito de longa data de ver nascer terminologias estranhas e de assistir a desvios de significado audciosos, ou até criminosos

7. What is called the Geometry of Koszul

An important part of works of Koszul has been devoted to the Geometry of bounded domains. I go to focus on a tarticular those between those, which impacts the refoundattion of the theory of statistical models of measurable sets. I go to overview the following subjects.

A- Affinely flat Geometry.

(A.1) The affiney flat Geometry.
Complete atltas whose local chart changes are affine transformations

(A.2) The locally flat Geometry.
Curvature free and torsion free gauge structure in tangent vector bundle

(A.3) The completeness of locally flat structures.
Developping mapping sends universal covering onto the Euclidean space

(A.4) The deformations of locally flat structures.
A long history. The point set topolog. Hyperbolicity and rigidity problem.
A theorem of Koszul)

(A.5) The existence of locally flat structures.
A long history. Many and long efforts. Koszul-Milnor-Matsushima-Vinberg and al. Recently brought in completion.
The main via the Hessian differential operator ∇^2

B- Main contributions of Jean- Louis Koszul.

(B.1) Affine representaions of Lie groups.
Pour ce qui est representations affine

(B.2) Non rigidity of hyperbolic locally flat structures.
Every locally flat hyperbolic manifold admits non trivial deformation: Koszul.
Proof based on the point set topology

(B.3) The Hessian Geometry.
Riemannaian Hessian defect $r^b(M, g)$. *Affine Hessian defect (*$r^b(M, \nabla)$. *Absolute Hessian defect* $r^b(M)$

(B.4) The Hessian Geometry and the hyperbolicity.
Handled with the (algebraic) topology of Koszul. See the versus KV cohomology

(B.5) The geometry of convex domains
Many best reference exist. Also other talks in the workshop

(B.5) Characteristic invariants of convex cones.
Large impacts: Fisher information. Lie group theory of heat

C- The theory of deformation of mathematical structures.

(C.1) Algebraic strtuctures:
Gerstenhaber. Nijenhuis Richardson,Piper

(C.2) Analytic structures.
Kodaira,Koszul, Kuranishi, Spencer and many others

(C.3) Geometric structures.

D- Theory of deformation and theory of cohomolgy.
A conjecture of Gerstenhaber:
Infinitesimal deformation : = cocycle.
Infinitesimal trivial deformation : = coboudary.
Rigidity : = Open orbite: = cohomology vanishing theorem

(D.1) Deformation and Extension of associative algebras.
The cohomology of Hochschild

(D.2) Deformation and Extension of Lie algebras.
The cohomology of Chevalley-Eilenberg.

(D.3) Deformation and Formal integrability of Geometric structures.
The cohomology of Koszul-Spencer.

E- A conjecture of Muray Gerstenhaber.
Every restrict theory of deformation generates its proper theory of cohomology

E- The deformation locally structures.

(E.1) The approach of J-L Koszul
Point set topology

(E.2) The pionnering work of Albert Nijenhuis.
Commutator Lie algebras of Vinberg agebra: CE cohomology

(E.3) *Versus deformation and extension of KV algebras:cohomology*.

8. What is called the topology of Koszul

F - Theory of KV cohomology and its impacts.
(F.1) *The notion of Koszul-Vinberg algebra*.
(F.2) *Two-sided KV-modules*.
(F.3) *The KV complex*

$$\delta_{KV}F(X_1\otimes..\otimes X_{q+1} = \Sigma_1^q(-1)^i[\nabla_{X_i}F(..\otimes\hat{X}_i\otimes..)$$

$$+\nabla_{F(..\otimes\hat{X}_i\otimes..\hat{X}_{q+1}\otimes X_i)}X_{q+1}$$

$$-\Sigma_{j\neq i}F(..\hat{X}_i\otimes..\otimes\nabla_{X_i}X_j\text{Å}times..)]$$

(F.4) The total KV complex.

$$\delta_\tau F(X_1\otimes..\otimes X_{q+1}) = \Sigma_1^{q+1}(-1)^i[\nabla_{X_i}F(..\otimes\hat{X}_i\otimes..)$$

$$+\nabla_{F(..\otimes\hat{X}X_i\otimes..\otimes\hat{X}_{q+1}\otimes X_i)}X_{q+1}$$

$$-\Sigma_{j\neq i}F(..\otimes\hat{X}_i\otimes..\otimes\nabla_{X_i}X_j\otimes..)]$$

(F.5) Relationships with the cohomology of Hochschild.
The Poisson structures.

(F.6) Relationships with the de Rham cohomology

The differential topology.

$$(F.6.1)\quad g(\nabla^g_X Y, Z) = Xg(Y,Z) - g(Y,\nabla_X Z$$

$$(F.6.2)\quad \nabla^g_X\psi(Y) - \psi(\nabla_X Y) = 0$$

$$(F.6.3)\quad 0\to\Omega_2^\nabla(M)\to\mathcal{J}(\nabla,\nabla^g)\to\mathcal{S}_2^\nabla(M)\to O$$

$$(F.6.4)\quad 0\to H^2_{dR}(M)\to H^2_\tau(\nabla)\to\mathcal{S}_2^\nabla(M)\to O$$

9. Geometry of Koszul and the Information Geometry

(G.1) The local theory of statistical models.

(G.2) The homological theory of statistical models.

(G.3) The topology of Koszul as the source of the information geometry.

(G.4) The Geometry of Koszul as a golbal vanishing theorem in the topology of Koszul.

(G.5) The local theory of statistical models as a local vanishinh theorem in the topology of Koszul.

10. A graph representation of the information geometry

(H.1) Random Hessian structure.

(H.2) The Lemma of Poincaré versus KV cohomology.

(H.3) The probability densities.

(H.4) The Fisher information.

(H.5) Relationships with the differential topology.

Fisher information g

α-connections ∇^{α}

$$Xg(Y,Z) - g(\nabla^{\alpha}_X Y, Z) - g(Y, \nabla^{-\alpha}_X Z) = 0$$

11. The source of the information geometry is the topology of Koszul

One represents this feature by a rooted tree whose root is a random KV cohomology class $[Q]$.

DT— CIG —————— AIG $[\mathcal{E}, \pi, M, Q]$

$\int p\nabla^2 log(p)$ – – – – – – – – – $\int plog(p)$

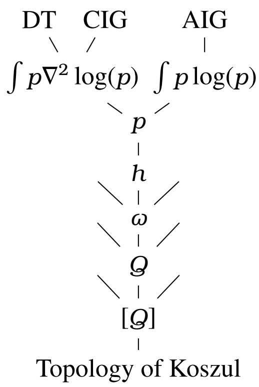
$[\mathcal{E}, \pi, M, p]$

[M,h]

$[\mathcal{E}, \pi, M, [Q]]$

$[M, \omega]$

Topology of Koszul: Homological data.



Local reading the tree above.

$d_{KV}Q = 0,$ KV random KV cocycle.

$Q = d_{KV}\omega,$ KV Poincaré Lemma.

$\omega = d_{dR}h,$ de Rham Poincaré Lemma.

$p = \frac{exp(h)}{\int exp(h)}$, Weak Jensen inequality.

$E = \int plog(p)$, entropy function.

$g = \int p\nabla^2 log(p)$, Fisher information.

DT : Differential Topology.

CIG : Classical Information Geometry.

AIG : Applied Information Geometry.

So the classical information geometry is a leaf of a rooted tree whose root lies in the Tpology of Koszul.

References

Amari 1990. Amari S-I: Differential Geometry Methods in Statistics. Lecture Notes in Statistics 28, Springer Verlag, NY 1990.

Amari-Armstrog 2014. Amari S-I and Armonstrong J. Curvature of Hessian Manifolds. Diff Geom Appl 33 (2014), 1-12.

Amari-Nagaoka 191. Amari-S-I and Nagaoka H. Methods of Information Geometry,Translations of Mathematical Monographs, AMS-OXFORD, vol 191.

Baudot-Bennequin 2015. Baudot P. and Bennequin D. The Homological Nature of Entropy, Proceedings of The Amer Institute of Physics (2014).

Baudot-Bennequin. Baudot P. and Bennequin D. The Homological Nature of entropy. Entropy Special Issue (2015)

Barndorff-Nielsen . Barndorff-Nielsen O.E. Informations and Exponential Families in Statistical Theory., Wiley New York.

Byande 2000. Byande P.M. Des structures affines à la Géométrie de l'Information. La notion de T-Plongement. Edition Omniscriptum (2011).

Gromov 2013. Grmov M. The Search of Structure. ECM6, Krakow (2012Âř.

Gromov. Gromov M. The Search of Structure. MaxEnt 2014. Proc Amer Institute of Phys, 2014
[Gromov]Gromov Gromov M. Geometric Structure, People.mpim.mpg.de/hwblmnn/arxiv/geostr00pdf

Koszul 1968. Koszul J-L. Déformation des variétés localement plates. Ann Insttut Fourier 18(1968), 103-114.

Koszul 1974. J-L Koszul, Homologie des complexes deformes différentielles d'ordre supérieur. Ann. Sci ENS, Série 4 (7) (1974) 139-153.

McCullagh. McCullagh P. What is a Statistical Model? Annals of Statistics (2002) Volume 3 NÂř5 1225-1310.

Moerdijk-Mrcun . Moerdijk and Mrcun. Introduction to Foliations and Lie Groupoids. Cambridge Studies in Advance Mathematics 91.

Molino. Molino P. Riemannian Foliations. Birkhauser, Boston.

Muray-Rice. Murray and Rice. Differential Geometry and Statistics. Monographs in Statistics and Applied Probability 48, ChapmanHall /CRC.

Nguiffo Boyom 2006. Nguiffo Boyom M. The cohomology of Koszul-Vinberg Algebras. Pacific Journal of Mathematics, vol 225, (2006), 119-153.

Nguiffo Boyom 2016. Nguffo Boyom M. Foliations-Webs-Hessian Geometry-Information Geometry-Entropy and Cohomology, Entropy 2016,vol 18,NÂř12, 433.

Nguiffo Boyom-Wolak 2016. Nguiffo Boyom M. and Wolak R. Transversely Hessian Foliations and Information Geometry. Intern Journ of Mathematics, vol 27, NÂř11 (2016).

Nijenhuis 1968. Nijenhuis A. Sur une Classe des Propriétés communes à divers types d'Algèbres. Enseign Mathem 1968, (14), 225-277.

Pennec 2014. Pennec X. Geometric Statistics on manifolds and Lie Groups. Springer Lectures Notes in Computer Science, Barbaresco-Nielsen. Ed. GSI 2013.

Shima 2007. Geometry of Hessian Manifolds. Word Scientific Publishing Co, 2007.

Wolf 1974. Wolf J. Spaces of constant curvature, Poblish of Perish,Boston Mass, 1974.

IMAG : Alexander Grothendieck Research Institute, UMR CNRS 5149 University of Montpellier/FRANCE

E-mail address: `boyom@math.univ-montp2.fr michel.nguiffo-boyom@umontpellier`