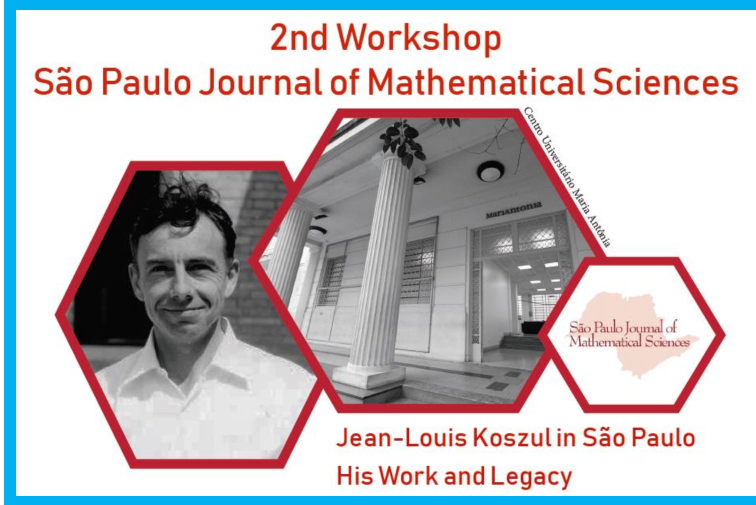
THALES

Jean-Louis KOSZUL & the Elementary Geometric Structure of Information: Koszul-Fisher Metric & Information Geometry

Michel Nguiffo Boyom & Frédéric Barbaresco



Napoleon & Koszul Ancestor (testimony of J.L. Koszul Daughter)

Jean-Louis Koszul ancestor was a Polish soldier incorporated in Napoleon's "Grande Armée". He returned to France and married in Strasbourg.

- Napoléon Grande Armée in Poland (http://heritage.bnf.fr/france-pologne/fr/napoleon-et-la-pologne-art) : The Polish soldiers will follow Napoleon throughout his career, until the period called "Les 100 jours" and he will be entirely devoted. The Polish legions are the most important foreign force of the Grand Army with 100,000 soldiers, 37,000 under the command of Prince Jozef Poniatowski. The retreat in Moscow will kill 70% of the Polish workforce

BIBLIOTEKA NARODOWA

L'Ombre de Napoléon visitant Les Héros de la Pologne

Prince Jozef Poniatowski - le maréchal de France

Mathieu Koszul

- Born about 1788 at Wola Radziszowska, Cracovie, Poland
- Deceased 18 November 1858 at Morschwiller-le-Bas, Haut-Rhin, Alsace, France
- Parents: Jacques Koszul & Marie-Anne Gregorin
- Son: Joseph Kosul (Father of Julien Koszul)

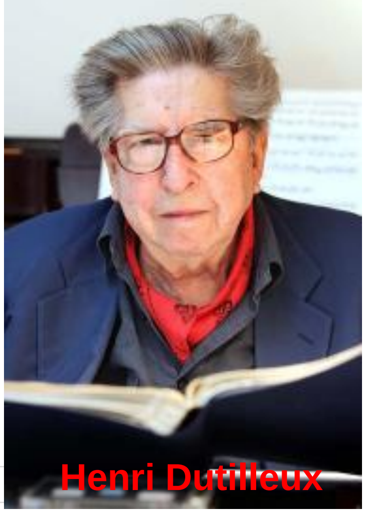
Jean-Louis & Julien Koszul: «Moment 1900» & «Ecole Niedermeyer»

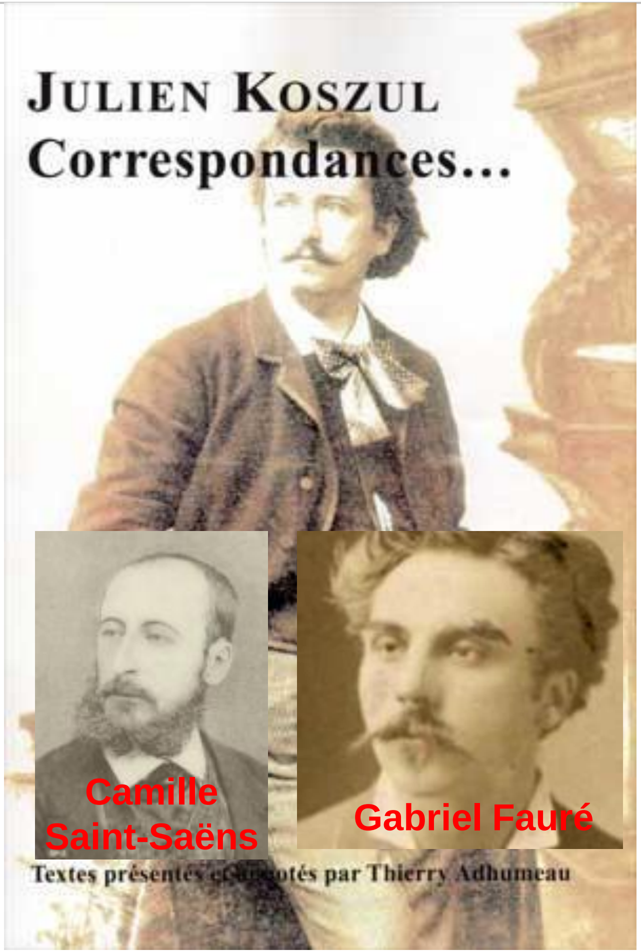
Julien KOSZUL (1844-1927)

- Grand father ofJean-Louis Koszul and composer Henri Dutilleux, student of Camille Saint-Saëns and friend of Gabriel Fauré. Professor of Albert Roussel.

Main Music writtings:

- Quo Vadis pour choeur d'hommes à 5 voix
- Pié Jesus en si m
- Pièces pour piano à deux mains et 1 pièce pour piano à 4 mains
- Mélodies de 1872 et 1879





Julien KOSZUL, Gabriel Fauré et Camille Saint-Saëns

G. Fauré à Julien Koszul[1]

Rue des Vignes 32 XVIe 21 avril 1924

Mon cher ami

Je te remercie de m'avoir envoyé une jolie Berceuse qui me donne le vif désir de connaître les autres mélodies ; je les demanderai à Hamelle.

Es-tu content de ta santé ? Ne viens-tu jamais à Paris ? Je serais tellement heureux de te revoir, de pouvoir bavarder un peu longuement avec toi ! Nous avons tant de bons souvenirs. Te souviens-tu que c'est toi qui introduisis *Schumann* à l'École Niedermeyer où il était si profondément inconnu et où n'avons pas tardé, tous, à l'adorer ?

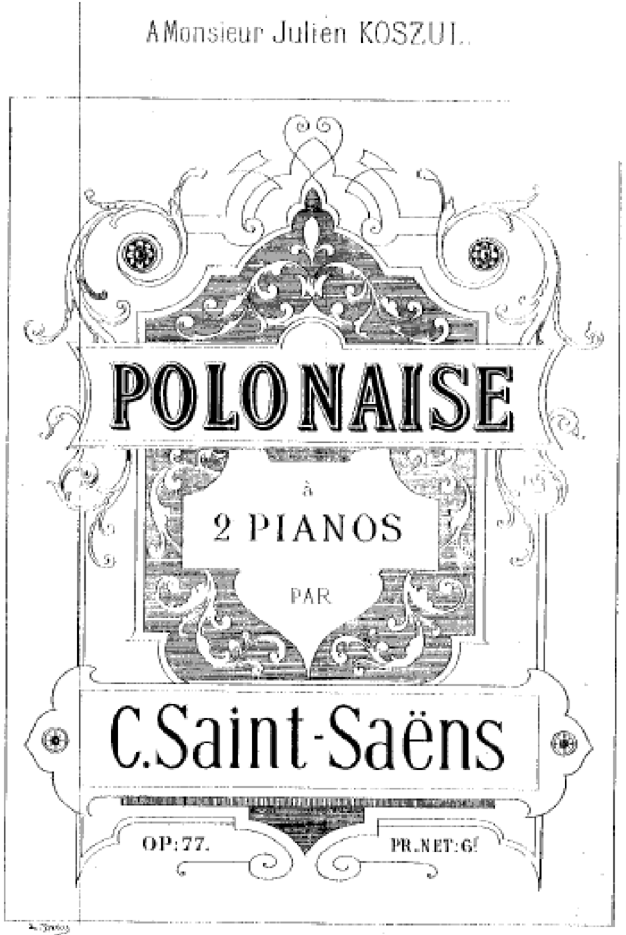
Et puis, autres moins lointains souvenirs, mes visites à Roubaix et l'accueil délicieux que je recevais dans ta chère maison !

Donne-moi de tes nouvelles ; parle-moi de tes enfants, et, si tu as une photo, envoie-la moi comme je t'envoie la mienne[2]. J'y joins, mon cher ami, toute ma vieille et bien fidèle amitié

Gabriel Fauré

Bibliothèque nationale, département de la Musique, don d'Henri Dutilleux

1. Voir plus haut la lettre que lui adressa le jeune Fauré en juin 1870 (lettre 7). Enveloppe portant le cachet postal Paris, 22.IV.1924 : « Monsieur J. Koszul, Ancien Directeur du Conservatoire de Roubaix, *Douai. Nord* »
2. Photo jointe : portrait de Fauré en 1924 par les frères Manuel.



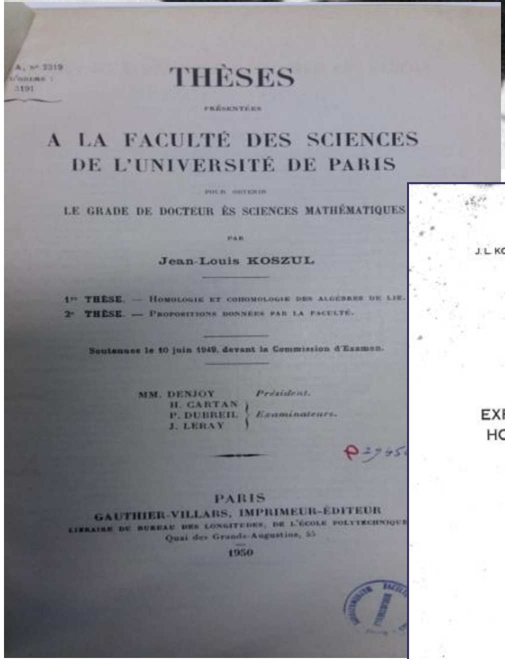
Jean-Louis Koszul 1921-2018

Jean-Louis Koszul: 1921-2018

Interview of Jean-Louis Koszul: https://www.youtube.com/watch?v=AzK5K7Q05sw

Jean-Louis Koszul Scientific Biography

- He entered ENS Ulm in the class of 1940 and defended his thesis with Henri Cartan.
- Koszul's thesis, defended in June 10th 1949 under the direction of Henri Cartan, dealt with the homology and cohomology of Lie algebras. The jury was composed of M. Denjoy (President), J. Leray, P. Dubreil and H. Cartan.
- He then taught in Strasbourg and was appointed Associate Professor at the University of Strasbourg in 1949, and had for colleagues R. Thom, M. Berger and B. Malgrange. He was promoted to professor status in 1956.
- He became a member of Bourbaki with the 2nd generation, J. Dixmier, R. Godement, S. Eilenberg, P. Samuel, J. P. Serre and L. Schwartz.
- Koszul was awarded by Jaffré Prize in 1975 and was elected correspondent at the Academy of Sciences on January 28th 1980. Koszul was one of the CIRM conference center founder at Luminy. The following year, he was elected to the Academy of São Paulo.



THÈSES

A LA FACULTÉ DES SCIENCES
DE L'UNIVERSITÉ DE PARIS

LE GRADE DE DOCTEUR ÈS SCIENCES MATHÉMATIQUES

Jean-Louis KOSZUL

1re THÈSE. — Homologie et cohomologie des algèbres de Lie.
2e THÈSE. — Propositions données par la Faculté.

Soutenues le 10 juin 1949, devant la Commission d'Examen.

MM. DENJOY Président.
H. CARTAN
P. DUBREIL
J. LERAY Examinateurs.

PARIS
GAUTHIER-VILLARS, IMPRIMEUR-ÉDITEUR
Quai des Grands-Augustins, 55
1950

J. L. KOSZUL

EXPOSÉS SUR LES ESPACES HOMOGENES SYMÉTRIQUES

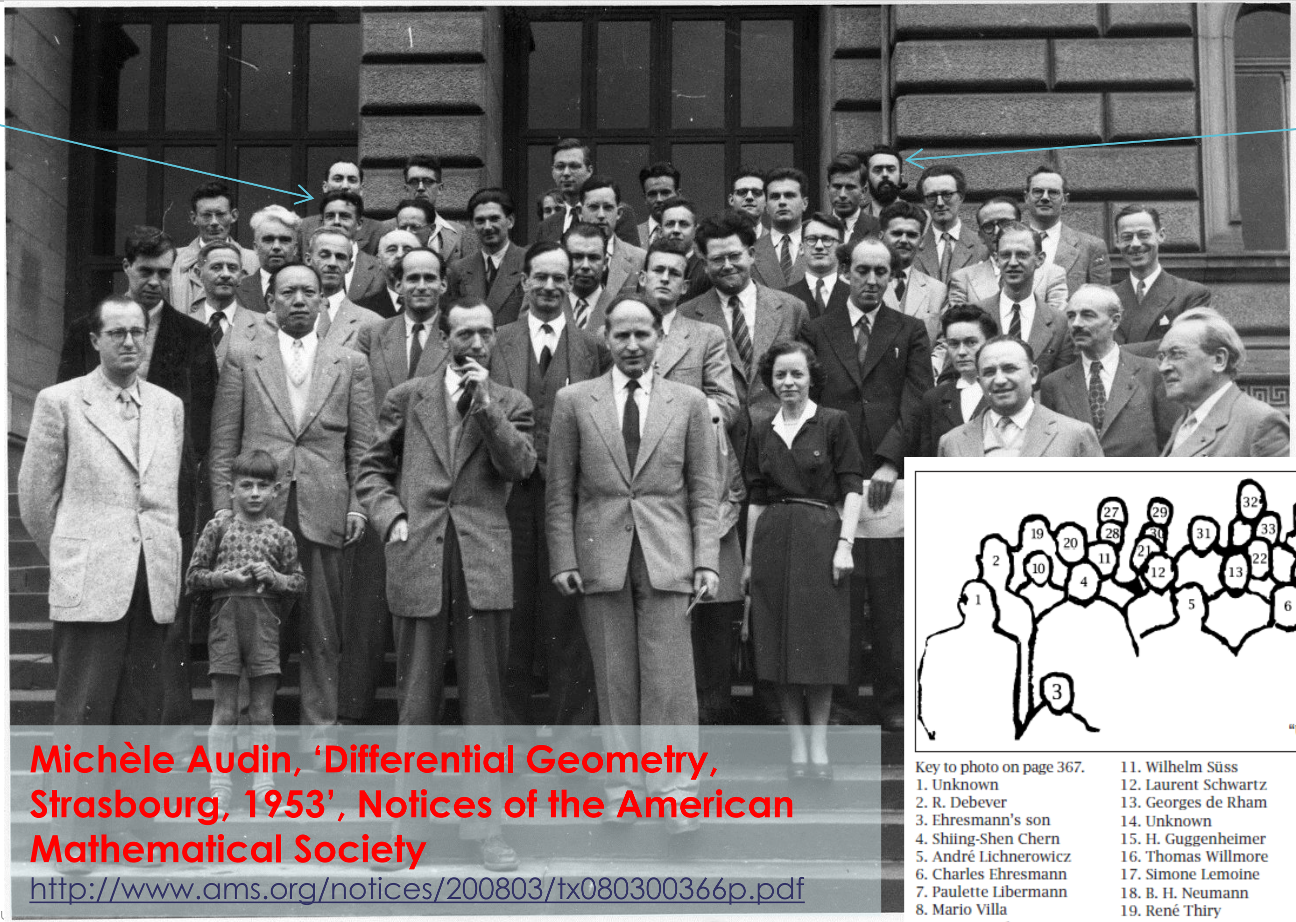
PUBLICAÇÃO DA SOCIEDADE DE MATEMÁTICA DE SÃO PAULO

SÃO PAULO
1959

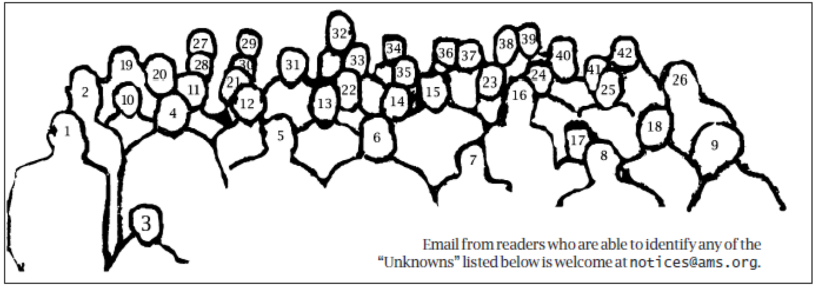
Souriau & Koszul at 1953 Conference « Topologie différentielle » in Strasbourg

J.L. Koszul

J.M. Souriau



Michèle Audin, ‘Differential Geometry, Strasbourg, 1953’, Notices of the American Mathematical Society
http://www.ams.org/notices/200803/tx080300366p.pdf



Key to photo on page 367.
1. Unknown
2. R. Debever
3. Ehresmann's son
4. Shiing-Shen Chern
5. André Lichnerowicz
6. Charles Ehresmann
7. Paulette Libermann
8. Mario Villa
9. Lucien Godeaux
10. Heinz Hopf
11. Wilhelm Süss
12. Laurent Schwartz
13. Georges de Rham
14. Unknown
15. H. Guggenheimer
16. Thomas Willmore
17. Simone Lemoine
18. B. H. Neumann
19. René Thiry
20. E. T. Davies
21. Unknown
22. Unknown
23. Unknown
24. Unknown
25. Nicolaas Kuiper
26. Beno Eckmann
27. Unknown
28. Jean-Louis Koszul
29. Unknown
30. André Weil
31. René Thom
32. John Milnor
33. Marcel Berger
34. Unknown
35. Bernard Malgrange
36. Daniel Bernard
37. André Aragnol
38. G. Legrand
39. Jean-Marie Souriau
40. Unknown
41. Georges Reeb
42. Unknown

Cooperation with Japanese School of Differential Geometry: Hirohiko Shima Book on « the geometry of Hessian Structures »

- The elementary geometric structures discovered by Jean-Louis Koszul are the foundations of Information Geometry. These links were first established by Professor Hirohiko Shima.
- These links were particularly crystallized in Shima book 2007 "The Geometry of Hessian Structures", which is dedicated to Professor Koszul.
- The origin of this work followed the visit of Koszul in Japan in 1964, for a mission coordinated with the French government.
- Koszul taught lectures on the theory of flat manifolds at Osaka University. Hirohiko Shima was then a student and attended these lectures with the teachers Matsushima and Murakami.
- This lecture was at the origin of the notion of Hessian structures and the beginning of the works of Hirohiko Shima.

Henri Cartan noted concerning Koszul's ties with Japan, "*Koszul has attracted eminent mathematicians from abroad to Strasbourg and Grenoble. I would like to mention in particular the links he has established with representatives of the Japanese School of Differential Geometry*".

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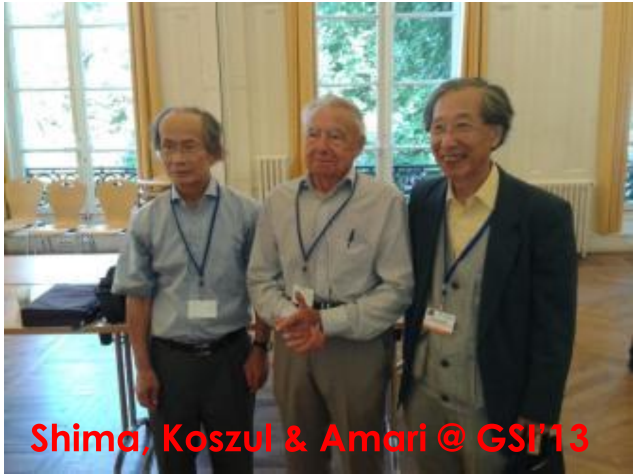
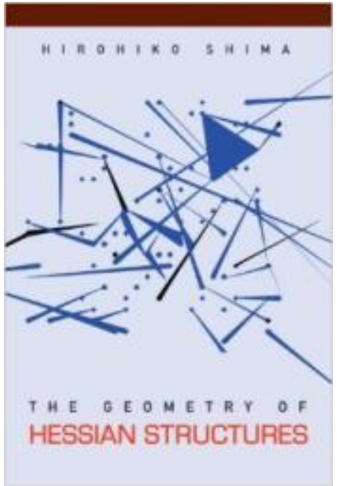
THALES

Jean-Louis Koszul (1921-2018) and Information Geometry


Koszul, Shima, Boyom @ GSI'13


H. Shima Talk « Koszul Geometry of Hessian Structure » @ GSI'13


Shima, Koszul & Amari @ GSI'13




Koszul & Shima @ GSI'13

Cartan & Koszul families

The family settled in Dolomieu. **Joseph Cartan was the village blacksmith**. Elie Cartan recalled that his childhood had passed under "**blows of the anvil, which started every morning from dawn**".

Origin of Koszul Family: Ancestor from Poland was engaged in **Napoléon "Grande Armée"**. He came back to France with Napoléon Army and Married with Woman from Strasbourg

ALLOCUTION DE MONSIEUR HENRI CARTAN

Je n'ai nullement l'intention de faire un « discours », contrairement à ce qu'annonçait le programme de ces journées. Je voudrais simplement évoquer ici brièvement quelques souvenirs qui, avec les années qui passent inexorablement, tendent malheureusement à s'estomper.

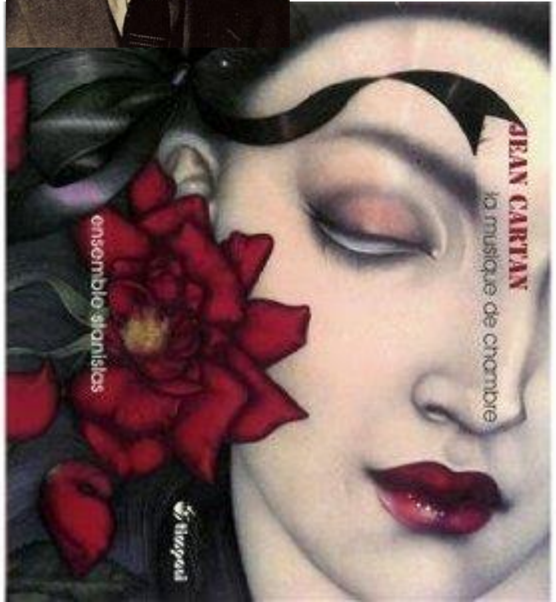
Ces souvenirs commencent, il est vrai, avant la naissance de Koszul. En effet, ma mère, dans sa jeunesse, avait été une amie intime de celle qui devait devenir la mère de Jean-Louis Koszul. Il arriva que ces deux amies se marièrent ; l'une épousa un mathématicien connu, l'autre un angliciste non moins connu. Malgré l'éloignement consécutif à leurs mariages, des liens d'amitié subsistèrent, qui expliquent pourquoi, lorsque beaucoup plus tard, au printemps de 1929, j'arrivai à Strasbourg comme jeune chargé de cours à la Faculté des Sciences, je fus reçu dans la famille du professeur Koszul de la Faculté des Lettres. J'ai oublié le menu du repas familial, mais je vois toujours un jeune garçonnet de 8 ans, nommé Jean-Louis, qui évoluait dans l'appartement au milieu de ses grandes sœurs. L'aînée d'entre elles était mariée à un agrégatif de mathématiques que j'avais comme élève à la Faculté. Je ne restai à Strasbourg que quelques mois et perdis donc de vue le jeune Jean-Louis.

Puisque j'ai évoqué le souvenir de ses parents, permettez-moi de nommer aussi le grand-père paternel de Jean-Louis. Je ne l'ai pas connu, certes ; mais comme directeur du Conservatoire de musique de Roubaix-Tourcoing, il joua un rôle historique, car c'est lui qui donna au jeune Albert Roussel, qui venait d'abandonner la carrière navale, les conseils décisifs qui lui permirent de devenir l'un des plus grands compositeurs de musique du début du siècle. On aura l'occasion d'en parler cette année, puisqu'on va célébrer le cinquantenaire de la mort d'Albert Roussel.

Jean Cartan (Elie Cartan'son & Henri Cartan's brother) and Julien Koszul (Jean-Louis Koszul's grand-father): MUSIC FILIATION

Julien Koszul => Albert Roussel => Jean Cartan

- Albert Roussel studied harmony in Roubaix with Julien Koszul.
- Condisciple of Olivier Messiaen and Maurice Duruflé, the career of Jean Cartan is followed attentively by Albert Roussel.



Chronologically, Jean Cartan composed between 1926 and 1930 the Introduction and Allegro for flute, oboe, clarinet, horn, bassoon and piano (the same training as Roussel's Op.6, his mentor who will follow his whole career not without his 'influence'), the Quartet No. 1 (1927) dedicated precisely to Roussel, the Quartet No. 2 (1930) and finally the Sonatine for flute and clarinet (1931).

- https://www.see.asso.fr/en/node/24148

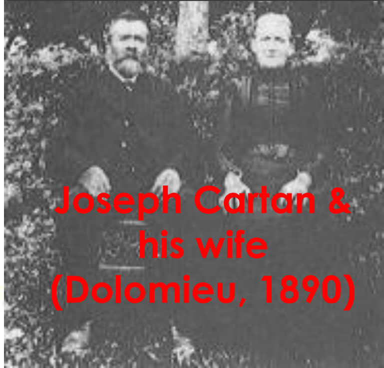
1869-2019 Elie Cartan 150th Bithday

150 years ago, April 9th 1869, was born a spirit, raised to the heat of the forge and the sound of the anvil and the hammer of his father Joseph, blacksmith of little Dolomieu village.

Henri Cartan Testimony on his father Elie Cartan:
https://www.youtube.com/watch?v=GJ9NwEwUcyY

Elie Cartan with his wife and 4 children

"des paysans sans prétention qui, au cours de leur longue vie, ont montré à leurs enfants un exemple de travail accompli avec joie et d'acceptation courageuse des fardeaux"
- Elie Cartan



Joseph Cartan & his wife (Dolomieu, 1890)

Elie Cartan Colloquium 1984

INTRODUCTION

Le séminaire conjoint NSF–CNRS "Élie Cartan et les mathématiques d'aujourd'hui" s'est tenu du 25 au 29 juin à l'Université de Lyon I. Il était centré sur la présentation de thèmes importants de la recherche actuelle en mathématiques et en physique mathématique dans des domaines où Élie CARTAN a joué un rôle de pionnier. Des discussions très animées ont eu lieu à propos de ces thèmes. La partie centrale du programme comprenait vingt-deux conférences qui ont été suivies par un public nombreux et enthousiaste de plus de deux cents mathématiciens et physiciens mathématiciens venus d'au moins dix-sept pays. Les conférences se sont tenues dans le grand amphithéâtre de mathématiques de l'Université de Lyon I, la salle Camille Jordan.

Ce volume regroupe les contributions écrites des conférenciers à l'exception de trois d'entre eux. Nous publions en annexe les résumés que ces auteurs ont bien voulu nous communiquer.

Le programme du séminaire avait été préparé par un Comité Scientifique sous la co-présidence de Shing-shen CHERN et d'Henri CARTAN. Les détails pratiques pour l'organisation du séminaire ont été réglés par un Comité mis sur pied par le Département de mathématiques de l'Université de Lyon I, sous la responsabilité d'Edmond COMBET. L'organisation a été très efficace et a créé une atmosphère dans laquelle la communication mathématique était stimulée. Le professeur GELFAND a reçu un diplôme de docteur honoris causa de l'Université de Lyon I lors de la séance de clôture du séminaire. Sa participation au séminaire, la participation simultanée de trois mathématiciens soviétiques émigrés parmi les plus éminents (Victor KAC du Massachusetts Institute of Technology de Boston, U.S.A., Ilya PIATETSKII-SHAPIRO de Tel-Aviv et Mikhail GROMOV de l'Institut des Hautes Études Scientifiques de Bures-sur-Yvette) ainsi que celle du physicien mathématicien polonais Andrzej TRAUTMAN de Varsovie ont élevé le séminaire au-delà du niveau d'une rencontre entre les écoles américaine et française à celui d'un événement mathématique réellement international.

4

Le séminaire n'a été rendu possible que par le soutien de la National Science Foundation, du Centre National de la Recherche Scientifique, de l'American Mathematical Society et de la Société mathématique de France, ainsi que celui de l'Université Claude-Bernard (Lyon I), des villes de Lyon et Villeurbanne et du conseil général du Rhône. Nous espérons que ces institutions trouveront dans ce volume une preuve concrète du bien-fondé de leur effort!

Special Issue in Astérisque en 1984 :

Elie Cartan et les mathématiques d'aujourd'hui

proceedings of ENS Lyon Colloquium from 25th to 29th June 1984 (200 attendees from Mathematics & Physics) .

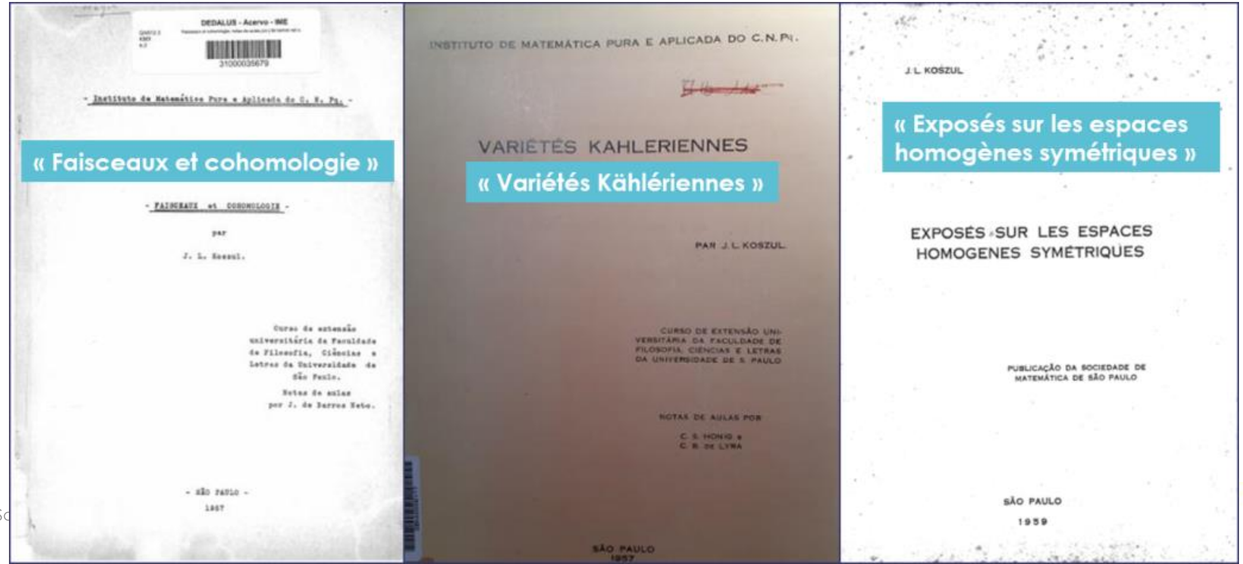
Among Speakers: **Jean-Louis Koszul et Jean-Marie Souriau**

Jean-Louis KOSZUL in Sao-Paulo

Jean-Louis Koszul was foreign member of São Paulo Academia of Sciences

Jean Louis Koszul Lectures at Sao Paulo:

- Faisceaux et Cohomologie
- Variétés Kählériennes
- Exposés sur les espaces homogènes symétriques



Koszul Works and links with Souriau Work

- In 1986, "**Introduction to symplectic geometry**“ book following a Chinese Koszul course in China (translated into English by Springer in 2019).
- This book takes up and develops works of Jean-Marie Souriau on homogeneous symplectic manifolds. Chuan Yu Ma writes in a review, on this book in Chinese, that "***This work coincided with developments in the field of analytical mechanics. Many new ideas have also been derived using a wide variety of notions of modern algebra, differential geometry, Lie groups, functional analysis, differentiable manifolds, and representation theory. [Koszul's book] emphasizes the differential-geometric and topological properties of symplectic manifolds. It gives a modern treatment of the subject that is useful for beginners as well as for experts***".

Table of Contents

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Koszul Book on Souriau Work

Jean-Louis Koszul · Yiming Zou

Introduction to Symplectic Geometry

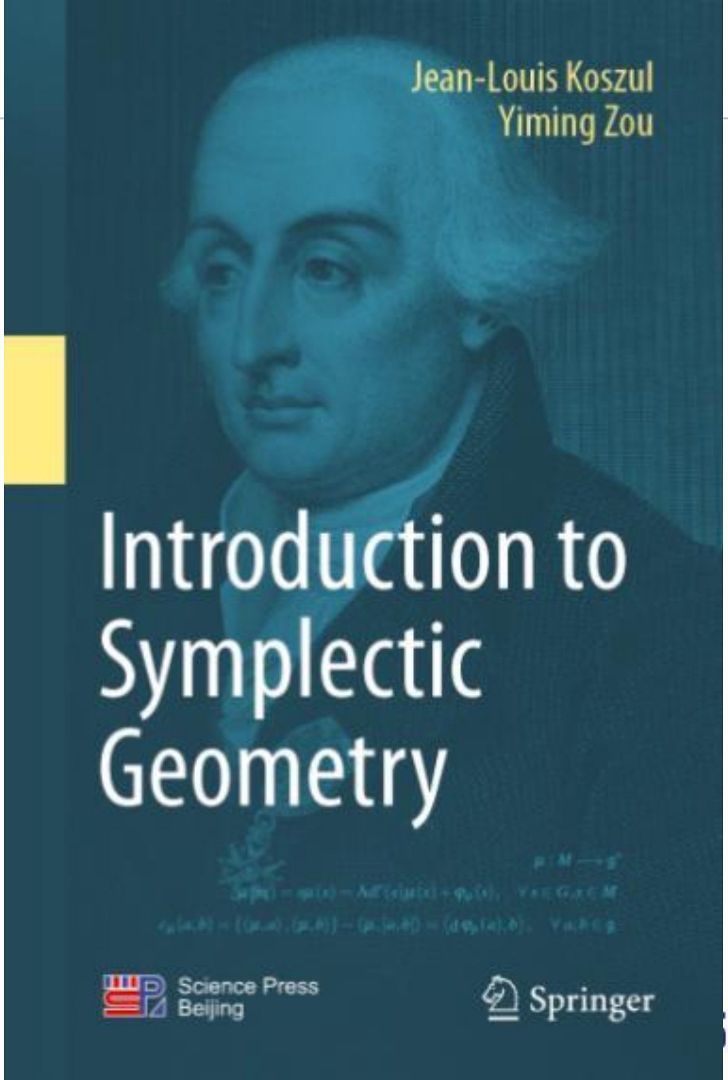
Forewords by Michel Nguiffo Boyom, Frédéric Barbaresco and Charles-Michel Marle

This introductory book offers a unique and unified overview of symplectic geometry, highlighting the differential properties of symplectic manifolds. It consists of six chapters: Some Algebra Basics, Symplectic Manifolds, Cotangent Bundles, Symplectic G-spaces, Poisson Manifolds, and A Graded Case, concluding with a discussion of the differential properties of graded symplectic manifolds of dimensions (o,n). It is a useful reference resource for students and researchers interested in geometry, group theory, analysis and differential equations.

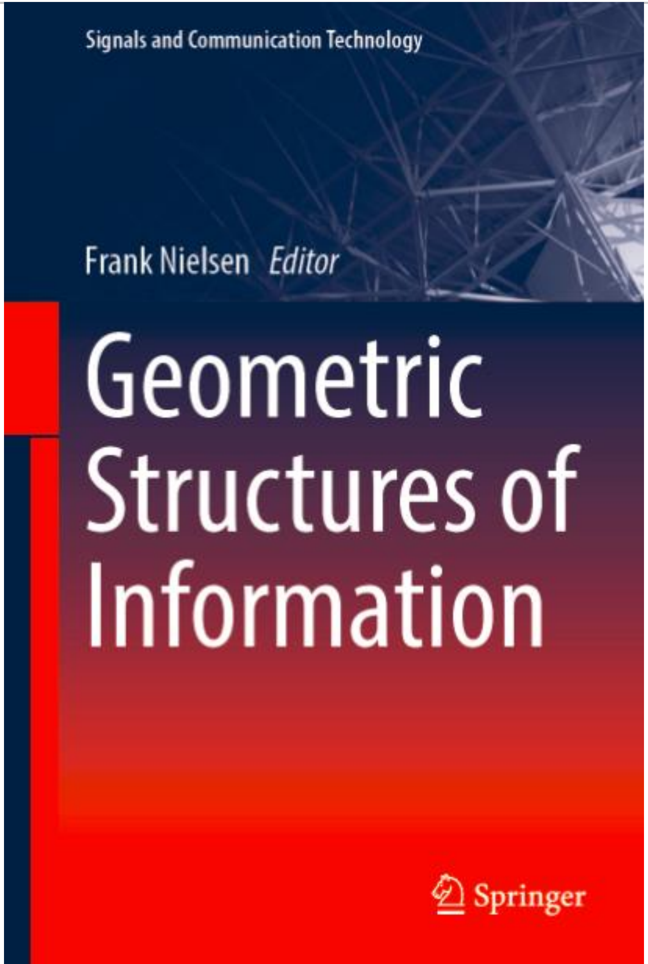
$$\mu : M \longrightarrow \mathfrak{g}^*$$

$$\mu(sx) = s\mu(x) = \mathrm{Ad}^*(s)\mu(x) + \varphi_\mu(s), \quad \forall s \in G, x \in M$$

$$c_\mu(a,b) = \{\langle\mu,a\rangle,\langle\mu,b\rangle\} - \langle\mu,[a,b]\rangle = \langle d\varphi_\mu(a), b\rangle, \quad \forall a,b \in \mathfrak{g}$$



Geometric Structures of Information, SPRINGER



Geometric Structures of Information

- https://www.springer.com/us/book/9783030025199

Paper on Jean-Louis Koszul

- Barbaresco, F. , Jean-Louis Koszul and the Elementary Structures of Information Geometry, Geometric Structures of Information, pp 333-392, SPRINGER, 2018
- https://link.springer.com/chapter/10.1007%2F978-3-030-02520-5_12

OPEN

Tribute to Jean-Louis Koszul

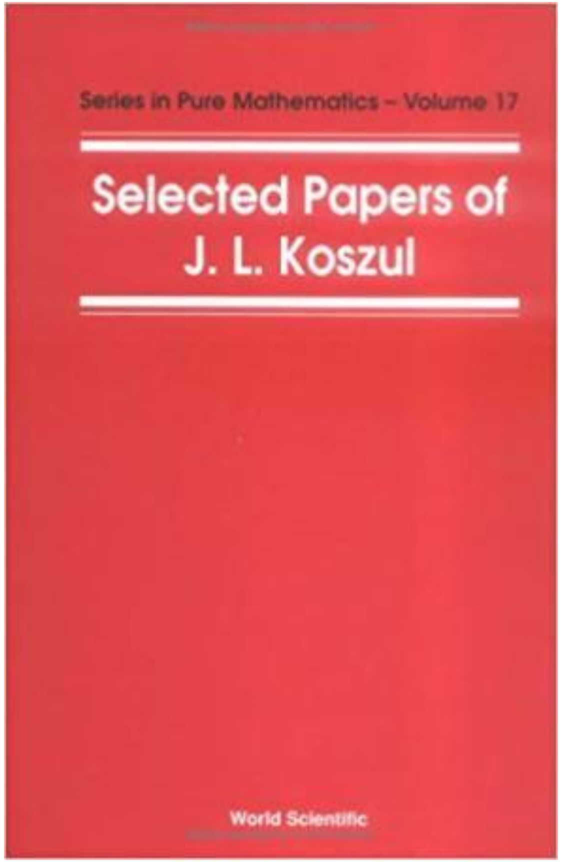
Jean-Louis Koszul and the elementary structures of Information Geometry

- Koszul has introduced fundamental tools to characterize the geometry of sharp convex cones, as Koszul-Vinberg characteristic Function, Koszul Forms, and affine representation of Lie Algebra and Lie Group.
- The 2nd Koszul form is linked to an extension of classical Fisher metric.
- Koszul theory of hessian structures and Koszul forms could be considered as main foundation and pillars of Information Geometry.

Affine Transformation Groups, Flat Manifolds & Invariant Forms convexity

Koszul works on Homogeneous Bounded Domains

- In the book "***Selected papers of JL Koszul***", Koszul summarizes his work on homogeneous bounded domains: "***It is with the problem of the determination of the homogeneous bounded domains posed by E. Cartan around 1935 that are related [my papers]. The idea of approaching the question through invariant Hermitian forms already appears explicitly in Cartan. This leads to an algebraic approach which constitutes the essence of Cartan's work and which, with the Lie J-algebras, was pushed much further by the Russian School. It is the work of Piatetski Shapiro on the Siegel domains, then those of E.B. Vinberg on the homogeneous cones that led me to the study of the affine transformation groups of the locally flat manifolds and in particular to the convexity criteria related to invariant forms***".



Koszul's papers

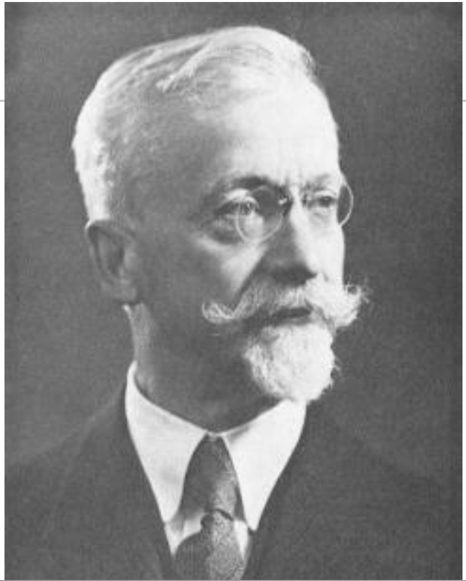
Koszul's paper at the foundation of the elementary structure of Information

- Koszul J.L., Sur la forme hermitienne canonique des espaces homogènes complexes. Can. J. Math., n°7, 562–576, 1955
- Koszul J.L., Exposés sur les Espaces Homogènes Symétriques; Publicação da Sociedade de Matematica de São Paulo: São Paulo, Brazil, 1959
- Koszul J.L., Domaines bornées homogènes et orbites de groupes de transformations affines, Bull. Soc. Math. France 89, pp. 515-533., 1961
- Koszul J.L., Ouverts convexes homogènes des espaces affines. Math. Z., n°79, 254–259, 1962
- Koszul J.L. Sous-groupes discrets des groupes de transformations affines admettant une trajectoire convexe, C.R. Acad. Sc. T.259, pp.3675-3677, 1964
- Koszul J.L., Variétés localement plates et convexité. Osaka. J. Math., n°2, 285–290, 1965
- Koszul J.L, Lectures on Groups of Transformations, Tata Institute of Fundamental Research, Bombay, 1965
- Koszul J.L., Déformations des variétés localement plates, .Ann Inst Fourier, n°18 , 103-114., 1968
- Koszul J.L., Trajectoires Convexes de Groupes Affines Unimodulaires. In Essays on Topology and Related Topics; Springer: Berlin, Germany, pp. 105–110, 1970
- Selected Papers of J L Koszul, Series in Pure Mathematics, Volume 17, World Scientific Publ, 1994

« La source » of Koszul inspiration: Elie Cartan

1935 Elie Cartan paper

- J.L. Koszul source of inspiration is given in this last sentence of Elie Cartan's 1935 paper where we can read in the last sentence:
- *"It is clear that if one could demonstrate that all homogeneous domains whose form* $\Phi = \sum_{i,j} \frac{\partial^2 \log K(z, z^*)}{\partial z_i \partial z_j^*} dz_i dz_j^*$ *is positive definite are symmetric, the whole theory of homogeneous bounded domains would be elucidated. This is a problem of Hermitian geometry certainly very interesting*" – Elie Cartan 1935
  - Cartan E., Sur les invariants intégraux de certains espaces homogènes clos et les propriétés topologiques de ces espaces, Ann. Soc. Pol. De Math., n°8, pp.181-225, 1929
  - Cartan E., Sur les domaines bornés de l'espace de n variables complexes, Abh. Math. Seminar Hamburg, vol. 1, p.116-162, 1935



$$\Phi = \sum \frac{\partial^2 \log K}{\partial z_i \partial \bar{z}_j} dz_i d\bar{z}_j$$

THÉORÈME. *Pour qu'un domaine homogène à $n \leq 3$ dimensions à groupe de stabilité clos soit équivalent à un domaine borné, il faut et il suffit que la forme différentielle Φ calculée, comme il a été dit plus haut, au moyen du groupe du domaine, soit définie positive.*

Il serait intéressant de savoir si ce théorème s'étend aux valeurs de n supérieures à 3.

Il est clair que si l'on parvenait à démontrer que tous les domaines homogènes dont la forme Φ est définie positive sont symétriques, toute la théorie des domaines bornés homogènes serait élucidée. C'est là un problème de géométrie hermitienne certainement très intéressant.

Filiation Poincaré/Cartan/Koszul

« Il est clair que si l'on parvenait à démontrer que tous les domaines homogènes dont la forme

$$\Phi = \sum_{i,j} \frac{\partial^2 \log K(z,\bar{z})}{\partial z_i \partial \bar{z}_j} dz_i d\bar{z}_j$$

est définie positive sont symétriques, toute la théorie des domaines bornés homogènes serait élucidée. C'est là un problème de géométrie hermitienne certainement très intéressant »

Dernière phrase de Elie Cartan, dans « Sur les domaines bornés de l'espace de n variables complexes », Abh. Math. Seminar Hamburg, 1935

Carl Ludwig Siegel
(Siegel space of 1st king and Symplectic Geometry)

Structure of Information Geometry (Koszul Hessian Geometry)

Lookeng Hua
(Bergman Kernel, Cauchy and Poisson of Siegel Domains

Ernest Vinberg
(Siegel Domains of 2nd kind)

Henri Poincaré
(half-plane) n=1

Elie Cartan
(classification in 6 classes of symmetric homogeneous bounded domains) n<=3

Jean-Louis Koszul
(hermitian canonical forms of complex homogeneous spaces, a complex homogenesous space with positive definite hermitian canonical form is isomorphe to a bounded domain, study of affine transformation groups of locally flat manifolds)

Koszul filiation with Elie Cartan

- “[***Détecter l'origine d'une notion ou la première apparition d'un résultat est souvent difficile. Je ne suis certainement pas le premier à avoir utilisé des représentations affines de groupes ou d'algèbres de Lie. On peut effectivement imaginer que cela se trouve chez Elie Cartan, mais je ne puis rien dire de précis. A propos d'Elie Cartan: je n'ai pas été son élève. C'est Henri Cartan qui a été mon maître pendant mes années de thèse. En 1941 ou 42 j'ai entendu une brève série de conférences données par Elie à l'Ecole Normale et ce sont des travaux d'Elie qui ont été le point de départ de mon travail de thèse***.] *There are many things that I would like to understand (too much perhaps!), If only the relationship between what I did and the work of Souriau. Detecting the origin of a notion or the first appearance of a result is often difficult. I am certainly not the first to have used affine representations of groups or Lie algebras. We can imagine that it is at Elie Cartan, but I cannot say anything specific. About Elie Cartan: I was not his student. It was Henri Cartan who was my master during my years of thesis. In 1941 or 42, I heard a brief series of lectures given by Elie at the Ecole Normale and it was Elie's work that was the starting point of my thesis work".*

Koszul's main papers related tothe elementary structures of information geometry

1955 « Sur la forme hermitienne canonique des espaces homogènes complexes »

- Koszul considers the Hermitian structure of a homogeneous G/B manifold (G related Lie group and B a closed subgroup of G, associated, up to a constant factor, to the single invariant G, and to the invariant complex structure by the operations of G).
- Koszul says "***The interest of this form for the determination of homogeneous bounded domains has been emphasized by Elie Cartan: a necessary condition for G/B to be a bounded domain is indeed that this form is positive definite***". Koszul calculated this canonical form from infinitesimal data Lie algebra of G, the sub-algebra corresponding to B and an endomorphism algebra defining the invariant complex structure of G/B. The results obtained by Koszul proved that the homogeneous bounded domains whose group of automorphisms is semi-simple are bounded symmetric domains in the sense of Elie Cartan. Koszul also refers to André Lichnerowicz's work on Kählerian homogeneous spaces. In this seminal paper, Koszul also introduced a left invariant form of degree 1 on G:

$$\Psi(X)=Tr_{g/b}\left[ad(JX)-J.ad(X)\right] \quad \forall X \in g$$

- with J an endomorphism of the Lie algebra space and the trace $Tr_{g/b}[\cdot]$ corresponding to that of the endomorphism *g/b*. The Kähler form of the canonical Hermitian form is given by the differential of $-\frac{1}{4}\Psi(X)$ of this form of degree 1.

Koszul's main papers, related to the elementary structures of information geometry

Koszul Forms for Homogeneous Bounded domains

- Koszul has developed his previously described theory for Homogenous Siegel Domains SD. He has proved that there is a subgroup G in the group of the complex affine automorphisms of these domains (Iwasawa subgroup), such that G acts on SD simply transitively. The Lie algebra $\mathfrak{g}$ of G has a structure that is an algebraic translation of the Kähler structure of SD.
- There is an integrable almost complex structure J on $\mathfrak{g}$ and there exists $\eta \in \mathfrak{g}^*$ such that $\langle X, Y \rangle_\eta = \langle [JX, Y], \eta \rangle$ defines a J-invariant positive definite inner product on $\mathfrak{g}$.

  Koszul has proposed as admissible form $\eta \in \mathfrak{g}^*$, the form ξ:

$$\Psi(X) = \langle X, \xi \rangle = Tr[ad(JX) - J.ad(X)] \quad \forall X \in \mathfrak{g}$$

- Koszul has proved that $\langle X, Y \rangle_\xi$ coincides, up to a positive number multiple with the real part of the Hermitian inner product obtained by the Bergman metric of SD by identifying $\mathfrak{g}$ with the tangent space of SD. The Koszul forms are then given by:

$$\alpha = -\frac{1}{4} d\Psi(X) \qquad \beta = D\alpha$$

Koszul's main papers, related to the elementary structures of information geometry

1959 « Exposés sur les espaces homogènes symétriques »

- It is a Lecture written as part of a seminar held in September and October 1958 at the University of Sao Paulo
- Koszul showed that any symmetric bounded domain is a direct product of irreducible symmetric bounded domains, determined by Elie Cartan (4 classes corresponding to classical groups and 2 exceptional domains).
- For the study of irreducible symmetric bounded domains, Koszul refered to Elie Cartan, Carl-Ludwig Siegel and Loo-Keng Hua.
- Koszul illustrated the subject with two particular cases, the half-plane of Poincaré and the half-space of Siegel, and showed that with its trace formula of endomorphism g/b, he found that the canonical Kähler hermitian form and the associated metrics are the same as those introduced by Henri Poincaré and Carl-Ludwig Siegel (who introduced them as invariant metric under action of the automorphisms of these spaces).

Koszul's main papers, related to the elementary structures of information geometry

1961 « Domaines bornées homogènes et orbites de groupes de transformations affines »

- It is written by Koszul at the Institute for Advanced Study at Princeton during a stay funded by the National Science Foundation.
- On a complex homogeneous space, an invariant volume defines with the complex structure the canonical invariant Hermitian form. If the homogeneous space is holomorphically isomorphic to a bounded domain of a space C^n, this Hermitian form is positive definite because it coincides with the Bergmann metric of the domain.
- Koszul demonstrated in this article the reciprocal of this proposition for a class of complex homogeneous spaces. This class consists of some open orbits of complex affine transformation groups and contains all homogeneous bounded domains. Koszul addressed again the problem of knowing if a complex homogeneous space, whose canonical Hermitian form is positive definite is isomorphic to a bounded domain, but via the study of the invariant bilinear form defined on a real homogeneous space by an invariant volume and an invariant flat connection.

Koszul's main papers, related to the elementary structures of information geometry

1961 « Domaines bornées homogènes et orbites de groupes de transformations affines »

- Koszul demonstrated that if this bilinear form is positive definite then the homogeneous space with its flat connection is isomorphic to a convex open domain containing no straight line in a real vector space and extended it to the initial problem for the complex homogeneous spaces obtained in defining a complex structure in the variety of vectors of a real homogeneous space provided with an invariant flat connection.
- It is in this article that Koszul used the affine representation of Lie groups and algebras. By studying the open orbits of the affine representations, he introduced an affine representation of G, written $(\mathbf{f},\mathbf{q})$, and the following equation setting f the linear representation of the Lie algebra $\mathfrak{g}$ of G, defined by $\mathbf{f}$ and q the restriction to $\mathfrak{g}$ and the differential of $\mathbf{q}$ (f and q are differential respectively of $\mathbf{f}$ and $\mathbf{q}$):

$$f(X)q(Y)-f(Y)q(X)=q\left([X,Y]\right)\quad \forall X,Y\in\mathfrak{g}$$

$$\text{with}\quad f:\mathfrak{g}\to gl(E)\quad\text{and}\quad q:\mathfrak{g}\mapsto E$$

Koszul's main papers, related to the elementary structures of information geometry

1962 « Ouverts convexes homogènes des espaces affines »

- Koszul is interested in this paper by the structure of the convex open non-degenerate Ω (with no straight line) and homogeneous (the group of affine transformations of E leaving stable Ω operates transitively in Ω) in a real affine space of finite dimension.
- Koszul demonstrated that they can be all deduced from non-degenerate and homogeneous convex open cones built previously.
- He used for this the properties of the group of affine transformations leaving stable a non-degenerate convex open domain and an homogeneous domain.

Koszul's main papers, related to the elementary structures of information geometry

1965 « Variétés localement plates et convexité »

- Koszul established the following **Koszul's theorem**:
- Let M *be* a locally related differentiable manifold. If the universal covering of M is isomorphic as a flat manifold with a convex open domain containing no straight line in a real affine space, then there exists on M a closed differential form α such that $D\alpha$ *(D* linear covariant derivative of zero torsion) is positive definite in all respects and which is invariant under every automorphism of M.
- If G is a group of automorphisms of M such that G\M is quasi-compact and if there exists on M a closed 1-differential form α invariant by G and such that $D\alpha$ is positive definite at any point, then the universal covering of M is isomorphic as a flat manifold with a convex open domain that does not contain a straight line in a real affine space.

1965 « Lectures on Groups of Transformations »

- This is lecture notes given by Koszul at Bombay "Tata Institute of Fundamental Research" on transformation groups.
- In particular in Chapter 6, Koszul studied discrete linear groups acting on convex open cones in vector spaces based on the work of C.L. Siegel (work on quadratic forms).
- Koszul used what we will call in the following Koszul-Vinberg characteristic function on convex sharp cone.

THALES

Koszul's main papers, related to the elementary structures of information geometry

1968 « Déformations des variétés localement plates »

- Koszul provided other proofs of theorems introduced previously.
- Koszul considered related differentiable manifolds of dimension n and TM the fibered space of M. The linear connections on M constitute a subspace of the space of the differentiable applications of the TMxTM fiber product in the space T(TM) of the TM vectors.
- Any locally flat connection D (the curvature and the torsion are zero) defines a locally flat connection on the covering of M, and is hyperbolic when universal covering of M, with this connection, is isomorphic to a sharp convex open domain (without straight lines) in R^n.
- Koszul showed that, if M is a compact manifold, for a locally flat connection on M to be hyperbolic, it is necessary and sufficient that there exists a closed differential form of degree 1 on M whose covariant differential is positive definite.

1970 « Trajectoires Convexes de Groupes Affines Unimodulaires »

- Koszul demonstrated that a convex sharp open domain in R^n that admits a unimodular transitive group of affine automorphisms is an auto-dual cone.
- This is a more geometric demonstration of the results shown by Ernest Vinberg on the automorphisms of convex cones.

The Geometry of Hessian Geometry and Koszul forms

The Elementary Structures:

- Codazzi Structure (D,g), *D* is a connexion without torsion: $(D_X g)(Y,Z)=(D_Y g)(X,Z)$
- Hessian structure: (D,g) Codazzi with *D* is flat => dual structure (D',g) with : $D'=\nabla-D$ (with ∇ la Levi-Civita connexion)
- For a hessian structure (D,g) with $g=Dd\varphi$, $g=D'd\varphi'$ and the dual Codazzi structure (D',g) is also a hessian structure
- We have the property that φ' is the Legendre transform of φ : $\varphi'=\sum_i x^i \frac{\partial\varphi}{\partial x^i}-\varphi$
- A hessian structure (D,g) is a Koszul structure, if there is a closed 1-form ω such that $g=D\omega$
- Koszul has introduced a 2-form, that plays same role than Ricci tensor for a kahlerian metric: $\gamma=D\alpha$ with1-form α , such that $D_X\upsilon=\alpha(X)\upsilon$ with volume element υ , and for (D',g) : $\alpha'=-\alpha$ and $\gamma'=\gamma-2\nabla\alpha$
- For an homogeneous regular convex cone Ω , the hessian structure (D,g) is given by $g=Dd\psi$ with Koszul forms $\alpha=d\log\psi$ and $\gamma=g$ Volume element υ is invariant under the action of automorphisms of Ω .

Koszul-Vinberg Characteristic Function/Metric of convex cone

- J.L. Koszul and E. Vinberg have introduced an affinely invariant Hessian metric on a sharp convex cone through its characteristic function.

- Ω is a sharp open convex cone in a vector space E of finite dimension on R (a convex cone is sharp if it does not contain any full straight line).

- Ω^* is the dual cone of Ω and is a sharp open convex cone.

- Let $d\xi$ the Lebesgue measure on E^* dual space of E, the following integral:

$$\psi_\Omega(x) = \int_{\Omega^*} e^{-\langle \xi, x \rangle} d\xi \quad \forall x \in \Omega$$

is called the **Koszul-Vinberg characteristic function**

Koszul-Vinberg Characteristic Function/Metric of convex cone

- Koszul-Vinberg Metric : $g = d^2 \log \psi_\Omega$

$$d^2 \log \psi(x) = d^2 \left[\log \int \psi_u du \right] = \frac{\int \psi_u d^2 \log \psi_u du}{\int \psi_u du} + \frac{1}{2} \frac{\iint \psi_u \psi_v (d \log \psi_u - d \log \psi_v)^2 dudv}{\iint \psi_u \psi_v dudv}$$

- We can define a diffeomorphism by: $x^* = -\alpha_x = -d \log \psi_\Omega(x)$

with $\langle df(x), u \rangle = D_u f(x) = \left. \frac{d}{dt} \right|_{t=0} f(x+tu)$

- When the cone Ω is symmetric, the map $x^* = -\alpha_x$ is a bijection and an isometry with a unique fixed point (the manifold is a Riemannian Symmetric Space given by this isometry):

$$(x^*)^* = x \quad \langle x, x^* \rangle = n \qquad \psi_\Omega(x) \psi_{\Omega^*}(x^*) = cste$$

- x^* is characterized by $x^* = \arg\min \left\{ \psi(y) / y \in \Omega^*, \langle x, y \rangle = n \right\}$

- x^* is the center of gravity of the cross section $\left\{ y \in \Omega^*, \langle x, y \rangle = n \right\}$ of Ω^* :

$$x^* = \int_{\Omega^*} \xi . e^{-\langle \xi, x \rangle} d\xi \Big/ \int_{\Omega^*} e^{-\langle \xi, x \rangle} d\xi$$

THALES

Koszul metric & Fisher Metric

- To make the link with Fisher metric given by matrix $I(x)$, we can observe that the second derivative of $\log p_x(\xi)$ is given by:

$$\log p_x(\xi) = -\Phi^*(\xi) = \Phi(x) - \langle x, \xi \rangle$$

$$\frac{\partial^2 \log p_x(\xi)}{\partial x^2} = \frac{\partial^2 \left[\Phi(x) - \langle x, \xi \rangle\right]}{\partial x^2} = \frac{\partial^2 \Phi(x)}{\partial x^2}$$

$$\Rightarrow I(x) = -E_\xi\left[\frac{\partial^2 \log p_x(\xi)}{\partial x^2}\right] = -\frac{\partial^2 \Phi(x)}{\partial x^2} = \frac{\partial^2 \log \psi_\Omega(x)}{\partial x^2}$$

- We could then deduce the close interrelation between Fisher metric and hessian of Koszul-Vinberg characteristic logarithm.

$$I(x) = -E_\xi\left[\frac{\partial^2 \log p_x(\xi)}{\partial x^2}\right] = \frac{\partial^2 \log \psi_\Omega(x)}{\partial x^2}$$

Koszul Metric and Fisher Metric as Variance

- We can also observed that the Fisher metric or hessian of KVCF logarithm is related to the variance of ξ :

$$\log \Psi_\Omega(x) = \log \int_{\Omega^*} e^{-\langle \xi, x\rangle} d\xi \Rightarrow \frac{\partial \log \Psi_\Omega(x)}{\partial x} = -\frac{1}{\int_{\Omega^*} e^{-\langle \xi, x\rangle} d\xi} \int_{\Omega^*} \xi . e^{-\langle \xi, x\rangle} d\xi$$

$$\frac{\partial^2 \log \Psi_\Omega(x)}{\partial x^2} = -\frac{1}{\left(\int_{\Omega^*} e^{-\langle \xi, x\rangle} d\xi\right)^2} \left[-\int_{\Omega^*} \xi^2 . e^{-\langle \xi, x\rangle} d\xi . \int_{\Omega^*} e^{-\langle \xi, x\rangle} d\xi + \left(\int_{\Omega^*} \xi . e^{-\langle \xi, x\rangle} d\xi \right)^2 \right]$$

$$\frac{\partial^2 \log \Psi_\Omega(x)}{\partial x^2} = \int_{\Omega^*} \xi^2 . \frac{e^{-\langle \xi, x\rangle}}{\int_{\Omega^*} e^{-\langle \xi, x\rangle} d\xi} d\xi - \left(\int_{\Omega^*} \xi . \frac{e^{-\langle \xi, x\rangle}}{\int_{\Omega^*} e^{-\langle \xi, x\rangle} d\xi} d\xi \right)^2 = \int_{\Omega^*} \xi^2 . p_x(\xi) d\xi - \left(\int_{\Omega^*} \xi . p_x(\xi) d\xi \right)^2$$

$$I(x) = -E_\xi \left[\frac{\partial^2 \log p_x(\xi)}{\partial x^2} \right] = \frac{\partial^2 \log \psi_\Omega(x)}{\partial x^2} = E_\xi\left[\xi^2\right] - E_\xi[\xi]^2 = Var(\xi)$$

J.L. Koszul and J. Vey have proved the following theorem:

- Koszul J.L., Variétés localement plates et convexité, Osaka J. Math.,n°2 ,p.285-290, 1965
- Vey J., Sur les automorphismes affines des ouverts convexes saillants, Annali della Scuola Normale Superiore di Pisa, Classe di Science, 3e série, tome 24,n°4, p.641-665, 1970

Koszul-Vey Theorem:

Let M be a connected Hessian manifold with Hessian metric g .
Suppose that admits a closed 1-form α such that $D\alpha = g$ and
there exists a group G of affine automorphisms of M preserving α:

- If M/G is quasi-compact, then the universal covering manifold of M is affinely isomorphic to a convex domain Ω real affine space not containing any full straight line.
- If M/G is compact, then Ω is a sharp convex cone.

- Koszul J.L., Variétés localement plates et convexité, Osaka J. Math. , n°2, p.285-290, 1965
- Vey J., Sur les automorphismes affines des ouverts convexes saillants, Annali della Scuola Normale Superiore di Pisa, Classe di Science, 3e série, tome 24,n°4, p.641-665, 1970

KOSZUL Works at Foundation of Statistical Physics & Information Geometry

THERMODYNAMICS
- ➔ Clausius/Boltmann Entropy
- ➔ Massieu Functions
- ➔ Gibbs-Duhem Potentials
- ➔ Gibbs Density
- ➔ Capacities

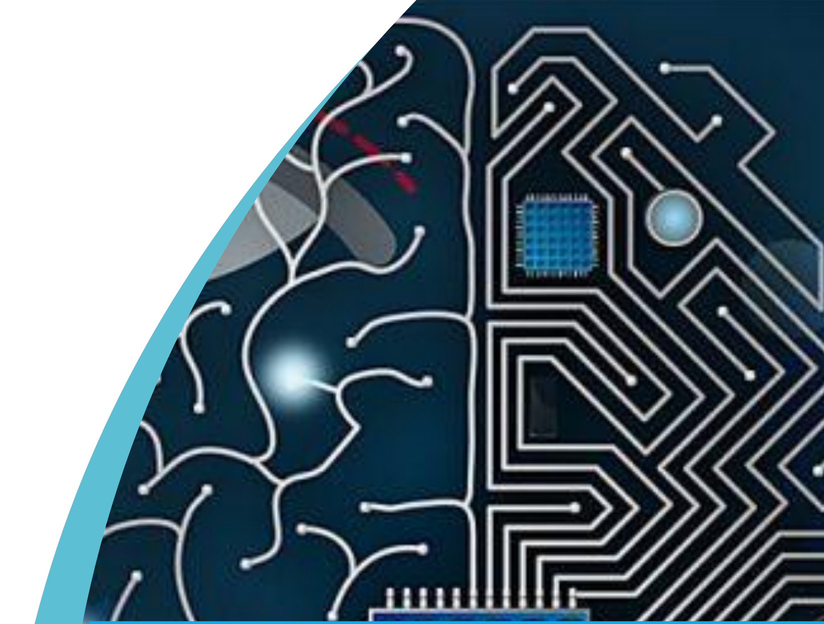
STATISICAL PHYSICS
- ➔ Legendre Structure
- ➔ Contact/Symplectic models
- ➔ Quantum Fisher-Balian metric

INFORMATION GEOMETRY
- ➔ Clairaut-Legendre Transform
- ➔ Fisher Information Metric
- ➔ Natural Gradient

LIE GROUP & COHOMOLOGY
- ➔ Symplectic Geometry
- ➔ Souriau Moment Map
- ➔ Lie Group Thermodynamics
- ➔ Kirillov Representation
- ➔ KKS 2-Form

THALES

Fisher-Koszul-Souriau Metric and Geometric Structures of Inference and Learning



ENS Ulm 1940

ENS Ulm 1942

Fisher Metric and Fréchet-Darmois (Cramer-Rao) Bound

- Cramer-Rao –Fréchet-Darmois Bound has been introduced by Fréchet in 1939 and by Rao in 1945 as inverse of the Fisher Information Matrix: $I(\theta)$

$$R_{\hat{\theta}} = E\left[\left(\theta-\hat{\theta}\right)\left(\theta-\hat{\theta}\right)^{+}\right] \geq I(\theta)^{-1} \qquad \left[I(\theta)\right]_{i,j} = -E\left[\frac{\partial^2 \log p_\theta(z)}{\partial\theta_i \partial\theta_j^*}\right]$$

- Rao has proposed to introduced an invariant metric in parameter space of density of probabilities (axiomatised by N. Chentsov):

$$ds_\theta^2 = Kullback\_Divergence\left(p_\theta(z), p_{\theta+d\theta}(z)\right)$$

$$ds_\theta^2 = -\int p_\theta(z)\log\frac{p_{\theta+d\theta}(z)}{p_\theta(z)}dz$$

$$w = W(\theta) \Rightarrow ds_w^2 = ds_\theta^2$$

$$ds_\theta^2 \underset{Taylor}{\approx} \sum_{i,j} g_{ij}\, d\theta_i d\theta_j^* = \sum_{i,j}\left[I(\theta)\right]_{i,j} d\theta_i d\theta_j^* = d\theta^+ . I(\theta) . d\theta$$

Distance Between Gaussian Density with Fisher Metric

Fisher Matrix for Gaussian Densities:

$$I(\theta)=\begin{bmatrix}\dfrac{1}{\sigma^2} & 0\\ 0 & \dfrac{2}{\sigma^2}\end{bmatrix}\quad \text{avec}\quad E\left[\left(\theta-\hat{\theta}\right)\left(\theta-\hat{\theta}\right)^T\right]\geq I(\theta)^{-1}\quad \text{et}\quad \theta=\begin{pmatrix}m\\ \sigma\end{pmatrix}$$

- Fisher matrix induced the following differential metric :

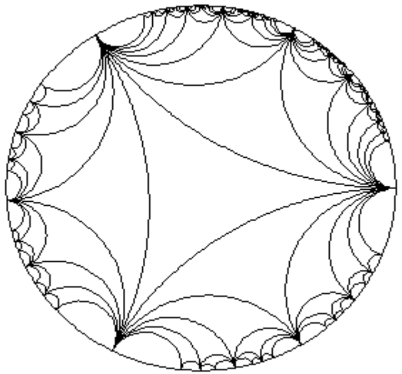
$$ds^2=d\theta^T.I(\theta).d\theta=\frac{dm^2}{\sigma^2}+2.\frac{d\sigma^2}{\sigma^2}=\frac{2}{\sigma^2}\left[\left(\frac{dm}{\sqrt{2}}\right)^2+(d\sigma)^2\right]$$

- Poincaré Model of upper half-plane and unit disk

$$z=\frac{m}{\sqrt{2}}+i.\sigma\qquad \omega=\frac{z-i}{z+i}\quad \left(|\omega|<1\right)$$

$$\Rightarrow ds^2=8.\frac{|d\omega|^2}{\left(1-|\omega|^2\right)^2}$$

La géométrie paramètres des gaussiennes est la géométrie du disque de Poincaré



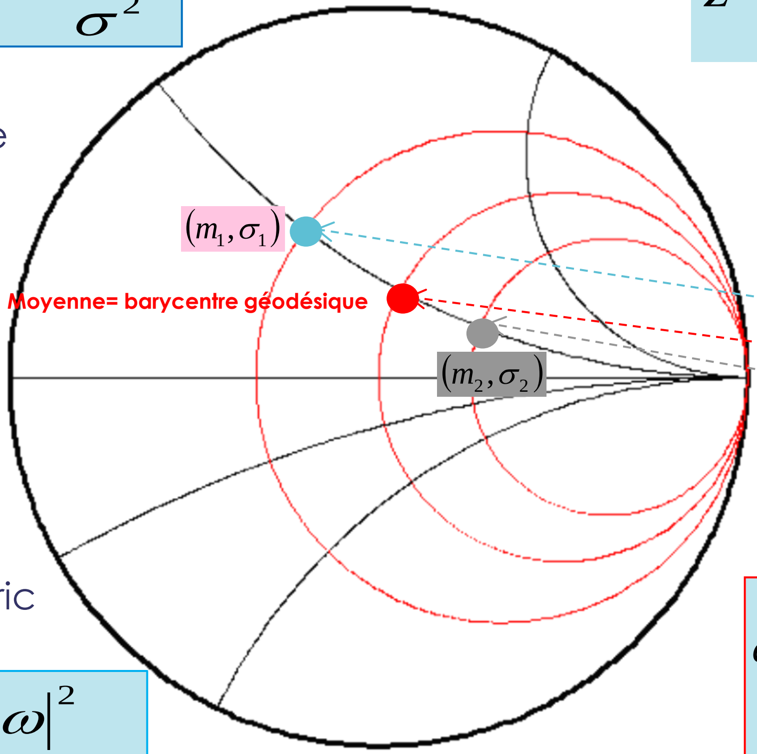
1 monovariate gaussian = 1 point in Poincaré unit disk

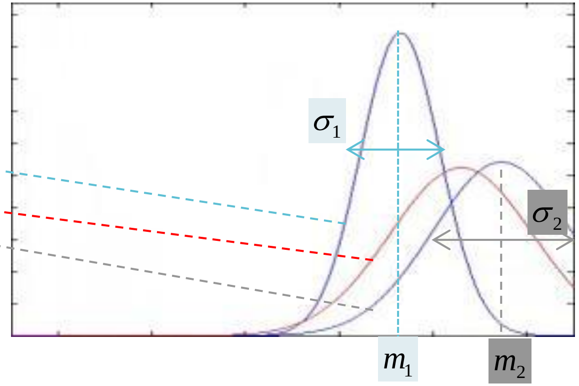
$$ds^2 = \frac{dm^2}{\sigma^2} + 2.\frac{d\sigma^2}{\sigma^2}$$

Fisher Metric in
Poincaré Half-Plane

$$z = \frac{m}{\sqrt{2}} + i.\sigma$$

$$\omega = \frac{z-i}{z+i} \quad (|\omega|<1)$$





Poincaré-Fisher metric
In Unit Disk

$$ds^2 = 8.\frac{|d\omega|^2}{\left(1-|\omega|^2\right)^2}$$

$$d^2\left(\{m_1,\sigma_1\},\{m_2,\sigma_2\}\right) = 2.\left(\log\frac{1+\delta(\omega^{(1)},\omega^{(2)})}{1-\delta(\omega^{(1)},\omega^{(2)})}\right)^2$$

$$\text{with} \quad \delta(\omega^{(1)},\omega^{(2)}) = \left|\frac{\omega^{(1)}-\omega^{(2)}}{1-\omega^{(1)}\omega^{(2)*}}\right|$$

Information Geometry & Natural Gradient

- This simple gradient descent has a first drawback of using the same non-adaptive learning rate for all parameter components, and a second drawback of non invariance with respect to parameter re-encoding inducing different learning rates. **S.I. Amari** has introduced the **natural gradient** to preserve this invariance to be insensitive to the characteristic scale of each parameter direction. The gradient descent could be corrected by $I(\theta)^{-1}$ where I is the **Fisher information matrix** with respect to parameter θ , given by:

$$\theta_t \leftarrow \theta_{t-1} - \eta_t I(\theta_{t-1})^{-1} \frac{\partial l_t(y_t)^T}{\partial \theta}$$

$$I(\theta) = [g_{ij}]$$

$$\text{with } g_{ij} = \left[-E_{y \sim p(y/\theta)} \left[\frac{\partial^2 \log p(y/\theta)}{\partial \theta_i \partial \theta_j} \right] \right]_{ij} = \left[E_{y \sim p(y/\theta)} \left[\frac{\partial \log p(y/\theta)}{\partial \theta_i} \frac{\partial \log p(y/\theta)}{\partial \theta_j} \right] \right]_{ij}$$

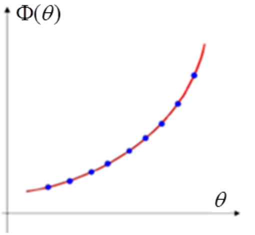
Legendre Transform, Dual Potentials & Fisher Metric

- **S.I. Amari** has proved that the Riemannian metric in an exponential family is the **Fisher information matrix** defined by:

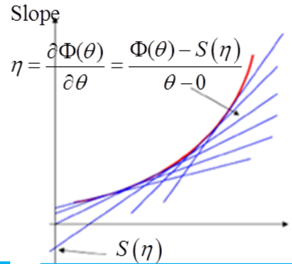
$$g_{ij} = -\left[\frac{\partial^2 \Phi}{\partial \theta_i \partial \theta_j}\right]_{ij} \quad \text{with} \quad \Phi(\theta) = -\log \int_{\mathbb{R}} e^{-\langle \theta, y\rangle} dy$$

- and the dual potential, the **Shannon entropy**, is given by the **Legendre transform**:

$$S(\eta) = \langle \theta, \eta \rangle - \Phi(\theta) \quad \text{with} \quad \eta_i = \frac{\partial \Phi(\theta)}{\partial \theta_i} \quad \text{and} \quad \theta_i = \frac{\partial S(\eta)}{\partial \eta_i}$$



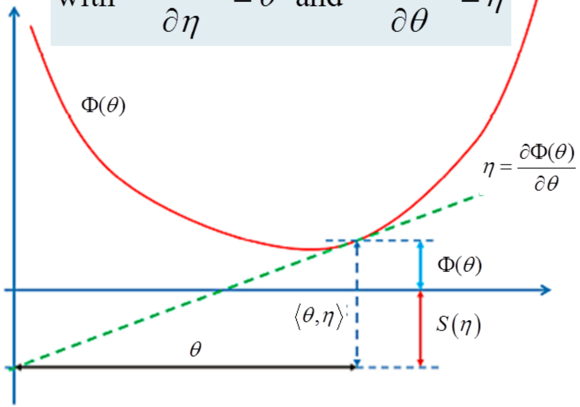
Classical Geometry
(curve is given by a continuum of points)



Plücker Geometry
(curve is given by the envelop of its tangents)

$$S(\eta) = \langle \theta, \eta \rangle - \Phi(\theta)$$

$$\text{with} \quad \frac{\partial S(\eta)}{\partial \eta} = \theta \quad \text{and} \quad \frac{\partial \Phi(\theta)}{\partial \theta} = \eta$$



Fisher Metric and Koszul 2 form on sharp convex cones

Koszul-Vinberg Characteristic Function, Koszul Forms

- **J.L. Koszul** and **E. Vinberg** have introduced an affinely invariant Hessian metric on a sharp convex cone through its **characteristic function**

$$\Phi_\Omega(\theta) = -\log \int_{\Omega^*} e^{-\langle \theta, y\rangle} dy = -\log \psi_\Omega(\theta) \quad \text{with } \theta \in \Omega \text{ sharp convex cone}$$

$$\psi_\Omega(\theta) = \int_{\Omega^*} e^{-\langle \theta, y\rangle} dy \quad \text{with Koszul-Vinberg Characteristic function}$$

Jean-Louis Koszul

- **1st Koszul form α**: $\alpha = d\Phi_\Omega(\theta) = -d\log \psi_\Omega(\theta)$
- **2nd Koszul form γ**: $\gamma = D\alpha = Dd\log \psi_\Omega(\theta)$

$$\left(Dd\log \psi_\Omega(x)\right)(u) = \frac{1}{\psi_\Omega(u)^2}\left[\int_{\Omega^*} F(\xi)^2 d\xi . \int_{\Omega^*} G(\xi)^2 d\xi - \left(\int_{\Omega^*} F(\xi).G(\xi) d\xi\right)^2\right] > 0 \text{ with } F(\xi) = e^{-\frac{1}{2}\langle x,\xi\rangle} \text{ and } G(\xi) = e^{-\frac{1}{2}\langle x,\xi\rangle}\langle u,\xi\rangle$$

- Diffeomorphism: $\eta = \alpha = -d\log \psi_\Omega(\theta) = \int_{\Omega^*} \xi p_\theta(\xi) d\xi$ with $p_\theta(\xi) = \dfrac{e^{-\langle \xi,\theta\rangle}}{\int_{\Omega^*} e^{-\langle \xi,\theta\rangle} d\xi}$
- Legendre transform: $S_\Omega(\eta) = \langle \theta, \eta\rangle - \Phi_\Omega(\theta)$ with $\eta = d\Phi_\Omega(\theta)$ and $\theta = dS_\Omega(\eta)$

Statistical Mechanics, Dual Potentials & Fisher Metric

- In geometric statistical mechanics, **J.M. Souriau** has developed a "**Lie groups thermodynamics**" of dynamical systems where the (maximum entropy) **Gibbs density is covariant** with respect to the action of the Lie group. In the Souriau model, previous structures of information geometry are preserved:

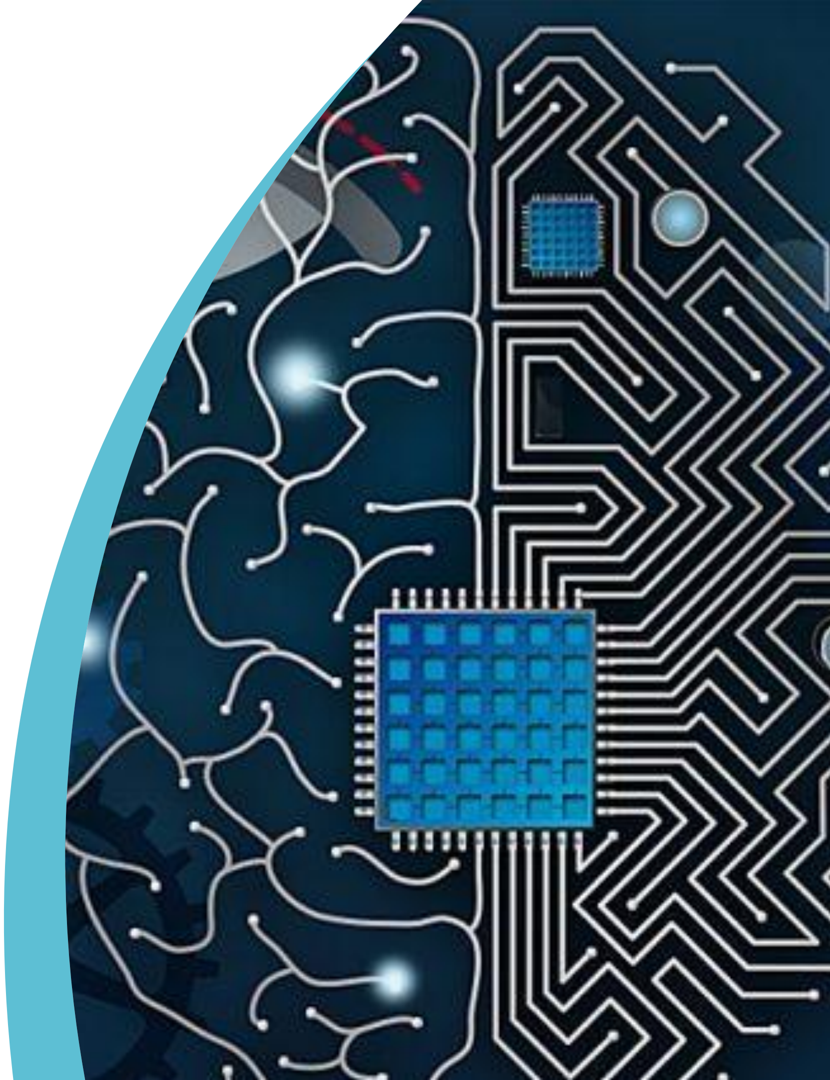
$$I(\beta) = -\frac{\partial^2 \Phi}{\partial \beta^2} \text{ with } \Phi(\beta) = -\log \int_M e^{-\langle \beta, U(\xi) \rangle} d\lambda \qquad U : M \to \mathfrak{g}^*$$

$$S(Q) = \langle \beta, Q \rangle - \Phi(\beta) \text{ with } Q = \frac{\partial \Phi(\beta)}{\partial \beta} \in \mathfrak{g}^* \text{ and } \beta = \frac{\partial S(Q)}{\partial Q} \in \mathfrak{g}$$


Jean-Marie Souriau

- In the Souriau **Lie groups thermodynamics** model, β is a "geometric" (Planck) temperature, element of Lie algebra $\mathfrak{g}$ of the group, and Q is a "geometric" heat, element of dual Lie algebra $\mathfrak{g}^*$ of the group.

Events



FGSI'19 Cartan-Koszul-Souriau Foundations of Geometric Structures of Information

https://fgsi2019.sciencesconf.org/
https://fgsi2019.sciencesconf.org/resource/page/id/5

OPEN

https://franknielsen.github.io/SPIG-LesHouches2020/

ÉCOLE DE PHYSIQUE DES HOUCHES

Joint Structures and Common Foundation of Statistical Physics, Information Geometry and Inference for Learning

26th July to 31st July 2020

> **Geometric Structures of Statistical Physics & Information**

- Statistical Mechanics and Geometric Mechanics
- Thermodynamics, Symplectic and Contact Geometries
- Lie groups Thermodynamics
- Relativistic and continuous media Thermodynamics
- Symplectic Integrators

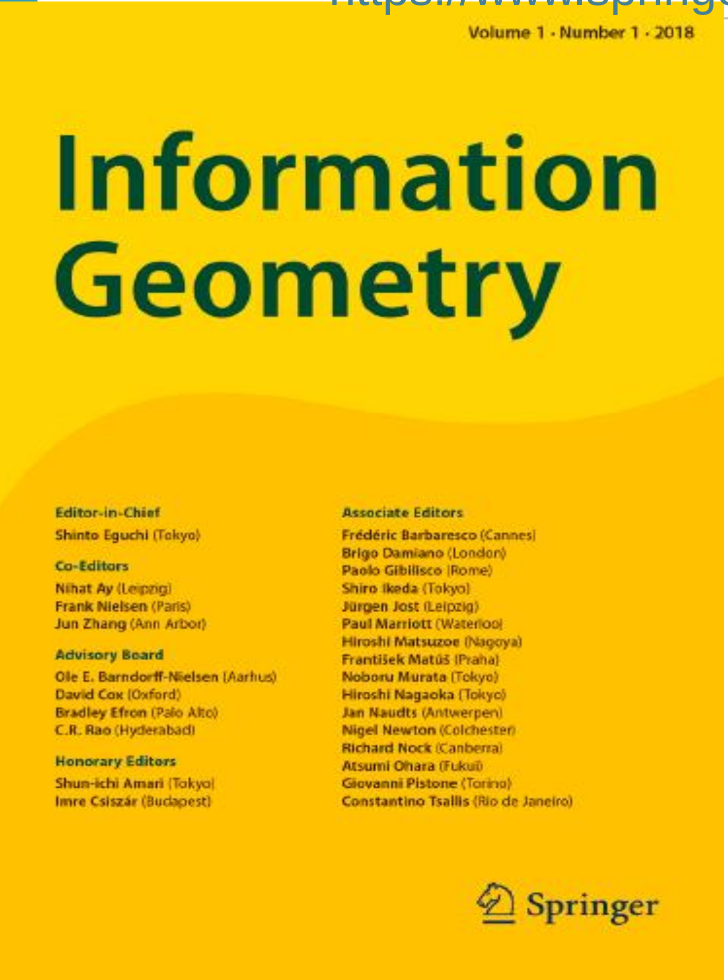
> **Physical structures of inference and learning**

- Stochastic gradient of Langevin's dynamics
- Information geometry, Fisher metric and natural gradient
- Monte-Carlo Hamiltonian methods
- Varational inference and Hamiltonian controls
- Boltzmann machine

THALES

INFORMATION GEOMETRY, SPRINGER JOURNAL

https://www.springer.com/mathematics/geometry/journal/41884



Special Issue:
Affine Differential Geometry and Hesse Geometry:
A Tribute and Memorial to Jean-Louis Koszul
Submission Deadline: 30th November 2019

Jean-Louis Koszul (January 3, 1921 – January 12, 2018) was a French mathematician with prominent influence to a wide range of mathematical fields. He was a second generation member of Bourbaki, with several notions in geometry and algebra named after him. He made a great contribution to the fundamental theory of Differential Geometry, which is foundation of Information Geometry. The special issue is dedicated to Koszul for the mathematics he developed that bear on information sciences.

Both original contributions and review articles are solicited. Topics include but are not limited to:

- Affine differential geometry over statistical manifolds
- Hessian and Kahler geometry
- Divergence geometry
- Convex geometry and analysis
- Differential geometry over homogeneous and symmetric spaces
- Jordan algebras and graded Lie algebras
- Pre-Lie algebras and their cohomology
- Geometric mechanics and Thermodynamics over homogeneous spaces

Guest Editor:

Hideyuki Ishi (Graduate School of Mathematics, Nagoya University)

THALES