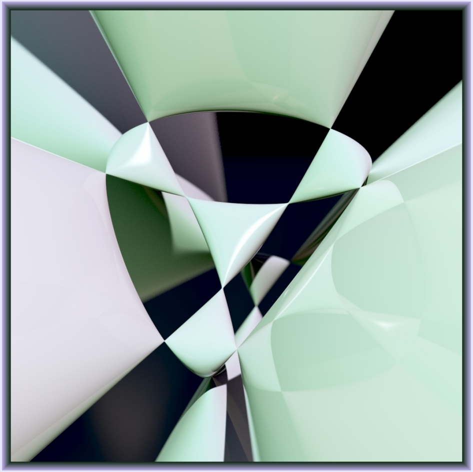
From Classical to Quantum and Back



John Baez

J.-L. Koszul in São Paulo, His Work and Legacy
November 13, 2019

First quantization is a mystery, but second quantization is a functor. — Edward Nelson

First quantization is a mystery, but second quantization is a functor. — Edward Nelson

Naively speaking, geometric quantization seeks to turn classical state spaces — symplectic manifolds — into quantum state spaces.

If we could do this *functorially*, this process would respect symmetries — and thanks to Noether's theorem, observables.

First quantization is a mystery, but second quantization is a functor. — Edward Nelson

Naively speaking, geometric quantization seeks to turn classical state spaces — symplectic manifolds — into quantum state spaces.

If we could do this *functorially*, this process would respect symmetries — and thanks to Noether's theorem, observables.

But in reality, we need to equip a classical state space with a lot of extra structure before we can quantize it.

First quantization is a mystery, but second quantization is a functor. — Edward Nelson

Naively speaking, geometric quantization seeks to turn classical state spaces — symplectic manifolds — into quantum state spaces.

If we could do this *functorially*, this process would respect symmetries — and thanks to Noether's theorem, observables.

But in reality, we need to equip a classical state space with a lot of extra structure before we can quantize it.

Indeed, there's no reason to think a classical description of the world can be systematically quantized. God did not create the world classically on the first day and quantize it on the second!

It’s actually easier to ‘classicize’ a quantum state space—that is, reinterpret it as a classical state space.

It's actually easier to 'classicize' a quantum state space—that is, reinterpret it as a classical state space.

If H is a finite-dimensional Hilbert space, the projective space $P(H)$ is the state space of the associated quantum system. Each point $x \in P(H)$ is a 1d subspace $x \subseteq H$.

In fact $P(H)$ is a symplectic manifold! Given

$$u, v \in T_x P(H) \cong x^\perp$$

we can define

$$\omega(u, v) = \mathrm{Im}\langle u, v \rangle$$

and this gives $P(H)$ a symplectic structure.

Moreover, any quantum observable gives a classical one!

Moreover, any quantum observable gives a classical one!

Any self-adjoint operator $A\colon H \to H$ gives a smooth function $f_A\colon P(H) \to \mathbb{R}$, namely

$$f_A(x) = \langle \psi, A\psi \rangle$$

for any $\psi \in x \subseteq H$ with $\|\psi\| = 1$.

Moreover, any quantum observable gives a classical one!

Any self-adjoint operator $A\colon H \to H$ gives a smooth function $f_A\colon P(H) \to \mathbb{R}$, namely

$$f_A(x) = \langle \psi, A\psi \rangle$$

for any $\psi \in x \subseteq H$ with $\|\psi\| = 1$.

The *classical* time evolution generated by f_A using Hamilton's equations matches the *quantum* time evolution coming from A, because

$$\{f_A, f_B\} = f_{[A,B]/i}$$

where the Poisson brackets come from the symplectic structure.

So, *quantum systems can be seen as specially well-behaved classical systems, whose state spaces are projective spaces.*

This is nicely explained here:

- Abhay Ashtekhar and Troy Schilling, Geometrical formulation of quantum mechanics.

If quantum systems are classical systems with special properties, geometric quantization should consist of *imposing those properties*.

I'll describe a functor called 'projectivization':

$$P\colon \text{Quant} \to \text{Class}$$

Geometric quantization will then be a left adjoint:

$$Q\colon \text{Class} \to \text{Quant}$$

Moreover this will make Quant into a 'reflective subcategory' of Class, meaning

$$Q(P(H)) \cong H \qquad \forall H \in \text{Quant}$$

If you take a finite-dimensional Hilbert space H, projectivize it, then quantize the result, you get H back!

Quantization is thus analogous to abelianizing a group. The forgetful functor

$$U\colon \mathrm{AbGp} \to \mathrm{Gp}$$

has left adjoint

$$Ab\colon \mathrm{Gp} \to \mathrm{AbGp}$$

and

$$Ab(U(A)) \cong A \qquad \forall A \in \mathrm{AbGp}$$

If you take an abelian group A, forget it's abelian, and abelianize it, you get A back!

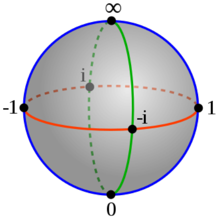
In Kähler quantization, we start with a compact symplectic manifold (M, ω) equipped with a **Kähler structure** on M: a smoothly varying way to make each tangent space $T_x M$ into a complex Hilbert space such that

$$\mathrm{Im}\langle u, v\rangle = \omega(u, v) \qquad \forall u, v \in T_x M.$$

We then choose a holomorphic line bundle $L \to M$. We then choose an inner product on L and a hermitian connection ∇ on L whose curvature is $i\omega$.

Our Hilbert space then consists of holomorphic sections of L.

For example, the phase space of a classical spin-$1/2$ particle is the Riemann sphere:



with the rotationally-invariant symplectic structure ω such that

$$\int_{\mathbb{C}\mathrm{P}^1} \omega = 2\pi$$

There is a holomorphic line bundle $L \to \mathbb{C}\mathrm{P}^1$, the dual of the tautological line bundle, with $L_x = x^*$. This has a hermitian structure coming from the usual inner product on $\mathbb{C}^2$, and a hermitian connection ∇ with curvature $i\omega$.

Holomorphic sections of L are the same as linear functionals on $\mathbb{C}^2$, so when we geometrically quantize we get

$$\Gamma(L) \cong \mathbb{C}^2$$

which is the Hilbert space for the quantum spin-$1/2$ particle.

More generally: suppose G is any complex simple Lie group and ρ is an irreducible representation of G on H. We can choose an inner product on H making ρ unitary for the maximal compact $K \subset G$.

If $v \in H$ is a highest weight vector then

$$M = \{[\rho(g)v] : \ g \in G\} \subseteq P(H)$$

is a Kähler manifold. Pulling back the dual of the tautological line bundle to M we get a holomorphic line bundle $L \to M$ with a connection ∇ having curvature $i\omega$. The space of sections is

$$\Gamma(L) \cong H$$

More generally: suppose G is any complex simple Lie group and ρ is an irreducible representation of G on H. We can choose an inner product on H making ρ unitary for the maximal compact $K \subset G$.

If $v \in H$ is a highest weight vector then

$$M = \{[\rho(g)v] : \ g \in G\} \subseteq P(H)$$

is a Kähler manifold. Pulling back the dual of the tautological line bundle to M we get a holomorphic line bundle $L \to M$ with a connection ∇ having curvature $i\omega$. The space of sections is

$$\Gamma(L) \cong H$$

Points in $M \subseteq P(H)$ are called **coherent states.**

Kähler quantization is like the parable of “stone soup”.

You can make soup just by boiling a large stone in some water! But it tastes better if we flavor it with carrots, leeks, broth, herbs, and some chicken.

Kähler quantization is like the parable of “stone soup”.

You can make soup just by boiling a large stone in some water! But it tastes better if we flavor it with carrots, leeks, broth, herbs, and some chicken. In fact, the stone is optional!

We are trying to make a Hilbert space starting with just a symplectic manifold. But to do this, we equip with the symplectic manifold with structures that make it look more and more like the projective space of a Hilbert space... or a subvariety of this.

If H is a finite-dimensional Hilbert space, $P(H)$ has a lot of structure:

- It is has a Kähler structure: we can define $\langle u, v \rangle$ where $u, v \in T_x P(H) \subseteq x^\perp$.
- It is thus symplectic with $\omega(u, v) = \mathrm{Im}\langle u, v \rangle$.
- It comes with a holomorphic line bundle $L \to P(H)$, where $L_x = x^*$.
- L has an inner product: we can also define $\langle u, v \rangle$ for $u, v \in L_x = x^*$.
- L has a hermitian connection ∇ with curvature $i\omega$.

If H is a finite-dimensional Hilbert space, $P(H)$ has a lot of structure:

- It is has a Kähler structure: we can define $\langle u, v \rangle$ where $u, v \in T_x P(H) \subseteq x^\perp$.
- It is thus symplectic with $\omega(u, v) = \mathrm{Im}\langle u, v \rangle$.
- It comes with a holomorphic line bundle $L \to P(H)$, where $L_x = x^*$.
- L has an inner product: we can also define $\langle u, v \rangle$ for $u, v \in L_x = x^*$.
- L has a hermitian connection ∇ with curvature $i\omega$.

All these structures are precisely what we put on a symplectic manifold to apply Kähler quantization!

This leads to the following first try:

Definition. Let Class_n be the category where an object is a linearly normal projective variety $M \subseteq P(\mathbb{C}^n)$ and a morphism $M \to M'$ is an inclusion $M \subseteq M'$.

Definition. Let Quant_n be the category where an object is a linear subspace $H \subseteq \mathbb{C}^n$ and a morphism $H \to H'$ is an inclusion $H \subseteq H'$.

Remember a projective variety $M \subseteq P(\mathbb{C}^n)$ is **linearly normal** if the restriction map

$$\Gamma(L) \to \Gamma(L|_M)$$

is surjective, where $L \to P(\mathbb{C}^n)$ is the bundle with $L_x = x^*$. This makes M easy to geometrically quantize, since then

$$\Gamma(L|_M) \cong H$$

where $H \subseteq \mathbb{C}^n$ is the smallest linear subspace such that $P(H)$ contains M.

Definition. Let $Q\colon \mathrm{Class}_n \to \mathrm{Quant}_n$ be the functor sending any linearly normal $M \subseteq P(\mathbb{C}^n)$ to $H \subseteq \mathbb{C}^n$, the smallest subspace such that $M \subseteq P(H)$.

Definition. Let $P\colon \mathrm{Quant}_n \to \mathrm{Class}_n$ be the functor sending any linear subspace $H \subseteq \mathbb{C}^n$ to its projectivization $P(H) \subseteq P(\mathbb{C}^n)$.

Then P and Q are adjoint functors between posets, also called a **Galois connection**:

$$Q(M) \subseteq H \quad \iff \quad M \subseteq P(H)$$

for all $M \subseteq P(\mathbb{C}^n)$, $H \subseteq \mathbb{C}^n$.

Then P and Q are adjoint functors between posets, also called a **Galois connection**:

$$Q(M) \subseteq H \quad \Longleftrightarrow \quad M \subseteq P(H)$$

for all $M \subseteq P(\mathbb{C}^n)$, $H \subseteq \mathbb{C}^n$.

Taking $H = Q(M)$ we get

$$M \subseteq P(Q(M))$$

Thus every classical state $x \in M$ gives a quantum state $x \in P(Q(M))$. Such a quantum state is called a **coherent state**: the *"best quantum approximation to a classical state"*.

Then P and Q are adjoint functors between posets, also called a **Galois connection**:

$$Q(M) \subseteq H \quad \iff \quad M \subseteq P(H)$$

for all $M \subseteq P(\mathbb{C}^n)$, $H \subseteq \mathbb{C}^n$.

Taking $H = Q(M)$ we get

$$M \subseteq P(Q(M))$$

Thus every classical state $x \in M$ gives a quantum state $x \in P(Q(M))$. Such a quantum state is called a **coherent state**: the *"best quantum approximation to a classical state"*.

Moreover we have

$$Q(P(H)) \cong H$$

for all $H \subseteq \mathbb{C}^n$. If you take a finite-dimensional Hilbert space H, projectivize it, then quantize the result, you get H back.

Three problems:

- We're studying the geometry of classical state spaces *extrinsically*, treating them as projective subvarieties of $P(\mathbb{C}^n)$. The "intrinsic" approach to geometry is far more fashionable.

Three problems:

- We're studying the geometry of classical state spaces *extrinsically*, treating them as projective subvarieties of $P(\mathbb{C}^n)$. The "intrinsic" approach to geometry is far more fashionable.
- We're fixing n.

Three problems:

- We're studying the geometry of classical state spaces *extrinsically*, treating them as projective subvarieties of $P(\mathbb{C}^n)$. The "intrinsic" approach to geometry is far more fashionable.
- We're fixing n.
- Our only morphisms between classical systems, or quantum systems, are *inclusions*. Thus, we cannot describe symmetries as morphisms: our categories are mere posets.

The last two problems are more important, so let's fix those.

Definition. Let Class be the category where:

- An object (M, V) is a finite-dimensional complex vector space V and a projective variety $M \subseteq P(V)$ such that no proper subspace $W \subset V$ has $M \subseteq P(W)$.
- A morphism $f\colon (M, V) \to (M', V')$ is an injective linear map $f\colon V \to V'$ such that $P(f)\colon P(V) \to P(V')$ maps M into M'.

Definition. Let Class be the category where:

- An object (M, V) is a finite-dimensional complex vector space V and a projective variety $M \subseteq P(V)$ such that no proper subspace $W \subset V$ has $M \subseteq P(W)$.
- A morphism $f\colon (M, V) \to (M', V')$ is an injective linear map $f\colon V \to V'$ such that $P(f)\colon P(V) \to P(V')$ maps M into M'.

Note: we've dropped the condition that M be linearly normal! You can include it if you want.

Definition. Let Class be the category where:

- An object (M, V) is a finite-dimensional complex vector space V and a projective variety $M \subseteq P(V)$ such that no proper subspace $W \subset V$ has $M \subseteq P(W)$.
- A morphism $f\colon (M, V) \to (M', V')$ is an injective linear map $f\colon V \to V'$ such that $P(f)\colon P(V) \to P(V')$ maps M into M'.

Note: we've dropped the condition that M be linearly normal! You can include it if you want.

Definition. Let Quant be the category where:

- An object V is a finite-dimensional complex vector space
- A morphism $f\colon V \to V'$ is an injective linear map.

Definition. Let $Q\colon \text{Class} \to \text{Quant}$ be the functor sending any object (M, V) to V, and any morphism $f\colon (M, V) \to (M', V')$ to its underlying injective linear map $f\colon V \to V'$.

Definition. Let $P\colon \text{Quant} \to \text{Class}$ be the functor sending any vector space V to $(P(V), V)$, and any injective linear map $f\colon V \to V'$ to $f\colon (P(V), V) \to (P(V'), V')$.

Theorem. Q is left adjoint to P, and Quant is a reflective subcategory of Class: $P \circ Q \cong 1_{\text{Class}}$.

Definition. Let $Q\colon \mathrm{Class} \to \mathrm{Quant}$ be the functor sending any object (M, V) to V, and any morphism $f\colon (M, V) \to (M', V')$ to its underlying injective linear map $f\colon V \to V'$.

Definition. Let $P\colon \mathrm{Quant} \to \mathrm{Class}$ be the functor sending any vector space V to $(P(V), V)$, and any injective linear map $f\colon V \to V'$ to $f\colon (P(V), V) \to (P(V'), V')$.

Theorem. Q is left adjoint to P, and Quant is a reflective subcategory of Class: $P \circ Q \cong 1_{\mathrm{Class}}$.

Note: we've removed the Hilbert space structure from our 'quantum systems': they are mere vector spaces. We've also removed the Kähler and even symplectic structure from our 'classical systems': they are mere projective varieties.

We have made stone soup without the stone!

But we can easily reinstate these extra structures: taking V to have an inner product makes $M \subseteq P(V)$ Kähler.